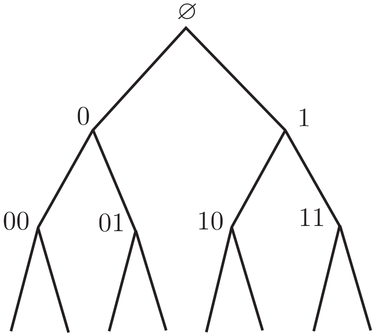
A Cantor set in the space of 3-generated groups

Volodymyr Nekrashevych

May 6, 2006,
Vanderbilt

Binary tree



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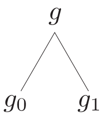

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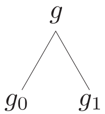

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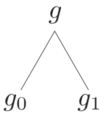

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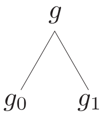

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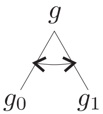

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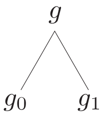

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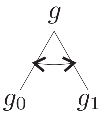

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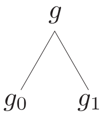

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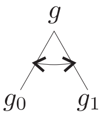

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Let $w \in \{0, 1\}^\infty$ and $w = x\overline{w}$.

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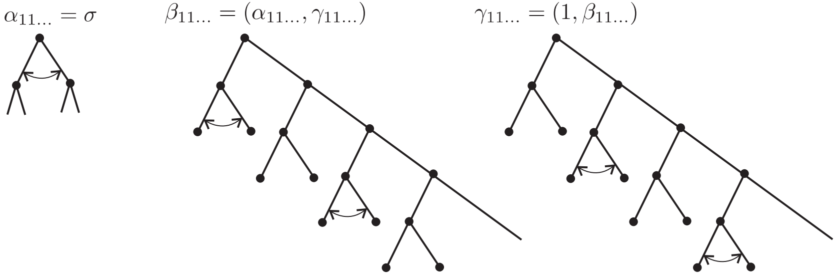
$\alpha_{11\ldots}, \beta_{11\ldots}, \gamma_{11\ldots}$



Proposition

Suppose that h_0, h_1, h_2 *are conjugate to* $\alpha_w, \beta_w, \gamma_w$ *in* $\mathrm{Aut}(X^*)$. *Then there exists a unique* $w' \in \{0,1\}^\infty$ *such that* h_0, h_1, h_2 *are simultaneously conjugate to* $\alpha_{w'}, \beta_{w'}, \gamma_{w'}$.

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Theorem

Groups $\mathcal{D}_{w_1}$ *and* $\mathcal{D}_{w_2}$ *are isomorphic if and only if they are conjugate in* $\mathrm{Aut}(X^*)$.

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Let $\mathcal{R}_w = \langle \alpha_w, \beta_w, \gamma_w \rangle$.
$\mathcal{R}_{11\ldots} = \mathrm{IMG}\left(z^2 + (-0.1226\ldots + 0.7449\ldots i)\right)$ and
$\mathcal{R}_{00\ldots} = \mathrm{IMG}\left(z^2 - 1.7549\ldots\right)$.

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Two groups are close if their Cayley graphs coincide on a large ball.
It is induced from the direct product topology on 2^{F_n}, if we identify a group with the kernel of the canonical epimorphism.

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Y. Stadler and L. Guyot studied the set of limit points of $B(m, n)$ as $n \to \infty$.

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Let $\Omega \subset \{0,1\}^\infty$ *be the set of sequences which have infinitely many zeros. Then the map* $\Omega \to \mathfrak{G}_3$

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Two groups $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are isomorphic if and only if the sequences w_1 and w_2 are cofinal, i.e., if they are of the form $w_1 = v_1 u$ and $w_2 = v_2 u$ for $|v_1| = |v_2|$.

