Combinatorial equivalence of topological polynomials and group theory

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(joint work with L. Bartholdi)

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Topological polynomials

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A *topological polynomial* is a branched covering $f : S^2 \to S^2$ such that $f^{-1}(\infty) = \{\infty\}$, where $\infty \in S^2$ is a distinguished "point at infinity".

A *post-critically finite branched covering* (a *Thurston map*) is an orientation-preserving branched covering

$$f : S^2 \longrightarrow S^2$$

such that the *post-critical set*

$$P_f = \bigcup_{n \geq 1} f^n(C_f)$$

is finite.

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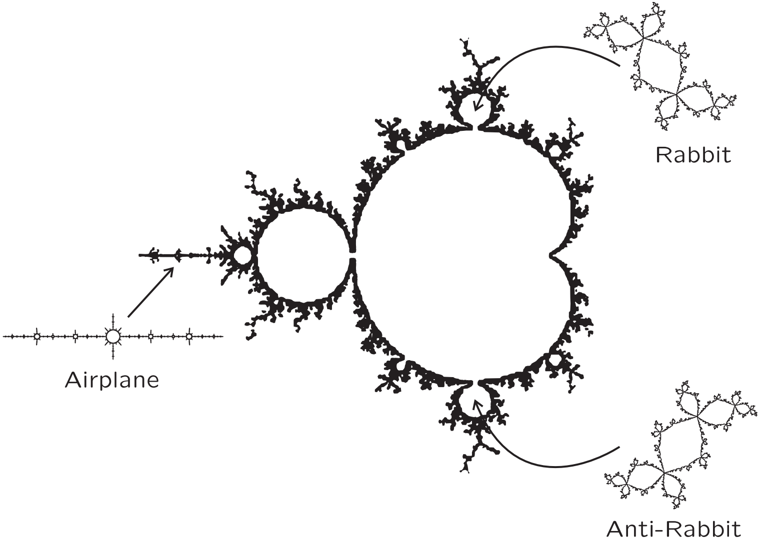
there exist homeomorphisms $h_1, h_2 : S^2 \to S^2$ such that $h_i(P_{f_1}) = P_{f_2}$, the diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{f_1} & S^2 \\ \downarrow h_1 & & \downarrow h_2 \\ S^2 & \xrightarrow{f_2} & S^2 \end{array}$$

is commutative and h_1 is isotopic to h_2 rel P_{f_1}.

Rabbit and Airplane



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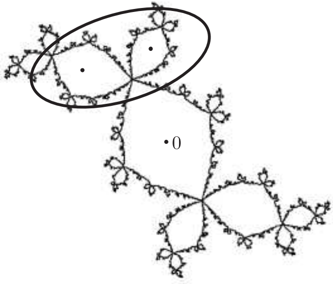

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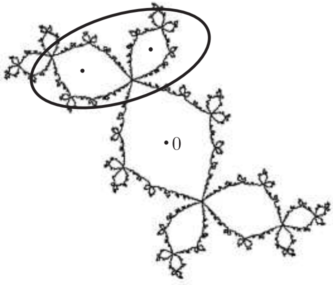

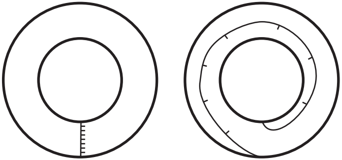

“Twisted rabbit” question of J. H. Hubbard

The Thurston’s theorem implies that the composition

$$f_r \circ T^m$$

is combinatorially equivalent either to the “rabbit” f_r or to the “anti-rabbit”, or to the “airplane”.

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Give an answer as a function of m.

More generally, give an answer for $f_r \circ g$, where g is any homeomorphism fixing $\{0, c, c^2 + c\}$ pointwise.

Theorem

If the 4-adic expansion of m has digits 1 or 2, then $f_r \circ T^m$ is equivalent to the "airplane", otherwise it is equivalent to the "rabbit" for $m \geq 0$ and to the "anti-rabbit" for $m < 0$.

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Here we use 4-adic expansions without sign. For example,

$$-1 = \ldots 333,$$

so that $f_r \circ T^{-1}$ is equivalent to the "anti-rabbit".

