Self-similar and branch groups II

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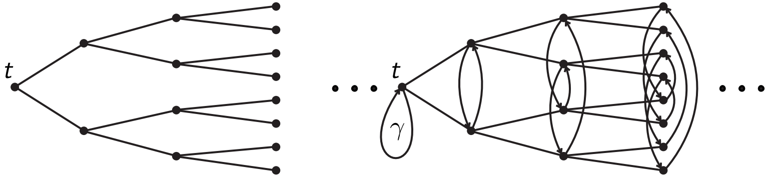
October 5, 2010,
Montreal

Iterated monodromy groups

Let $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be a finite degree covering map of a space by its subset. Iterate it as a partial map: $f^n : \mathcal{M}_n \longrightarrow \mathcal{M}$, and consider the *tree of preimages* T



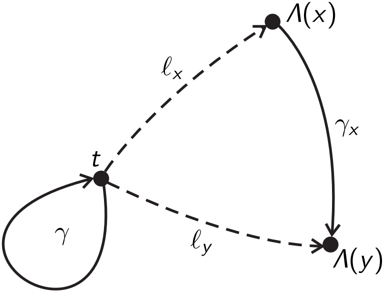
The fundamental group $\pi_1(\mathcal{M}, t)$ acts on it by automorphisms. The obtained group $\mathrm{IMG}(f) < \mathrm{Aut}(T)$ is the *iterated monodromy group* $\mathrm{IMG}(f)$.

Recurrent formula

Find a bijection $\Lambda : \mathsf{X} \longrightarrow f^{-1}(t)$ and a collection of paths ℓ_x from t to $\Lambda(x)$. Define $\Lambda : \mathsf{X}^* \longrightarrow \bigsqcup f^{-n}(t)$ by the rule

$$\Lambda(xv) \text{ is the end of the } f^{|v|}\text{-lift of } \ell_x \text{ starting at } \Lambda(v)$$

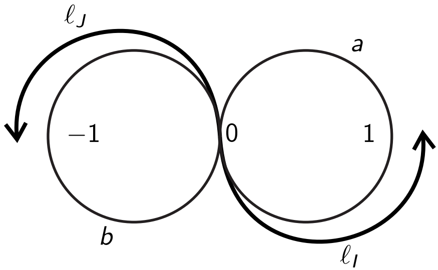
Then Λ is an isomorphism conjugating $\mathrm{IMG}(f)$ with a self-similar group. The recursive definition of the self-similar group:



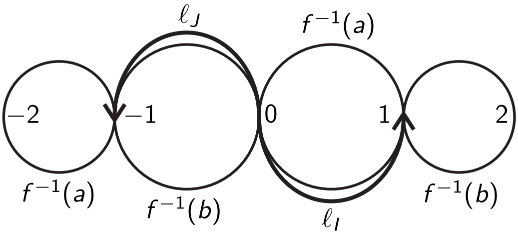
$$\gamma(xw) = y(\ell_x \gamma_x \ell_y^{-1})(w).$$

Examples

1. I.m.g. of an orientation-preserving double self-covering of the circle is the adding machine action of $\mathbb{Z}$.
2. The "interlaced adding machine" is the i.m.g. of $f(z) = -\frac{z^3}{2} + \frac{3z}{2}$ seen as the map $\mathbb{C} \setminus \{\pm 1, \pm 2\} \longrightarrow \mathbb{C} \setminus \{\pm 1\}$.



$\mathrm{IMG}(-z^3/2+3z/2)$ continued



$$a(Ow) = Iw, \quad a(Iw) = Oa(w), \quad a(Jw) = Jw$$

$$b(Ow) = Jw, \quad b(Iw) = Iw, \quad b(Jw) = Ob(w).$$

A multi-dimensional example

Consider the map F of $\mathbb{C}^2$:

$$(x, y) \mapsto \left(1 - \frac{y^2}{x^2}, 1 - \frac{1}{x^2}\right)$$

It can be naturally extended to the projective plane.

$$(x : y : z) \mapsto (x^2 - y^2 : x^2 - z^2 : x^2).$$

The set $\{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$ is the critical locus. The post-critical set is the union of the line at infinity with the lines $x = 0, x = 1, y = 0, y = 1, x = y$.
They are permuted as follows:

$$\{x = 0\} \mapsto \{z = 0\} \mapsto \{y = 1\} \mapsto \{x = y\} \mapsto \{x = 0\}$$

$$\{y = 0\} \mapsto \{x = 1\} \mapsto \{y = 0\}.$$

The iterated monodromy group of F (as computed by J. Belk and S. Koch) is generated by the transformations:

$$a(1v) = 1b(v), \quad a(2v) = 2v, \quad a(3v) = 3v, \quad a(4v) = 4b(v),$$

$$b(1v) = 1c(v), \quad b(2v) = 2c(v), \quad b(3v) = 3v, \quad b(4v) = 4v,$$

$$c(1v) = 4d(v), c(2v) = 3(ceb)^{-1}(v), c(3v) = 2(fa)^{-1}(v), c(4v) = 1v,$$

$$d(1v) = 2v, \quad d(2v) = 1a(v), \quad d(3v) = 4v, \quad d(4v) = 3a(v),$$

$$e(1v) = 1f(v), \quad e(2v) = 2v, \quad e(3v) = 3f(v), \quad e(4v) = 4v,$$

$$f(1v) = 3b^{-1}(v), \quad f(2v) = 4v, \quad f(3v) = 1eb(v), \quad f(4v) = 2e(v).$$

