Iterated Monodromy Groups
Lecture 3

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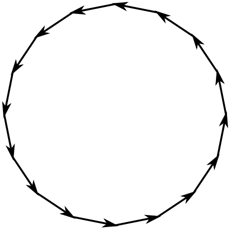
Bath

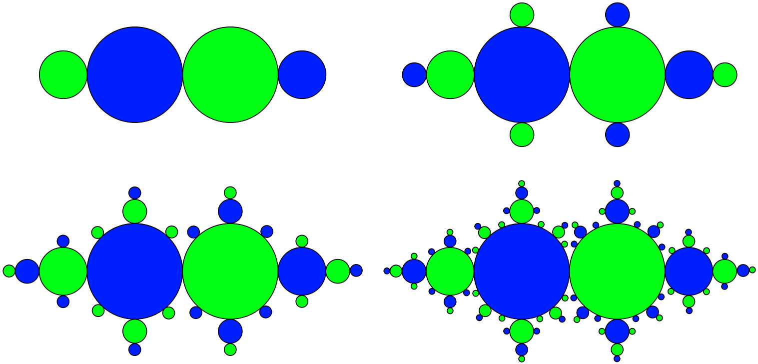
Schreier graphs of iterated monodromy groups

Let us draw the graphs of the action of the generators on the levels X^n of the tree. For example, for the adding machine:



The map $vx \mapsto v : \mathsf{X}^{n+1} \longrightarrow \mathsf{X}^n$ is a covering of the corresponding graphs. We get double self-coverings of the circle.

The Schreier graphs of $\mathrm{IMG}\left(-\frac{z^3}{2}+\frac{3z}{2}\right)$



Wir gehen von zwei gleichseitigen Dreiecken $\triangle A_1 A_2 A_3$ und $\triangle A_1 A_4 A_5$ mit der Seite a aus, die an der Ecke A_1 aneinanderstoßen (Fig. 2). Sie bilden zusammen den geschlossenen polygonalen Zug $p_1 = A_1 A_2 A_3 A_4 A_5$, der die Ebene in 3 Bereiche teilt:

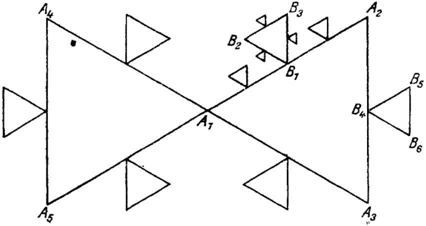


Fig. 2.

1. Das Innere von $\triangle A_1 A_2 A_3$: $\mathfrak{B}_1$.

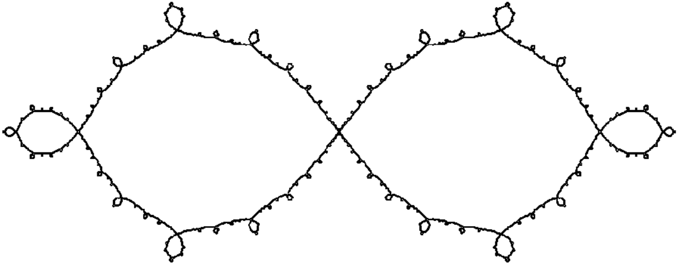
2. Das Innere von $\triangle A_1 A_4 A_5$: $\mathfrak{B}_1'$.

3. Den Bereich $\mathfrak{B}_1''$, der den unendlich fernen Punkt enthält und vom ganzen polygonalen Zug p_1 begrenzt wird.

In die Mitte jeder der Seiten von p_1 setzen wir die Spitze eines

The original picture appears in a paper of Gaston Julia in 1918.

The Julia set of $-\frac{z^3}{2} + \frac{3z}{2}$



It looks like the Schreier graphs converge to some limit.

Definition

Let G be a contracting self-similar group acting on X^*. Let $\mathsf{X}^{-\omega}$ be the space of left-infinite sequences $\ldots x_2x_1$, $x_i \in \mathsf{X}$. Two sequences $\ldots x_2x_1, \ldots y_2y_1 \in \mathsf{X}^{-\omega}$ are equivalent if there exists a bounded sequence $g_k \in G$ such that

$$g_k(x_k \ldots x_1) = y_k \ldots y_1$$

for every k. The quotient of $\mathsf{X}^{-\omega}$ by the equivalence relation is the *limit space* $\mathcal{J}_G$.

The shift $\ldots x_2x_1 \mapsto \ldots x_3x_2$ agrees with the equivalence relation, hence it defines a continuous map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$.