Iterated monodromy groups

Let $f : \mathcal{M}_1 \to \mathcal{M}$ be a d-fold covering map, where $\mathcal{M}_1$ is an open subset of $\mathcal{M}$.

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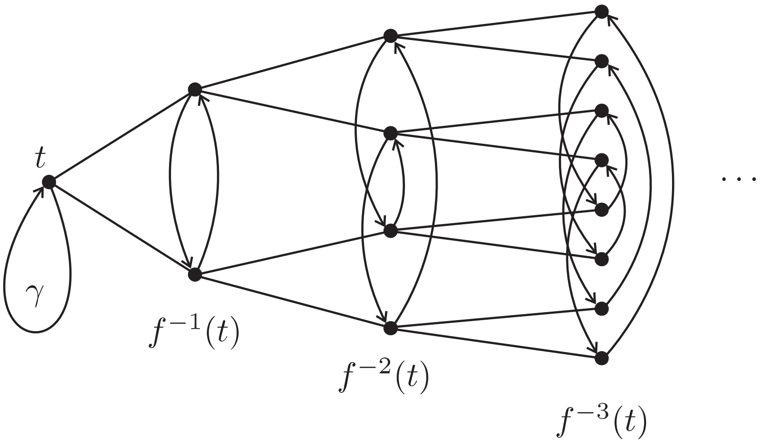
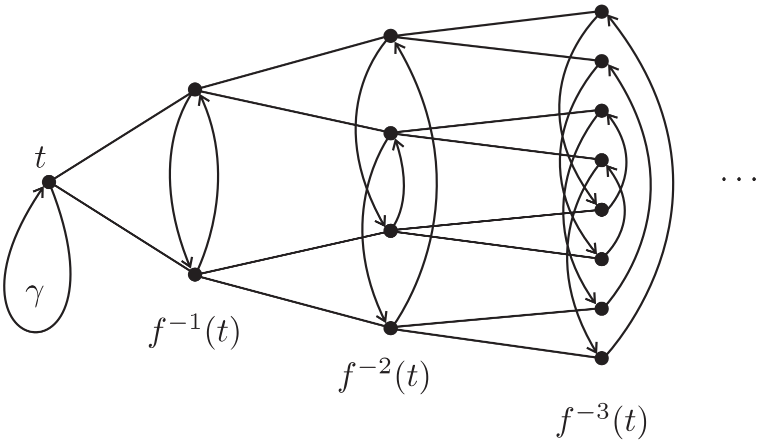
on which the fundamental group $\pi_1(\mathcal{M}, t)$ acts.


t
γ
$f^{-1}(t)$
$f^{-2}(t)$
$f^{-3}(t)$
$\cdots$



The obtained automorphism group of the rooted tree is called the *iterated monodromy group* of f.

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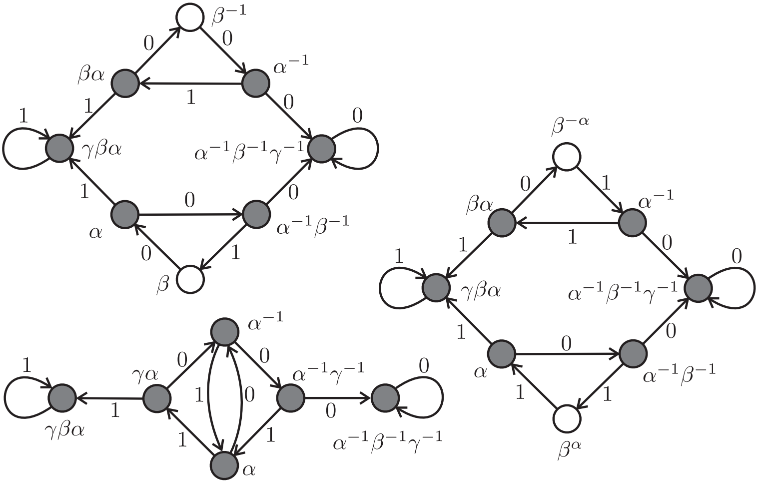
This makes it possible to distinguish specific Thurston maps.


