J. reine angew. Math. **545** (2002), 19—65

Journal für die reine und angewandte Mathematik

Pointwise estimates and stability for degenerate viscous shock waves

By *Peter Howard* at College Station

Abstract. We study the pointwise behavior of perturbed degenerate (sonic) shock waves for scalar conservation laws with constant diffusion. Building on the pointwise Green's function approach of [H1–3], [ZH], we extend the linear analysis to an equation with non-integrable coefficients, arriving at an estimate on linearized perturbations believed sharp to a possible error of size $\log t$. Nonlinear stability for degenerate waves follows in all L^p norms, $p \geqq 1$.

1. Introduction

We consider the scalar viscous conservation law

$$u_t + f(u)_x = u_{xx}; \quad u, x, f \in \mathbb{R}, t \in \mathbb{R}_+, \tag{1.1}$$
$$u(0, x) = u_0(x),$$

where $u_0(\pm\infty) = u_\pm$, $f \in C^2(\mathbb{R})$. In particular, we will study the stability of degenerate, or sonic, shock solutions to (1.1); that is solutions of the form $\bar{u}(x - st)$ which satisfy the Rankine-Hugoniot condition

$$s(u_+ - u_-) = f(u_+) - f(u_-),$$

as well as the degenerate condition that $f'(u_+) = s < f'(u_-)$ (or symmetrically $f'(u_+) < s = f'(u_-)$). Without loss of generality, we may take $s = 0$ and thus $f'(u_+) = 0 < f'(u_-)$.

This problem has recently been studied in some detail through energy methods by Mei, Matsumura-Nishihara, and Nishikawa [M1], [MN4], [Ni]. While we extend their results to slower-decaying initial perturbations, by factor $|x|$, and gain a greater rate of decay in time on the perturbation by a factor of $t^{-1/2} \log t$ (believed sharp to a possible error of

Research supported by the NSF under grant No. DMS-9804390.

size $\log t$), the proper aim of this paper is to introduce a method of study based on sharp ODE estimates, similar to that of [H1–3], [ZH], [HZ1–2]. This method has already proven quite robust in applications to scalar conservation laws of high order [HZ1], systems of conservation laws [ZH], and most recently multi-dimensional conservation laws [HoZ1–2], [Z3], [ZS]. We expect that similar extensions may also be viable in the case of degenerate waves, through application of the basic techniques presented here.

Our interest in the degenerate case is motivated by several concerns. Degenerate shock waves serve as a boundary case between Lax and undercompressive waves, and an understanding of such boundary cases often illuminates the general analysis. In the absence of such guiding example problems as those of Nishihara and Zumbrun for nondegenerate waves [N1], [Z1], (1.1) appears to be the simplest case from which we can extract definitive results. Additionally, the Green's function estimates of Theorem 1.1 are quite interesting apart from their application to viscous shock waves. It is notable that the Laplace transform analysis of [H1–2], [ZH], developed in the context of equations with exponentially decaying coefficients, can be extended to the case of non-integrable coefficients.

Finally, we have been motivated by the physical significance of degenerate waves in the study of detonation (see, for example, [FD]). Briefly, in the case that a pressure shock front impinges on an explosive, instigating detonation, classical Chapman-Jouguet theory asserts that the velocity of the impinging shock front must be chosen as the minimum possible velocity for which conservation of mass, momentum and energy allow a final-state solution. At this velocity, the flow of pressure takes the form of a degenerate wave, followed by a rarefaction wave.

Such detonation waves have been the subject of considerable research, and we mention a few results most closely related to our analysis. In the context of a model of Majda [M], Liu and Ying have shown the stability of nondegenerate, or "strong" detonation waves [LY]. Recently, in the context of a model of Rosales and Majda [RM], Tong Li has employed the methods of [MN4] to gain rates of decay on the perturbation in both the nondegenerate and degenerate (Chapman-Jouguet) cases [Li1–4]. In the case of nondegenerate detonation waves, Zumbrun has shown that even in the case of higher dimensions, stability can be analyzed via the methods of [H1–3], [ZH] (see [Z3]). In particular, Zumbrun's analysis suggests that the methods employed here may extend to the study of CJ waves.

One critical feature of degenerate shock solutions, $\bar{u}(x)$, to (1.1) is that when (1.1) is linearized about $\bar{u}(x)$, the linearized eigenvalue problem $L(\bar{u}(x))v = \lambda v$ has the property that zero lies not only in both the point spectrum and the essential spectrum of $L(\bar{u}(x))$ (as is generically the case for traveling wave solutions to viscous conservation laws), but is also a branch point of the Evans function (see [AGJ] for example, and below). In the case of nondegenerate viscous shock waves, Gardner and Zumbrun have shown that for branch points near the origin (within the gap of their Gap Lemma), the Evans function can be analytically extended through the branch on an appropriate Riemann manifold [GZ]. Kapitula and Rubin have recently employed a similar extension in the cases of the cubic nonlinear Schrödinger equation and the Ginzburg-Landau equation [KR]. The algebraic decay of coefficients of $L(\bar{u}(x))$ in the case of degenerate shock waves, however, seems to preclude the possiblity of a similar analysis. Without analyticity of the Evans function, even in this extended sense, the usual manner of analysis near the origin by Taylor expansion cannot apply [GZ], [HZ2], appendix, [PW]. Rather, the Evans function must by understood here through sharp ODE estimates. Such estimates, however, are difficult in themselves in the case of equations with non-integrable coefficients. In addition, previous analyses using these methods have made extensive use of contours passing through the negative-real axis [H1–3], [ZH]—contours disallowed by this branch of the Evans function. The heart of the method, however, lies not in particular contour shifts, but rather in sufficiently sharp ODE estimates, and we find here that considerably different contours suffice.

Our method of study will be to let $u(t,x)$ denote a second solution of (1.1) and consider the perturbation $v(t,x) = u(t,x) - \bar{u}(x)$. It is well known that a solution initialized by $u(0,x)$ near $\bar{u}(x)$ will not generally approach $\bar{u}(x)$, but rather will approach a translate of $\bar{u}(x)$ determined uniquely by the mass of $v(t,x)$, $\int_{\mathbb{R}} v(0,x)\,dx$ (see discussion, for example, in [HZ2]). Proceeding as in [G], [L], [MN4], we will study convergence to this asymptotically selected translate, in which case v will have zero mass. In particular, we take $v(t,x) := u(t,x) - \bar{u}(x-l)$, where l is chosen such that $\int_{\mathbb{R}} u(t,x) - \bar{u}(x-l)\,dx = 0$. As the rest of the paper will be concerned with $\bar{u}(x-l)$, we will henceforth denote it $\bar{u}(x)$.

We mention before proceeding that recent advances in the study of viscous shock wave stability have shown that local tracking rather than asymptotic tracking typically leads to a sharper understanding of the wave's underlying behavior [HZ1–2], [Z2], [ZH]. In fact, for undercompressive viscous shock waves, whose asymptotic location cannot generally be determined by conservation of mass, local tracking is essential. Such an analysis is hindered, however, in the case of degenerate shock waves by the branch point of the Evans function at $\lambda = 0$, as will be discussed in Section 6 on further work.

Substituting $u(t,x) = \bar{u}(x) + v(t,x)$ into (1.1) we arrive at the linearized equation

$$v_t + \big(a(x)v\big)_x = v_{xx} + \mathbf{O}(v^2)_x, \tag{1.2}$$

where $a(x) := f'\big(\bar{u}(x)\big)$ and $\mathbf{O}(v^2)$ is smooth. We will work with the integrated equation, obtained through defining $w(t,x) := \int^x v(t,y)\,dy$. We have

$$w_t + a(x)w_x = w_{xx} + \mathbf{O}(v^2). \tag{1.3}$$

Allowing $G(t,x;y)$ to represent the Green's function for $w_t = Lw, Lw := w_{xx} - a(x)w_x$, we have the integral equation

$$w(t,x) = \int_{-\infty}^{+\infty} G(t,x;y)w_0(y)\,dy + \int_0^t \int_{-\infty}^{+\infty} G(t-s,x;y)\mathbf{O}(v^2)\,dy\,ds,$$

or taking the x-derivative

$$v(t,x) = \int_{-\infty}^{+\infty} G_x(t,x;y)\int^y v_0(\xi)\,d\xi\,dy + \int_0^t \int_{-\infty}^{+\infty} G_x(t-s,x;y)\mathbf{O}(v^2)\,dy\,ds. \tag{1.4}$$

Our first theorem consists of pointwise estimates on $G(t,x;y)$ and rests upon the following observations (see Figure 4.1):

(1) The essential spectrum of L lies on and to the left of a parabola passing through the origin and opening into the negative real complex plane. We will denote this parabola by Γ_e and represent it through

$$\lambda_e(k) = -k^2 - ia_- k.$$

(2) The point spectrum of L lies to the left of a parabola Γ_d defined through

$$\lambda_d(k) = -d_2 k^2 - id_1 k - d_0,$$

where $d_0, d_1, d_2 > 0$ will be chosen sufficiently small during the analysis.