Topological models of virtual endomorphisms

Let $\phi : G \dashrightarrow G$ be a surjective contracting virtual endomorphism. A *topological model* of ϕ is a metric space $\mathcal{X}$ with a right proper co-compact G-action by isometries and a contracting map $F : \mathcal{X} \longrightarrow \mathcal{X}$ such that

$$F(\xi \cdot g) = F(\xi) \cdot \phi(g)$$

for all $\xi \in \mathcal{X}$ and $g \in \mathrm{Dom}\,\phi$.

Note that F induces a well-defined map $F_0 : \mathcal{X}/\ker\phi \longrightarrow \mathcal{X}$, since $F(\xi \cdot g) = F(\xi)$ for $g \in \ker\phi$.

More generally, it induces a map $F_n : \mathcal{X}/\ker\phi^{n+1} \longrightarrow \mathcal{X}/\ker\phi^n$.

$\mathcal{X}/\ker\phi^n$ is a G-space, for $(\xi \ker\phi^n) \cdot g = (\xi \cdot h) \ker\phi^n$, where $\phi^n(h) = g$.

Theorem

Let $F : \mathcal{X} \longrightarrow \mathcal{X}$ be a topological model of a virtual endomorphism $\phi : G \dashrightarrow G$. Then the inverse limit $\mathcal{X}_G$ of the G-spaces

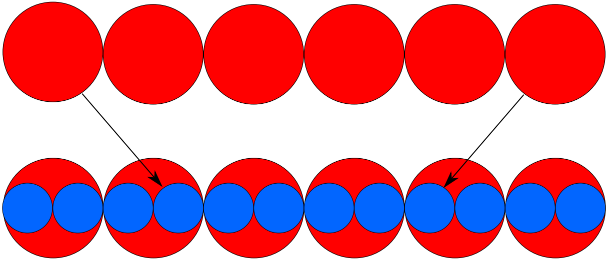
$$\ldots \xrightarrow{F_3} \mathcal{X}/\ker\phi^3 \xrightarrow{F_2} \mathcal{X}/\ker\phi^2 \xrightarrow{F_1} \mathcal{X}/\ker\phi \xrightarrow{F_0} \mathcal{X}$$

depends only on G and ϕ. The space of orbits $\mathcal{X}_G/G$ is homeomorphic to $\mathcal{J}_G$.

Corollary

The limit space of $\mathrm{IMG}(f)$ for a post-critically finite rational function f is homeomorphic to the Julia set of f.

Example: a model of $n \mapsto n/2$



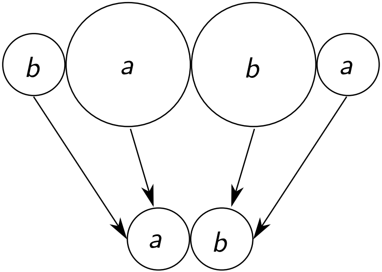
It follows that $\mathcal{X}_{\mathbb{Z}}$ in this case is the line $\mathbb{R}$ and the limit space $\mathcal{J}_{\mathbb{Z}}$ is the circle $\mathbb{R}/\mathbb{Z}$.

A model of the Julia set of $-\frac{z^3}{2} + \frac{3z}{2}$

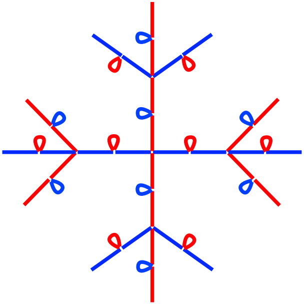
The virtual endomorphism associated with $\mathrm{IMG}\left(-\frac{z^3}{2} + \frac{3z}{2}\right)$ is

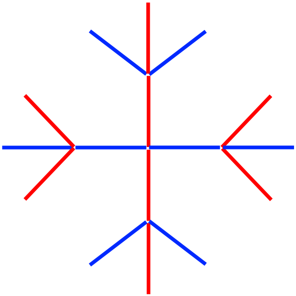
$$a^2 \mapsto a, \quad b^2 \mapsto b, \quad a^b \mapsto 1, \quad b^a \mapsto 1.$$