β^{-1}
$\beta\alpha$
α^{-1}
$\gamma\beta\alpha$
$\alpha^{-1}\beta^{-1}\gamma^{-1}$
α
$\alpha^{-1}\beta^{-1}$
β
$\beta^{-\alpha}$
$\beta\alpha$
α^{-1}
$\gamma\beta\alpha$
$\alpha^{-1}\beta^{-1}\gamma^{-1}$
α
$\alpha^{-1}\beta^{-1}$
β^{α}
α^{-1}
$\gamma\alpha$
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α
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0
1

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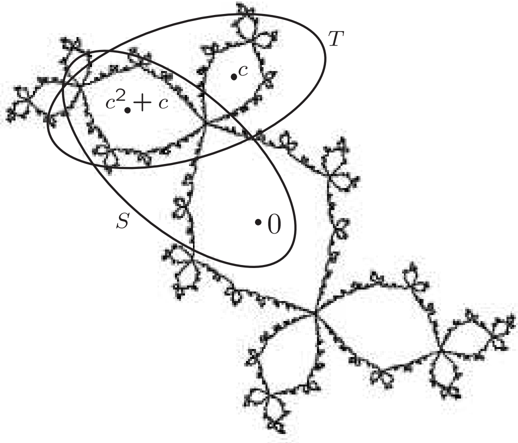

Proposition

Let ψ *be defined on* $H = \langle T^2, S, S^T \rangle < \mathcal{G}$ *by*

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Then for every $g \in \mathcal{G}$ the branched coverings $f_r \circ g$ and $f_r \circ \overline{\psi}(g)$ are combinatorially equivalent.

- $f_r \circ T^{4n} \sim f_r \circ T^n$,
- $f_r \circ T^{4n+1} \sim f_r \circ T$,
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This solves the problem for every $g \in \mathcal{G}$.

Dynamics on the Teichmüller space

The *Teichmüller space* $\mathcal{T}_{P_f}$ modelled on (S^2, P_f) is the space of homeomorphisms

$$\tau : S^2 \longrightarrow \widehat{\mathbb{C}},$$

where $\tau_1 \sim \tau_2$ if $\exists$ an automorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $h \circ \tau_1$ is isotopic to τ_2 rel P_f.

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$\mathcal{T}_{P_f}$ is the universal covering of the *moduli space* $\mathcal{M}_{P_f}$, i.e., the space of injective maps

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Hence, $\mathcal{M}_{P_f} \cong \widehat{\mathbb{C}} \setminus \{\infty, 1, 0\}$.

For every $\tau \in \mathcal{T}_{P_f}$ there exist unique $\tau' \in \mathcal{T}_{P_f}$ and $f_\tau \in \mathbb{C}(z)$ such that the diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{f} & S^2 \\ \downarrow \tau' & & \downarrow \tau \\ \widehat{\mathbb{C}} & \xrightarrow{f_\tau} & \widehat{\mathbb{C}} \end{array}$$

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The fixed point (if exists) is

$$\lim_{n\to\infty} \sigma_f^n(\tau_0).$$

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We get $f_\tau(z) = az^2 + 1$ and $ap_0^2 + 1 = 0$, hence $a = -\frac{1}{p_0^2}$ and

$$p_1 = 1 - \frac{1}{p_0^2}, \qquad f_\tau(z) = 1 - \frac{z^2}{p_0^2}.$$

We have proved

Proposition

The correspondence $\sigma_f(\tau) \mapsto \tau$ *on* $\mathcal{T}_{P_f}$ *is projected by the universal covering map to the rational function*

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If τ is the fixed point of σ_f, then f is equivalent to $f_\tau = 1 - \frac{z^2}{p^2}$, where p is the corresponding fixed point of P in the moduli space $\widehat{\mathbb{C}} \setminus \{\infty, 0, 1\}$.