Theorem 1.1. *Let $f''(u_+) \neq 0$ for f as in (1.1) (first order degeneracy). For some constants $C, M, T > 1$ and $\eta > 0$, depending on $a(x) = f'(\bar{u}(x))$ and the spectrum of L, the Green's function, $G(t, x; y)$, for $v_t = Lv$ satisfies the following estimates.*

(I) *For $|x - y| \geqq Kt$, K sufficiently large, and also for $t \leqq T$, all x, y $(n = 0, 1)$*

$$\partial_x^n G(t, x; y) = \mathbf{O}(t^{-\frac{n+1}{2}}) e^{-\frac{(x-y)^2}{Mt}}.$$

(II) *For $|x - y| \leqq Kt$, K as above, $t \geqq T$:*

(i) $y \leqq x \leqq 0$:

$$G(t, x; y) = \mathbf{O}(t^{-1/2}) e^{-\frac{(x-y-a_- t)^2}{Mt}} + \mathbf{O}(e^{-\eta|x|}) e^{-\frac{(x-y-a_- t)^2}{Mt}}$$
$$+ \mathbf{O}(e^{-\eta|x|})(|x - y - a_- y| + 1)^{-1} I_{\{|x-y| \leqq a_- t\}},$$

$$G_x(t, x; y) = \mathbf{O}(t^{-1}) e^{-\frac{(x-y-a_- t)^2}{Mt}} + \mathbf{O}(e^{-\eta|x|}) \mathbf{O}(t^{-1/2}) e^{-\frac{(x-y-a_- t)^2}{Mt}}$$
$$+ \mathbf{O}(e^{-\eta|x|})(|x - y - a_- t| + 1)^{-1} I_{\{|x-y| \leqq a_- t\}}.$$

(ii) $x \leqq y \leqq 0$:

$$G(t, x; y) = \mathbf{O}(t^{-1/2}) e^{-\frac{(x-y-a_- t)^2}{Mt}} + \mathbf{O}(e^{-\eta|x|}) e^{-\frac{(x-y-a_- t)^2}{Mt}}$$
$$+ \mathbf{O}(t^{-1}) \mathbf{O}(e^{-\eta|x|}) e^{-\frac{(x-y)^2}{Mt}} I_{\{|x-y| \leqq a_- t\}},$$

$$G_x(t, x; y) = \mathbf{O}(t^{-1}) e^{-\frac{(x-y-a_- t)^2}{Mt}} + \mathbf{O}(e^{-\eta|x|}) \mathbf{O}(t^{-1/2}) e^{-\frac{(x-y-a_- t)^2}{Mt}}$$
$$+ \mathbf{O}(t^{-1}) \mathbf{O}(e^{-\eta|x|}) e^{-\frac{(x-y)^2}{Mt}} I_{\{|x-y| \leqq a_- t\}}.$$

(iii) $x \leqq 0 \leqq y$:

$$G(t, x; y) = \mathbf{O}(t^{-1}) \mathbf{O}_1(|y|) \mathbf{O}(e^{-\eta|x|}) e^{-\frac{y^2}{Mt}},$$

$$G_x(t, x; y) = \mathbf{O}(t^{-1}) \mathbf{O}_1(|y|) \mathbf{O}(e^{-\eta|x|}) e^{-\frac{y^2}{Mt}}.$$

(iv) $y \leqq 0 \leqq x$:

$$G(t,x;y) = \mathbf{O}(t^{-1/2})\mathbf{O}_1(|x|^{-1})e^{-\frac{(x-y-a_-t)^2}{Mt}} I_{\{|y| \geqq a_-t\}}$$

$$+ \mathbf{O}_1(|x|^{-1})\left(t + \frac{y}{a_-} - \frac{3}{2a_-^3}\frac{x^2}{t^2}y\right)^{-1} e^{-\frac{x^2}{Mt}} I_{\{\{|y| \leqq a_-t\} \cap \{x \geqq 1\}\}}$$

$$+ \left(t + \frac{y}{a_-} - \frac{3}{2a_-^3 t}y\right)^{-1} I_{\{\{|y| \leqq a_-t\} \cap \{x \leqq 1\}\}},$$

$$G_x(t,x;y) = \mathbf{O}(t^{-1/2})\mathbf{O}_1(|x|^{-2})e^{-\frac{(x-y-a_-t)^2}{Mt}} I_{\{|y| \geqq a_-t\}}$$

$$+ \mathbf{O}_1(|x|^{-2})\left(t + \frac{y}{a_-} - \frac{3}{2a_-^3}\frac{x^2}{t^2}y\right)^{-1} e^{-\frac{x^2}{Mt}} I_{\{\{|y| \leqq a_-t\} \cap \{x \geqq 1\}\}}$$

$$+ \mathbf{O}(1)\left(t + \frac{y}{a_-} - \frac{3}{2a_-^3 t}y\right)^{-1} I_{\{\{|y| \leqq a_-t\} \cap \{x \leqq 1\}\}}.$$

(v) $0 \leqq y \leqq x$:

$$G(t,x;y) = \left[\mathbf{O}(t^{-1/2}) \wedge \left(\mathbf{O}(t^{-1})\mathbf{O}_1(|y|) + \mathbf{O}(t^{-1}\log t)\right)\right]\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}},$$

$$G_x(t,x;y) = \mathbf{O}(t^{-1})\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}}$$

$$+ \mathbf{O}(t^{-1}\log t)\mathbf{O}_1(|x|^{-2})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}}.$$

(vi) $0 \leqq x \leqq y$:

$$G(t,x;y) = \left[\mathbf{O}(t^{-1/2}) \wedge \left(\mathbf{O}(t^{-1})\mathbf{O}_1(|x|) + \mathbf{O}(t^{-1}\log t)\right)\right]\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}},$$

$$G_x(t,x;y) = \mathbf{O}(t^{-1})\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}}$$

$$+ \mathbf{O}(t^{-1}\log t)\mathbf{O}_1(|x|^{-2})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}},$$

where $\wedge$ *denotes minimum and estimates of form* $\mathbf{O}_1\big(f(\cdot)\big)$ *satisfy* $\mathbf{O}_1\big(f(\cdot)\big) \leqq Cf(1+\cdot)$ (*allowing numerous expressions that would otherwise extend over two lines to be completed on one*).

A detailed discussion of estimates of the form of those from Theorem 1.1 appears in [H2]. We mention here only that the estimates on G, G_x for Cases (i)–(iv) are not assumed sharp, and should be compared with the more natural estimates of [H1–2], [ZH]. The difficulty in obtaining sharp Green's function estimates in these cases centers around our inability to extend contours through the negative real axis, and also upon our inability to expand functions of $\sqrt{\lambda}$ in a Taylor series about the crucial point $\lambda = 0$. As the analysis is essentially dictated by the purely degenerate case, $x, y \geqq 0$, however, these estimates do not effect our final result. On the other hand, it would also appear that the $\log t$ terms in Cases (v)–(vi) are not sharp, and these contribute a final error on our perturbation of size $\log t$. In the absence of an exact analysis, it is unclear whether or not this could be removed. Finally, we observe that the estimates in (v)–(vi) of algebraic form y/x (i.e., those of the form $\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)$) are similar to those obtained in the rarefaction analysis of [SZ], by an explicit Hopf-Cole calculation. It is critical that such refined behavior can be understood through these methods.

Before stating our main theorem we make the following definitions.

Definition 1.1 (Class of initial data)**.** Denote by Δ_r the space of functions $d(\cdot) \geqq 0$ such that $d(x) \leqq C(1+|x|)^{-r}$, $r > 1$. Denote by $D(\cdot)$ the asymptotically decaying antiderivative of $d(\cdot)$,

$$D(x) := \begin{cases} \int_{-\infty}^{x} d(y)\,dy, & x < 0, \\ \int_{x}^{+\infty} d(y)\,dy, & x \geqq 0. \end{cases}$$

Also let

$$D_-(t) := \begin{cases} D(\sqrt{t})(\sim (1+t)^{\frac{1-r}{2}}), & 1 < r < 3, \\ (1+t)^{-1}\log(2+t), & r = 3, \\ (1+t)^{-1}, & r > 3. \end{cases}$$

and $D_+(t) := \log(2+t)D_-(t)$.

Remark. $D_-(t)$ will dictate the t-decay of the perturbation on the Lax side ($x \leqq 0$) and $D_+(t)$ will dictate the t-decay of the perturbation on the degenerate side ($x \geqq 0$). Terms of form $\log(2+t)$ will serve to represent generic $\log t$ growth that neither blows up nor vanishes as $t \to 0$. That such errors do not arise as $t \to 0$ is clear from the preceding discussion and the observation that small-t behavior is governed by large-$|\lambda|$ behavior, which is the same for degenerate waves as for the Lax waves of previous analyses [H1–3]. We mention that in their analysis of rarefaction wave stability for systems of conservation laws, Szepessy and Zumbrun observed a similar discrepancy by $\log t$ of decay on the degenerate side versus decay on the non-degenerate side [SZ].

Definition 1.2 (Orbital stability)**.** We say that a standing wave solution $\bar{u}(x)$ to (1.1) is orbitally stable in norm $\|\cdot\|$ if there exists an $\varepsilon > 0$ and a translate of $\bar{u}$, say $\bar{u}_l = \bar{u}(x-l)$, such that if another solution, u, to (1.1) satisfies $\|u(0,x) - \bar{u}_l(x)\| \leqq \varepsilon$, then $\|u(t,x) - \bar{u}_l(x)\|$ decays to zero in time.

We now state the main result of the paper, from which orbital stability follows in every L^p norm.

Theorem 1.2. *Suppose $\bar{u}(x)$ is a degenerate standing wave solution to* (1.1) *with $f''(u_+) \neq 0$ (first order degeneracy). Then for another solution to* (1.1)*, $u(t,x)$, with initial data, $u_0(x)$, satisfying $u_0(x) - \bar{u}(x) \in \mathscr{A}_\zeta^r$, with*

$$\mathscr{A}_\zeta^r := \left\{ v_0(x) \colon |v_0(x)| \leqq \zeta\, d(x),\ \textit{some } d \in \Delta_r,\ \int_{\mathbb{R}} v_0(x)\,dx = 0 \right\},$$

ζ sufficiently small and $r > 1$, we obtain the estimates

(I) $x \leqq 0$:

$$|u(t,x) - \bar{u}(x)| \leqq C\zeta[e^{-\frac{\eta}{2}|x|}D_-(t) + d(x - a_- t)],$$

(II) $x \geqq 0$:

$$|u(t,x) - \bar{u}(x)| \leqq C\zeta[(1+x)^{-1}e^{-\frac{x^2}{2Mt}}D_-(t) + (1+x)^{-2}e^{-\frac{x^2}{2Mt}}D_+(t) + d_+(t,x)],$$

where

$$d_+(t,x) := \begin{cases} (1+x)^{-r} \wedge t^{-1}(1+x)^{2-r}, & 1 < r < 2, \\ (1+x)^{-2}\log(2+x) \wedge t^{-1}, & r = 2, \\ (1+t)^{-1/2}(1+x)^{1-r} \wedge t^{-1}(1+x)^{2-r}, & r > 2, \end{cases}$$

η and M are as in Theorem 1.1 and $\wedge$ represents minimum.

