Iterated Monodromy Groups
Lecture 2

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Bath

Let $\Phi : G \longrightarrow S_d \wr G$ be a wreath recursion. Denote by K_Φ the kernel of the associated action on the tree.
If $g \notin K_\Phi$, then there exists a finite word $v \in \mathsf{X}^*$ moved by g. Hence there exists an algorithm, which stops if and only if g is not trivial in the self-similar group defined by Φ.

Let E_1 be the kernel of Φ. Denote

$$E_{n+1} = \Phi^{-1}(\{1\} \cdot E_n^d),$$

and $E_\infty = \bigcup_{n\geq 1} E_n$.
If $E_\infty = K_\Phi$ and the word problem is solvable in G, then the word problem is solvable in G/K_Φ.

Sections

Fix a wreath recursion $\Phi : G \longrightarrow S_d \wr G$. Define $g|_v$ for $g \in G$ and $v \in \mathsf{X}^*$ inductively by $g|_\varnothing = g$ and

$$\Phi(g|_v) = \pi(g|_{v1}, g|_{v2}, \ldots, g|_{vd}).$$

We have

$$g(vw) = g(v)g|_v(w)$$

for all $v, w \in \mathsf{X}^*$.
For all $g, h \in G$ and $v, u \in \mathsf{X}^*$ we have

$$(gh)|_v = g|_{h(v)}h|_v, \quad g|_{vu} = g|_v|_u.$$

Virtual endomorphisms

Let $\Phi : G \longrightarrow S_d \wr G$ be a wreath recursion. Suppose that the projection of $\Phi(G)$ onto S_d is transitive.
The *associated virtual endomorphism* of G is the map $\phi : g \mapsto g|_x$ from the stabilizer of $x \in \mathsf{X}$ to G.

The virtual endomorphism uniquely determines the wreath recursion (up to an equivalence). If $\{r_1, r_2, \ldots, r_d\}$ is a coset representative system for $\mathrm{Dom}\,\phi < G$, then we define

$$\Phi_1(g) = \pi(g_1, \ldots, g_d),$$

where $\pi(i) = j$ iff $gr_i\,\mathrm{Dom}\,\phi = r_j\,\mathrm{Dom}\,\phi$; and $g_i = \phi(r_j^{-1} g r_i)$. Then Φ_1 is equivalent to Φ.

The virtual endomorphism of $\mathbb{Z}$ associated with the wreath recursion

$$\Phi(\gamma) = (\mathit{01})(1, \gamma)$$

associated with $\mathrm{IMG}\left(z^2\right)$ is

$$\phi(\gamma^2) = \gamma,$$

i.e.,

$$n \mapsto n/2$$

on $\mathbb{Z}$.

The virtual endomorphism associated with $\mathrm{IMG}\left(-z^3/2 + 3z/2\right)$, i.e., with the wreath recursion

$$\Phi(a) = (\mathit{01})(1, a, 1), \quad \Phi(b) = (\mathit{02})(1, 1, b)$$

is

$$a^2 \mapsto a, \quad b^{-1}ab \mapsto 1$$
$$b^2 \mapsto b, \quad a^{-1}ba \mapsto 1.$$

If $\phi : G \dashrightarrow G$ is the virtual endomorphism associated with a wreath recursion Φ, then the kernel K_Φ of the self-similar action is

$$K_\Phi = \bigcap_{g \in G, n \geq 1} g^{-1} \cdot \mathrm{Dom}\,\phi^n \cdot g.$$

If $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a partial self-covering, then $\pi_1(\mathcal{M}_1)$ is a subgroup of finite index in $\pi_1(\mathcal{M})$ and the virtual endomorphism associated with the iterated monodromy action is the map $\pi_1(\mathcal{M}_1) \longrightarrow \pi_1(\mathcal{M})$ induced by the inclusion $\mathcal{M}_1 \hookrightarrow \mathcal{M}$.

Contracting groups

Definition

A wreath recursion over G is *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ such that

$$g|_v \in \mathcal{N}$$

for all words v of length at least n.

Theorem

Let $\phi : G \dashrightarrow G$ be a virtual endomorphism of a finitely generated group. The number

$$\rho = \limsup_{n\to\infty} \sqrt[n]{\limsup_{g\in \mathrm{Dom}\,\phi^n, l(g)\to\infty} \frac{l(\phi^n(g))}{l(g)}}$$

does not depend on the choice of the generating set and is less than one iff the associated wreath recursion is contracting.