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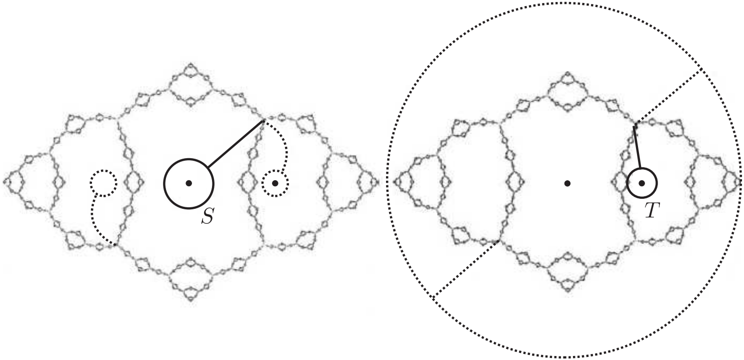

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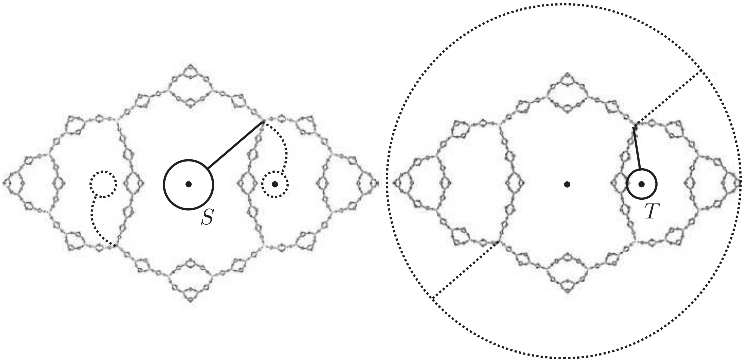

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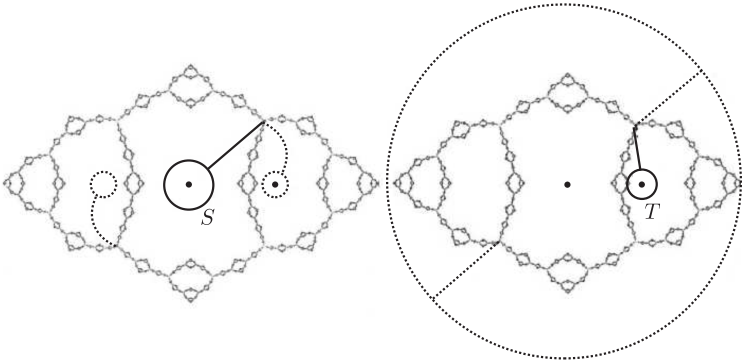

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Proposition

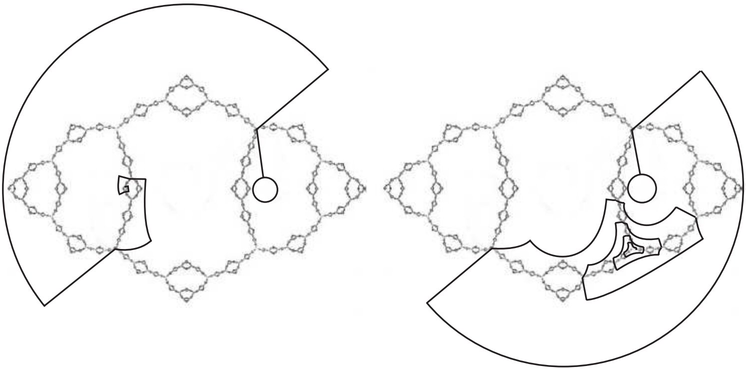
Let $g \in \mathcal{G}$ be represented by a loop $\gamma \in \pi_1(\mathcal{M}_{P_f}, t_0)$. Then

$$\lim_{n\to\infty} \sigma^n_{f_r \circ g}(\tau_0)$$

is projected onto the end of the path

$$\gamma\gamma_1\gamma_2\cdots$$

in $\mathcal{M}_{P_f}$, where γ_n continues γ_{n-1} and is a preimage of γ_{n-1} under $1 - \frac{1}{z^2}$.



A 2-dimensional iteration

Let us put together iteration on the moduli space and on the plane in one map on $\mathbb{P}^2$.

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or

$$F([z : p : u]) = [p^2 - z^2 : p^2 - u^2 : p^2].$$

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Its post-critical set is $\{z = 0\} \cup \{z = 1\} \cup \{z = p\} \cup \{p = 0\} \cup \{p = 1\}$ and the line at infinity.

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The fibers of the projection are the Julia sets of the iteration

$$z, \quad f_{p_1}(z), \quad f_{p_2} \circ f_{p_1}(z), \quad f_{p_3} \circ f_{p_2} \circ f_{p_1}(z), \ldots,$$

where $p_{n+1} = P(p_n)$.