Remark. By comparing Theorem 1.2 above with the similar Theorem 1.2 of [HZ1], one can see how the estimates for degenerate waves lie between those for Lax and undercompressive waves (undercompressive waves arising in scalar equations of higher order). For example, we see that while perturbations from Lax waves decay at a rate only limited by the decay of initial data, perturbations from undercompressive waves decay in L^∞ norm at a maximum rate of $t^{-1/2}$, while perturbations from degenerate waves decay at a maximum rate of $t^{-1}\log t$. Such maximum rates are formally determined directly from the Green's function estimates of Theorem 1.1 (and the related Theorem 1.1 of [HZ1]). Since we may assume rapidly decaying data, these are simply the t-decay rates of G_x, where in the Lax case such t-decay is essentially exponential. We also point out that aside from the $\log t$ terms, the estimates of Theorem 1.2 are essentially those obtained through a Duhamel iteration study of Burgers equation ($u_t + uu_x = u_{xx}$) with initial data $u_0(x)$ assumed to lie in $\mathscr{A}_\zeta^r$. Alternatively, such estimates on Burgers equation can be obtained directly from Hopf's solution [H].

Theorem 1.2 provides the following immediate corollary on stability.

Corollary 1.3 (Nonlinear stability). *Under the assumptions of Theorem 1.2 and with $u_0(x) - \bar{u}(x) \in \mathscr{A}_\zeta^r$ as there, we have*

$$\|u(t,x) - \bar{u}(x)\|_{L^p} \leqq CD_+(t), \quad p > 1,$$

and

$$\|u(t,x) - \bar{u}(x)\|_{L^1} \leqq CD_+(t)t^\varepsilon,$$

for $\varepsilon > 0$ arbitrarily small when $1 < r \leqq 3$, and for $\varepsilon = 0$ when $r > 3$.

Proof. For $p = 1$ we have from Theorem 1.2 ($1 < r < 2$)

$$\frac{1}{C\zeta}\int_{-\infty}^{+\infty}|u(t,x)-\bar{u}(x)|\,dx \leqq \int_{-\infty}^{0} e^{-\frac{\eta}{2}|x|}D_-(t)\,dx + \int_{-\infty}^{0} d(x-a_-t)\,dx$$
$$+\int_{0}^{+\infty}(1+x)^{-1}e^{-\frac{x^2}{2Mt}}D_-(t)\,dx$$
$$+\int_{0}^{+\infty}(1+x)^{-2}e^{-\frac{x^2}{2Mt}}D_+(t)\,dx$$
$$+\int_{0}^{+\infty}(1+x)^{-r}\wedge t^{-1}(1+x)^{2-r}\,dx$$
$$\overset{\text{resp.}}{\leqq} CD_-(t)+CD(t)+CD_-(t)\log(2+t)+D_+(t)$$
$$+\int_{0}^{+\infty}\left((1+x)^{-r}\right)^{\frac{3-r+\varepsilon}{2}}\left(t^{-1}(1+x)^{2-r}\right)^{1-\frac{3-r+\varepsilon}{2}}dx$$
$$\leqq CD_+(t)(1+t)^{\varepsilon/2}.$$

For $p>1$ the analysis is similar, and we consider only $(0<\varepsilon<r(p-1))$

$$\left(\int_{0}^{+\infty}\left((1+x)^{-r}\wedge t^{-1}(1+x)^{2-r}\right)^p dx\right)^{1/p}$$
$$\leqq \left(\int_{0}^{+\infty}\left((1+x)^{-r}\right)^{\frac{1+2p-rp+\varepsilon}{2}}\left(t^{-1}(1+x)^{2-r}\right)^{p-\frac{1+2p-rp+\varepsilon}{2}}dx\right)^{1/p}$$
$$\leqq Ct^{\frac{1}{2}-\frac{rp}{2}+\frac{\varepsilon}{2}} \leqq CD_+(t). \quad \square$$

Remark. We conclude the introduction by noting that the energy methods of [M1], [MN4], [Ni] yield estimates that, put in the present notation, compare as (taken from [Ni], Theorem 2.3):

For $r>2$ (required for decay)

$$\|u(t,x)-\bar{u}(x)\|_{L^\infty} \leqq C\begin{cases} t^{1-r/2}, & 2<r\leqq 3,\\ t^{-1/2}, & r\geqq 3\end{cases}$$

(no bounds in other L^p norms). In particular, our methods accomodate slower decaying data, by factor $|x|$, and realize faster decay on the perturbation by a rate $t^{-1/2}\log t$.

Caveat. For brevity, I have left off a full statement of the results of [M1], [MN4], [Ni], which involve L^2 spaces unsymmetrically weighted so that slower-decaying data ($|x|^{-r}, r>3/2$) is accomodated on the compressive side. I would mention also that (unlike here) the papers of Matsumura and Nishihara, and of Nishikawa, developed results for *all* orders of degeneracy. The paper of Mei, while less general, warrants note as the first such result for degenerate waves.

2. Structure of degenerate shock waves

We begin by collecting some observations regarding the behavior of degenerate shock waves, most of which appear (in more general forms) in [MN4] or [M1]. For definiteness we will assume throughout that $f'(u_-) > 0$ and $f'(u_+) = 0$.

Proposition 2.1. *Suppose* $f(u)$ *and* $u_\pm$ *in* (1.1) *satisfy the Rankine-Hugoniot condition* $(s = 0)$, $f \in C^{k+1}(\mathbb{R})$, $k \geqq 1$, *and* $f'(u_+) = f''(u_+) = \cdots = f^{(k)}(u_+) = 0$, *with* $f^{(k+1)}(u_+) \neq 0$. *Suppose also that Oleĭnik's entropy condition holds*:

$$f(u) - f(u_\pm)\begin{cases} < 0, & u_+ < u < u_-, \\ > 0, & u_- < u < u_+. \end{cases}$$

Then there exists a traveling wave solution $\bar{u}(x)$ *of* (1.1) *so that* $\bar{u}(\pm\infty) = u_\pm$, *unique up to a shift. Moreover, we have*

$$|\bar{u}(x) - u_-| = \mathbf{O}(e^{-\alpha|x|}), \quad x \leqq 0, \quad \text{and} \quad |\bar{u}(x) - u_+| = \mathbf{O}(|x|^{-1/k}), \quad x \geqq 0.$$

Proof. We have from (1.1) that $\bar{u}_x$ must satisfy $\bar{u}_{xx} = f(\bar{u}(x))_x$, so that $\bar{u}_x = f(\bar{u}) - f(u_\pm)$, which admits a unique solution up to shift by standard ODE results [MM]. For f sufficiently smooth, we obtain the Taylor expansion as $x \to \infty$,

$$\bar{u}_x = f(\bar{u}) - f(u_+) = f'(u_+)(\bar{u}(x) - u_+) + \frac{f''(u_+)}{2}(\bar{u}(x) - u_+)^2 + \cdots.$$

In the event that the order of degeneracy is $k = 1$, we have

$$(\bar{u} - u_+)_x = \frac{f''(u_+)}{2}(\bar{u}(x) - u_+)^2 + \mathbf{O}((\bar{u} - u_+)^3),$$

so that $\bar{u}(x) = \mathbf{O}(|x|^{-1})$. Each assertion of Proposition 2.1 is proved similarly. $\square$

Proposition 2.2. *Under the hypotheses of Proposition* 2.1, *for* $a(x) := f'(\bar{u}(x))$, $n = 0, 1$, *we have* (*for any order of degeneracy*)

(i) $$\left|\frac{\partial^n}{\partial x^n}(a(x) - a_-)\right| = \mathbf{O}(e^{-\alpha|x|}), \quad x \leqq 0,$$

(ii) $$\left|\frac{\partial^n}{\partial x^n} a(x)\right| = \mathbf{O}(|x|^{-1-n}), \quad x \geqq 0.$$

Moreover, in the case of degeneracy of order 1,

(iii) $$\gamma_+(x) := \frac{2\bar{u}_x}{\bar{u} - u_+} - f'(\bar{u}(x)) = \mathbf{O}(|x|^{-2}),$$

(iv) $$\bar{u}_x(x)x + (\bar{u}(x) - u_+) = \mathbf{O}(|x|^{-2}).$$

Proof. Proposition 2.2 is proved by Taylor expansion of $f'(\bar{u}(x))$ about u_+. $\square$

3. ODE estimates

Equations of type (1.3) readily lend themselves to study through the behavior of solutions of the eigenvalue ODE

$$Lv = \lambda v, \tag{3.1}$$

where we recall that L represents the linear operator for the integrated equation, $Lv := v_{xx} - a(x)v_x$. Following [H1–3], [ZH] and others, our approach will be to solve the associated Green's function equation

$$(L - \lambda)v = -\delta_y(x). \tag{3.2}$$

If we let $R(\lambda) := (\lambda I - L)^{-1}$ denote the resolvent of L, then (3.2) is solved by the Green's function

$$G_\lambda(x, y) = R(\lambda)\delta_y(x)$$

wherever $R(\lambda)$ is defined.