Schreier graphs of $\mathrm{IMG}(f)$

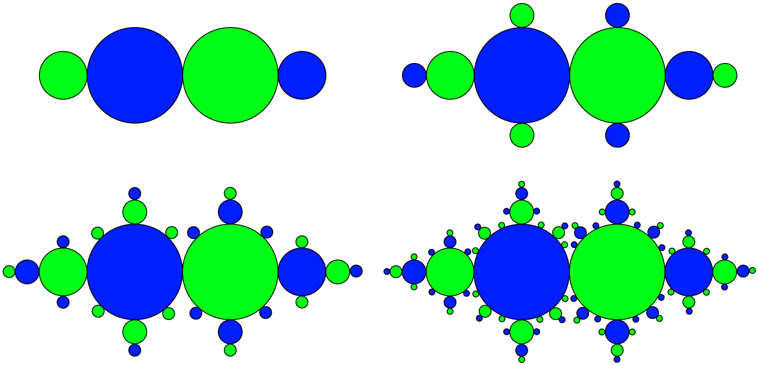
Let S be the generating set of $\pi_1(\mathcal{M}, t)$ as a graph in $\mathcal{M}$. Then $f^{-n}(S)$ is the Schreier graph $\Gamma_n(\mathrm{IMG}(f), S)$.

The natural covering map $\Gamma_{n+1}(\mathrm{IMG}(f), S) \longrightarrow \Gamma_n(\mathrm{IMG}(f), S)$ is the map $f : f^{-(n+1)}(S) \longrightarrow f^{-n}(S)$.

The map $xv \mapsto v$ corresponds to the map from the end to the beginning of lifts of $f^{-n}(\ell_x)$.

If f is expanding, then lengths of the edges of $f^{-n}(S)$ and of $f^{-n}(\ell_x)$ exponentially decrease. The sets $f^{-n}(S)$ converge to the *Julia set* of f.

The Schreier graphs of $\mathrm{IMG}(-\frac{z^3}{2} + \frac{3z}{2})$



Wir gehen von zwei gleichseitigen Dreiecken $\triangle A_1 A_2 A_3$ und $\triangle A_1 A_4 A_5$ mit der Seite a aus, die an der Ecke A_1 aneinanderstoßen (Fig. 2). Sie bilden zusammen den geschlossenen polygonalen Zug $p_1 = A_1 A_2 A_3 A_4 A_5$, der die Ebene in 3 Bereiche teilt:

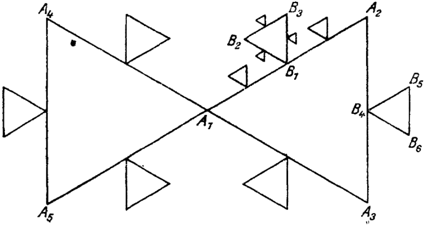


Fig. 2.

1. Das Innere von $\triangle A_1 A_2 A_3$: $\mathfrak{B}_1$.

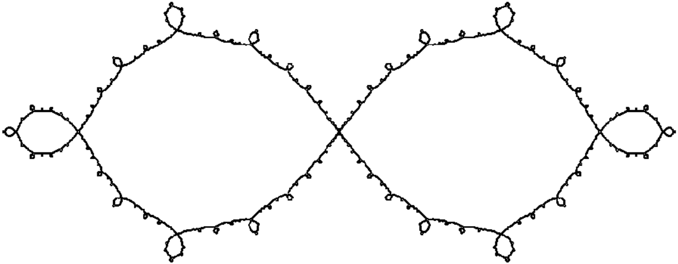
2. Das Innere von $\triangle A_1 A_4 A_5$: $\mathfrak{B}_1'$.

3. Den Bereich $\mathfrak{B}_1''$, der den unendlich fernen Punkt enthält und vom ganzen polygonalen Zug p_1 begrenzt wird.

In die Mitte jeder der Seiten von p_1 setzen wir die Spitze eines

The original picture appears in a paper of Gaston Julia in 1918.

The Julia set of $-\frac{z^3}{2} + \frac{3z}{2}$



Contracting self-similar groups

Definition

A self-similar group G is *contracting* if there exists a finite subset $\mathcal{N} \subset G$ such that for every $g \in G$ there exists n such that $g|_v \in \mathcal{N}$ for all $v \in X^*$ of length $\geq n$.