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Corollary

For any $w_1, w_2 \in \{0,1\}^\infty$ and any finite set of relations and inequalities between the generators of $\mathcal{R}_{w_1}$ there are generators of $\mathcal{R}_{w_2}$ such that the same relations and inequalities hold.

Theorem

Let

$$R_i = \left\{ \left[\beta^{\alpha^{2n+i}}, \gamma\right], \left[\beta^{\alpha^{2n+1}}, \beta\right], \left[\gamma^{\alpha^{2n+1}}, \gamma\right] \ : \ n \in \mathbb{Z} \right\}$$

for $i = 0, 1$, *and*

$$\begin{array}{rclcrcl} \varphi_0(\alpha) & = & \alpha\beta\alpha^{-1}, & \quad & \varphi_1(\alpha) & = & \beta, \\ \varphi_0(\beta) & = & \gamma, & & \varphi_1(\beta) & = & \gamma, \\ \varphi_0(\gamma) & = & \alpha^2, & & \varphi_1(\gamma) & = & \alpha^2. \end{array}$$

Then for every $w = x_1 x_2 \ldots \in \{0, 1\}^\infty$

$$\bigcup_{n=1}^{\infty} \varphi_{x_1} \circ \varphi_{x_2} \circ \cdots \circ \varphi_{x_{n-1}}(R_{x_n})$$

is a set of defining relations of $\mathcal{R}_w$.

Universal Groups of the Families

Let $\mathcal{D}$ be the subgroup of $\prod_{w\in\{0,1,2\}^\infty} \mathcal{D}_w$ generated by the "diagonal" elements

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This group can be defined as

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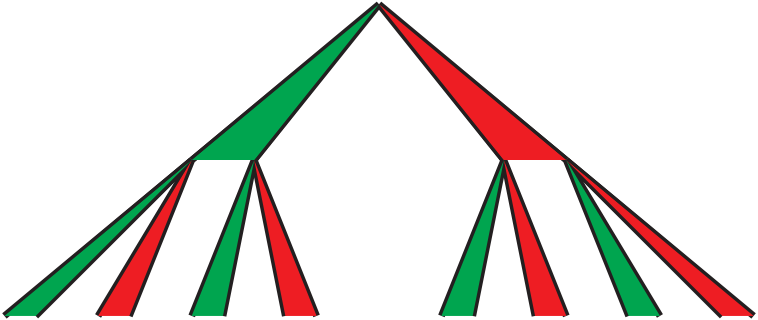
Restriction of $\mathcal{D}$ onto T_w is $\mathcal{D}_w$.



A bigger group

Let $\widetilde{\mathcal{D}}$ be the group generated by

$$\begin{aligned} \alpha &= (12)(34), & a &= (13)(24), \\ \beta &= (\alpha, \gamma, \alpha, \gamma), & b &= (a\alpha, a\alpha, c, c), \\ \gamma &= (\beta, 1, 1, \beta), & c &= (b\beta, b\beta, b, b). \end{aligned}$$

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Note that the group $\widetilde{\mathcal{D}}$ permutes the subtrees T_w.

Proposition

The following relations hold.

$$\begin{array}{lll} \alpha^a = \alpha, & \alpha^b = \alpha, & \alpha^c = \alpha, \\ \beta^a = \beta, & \beta^b = \beta, & \beta^c = \beta^\gamma, \\ \gamma^a = \gamma^\alpha, & \gamma^b = \gamma^\beta, & \gamma^c = \gamma. \end{array}$$

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Consequently, if w_1 and w_2 belong to one H-orbit, then $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are isomorphic.