The computation of $G_\lambda(x, y)$ is standard (see [CH], [BF], for example) in terms of decaying solutions of (3.1). Our notation will be to let $\phi^\pm$ denote the (unique) decay modes of (3.1) at $\pm\infty$, and $\psi^\pm$ a choice of (nonunique) growth solutions at $\pm\infty$. We can directly compute the asymptotic growth and decay rates of ϕ and ψ from (3.1) by noting that at $\pm\infty$ (3.1) becomes

$$v_{xx} - a_\pm v_x - \lambda v = 0,$$

where $a(-\infty) = a_- > 0$ and $a(+\infty) = a_+ = 0$, so that solutions of the form $v \sim e^{\mu x}$ give $\mu^2 - a_\pm \mu - \lambda = 0$, which can readily be solved for the $-\infty$ modes

$$\mu_1^-(\lambda) = \frac{a_- - \sqrt{a_-^2 + 4\lambda}}{2}; \quad \mu_2^-(\lambda) = \frac{a_- + \sqrt{a_-^2 + 4\lambda}}{2},$$

and the $+\infty$ modes $\pm\sqrt{\lambda}$. We immediately see that a crucial aspect of the analysis is that while the modes at $-\infty$ are analytic in a neighborhood of the origin, the modes at $+\infty$ are not.

In terms of the above notation, the Green's function $G_\lambda(x, y)$ for (3.1) takes the form

$$G_\lambda(x, y) = \begin{cases} \dfrac{\phi^+(x)\phi^-(y)}{W_\lambda(y)}, & x \geqq y, \\[2ex] \dfrac{\phi^+(y)\phi^-(x)}{W_\lambda(y)}, & x \leqq y, \end{cases}$$

where $W_\lambda(y)$ denotes the usual Wronskian,

$$W_\lambda(y) = \phi^+(y)\phi^{-\prime}(y) - \phi^{+\prime}(y)\phi^-(y),$$

and consequently satisifies Abel's equation,

$$\partial_y W_\lambda(y) = a(y) W_\lambda(y).$$

The Evans function of [AGJ] is here $W_\lambda(0)$.

Finally, we will achieve the desired estimates on $G(t,x;y)$ from Dunford's Integral (the resolvent formula for the semigroup, or simply the Laplace transform under certain conditions) [Y], which gives

$$G(t,x;y) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} G_\lambda(x,y)\, d\lambda,$$

where Γ is a contour enclosing the entire spectrum of L (possibly passing through the point at ∞).

The challenge of particular interest in extending the pointwise analysis of [H1–3], [ZH] to degenerate shock waves revolves around the algebraic (non-integrable) decay of the coefficients of the linearized equation (3.1). In the case of Lax shocks (and similarly for undercompressive shocks, which arise in scalar equations of higher order [HZ1]), we have $f'(u_\pm) \neq 0$, and consequently $\bar{u}(x)$ decays to its endstates at an exponential rate (see Proposition 2.1). In such cases, standard methods yield sufficiently sharp estimates for solutions of (3.1) [C], [H1–2], [S]. Two new difficulties arise in the case of degenerate waves. First, $\bar{u}(x)$ decays to its endstates at algebraic rate, so that (in (3.1)) $a(x) = \mathbf{O}(|x|^{-1})$; and second, the growth and decay modes of (3.1) coalesce as $\lambda \to 0$. While the non-integrability of $a(x)$ can generally be treated by the methods of Levinson [Le], also [C], [F], this coalescence of eigenvalues degrades the description in the crucial limit as $\lambda \to 0$. Hence, we provide estimates of the required accuracy in Lemma 3.1, below.

The extension of our analysis to the case of degenerate shock waves of higher order and to the case of degenerate shock waves in the presence of non-constant viscosity $b(\cdot)$ depends upon the derivation of similar ODE estimates in these cases. The essential problem involved with obtaining such estimates is the transition of behavior as $\lambda \to 0$. For $|\lambda| \geqq \delta > 0$, equations of form $v_{xx} - a(x)v_x = \lambda v$, where $a(x) = \mathbf{O}(|x|^{-1})$ (true for all orders of degeneracy) may be scaled as $v(x) \sim \sqrt{\bar{u}_x(x)} e^{\pm\sqrt{\lambda}x} w(x)$, where $w(+\infty) \sim 1$ (see, for example, [Z1]). That is, if we make the substitution $v(x) = e^{\int^x a(s)/2\, ds} u(x)$, we arrive at an equation of the form $u_{xx} - \big(a(x)^2/4 - a'(x)/2\big)u = \lambda u$, with an integrable coefficient. On the other hand, it is clear that at $\lambda = 0$, $v(x) = \bar{u}(x) - u_+$. We take advantage in the case of degeneracy of order one of the observation that $\bar{u}_x(x) \sim \big(\bar{u}(x) - u_+\big)^2$ so that these scalings are the same; however, for higher order degeneracies, the behavior is more difficult to match and the appropriate scaling more cumbersome to work with (see discussion in Section 6 on further work).

Lemma 3.1. *Under the assumptions of Theorem* 1.1 *and for some constant* M_s, *we have the following estimates on the growth and decay modes* ($\phi^\pm$ *and* $\psi^\pm$) *of* (3.1).

(i) $(x \leqq 0)$ *For all* $|\lambda| \leqq M_s$ *and to the right of* Γ_d,

$$\phi^-(x) = e^{-\mu_1^- x}\big(\bar{u}(x) - u_-\big)\big(1 + e_1^-(x,\lambda)\big),$$

$$\phi^{-\prime}(x) = e^{-\mu_1^- x}\big(\bar{u}(x) - u_-\big)\left(-\mu_1^- + \frac{\bar{u}_x}{\bar{u}(x) - u_-} + e_2^-(x,\lambda) + \frac{\bar{u}_x}{\bar{u}(x) - u_-} e_1^-(x,\lambda)\right),$$

$$\frac{\partial^n}{\partial x^n}\psi^-(x) = e^{\mu_1^- x}\big((\mu_1^-)^n + \mathbf{O}(e^{-\alpha|x|})\big).$$

(ii) $(x \geqq 0)$ *For all* $|\lambda| \leqq M_s$, *to the right of* Γ_d *and off the negative real axis,*

$$\phi^+(x) = e^{-\sqrt{\lambda}x}\big(\bar{u}(x) - u_+\big)\big(1 + e_1^+(x,\lambda)\big),$$

$$\phi^{+\prime}(x) = e^{-\sqrt{\lambda}x}\big(\bar{u}(x) - u_+\big)\left(-\sqrt{\lambda} + \frac{\bar{u}_x}{\bar{u}(x) - u_+} + e_2^+(x,\lambda) + \frac{\bar{u}_x}{\bar{u}(x) - u_+} e_1^+(x,\lambda)\right),$$

$$\psi^+(x) = e^{\sqrt{\lambda}x}\big(\bar{u}(x) - u_+\big)\big(1 + \tilde{e}_1^+(x,\lambda)\big),$$

$$\psi^{+\prime}(x) = e^{\sqrt{\lambda}x}\big(\bar{u}(x) - u_+\big)\left(\sqrt{\lambda} + \frac{\bar{u}_x}{\bar{u}(x) - u_+} + \tilde{e}_2^+(x,\lambda) + \frac{\bar{u}_x}{\bar{u}(x) - u_+} \tilde{e}_1^+(x,\lambda)\right),$$

where

$$e_1^-(x,\lambda), e_2^-(x,\lambda) = \mathbf{O}(\lambda)\mathbf{O}(e^{-\eta|x|}),$$

while $(\wedge = \min)$

$$e_1^+(x,\lambda), \tilde{e}_1^+(x,\lambda) = \mathbf{O}(\sqrt{\lambda}\log\lambda) \wedge \mathbf{O}_1(|x|^{-1}),$$

and

$$e_2^+(x,\lambda), \tilde{e}_2^+(x,\lambda) = \mathbf{O}(\sqrt{\lambda})\mathbf{O}_1(|x|^{-1}).$$

Remark. The odd form of the error estimate $e_1^+(x,\lambda)$ is a consequence of the fact that equations of form $v_{xx} + (\kappa/x)v_x = \lambda v$—which essentially govern ϕ and ψ, since $a(x) \sim x^{-1}$—have the property that they can be scaled to depend only upon $\xi := \sqrt{\lambda}x$, leading to trade-off of $\lambda \to 0$ decay versus $x \to \infty$ blow-up, or vice versa. The $\log t$ term in $G(t,x;y)$ of Theorem 1.1 can be traced back to the $\log\lambda$ term arising in these estimates.

Proof. As the analysis of each case is similar, we will develop the result only in the case of $\phi^+(y)$. We begin with the ODE

$$v_{xx} - a(x)v_x = \lambda v, \tag{3.3}$$

and look for solutions of the form $v(x) = \big(\bar{u}(x) - u_+\big)u(x)$, natural in this context since the eigenvector at $\lambda = 0$ of (3.3) is $\bar{u}(x)$. We obtain

$$\big((\bar{u}(x) - u_+)u\big)_{xx} - a(x)\big((\bar{u} - u_+)u\big)_x = \lambda\big(\bar{u}(x) - u_+\big)u,$$

or

$$\big(\bar{u}(x) - u_+\big)u_{xx} + 2\bar{u}_x u_x + \bar{u}_{xx}u - a(x)\big(\bar{u}(x) - u_+\big)u_x - a(x)\bar{u}_x u = \lambda(\bar{u} - u_+)u.$$

Observing that $\bar{u}_{xx} - a(x)\bar{u}_x = 0$ and dividing by $\bar{u}(x) - u_+$, we obtain

$$u_{xx} + \gamma_+(x)u_x = \lambda u, \tag{3.4}$$

where $\gamma_+(x) := \dfrac{2\bar{u}_x}{\bar{u} - u_+} - f'(\bar{u})$, which according to Proposition 2.2 is $\mathbf{O}(|x|^{-2})$.