A minimal Cantor set of 3-generated groups

The iterated monodromy groups of the backward iterations

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where $p_{n-1} = P(p_n)$, is a Cantor set of 3-generated groups G_w with countable dense isomorphism classes.

A minimal Cantor set of 3-generated groups

The iterated monodromy groups of the backward iterations

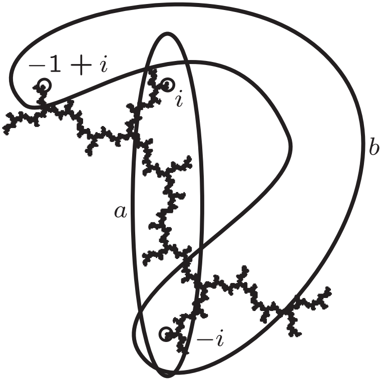
$$\cdots \xrightarrow{f_{p_3}} z_2 \xrightarrow{f_{p_2}} z_1 \xrightarrow{f_{p_1}} z_0,$$

where $p_{n-1} = P(p_n)$, is a Cantor set of 3-generated groups G_w with countable dense isomorphism classes.

For any finite set of relations between the generators of G_{w_1} there exists a generating set of G_{w_2} with the same relations.

Twisting $z^2 + i$

Let *a* and *b* be the Dehn twists



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This map was studied by J. E. Fornæss and N. Sibony (1992).

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Consider the group $\mathbb{Z}^2 \rtimes C_4$ of affine transformations of $\mathbb{C}$

$$z \mapsto i^k z + z_0,$$

where $k \in \mathbb{Z}$ and $z_0 \in \mathbb{Z}[i]$.

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We have a natural homomorphism $\phi : \mathcal{G} \longrightarrow \mathbb{Z}^2 \rtimes C_4$

$$a \mapsto -z + 1, \qquad b \mapsto iz$$

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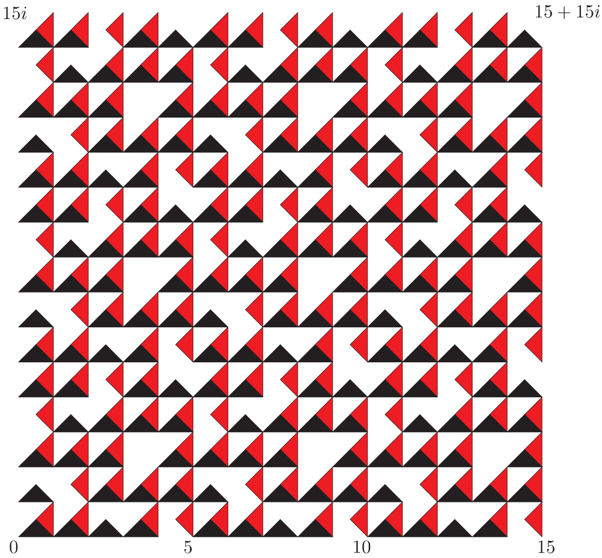
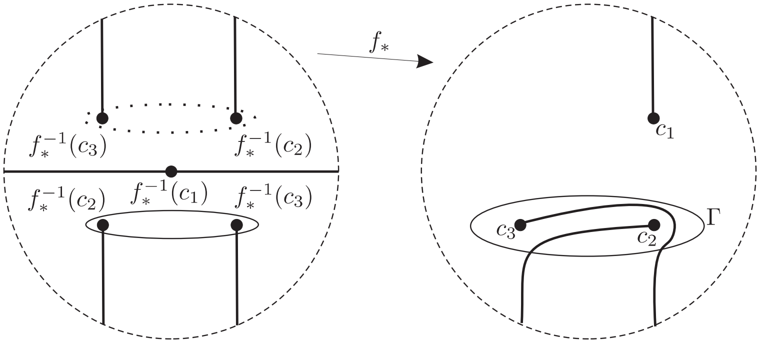
It depends only on $\phi(g)$, which of these cases takes place.

The answer



Obstructed polynomials



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A. Erschler proved that the growth of this group has bounds

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$\mathrm{IMG}\left(z^2 + i\right)$ also has intermediate growth (Kai-Uwe Bux and Rodrigo Pérez, 2004)