An overgroup of $\mathcal{R}$

Let $\widetilde{\mathcal{R}} \rhd \mathcal{R}$ be generated by

$$\begin{aligned}
\alpha &= \sigma\,(1, \gamma, 1, \gamma)\,, & a &= \pi\,(c, c, 1, 1)\,, & l_0 &= \left(l_2 c\gamma^{-1}, l_2 c, l_2\gamma^{-1}, l_2\right) \\
\beta &= (\alpha, 1, 1, \alpha)\,, & b &= (1, 1, a, a)\,, & l_1 &= (l_0, l_0, l_0, l_0) \\
\gamma &= (1, \beta, 1, \beta)\,, & c &= \left(1, \beta, b\beta^{-1}, b\right), & l_2 &= (l_1, l_1, l_1, l_1)\,,
\end{aligned}$$

where $\sigma = (12)(34) : (0, y) \leftrightarrow (1, y)$ and $\pi = (13)(24) : (x, 0) \leftrightarrow (x, 1)$.

An overgroup of $\mathcal{R}$

Let $\widetilde{\mathcal{R}} \triangleright \mathcal{R}$ be generated by

$$\begin{aligned} \alpha &= \sigma\,(1,\gamma,1,\gamma)\,, & a &= \pi\,(c,c,1,1)\,, & l_0 &= \left(l_2 c\gamma^{-1}, l_2 c, l_2\gamma^{-1}, l_2\right) \\ \beta &= (\alpha,1,1,\alpha)\,, & b &= (1,1,a,a)\,, & l_1 &= (l_0,l_0,l_0,l_0) \\ \gamma &= (1,\beta,1,\beta)\,, & c &= \left(1,\beta,b\beta^{-1},b\right), & l_2 &= (l_1,l_1,l_1,l_1)\,, \end{aligned}$$

where $\sigma = (12)(34) : (0,y) \leftrightarrow (1,y)$ and $\pi = (13)(24) : (x,0) \leftrightarrow (x,1)$. The group $\widetilde{\mathcal{R}}$ acts on the second coordinates as

$$a = \sigma(c,1), b = (1,a), c = (1,b),$$

$$r_0 = (r_2 c, r_2), r_1 = (r_0, r_0), r_2 = (r_1, r_1).$$

$\mathcal{D}_w$ as Iterated Monodromy Groups

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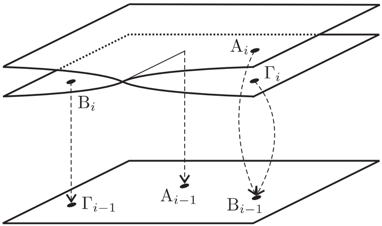

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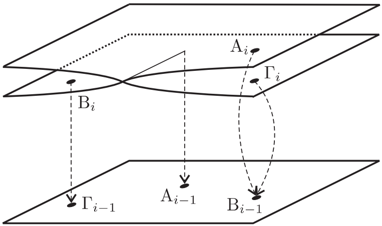

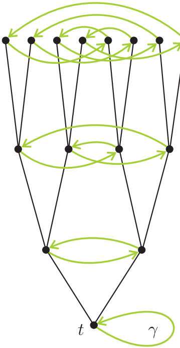

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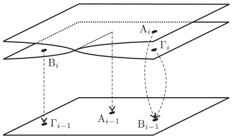

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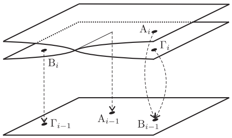

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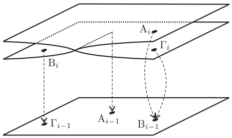

We get $f_i = (az+1)^2$ and $ap_i + 1 = -1$, hence $f_i(z) = \left(1 - \frac{2z}{p_i}\right)^2$, $p_{i-1} = \left(1 - \frac{2}{p_i}\right)^2$.

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$\mathrm{IMG}\,(F)\,/\mathcal{D} \cong \mathrm{IMG}\left(\left(1 - \frac{2}{p}\right)^2\right)$.

The family $\mathcal{R}_w$ can be defined in the similar way, but starting from the map

$$\begin{pmatrix} z \\ p \end{pmatrix} \mapsto \begin{pmatrix} 1 - \frac{z^2}{p^2} \\ 1 - \frac{1}{p^2} \end{pmatrix}$$