Lemma 3.1 is now straightforward to prove, as in the ODE results of [ZH], by writing (3.4) as a system and proceeding via iteration and the contraction mapping principle.

Let $U_1(x) := u(x)$ and $U_2(x) := u_x(x)$, thus rendering (3.4) the system $\left(U = \begin{pmatrix} U_1(x) \\ U_2(x) \end{pmatrix}\right)$

$$U'(x) = A(\lambda)U(x) + E(x)U(x),$$

where

$$A(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \quad \text{and} \quad E(x) = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma_+(x) \end{pmatrix}.$$

We look for solutions of the form $U(x) = e^{-\sqrt{\lambda}x}Z(x)$ and observe that $Z(x)$ satisfies

$$Z'(x) - \big(A(\lambda) + I\sqrt{\lambda}\big)Z(x) = E(x)Z(x),$$

and hence the integral equation

$$Z(x) = Z(+\infty) - \int_x^{+\infty} e^{-(A(\lambda)+I\sqrt{\lambda})(\xi - x)}E(\xi)Z(\xi)\,d\xi,$$

where $Z(+\infty) = \begin{pmatrix} 1 \\ -\sqrt{\lambda} \end{pmatrix}$ and

$$e^{-(A(\lambda)+I\sqrt{\lambda})(\xi-x)} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}e^{2\sqrt{\lambda}(x-\xi)} & -\frac{1}{2\sqrt{\lambda}} + \frac{1}{2\sqrt{\lambda}}e^{2\sqrt{\lambda}(x-\xi)} \\ -\frac{\sqrt{\lambda}}{2} + \frac{\sqrt{\lambda}}{2}e^{2\sqrt{\lambda}(x-\xi)} & \frac{1}{2} + \frac{1}{2}e^{2\sqrt{\lambda}(x-\xi)} \end{pmatrix}.$$

Combining the above expressions, we have

$$\begin{pmatrix} Z_1(x) \\ Z_2(x) \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{\lambda} \end{pmatrix} + \int_x^{+\infty} \begin{pmatrix} \gamma_+(\xi)Z_2(\xi)\left(-\frac{1}{2\sqrt{\lambda}} + \frac{1}{2\sqrt{\lambda}}e^{-2\sqrt{\lambda}(\xi-x)}\right) \\ \gamma_+(\xi)Z_2(\xi)\left(\frac{1}{2} + \frac{1}{2}e^{-2\sqrt{\lambda}(\xi-x)}\right) \end{pmatrix} d\xi.$$

Noting that $Z_2(x)$ is decoupled, we look for solutions of the form $Z_2(x)/\sqrt{\lambda} \in L^\infty[N, +\infty)$ for some N sufficiently large. The contraction mapping theorem combined with the observation from Proposition 2.2 that $\gamma_+(x) = \mathbf{O}(|x|^{-2})$ for first order degenerate shock waves implies that such a $Z_2(x)$ exists. $Z_1(x) \in L^\infty[N, +\infty)$ is immediate.

For x sufficiently large, then, we have verified the $Z_1(x)$ representation

$$Z_1(x) = 1 + \int_x^{+\infty} \gamma_+(\xi) Z_2(\xi) \left(-\frac{1}{2\sqrt{\lambda}} + \frac{1}{2\sqrt{\lambda}} e^{2\sqrt{\lambda}(x-\xi)} \right) d\xi.$$

Noting that clearly $Z_2(x) = -\sqrt{\lambda} + \sqrt{\lambda}\mathbf{O}(|x|^{-1})$ for all $x \in [0, +\infty)$ (by standard Volterra iteration on the bounded x) we study the error

$$\int_x^{+\infty} \gamma_+(\xi) \left(\frac{1}{2} - \frac{1}{2} e^{2\sqrt{\lambda}(x-\xi)} \right) d\xi$$
$$\overset{\text{parts}}{=} \int_x^{+\infty} \left(\int \gamma_+ \right) \left(\sqrt{\lambda} e^{\sqrt{\lambda}(x-\xi)} \right) d\xi.$$

The estimates on $e_1^+(x, \lambda)$ are now clear from the integrability of

$$\sqrt{\lambda} e^{-\sqrt{\lambda}(\xi - x)} \left(\mathbf{O}_1(|x|^{-1}) \right),$$

and from the observation that

$$\int_x^{+\infty} \mathbf{O}(x^{-1}) e^{-\sqrt{\lambda}(\xi - x)}\, d\xi \leqq C \lambda^{-1/2} \log \lambda. \quad \square$$

Remark. In the presence of exponentially decaying coefficients, it has been shown that near certain critical branch points, the Evans function can be extended analytically onto an appropriate Riemann manifold [GZ], [KR]. Such extensions have been carried out for general systems by insuring that a certain wedge product of ODE solutions that decay at $+\infty$ $(\eta(\lambda))$ and a certain wedge product of ODE solutions that decay at $-\infty$ $(\zeta(\lambda))$ can be so extended. The Evans function, then, which is itself a wedge product of $\eta(\lambda)$ and $\zeta(\lambda)$, inherits this analyticity. In the scalar case considered here, η reduces to $\phi^+(0)$ and ζ reduces to $\phi^-(0)$, so that the Evans function becomes

$$D(\lambda) := \phi^+(0) \wedge \phi^-(0) = \phi^+(0)\phi^{-\prime}(0) - \phi^{+\prime}(0)\phi^-(0).$$

Extension to the obvious Riemann manifold $\sqrt{\lambda} = \zeta$ would require differentiability in ζ on $D(\zeta)$, which would require differentiability in ζ on the error

$$\int_0^{+\infty} \gamma_+(\xi) Z_2(\xi) \left(\frac{1}{2} + \frac{1}{2} e^{-2\zeta\xi} \right) d\xi. \tag{3.5}$$

Observing that each ζ-derivative on (3.5) would produce algebraic growth of size ξ, we see that for $\gamma_+(\xi) = \mathbf{O}(e^{-\alpha|\xi|})$ such an analysis could be carried out. For $\gamma_+(\xi) = \mathbf{O}_1(|\xi|^2)$, however, such an analysis fails. In light of the observation that the degenerate case is similar to the heat equation, in which a Fourier tranform analysis never requires that a contour pass through the branch on the negative-real axis, we will assume that no such extension is possible here, and arrange our contours accordingly. In fact, we find that no greater decay rate on the perturbation could be obtained through assuming the Evans function analytic. As to whether or not the Evans function indeed is analytic in this case, we make no claim.

Before applying Lemma 3.1 toward some non-trivial observations regarding the Wronskian, we state without proof the following large $|\lambda|$ ODE estimates, which may be proved as in [H1–2], [ZH].

Lemma 3.2. *For $|\lambda| \geqq M_l$ and to the right of Γ_d, we have $(k = 0, 1)$*

$$\frac{\partial^k}{\partial x^k}\phi^{\pm}(x) = (\mp\sqrt{\lambda})^k K_{\pm}(x)\big(1 + \mathbf{O}(|\lambda|^{-1/2})\big),$$

where $x \in \mathbb{R}$ and $K_{\pm}(x)$ is bounded in λ.

A critical feature of the degenerate shock case is that while $\lambda = 0$ lies in both the point and essential spectrum, as in the Lax case, it is also a branch point of the Evans function. In the following key lemma, we prove that while the Wronskian is not analytic at zero, its behavior remains $\mathbf{O}(1)$, as in the Lax case.

Lemma 3.3. *For λ sufficiently small, to the right of Γ_d, and off the negative real axis, we have*

$$D(\lambda) = -\bar{u}_x(0)(u_+ - u_-) + \mathbf{O}(\sqrt{\lambda}\log\lambda).$$

Moreover, for the scattering coefficients $A(\lambda)$ and $B(\lambda)$ for $\phi^-(x) = A(\lambda)\phi^+(x) + B(\lambda)\psi^+(x)$, $x \geqq 0$ we have

$$A(\lambda) = \frac{\bar{u}_x(0)(u_+ - u_-) + \mathbf{O}(\sqrt{\lambda}\log\lambda)}{G(\lambda)},$$

$$B(\lambda) = \frac{-\bar{u}_x(0)(u_+ - u_-) + \mathbf{O}(\sqrt{\lambda}\log\lambda)}{G(\lambda)},$$

where $G(\lambda) \sim \sqrt{\lambda}$, by which we mean there exists a $C > 0$ such that $(1/C)\sqrt{\lambda} \leqq G(\lambda) \leqq C\sqrt{\lambda}$. Additionally, for the scattering coefficients $E(\lambda), F(\lambda)$ for $\phi^+(x) = E(\lambda)\phi^-(x) + F(\lambda)\psi^-(x)$, $x \leqq 0$, $E(\lambda), F(\lambda) = \mathbf{O}(1)$ as $|\lambda| \to 0$, and $F(\lambda)/W_\lambda(y)$ is analytic in a neighborhood of $\lambda = 0$ (including the negative-real axis).