Hyperbolic dynamical systems

Theorem

If the partial self-covering $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is expanding, then $\mathrm{IMG}\,(p)$ is a contracting self-similar group.

In particular, the iterated monodromy groups of post-critically finite rational functions are contracting.

Algebraic properties of contracting groups

Theorem

The word problem in a contracting self-similar group is solvable in polynomial time.
If ρ is the contraction coefficient of the associated virtual endomorphism, then for every $\epsilon > 0$ there is an algorithm solving the word problem in degree $\frac{\log(|X|)}{-\log \rho} + \epsilon$ time.

Theorem

Contracting groups have no free subgroups.

It is a corollary of

Theorem

Let G be a group acting faithfully on a locally finite rooted tree T. Then one of the following is true

1. *G has no free subgroups;*
2. *there is a free subgroup $F \leq G$ and a point $\xi \in \partial T$ such that the stabilizer F_ξ is trivial;*
3. *there is a point $\xi \in \partial T$ and a free subgroup $F \leq G_\xi$ such that F acts faithfully on all neighborhoods of ξ.*

Open questions

- Is the conjugacy problem solvable in contracting groups? Most of the other algorithmic problems are open.
- Are contracting groups amenable?
- Which contracting groups are finitely presented? (Only if they are virtually nilpotent?)
- Given a wreath recursion decide if it defines a contracting group.

The following is a corollary of a more general result of L. Bartholdi, V. Kaimanovich and V. N.

Theorem

If f is a post-critically finite polynomial, then $\mathrm{IMG}(f)$ is amenable.

The first non-trivial partial case of this theorem ($\mathrm{IMG}(z^2 - 1)$) was shown by L. Bartholdi and B. Virag.

IMG of correspondences

There is no reason to restrict to the case of partial self-covering. A *topological correspondence* is a pair of maps $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$ and $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$, where p is a finite covering and ι is a continuous map.

The *associated virtual endomorphism* of $\pi_1(\mathcal{M})$ is the homomorphism $\iota_* : \pi_1(\mathcal{M}_1) \longrightarrow \pi_1(\mathcal{M})$, where $\pi_1(\mathcal{M}_1)$ is identified with a subgroup of $\pi_1(\mathcal{M})$ by the isomorphism p_*.

The virtual endomorphism ι_* defines a self-similar group, which is the *iterated monodromy group* of the correspondence.

Dual Moore diagrams

Every virtual endomorphism ϕ of the free group can be realized as ι_* for a map $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$, where $\mathcal{M}$ is a bouquet of circles, $\mathcal{M}_1$ is a finite covering graph of $\mathcal{M}$ defining the domain of ϕ.

Consequently, every self-similar group is an iterated monodromy group of a correspondence on graphs.

Arithmetic-geometric mean of Gauss

Consider the multivalued map on $\mathbb{PC}^1$:

$$[z_1 : z_2] \mapsto \left[\frac{z_1 + z_2}{2} : \sqrt{z_1 z_2}\right]$$

or, in non-homogeneous coordinates:

$$z \mapsto \frac{1+z}{2\sqrt{z}}$$

Gauss used iterations of this map (on the positive real axis) to compute the integrals

$$\int_0^{\pi/2} \frac{dt}{\sqrt{a^2 \cos^2 t + b^2 \sin^2 t}}$$

Arithmetic-geometric mean of Gauss

Uniformizing the square root, we get a correspondence

$$f : \mathbb{C} \setminus \{0, \pm 1\} \longrightarrow \mathbb{C} \setminus \{0, 1\} : z \mapsto \frac{(1+z)^2}{4z}$$

$$\iota : \mathbb{C} \setminus \{0, \pm 1\} \longrightarrow \mathbb{C} \setminus \{0, 1\} : z \mapsto z^2.$$

$\pi_1(\mathbb{C} \setminus \{0, 1\})$ is naturally identified with the free group $\langle \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \rangle$.
Then the associated virtual endomorphism is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}.$$

The action of the free group is faithful and is given by

$$\alpha = \sigma(1, \alpha), \quad \beta = (\beta^2, (\beta^{-1}\alpha)^2).$$

Lattices in Lie groups

Theorem (M. Kapovich)

Let Γ be an irreducible lattice in a semisimple algebraic Lie group G. Then the following are equivalent:

1. Γ *is virtually isomorphic to an arithmetic lattice in G, i.e., contains a finite index subgroup isomorphic to such arithmetic lattice.*
2. Γ *admits a faithful self-similar action, which is transitive on the first level.*

All such lattices are IMGs of correspondences on the corresponding symmetric spaces.