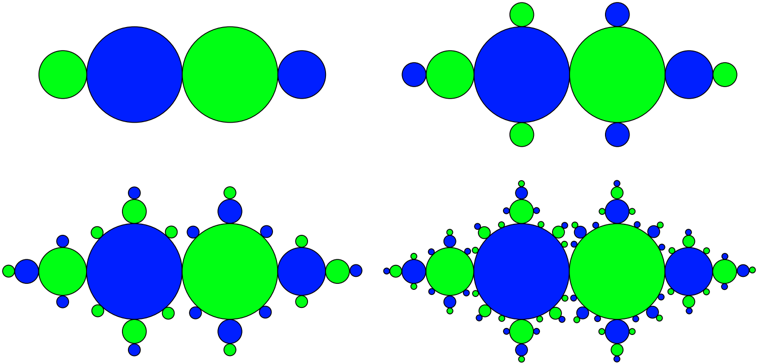
Take $\mathcal{X}$ to be the Cayley graph of the free group $\langle a, b\rangle$. Let $F : \mathcal{X} \longrightarrow \mathcal{X}$ be the lift to the universal covering of the map



It follows that the graph $\mathcal{X}/\ker\phi^{n+1}$ is obtained from the graph $\mathcal{X}/\ker\phi^{n}$ by adding a loop in the middle of every edge.







Equivalently, define a model of $\phi : G \dashrightarrow G$ as a correspondence $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ of compact spaces such that ι is contracting (with resp. to metrics making f a local isometry) and the virtual endomorphism ι_* of $\pi_1(\mathcal{M})$ is conjugate to ϕ.

Define a covering $f_n : \mathcal{M}_n \longrightarrow \mathcal{M}_{n-1}$ and a map $\iota_n : \mathcal{M}_n \longrightarrow \mathcal{M}_{n-1}$ by the pull-back diagram

$$\begin{array}{ccc} \mathcal{M}_{n+1} & \xrightarrow{\iota_{n+1}} & \mathcal{M}_n \\ \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} \\ \mathcal{M}_n & \xrightarrow{\iota_n} & \mathcal{M}_{n-1} \end{array}$$

Then $\mathcal{J}_G$ is the inverse limit of $\mathcal{M}_n$ w.r.t. to ι_n.

Examples: $\mathbb{Z}^n$

Let A be an $n \times n$-matrix with integer entries and let $\det A = d > 1$. Then the linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ induces a d-fold self-covering of the torus $\mathbb{R}^n/\mathbb{Z}^n$.

The associated virtual endomorphism is $A^{-1} : \mathbb{Z}^n \dashrightarrow \mathbb{Z}^n$ with the domain $A(\mathbb{Z}^n)$.

Choosing a coset representative system (equivalently a collection of connecting paths) we get the associated action on X^*, which corresponds to a *numeration system* on $\mathbb{Z}^n$.

Let $r_1, \dots, r_d$ be a coset-representative system of $\mathbb{Z}^n/A(\mathbb{Z}^n)$. Then every element $x \in \mathbb{Z}^n$ is uniquely written as a formal sum

$$x = r_{i_0} + A(r_{i_1}) + A^2(r_{i_2}) + \cdots ,$$

where r_{i_k} is defined by the condition

$$x - \left(r_{i_0} + A(r_{i_1}) + \cdots + A^k(r_{i_k})\right) \in A^{k+1}(\mathbb{Z}^n).$$

The associated action of $\mathbb{Z}^n = \pi_1(\mathbb{R}^n/\mathbb{Z}^n)$ describes addition of the elements of $\mathbb{Z}^n$ to such formal series.

Theorem (S. Sidki, V. N.)

If no eigenvalue of A^{-1} is an algebraic integer, then $\mathrm{IMG}\,(A) = \pi_1(\mathbb{R}^n/\mathbb{Z}^n) = \mathbb{Z}^n$.

Proposition

The IMG of $A : \mathbb{R}^n/\mathbb{Z}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n$ is contracting if and only if A is expanding (i.e., all eigenvalues > 1). In this case the limit space of the IMG is the torus $\mathbb{R}^n/\mathbb{Z}^n$.

The points of $\mathbb{R}^n/\mathbb{Z}^n$ are encoded by the sequences from $\mathsf{X}^{-\omega}$ according to "base A" numeration system:

$$\xi = A^{-1}(r_{i_1}) + A^{-2}(r_{i_2}) + A^{-3}(r_{i_3}) + \cdots$$

Twin Dragon

Take A to be multiplication by $(1+i)$ on $\mathbb{C}$.
We get a degree 2 self-covering of the torus $\mathbb{C}/\mathbb{Z}[i]$.

Take the coset representatives 0 and 1 of $\mathbb{Z}[i]$ by $(1+i)\mathbb{Z}[i]$. We get a "binary numeration system" on the Gaussian integers.

The torus $\mathbb{C}/\mathbb{Z}[i]$ is the limit space of the associated IMG. The images of the *cylindrical sets* $\mathsf{X}^{-\omega}v$, $v \in \mathsf{X}^n$ tile the torus by "twin dragons".

Twin Dragon tiling