Proof. For the Evans function, we compute directly from the definition and Lemma 3.1. We have

$$\begin{aligned}
D(\lambda) &:= \phi^+(0)\phi^{-\prime}(0) - \phi^{+\prime}(0)\phi^-(0) \\
&= \big(\bar{u}(0) - u_+\big)\big(\bar{u}(0) - u_-\big)\big(1 + e_1^+(0,\lambda)\big) \\
&\quad \times \left(-\mu_1^- + \frac{\bar{u}_x(0)}{\bar{u}(0) - u_-} + e_2^-(0,\lambda) + \frac{\bar{u}_x(0)}{\bar{u}(0) - u_-}e_1^-(0,\lambda)\right) \\
&\quad - \big(\bar{u}(0) - u_+\big)\big(\bar{u}(0) - u_-\big)\big(1 + e_1^-(0,\lambda)\big) \\
&\quad \times \left(-\sqrt{\lambda} + \frac{\bar{u}_x(0)}{\bar{u}(0) - u_+} + e_2^+(0,\lambda) + \frac{\bar{u}_x(0)}{\bar{u}(0) - u_+}e_1^+(0,\lambda)\right)
\end{aligned}$$

$$= \big(\bar{u}(0) - u_+\big)\big(\bar{u}(0) - u_-\big)\left(\frac{\bar{u}_x(0)}{\bar{u}(0) - u_-} - \frac{\bar{u}_x(0)}{\bar{u}(0) - u_+} + \mathbf{O}(\sqrt{\lambda}\log\lambda)\right)$$

$$= -\bar{u}_x(0)(u_+ - u_-) + \mathbf{O}(\sqrt{\lambda}\log\lambda).$$

For the computation of $A(\lambda)$ and $B(\lambda)$ we augment $\phi^-(x) = A(\lambda)\phi^+(x) + B(\lambda)\psi^+(x)$ with its derivative $\phi^{-\prime}(x) = A(\lambda)\phi^{+\prime}(x) + B(\lambda)\psi^{+\prime}(x)$, and apply Cramer's rule to the resulting system in order to obtain the representations

$$A(\lambda) = \frac{\det\begin{pmatrix} \phi^- & \psi^+ \\ \phi^{-\prime} & \psi^{+\prime} \end{pmatrix}}{\det\begin{pmatrix} \phi^+ & \psi^+ \\ \phi^{+\prime} & \psi^{+\prime} \end{pmatrix}} \quad \text{and} \quad B(\lambda) = \frac{\det\begin{pmatrix} \phi^+ & \phi^- \\ \phi^{+\prime} & \phi^{-\prime} \end{pmatrix}}{\det\begin{pmatrix} \phi^+ & \psi^+ \\ \phi^{+\prime} & \psi^{+\prime} \end{pmatrix}}.$$

Proceeding as above we observe that

$$\phi^-(0)\psi^{+\prime}(0) - \phi^{-\prime}(0)\psi^+(0) = \bar{u}_x(0)(u_+ - u_-) + \mathbf{O}(\sqrt{\lambda}\log\lambda),$$

and (by virtue of considering also the $x \to \infty$ limit)

$$\phi^+(0)\psi^{+\prime}(0) - \phi^{+\prime}(0)\psi^+(0) \sim \sqrt{\lambda}.$$

Finally, for $\phi^+(x) = E(\lambda)\phi^-(x) + F(\lambda)\psi^-(x)$, the claims can be observed from

$$E(\lambda) = \frac{\det\begin{pmatrix} \phi^+ & \psi^- \\ \phi^{+\prime} & \psi^{-\prime} \end{pmatrix}}{\det\begin{pmatrix} \phi^- & \psi^- \\ \phi^{-\prime} & \psi^{-\prime} \end{pmatrix}} \quad \text{and} \quad F(\lambda) = \frac{\det\begin{pmatrix} \phi^- & \phi^+ \\ \phi^{-\prime} & \phi^{+\prime} \end{pmatrix}}{\det\begin{pmatrix} \phi^- & \psi^- \\ \phi^{-\prime} & \psi^{-\prime} \end{pmatrix}},$$

and the analyticity of $\phi^-(x)$ and $\psi^-(x)$ to the right of Γ_d. □

We now develop estimates on the ODE Green's function $G_\lambda(x, y)$. Through the estimates of Lemma 3.1 these can be obtained with varying levels of precision. For convenience, we will state estimates here in exactly the form later required by the analysis.

Lemma 3.4. *Under the assumptions of Theorem* 1.1 *and for* $|\lambda| \leqq M_s$, *we have the following estimates, for which terms containing* $\mathbf{O}_a$ *are analytic to the right of* Γ_d, *while the remaining terms are analytic to the right of* Γ_d *and away from the negative real axis.*

(i) $y \leqq x \leqq 0$:

$$G_\lambda(x, y) = \mathbf{O}_a(1)e^{\mu_1^-(x-y)} + \mathbf{O}(e^{-\eta|x|})e^{\mu_1^-(x-y)},$$

$$\partial_x G_\lambda(x, y) = \mathbf{O}_a(1)\mu_1^- e^{\mu_1^-(x-y)} + \mathbf{O}(e^{-\eta|x|})e^{\mu_1^-(x-y)}.$$

(ii) $x \leqq y \leqq 0$:

$$G_\lambda(x, y) = \mathbf{O}_a(1)e^{\mu_2^-(x-y)} + \mathbf{O}(1)e^{\mu_2^- x - \mu_1^- y},$$

$$\partial_x G_\lambda(x, y) = \mathbf{O}_a(1)e^{\mu_2^-(x-y)} + \mathbf{O}(1)e^{\mu_2^- x - \mu_1^- y}.$$

(iii) $x \leqq 0 \leqq y$:

$$G_\lambda(x, y) = \mathbf{O}(e^{-\eta|x|})\mathbf{O}_1(|y|)e^{-\mu_1^- x - \sqrt{\lambda} y},$$

$$\partial_x G_\lambda(x, y) = \mathbf{O}(e^{-\eta|x|})\mathbf{O}_1(|y|)e^{-\mu_1^- x - \sqrt{\lambda} y}.$$

(iv) $y \leqq 0 \leqq x$:

$$G_\lambda(x, y) = \mathbf{O}_1(|x|^{-1})e^{-\sqrt{\lambda}x - \mu_1^- y},$$

$$\partial_x G_\lambda(x, y) = \mathbf{O}_1(|x|^{-2})e^{-\sqrt{\lambda}x - \mu_1^- y} + \mathbf{O}(\sqrt{\lambda})\mathbf{O}_1(|x|^{-1})e^{-\sqrt{\lambda}x - \mu_1^- y}.$$

(v) $0 \leqq y \leqq x$:

$$G_\lambda(x, y) = \big[\mathbf{O}(|\lambda|^{-1/2}) \wedge \big(\mathbf{O}_1(y) + \mathbf{O}(\log \lambda)\big)\big]\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\sqrt{\lambda}(x-y)},$$

$$\partial_x G_\lambda(x, y) = \mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\sqrt{\lambda}(x-y)} + \mathbf{O}(\log \lambda)\mathbf{O}_1(|x|^{-2})\mathbf{O}_1(|y|)e^{-\sqrt{\lambda}(x-y)}.$$

(vi) $0 \leqq x \leqq y$:

$$G_\lambda(x, y) = \big[\mathbf{O}(|\lambda|^{-1/2}) \wedge \big(\mathbf{O}_1(x) + \mathbf{O}(\log \lambda)\big)\big]\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\sqrt{\lambda}(y-x)},$$

$$\partial_x G_\lambda(x, y) = \mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\sqrt{\lambda}(y-x)} + \mathbf{O}(\log \lambda)\mathbf{O}_1(|x|^{-2})\mathbf{O}_1(|y|)e^{-\sqrt{\lambda}(y-x)}.$$

Proof. In Case (i) we have $y \leqq x \leqq 0$, thus

$$G_\lambda(x, y) = \frac{\phi^+(x)\phi^-(y)}{W_\lambda(y)},$$

where for $x \leqq 0$ we must write

$$\phi^+(x) = E(\lambda)\phi^-(x) + F(\lambda)\psi^-(x).$$

We recall that according to Lemma 3.3, $F(\lambda)/W_\lambda(y)$ is analytic in a neighborhood of $\lambda = 0$. We have

$$\begin{aligned} G_\lambda(x, y) &= \frac{E(\lambda)\phi^-(x)\phi^-(y)}{W_\lambda(y)} + \frac{F(\lambda)\psi^-(x)\phi^-(y)}{W_\lambda(y)} \\ &= \mathbf{O}(1)\frac{E(\lambda)}{W_\lambda(y)}e^{-\mu_1^-(x+y)}\big(\bar{u}(x) - u_-\big)\big(\bar{u}(y) - u_-\big) \\ &\quad + \mathbf{O}(1)\frac{F(\lambda)}{W_\lambda(y)}e^{-\mu_1^-(x-y)}\big(\bar{u}(y) - u_-\big). \end{aligned}$$

Observing that

$$W_\lambda(y) = W_\lambda(0)e^{\int_0^y a(s)\, ds} = W_\lambda(0)\mathbf{O}(1)e^{(\mu_1^- + \mu_2^-)y},$$

and similarly

$$\big(\bar{u}(x) - u_-\big) = \mathbf{O}(1)e^{(\mu_1^- + \mu_2^-)x},$$

we have our first claim.

It is worth mentioning that by proceeding slightly more carefully we could divide $G_\lambda(x, y)$ into *exactly* its constant-coefficient approximation plus an exponentially decaying error (see [HoZ1–2], [Z3–4]). Indeed, a careful inspection of Lemma 7.1 of [ZH], where the diagonals $e_{kk}(\lambda)$ are closely related to our scattering coefficients here, reveals that a similar observation may be made in the general case of systems. Such refinement on the term

$$\frac{E(\lambda)\phi_-(x)\phi^-(y)}{W_\lambda(y)}$$

would serve to expand it as a function analytic on the negative-real axis, plus an error with additional decay $\mathbf{O}(\sqrt{\lambda}\log\lambda)$—improving our eventual decay rate on this side by a factor of $t^{-1/2}$. As previously mentioned, however, the final decay rate is determined by the degenerate side, so no such refinement is made.

Cases (i)–(iv), with derivatives, follow similarly. For further details the reader is referred to [H1–2]. We only mention here that in these cases the $\log\lambda$ error does not arise.