The smallest set $\mathcal{N}$ is called the *nucleus* of G.
For the adding machine action we have

$$a^n|_0 = a^{\lfloor n/2 \rfloor}, \qquad a^n|_1 = a^{\lfloor (n+1)/2 \rfloor},$$

hence it is contracting (with nucleus $\{0, \pm 1\}$).
If $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is expanding, then the length of $\gamma|_x = \ell_x \gamma_x \ell_y^{-1}$ is $\lambda \cdot \text{length}(\gamma) + C$ for $0 < \lambda < 1$ and $C \geq 0$. It follows that $\mathrm{IMG}(f)$ is contracting.

Theorem

Contracting groups have no free subgroups.

Theorem (L. Bartholdi, V. Kaimanovich, V.N.)

Iterated monodromy groups of post-critically finite complex polynomials are amenable.

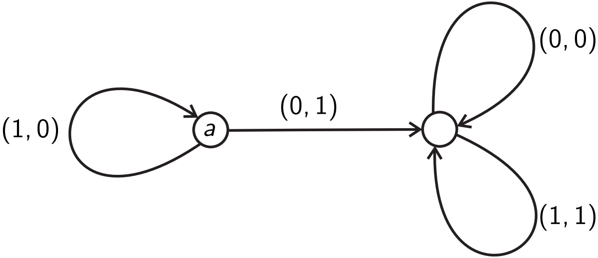
It is an open question if all contracting groups are amenable.

Limit spaces

Definition

Let G be a contracting group. Consider the space $\mathsf{X}^{-\omega}$ of left-infinite sequences. Sequences $\ldots x_2 x_1, \ldots y_2 y_1$ are G-equivalent if there is a finite set $A \subset G$ and a sequence $g_n \in A$ such that $g_n(x_n \ldots x_1) = y_n \ldots y_1$. The quotient of $\mathsf{X}^{-\omega}$ by this equivalence relation is the *limit space* of G.

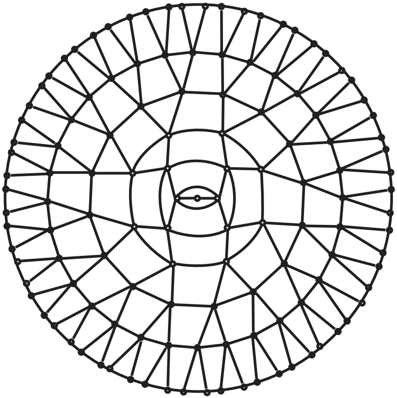
The equivalence relation is generated by pairs $(\ldots x_2 x_1, \ldots y_2 y_1)$ such that $\ldots(x_2, y_2)(x_1, y_1)$ can be read on a path in the Moore diagram of an automaton generating G.



$$\ldots 0001x_n \ldots x_1 \sim \ldots 1110x_n \ldots x_1, \qquad \ldots 000 \sim \ldots 111$$

Hence, the limit space of the adding machine is the circle $\mathbb{R}/\mathbb{Z}$.

Let Σ be the graph with the set of vertices X^* with edges $(v, s(v))$ and (v, xv). If G is contracting, then Σ is Gromov hyperbolic and $\partial\Sigma$ is the limit space. If $G = \mathrm{IMG}(f)$, then Σ is the graph $\bigcup f^{-n}(S \cup \{\ell_x\})$.



Julia sets

The equivalence relation on $\mathsf{X}^{-\omega}$ is invariant under the shift $\ldots x_2 x_1 \mapsto \ldots x_3 x_2$, hence the shift induces a continuous self-map of the limit space. This is called the *limit dynamical system of* G.

Theorem

If $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is expanding, then $\mathrm{IMG}(f)$ is contracting and the limit dynamical system of $\mathrm{IMG}(f)$ is topologically conjugate to action of f on its Julia set (defined as the set of accumulation points of $\bigcup f^{-n}(t)$).

Corollary

Let $f(z)$ be a post-critically finite complex rational function. Then the action of f on its Julia set is topologically conjugate with the limit dynamical system of $\mathrm{IMG}(f)$.

Simplicial approximations of the limit space

Theorem

Let G be a contracting group with nucleus $\mathcal{N}$. Let $\Delta_n(G, \mathcal{N})$ be the geometric realization of the flag complex of $\Gamma_n(G, \mathcal{N})$. There exists k such that

$$p_{n,k} : vw \mapsto w : \Delta_{n+k}(G, \mathcal{N}) \longrightarrow \Delta_n(G, \mathcal{N}), \qquad v \in \mathsf{X}^n$$

are homotopic to contracting maps, and the corresponding inverse limit is homeomorphic to the limit space.

Corollary

The topological dimension of the limit space is not greater than the size of the nucleus.