For $0 \leqq y \leqq x$, we have

$$G_\lambda(x, y) = \frac{A(\lambda)\phi^+(x)\phi^+(y)}{W_\lambda(y)} + \frac{B(\lambda)\phi^+(x)\psi^+(y)}{W_\lambda(y)},$$

where $A(\lambda)$ and $B(\lambda)$ are as in Lemma 3.3. Hence,

$$\begin{aligned} G_\lambda(x,y) = {} & \frac{A(\lambda)}{W_\lambda(y)} e^{-\sqrt{\lambda}(x+y)}\big(\bar{u}(x) - u_+\big)\big(\bar{u}(y) - u_+\big)\big(1 + e_1^+(x,\lambda)\big)\big(1 + e_1^+(y,\lambda)\big) \\ & + \frac{B(\lambda)}{W_\lambda(y)} e^{-\sqrt{\lambda}(x-y)}\big(\bar{u}(x) - u_+\big)\big(\bar{u}(y) - u_+\big)\big(1 + e_1^+(x,\lambda)\big)\big(1 + \tilde{e}^+(y,\lambda)\big). \end{aligned}$$

We observe from Lemma 3.3 that in this delicate situation, there is cancellation between these terms, and thus they cannot be treated independently. Dividing the analysis into terms that cancel and terms that do not, we have an estimate on $G_\lambda(x, y)$ of

$$\begin{aligned} & \big(1 + e_1^+(x,\lambda)\big)\frac{\big(\bar{u}(x) - u_+\big)\big(\bar{u}(y) - u_+\big)}{W_\lambda(y)}\big[A(\lambda)e^{-\sqrt{\lambda}(x+y)} + B(\lambda)e^{-\sqrt{\lambda}(x-y)}\big] \\ & \quad + \mathbf{O}(1)\frac{\big(\bar{u}(x) - u_+\big)\big(\bar{u}(y) - u_+\big)}{W_\lambda(y)}\big[A(\lambda)e_1^+(y,\lambda)e^{-\sqrt{\lambda}(x+y)} + B(\lambda)\tilde{e}_1^+(y,\lambda)e^{-\sqrt{\lambda}(x-y)}\big]. \end{aligned}$$

Recalling that $A(\lambda), B(\lambda) = \mathbf{O}(|\lambda|^{-1/2})$, the second term immediately yields an estimate by

$$\big[\big(\mathbf{O}(|\lambda|^{-1/2})\mathbf{O}_1(|y|^{-1})\big) \wedge \mathbf{O}(\log\lambda)\big]\mathbf{O}_1(|y|)\mathbf{O}_1(|x|^{-1})e^{-\sqrt{\lambda}(x-y)}.$$

For the first term we note the following interesting cancellation estimate.

$$\begin{aligned}A(\lambda)e^{-\sqrt{\lambda}(x+y)} + B(\lambda)e^{-\lambda(x-y)} &= \left[\frac{\bar{u}_x(0)(u_+ - u_-)}{G(\lambda)} + \mathbf{O}(\log\lambda)\right]e^{-\sqrt{\lambda}(x+y)}\\ &\quad + \left[\frac{-\bar{u}_x(0)(u_+ - u_-)}{G(\lambda)} + \mathbf{O}(\log\lambda)\right]e^{-\sqrt{\lambda}(x-y)}\\ &= \frac{\bar{u}_x(0)(u_+ - u_-)}{G(\lambda)}(e^{-\sqrt{\lambda}(x+y)} - e^{-\sqrt{\lambda}(x-y)}) + \mathbf{O}(\log\lambda)e^{-\sqrt{\lambda}(x-y)}\\ &= \left[\mathbf{O}(|\lambda|^{-1/2}) \wedge \left(\mathbf{O}_1(|y|) + \mathbf{O}(\log\lambda)\right)\right]e^{-\sqrt{\lambda}(x-y)}.\end{aligned}$$

We see that a key point of the analysis is a trade-off between y-growth and $|\lambda|$ blow-up. We shall observe that while either effect serves to decrease our final time decay by the same amount, y-growth yields optimal behavior in the case of fast-decaying initial data. Since $A(\lambda), B(\lambda) = \mathbf{O}(|\lambda|^{-1/2})$, in the case $|\lambda|^{-1/2} \leqq 1 + |y|$ the inequality is clear. If, on the other hand, $|\lambda|^{-1/2} \geqq 1 + |y|$, we have $|\sqrt{\lambda}|\,|y| < 1$, so that

$$e^{-\sqrt{\lambda}(x+y)} - e^{-\sqrt{\lambda}(x-y)} = e^{-\sqrt{\lambda}(x-y)}[e^{-2\sqrt{\lambda}y} - 1] = \mathbf{O}(|y|)\sqrt{\lambda}e^{-\sqrt{\lambda}(x-y)}.$$

Thus our estimate on the first term becomes

$$\left[\mathbf{O}(|\lambda|^{-1/2}) \wedge \left(\mathbf{O}_1(|y|) + \mathbf{O}(\log\lambda)\right)\right]\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\sqrt{\lambda}(x-y)},$$

which subsumes the previous estimate.

We develop derivative estimates similarly. We have

$$\begin{aligned}\partial_x G_\lambda(x,y) &= \frac{A(\lambda)}{W_\lambda(y)}e^{-\sqrt{\lambda}(x+y)}\left(\bar{u}(x) - u_+\right)\left(\bar{u}(y) - u_+\right)\\ &\quad \times \left(-\sqrt{\lambda} + \frac{\bar{u}_x}{\bar{u}(x) - u_+} + e_2^+(x,\lambda) + \frac{\bar{u}_x}{\bar{u}(x) - u_+}e_1^+(x,\lambda)\right)\left(1 + e_1^+(y,\lambda)\right)\\ &\quad + \frac{B(\lambda)}{W_\lambda(y)}e^{-\sqrt{\lambda}(x-y)}\left(\bar{u}(x) - u_+\right)\left(\bar{u}(y) - u_+\right)\\ &\quad \times \left(-\sqrt{\lambda} + \frac{\bar{u}_x}{\bar{u}(x) - u_+} + e_2^+(x,\lambda) + \frac{\bar{u}_x}{\bar{u}(x) - u_+}e_1^+(x,\lambda)\right)\left(1 + \tilde{e}_1^+(y,\lambda)\right)\\ &= \frac{\left(\bar{u}(x) - u_+\right)\left(\bar{u}(y) - u_+\right)}{W_\lambda(y)}\left(-\sqrt{\lambda} + \mathbf{O}_1(|x|^{-1})\right)[A(\lambda)e^{-\sqrt{\lambda}(x+y)} + B(\lambda)e^{-\sqrt{\lambda}(x-y)}]\\ &\quad + \frac{\left(\bar{u}(x) - u_+\right)\left(\bar{u}(y) - u_+\right)}{W_\lambda(y)}\left(-\sqrt{\lambda} + \mathbf{O}_1(|x|^{-1})\right)\mathbf{O}(|\lambda|^{-1/2})\\ &\quad \times [e_1^+(y,\lambda)e^{-\sqrt{\lambda}(x+y)} + \tilde{e}_1^+(y,\lambda)e^{-\sqrt{\lambda}(x-y)}].\end{aligned}$$

The cancelling terms are estimated as above by

$$\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\sqrt{\lambda}(x-y)} + \mathbf{O}(\log\lambda)\mathbf{O}_1(|x|^{-2})\mathbf{O}_1(|y|)e^{-\sqrt{\lambda}(x-y)},$$

and the remainder by terms that may be subsumed.

For the final case, $0 \leqq x \leqq y$, we have

$$\begin{aligned}
G_\lambda(x,y) &= \frac{A(\lambda)\phi^+(x)\phi^+(y)}{W_\lambda(y)} + \frac{B(\lambda)\psi^+(x)\phi^+(y)}{W_\lambda(y)} \\
&= \frac{A(\lambda)}{W_\lambda(y)}e^{-\sqrt{\lambda}(x+y)}\big(\bar{u}(x)-u_+\big)\big(\bar{u}(y)-u_+\big)\big(1+e_1^+(x,\lambda)\big)\big(1+e_1^+(y,\lambda)\big) \\
&\quad + \frac{B(\lambda)}{W_\lambda(y)}e^{-\sqrt{\lambda}(y-x)}\big(\bar{u}(x)-u_+\big)\big(\bar{u}(y)-u_+\big)\big(1+\tilde{e}^+(x,\lambda)\big)\big(1+e_1^+(y,\lambda)\big) \\
&= \frac{\mathbf{O}(1)\big(\bar{u}(x)-u_+\big)\big(\bar{u}(y)-u_+\big)}{W_\lambda(y)}[A(\lambda)e^{-\sqrt{\lambda}(x+y)} + B(\lambda)e^{-\sqrt{\lambda}(y-x)}] \\
&\quad + \frac{\mathbf{O}(1)\big(\bar{u}(x)-u_+\big)\big(\bar{u}(y)-u_+\big)}{W_\lambda(y)} \\
&\quad \times [A(\lambda)e_1^+(x,\lambda)e^{-\sqrt{\lambda}(x+y)} + B(\lambda)\tilde{e}_1^+(x,\lambda)e^{-\sqrt{\lambda}(y-x)}].
\end{aligned}$$

For the terms with cancellation, we now have the symmetric

$$[A(\lambda)e^{-\sqrt{\lambda}(x+y)} + B(\lambda)e^{-\sqrt{\lambda}(y-x)}] = \big[\mathbf{O}(\lambda^{-1/2}) \wedge \big(\mathbf{O}_1(|x|) + \mathbf{O}(\log\lambda)\big)\big]e^{-\sqrt{\lambda}(y-x)},$$

from which we obtain an estimate of the form

$$\big[\mathbf{O}(|\lambda|^{-1/2}) \wedge \big(\mathbf{O}_1(|x|) + \mathbf{O}(\log\lambda)\big)\big]\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\sqrt{\lambda}(y-x)},$$

symmetric with that obtained in the case $0 \leqq y \leqq x$. The terms without cancellation can again be subsumed into these terms.

We close the proof of Lemma 3.4 with derivative estimates for $0 \leqq x \leqq y$, for which we have

$$\begin{aligned}
\partial_x G_\lambda(x,y) &= \frac{A(\lambda)}{W_\lambda(y)}e^{-\sqrt{\lambda}(x+y)}\big(\bar{u}(x)-u_+\big)\big(\bar{u}(y)-u_+\big) \\
&\quad \times \left(-\sqrt{\lambda} + \frac{\bar{u}_x(x)}{\bar{u}(x)-u_+} + e_2^+(x,\lambda) + \frac{\bar{u}_x}{\bar{u}(x)-u_+}e_1^+(x,\lambda)\right)\big(1+e_1^+(y,\lambda)\big) \\
&\quad + \frac{B(\lambda)}{W_\lambda(y)}e^{-\sqrt{\lambda}|x-y|}\big(\bar{u}(x)-u_+\big)\big(\bar{u}(y)-u_+\big) \\
&\quad \times \left(\sqrt{\lambda} + \frac{\bar{u}_x(x)}{\bar{u}(x)-u_+} + \tilde{e}_2^+(x,\lambda) + \frac{\bar{u}_x(x)}{\bar{u}(x)-u_+}\tilde{e}_1^+(x,\lambda)\right)\big(1+e_1^+(y,\lambda)\big)
\end{aligned}$$

$$\overset{1}{=}\frac{\bar{u}_x(x)\big(\bar{u}(y)-u_+\big)}{W_\lambda(y)}\big(1+e_1^+(y,\lambda)\big)\big[A(\lambda)e^{-\sqrt{\lambda}(x+y)}+B(\lambda)e^{-\sqrt{\lambda}|x-y|}\big]$$

$$\overset{2}{+}\frac{\big(\bar{u}(x)-u_+\big)\big(\bar{u}(y)-u_+\big)\sqrt{\lambda}}{W_\lambda(y)}\big(1+e_1^+(y,\lambda)\big)\big[-A(\lambda)e^{-\sqrt{\lambda}(x+y)}+B(\lambda)e^{-\sqrt{\lambda}|x-y|}\big]$$

$$\overset{3}{+}\frac{\big(\bar{u}(x)-u_+\big)\big(\bar{u}(y)-u_+\big)}{W_\lambda(y)}\mathbf{O}_1(|x|^{-1})\mathbf{O}(\log\lambda).$$

For term 1 we have an estimate of

$$\big[\mathbf{O}(|\lambda|^{-1/2})\wedge\big(\mathbf{O}_1(|x|)+\mathbf{O}(\log\lambda)\big)\big]\mathbf{O}_1(|x|^{-2})\mathbf{O}_1(|y|)e^{-\sqrt{\lambda}|x-y|},$$

while for the second term we have

$$\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\sqrt{\lambda}|x-y|}.$$

As the third term can clearly be subsumed into these, we have the claim. □

The final lemma of this section regards large $|\lambda|$ estimates on $G_\lambda(x,y)$. The proof is exactly that of Lemma 3.5 of [H1] and is omitted.

Lemma 3.5. *For $|\lambda|\geqq M_l$, some $M_l>0$, and to the right of Γ_d, we have*

$$\left|\frac{\partial^k}{\partial x^k}G_\lambda(x,y)\right|=\mathbf{O}(|\lambda|^{\frac{k-1}{2}})e^{-\mathrm{Re}\frac{\sqrt{\lambda}}{2}|x-y|}.$$

4. Estimates on the time-propagating Green's function

We now employ the estimates of Lemmas 3.4 and 3.5 to derive estimates on the time-propagating Green's function $G(t,x;y)$. The analysis is governed by the observation that though we cannot extend the Evans function onto the negative real axis, we can still extend contours into the essential spectrum, provided they do not cross the negative real axis.

We begin with the observation that the large-$|\lambda|$, or small-time, analysis of [H1–2] remains virtually unchanged. Recalling our relation

$$G(t,x;y)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}G_\lambda(x,y)\,d\lambda,$$

where Γ encircles the point spectrum of L, we have for all $|\lambda|\geqq M_l$ integrals of the form

$$\int_\Gamma e^{\lambda t}\mathbf{O}(|\lambda|^{-1/2})e^{-\frac{\sqrt{\lambda}}{2}|x-y|}.$$

In the case $|x-y|\geqq Kt$, some K sufficiently large, we proceed as in [H1–2] with the contour, Γ_d^l, determined by

$$\sqrt{\lambda_l(k)}=\sqrt{\frac{|x-y|}{4t}}+ik,$$

for λ_l to the right of Γ_d and Γ_d—along which exponential time decay is clear—otherwise (see Figure 4.1). We develop, then, an estimate of the form $t^{-1/2}e^{-\frac{|x-y|^2}{Mt}}$, where the only effect of x or y derivatives is that of increasing the algebraic $t \to 0$ blow-up by a power of $t^{-1/2}$. Similarly, the bounded-time Green's function estimates of [H1] remain unchanged. We note that each contour we take in the following analysis will proceed similarly, following Γ_d out to the point at ∞. The contour Γ_d may be thought of as analogous to that which we would take were there a gap between essential spectrum and the imaginary axis.

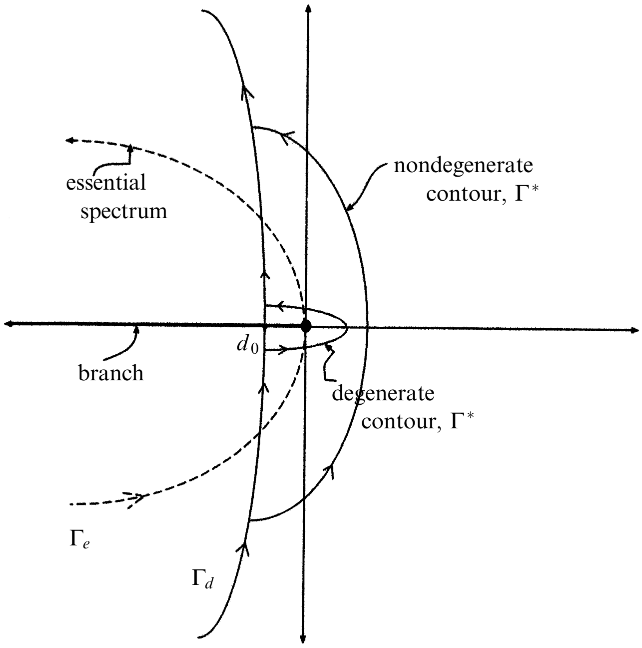

Figure 4.1. Contours Γ_d, Γ_e, Γ^*. In all cases, contours intersecting Γ_d follow it out to the point at ∞, avoiding possible point spectrum.

For $|x-y| \leqq Kt$ we divide the analysis into cases, similar to those of Lemma 3.4. We begin in the case $y \leqq x \leqq 0$, for which

$$G_\lambda(x,y) = \mathbf{O}_a(1)e^{\mu_1^-(x-y)} + \mathbf{O}(e^{-\eta|x|})e^{\mu_1^-(x-y)}.$$

The term $\mathbf{O}_a(1)$ is analytic in a neighborhood of the origin, so that its analysis may be carried into the essential spectrum, as in that of [H1–3], [ZH]. We obtain immediately an estimate by $\mathbf{O}(t^{-1/2})e^{-\frac{(x-y-a_-t)^2}{Mt}}$. For $|x-y| \geqq a_-t$ ($|x-y| \leqq Kt$), we note that the analysis of [H1–3], [ZH] remains to the right of essential spectrum (away from $\lambda = 0$) and can also be applied to the second term of $G_\lambda(x,y)$ to yield a similar estimate by $\mathbf{O}(t^{-1/2})\mathbf{O}(e^{-\eta|x|})e^{-\frac{(x-y-a_-t)^2}{Mt}}$. For $|x-y| \leqq a_-t$, we shall take the heat equation-like contour described through

$$\sqrt{\lambda(k)} = t^{-1/2} + ik,$$

for k such that $t^{-1/2} + ik$ lies to the right of Γ_d ($k \leqq k^*$), and Γ_d otherwise (denoted by Γ^*, see Figure 4.1). Note in particular that though this contour does not cross the negative real axis, it moves rapidly into essential spectrum. Observing that

$$\lambda(k) = t^{-1} + 2ikt^{-1/2} - k^2,$$

and by Taylor expansion near $\lambda = 0$ that

$$\mu_1^-(\lambda) = -\frac{1}{a_-}\lambda + \frac{1}{a_-^3}\lambda^2 + \mathbf{O}(\lambda^3),$$

we have the computation

$$\begin{aligned} &\left|\int_{\Gamma^*} e^{\lambda t + \mu_1^-(x-y)}\, d\lambda\right| \\ &\leqq C\int_{-k^*}^{k^*} e^{-k^2 t + \frac{1}{a_-}k^2(x-y) - \frac{6}{a_-^3}t^{-1}(x-y)k^2} |2it^{-1/2} - 2k|\,|dk| \\ &\quad + C\int_{[-k^*,k^*]^c} e^{-d_0 t - d_2 k^2 t + \frac{1}{a_-}d_0(x-y) + \frac{1}{a_-}d_2 k^2(x-y)} |dk| \\ &\leqq C\left(t - \frac{1}{a_-}(x-y) + \frac{6}{a_-^3}\frac{x-y}{t}\right)^{-1} \\ &\leqq C(|x-y-a_-t| + 1)^{-1}, \end{aligned}$$

where the final inequality is a result of the observation that for $x - y \leqq (a_-/2)t$, $\big(t - (x-y)/a_- + \big(6(x-y)/(a_-^3 t)\big)\big) \geqq (a_-/2)t$, while for $(a_-/2)t \leqq x - y \leqq a_- t$,

$$\big(t - (x-y)/a_- + \big(6(x-y)/(a_-^3 t)\big)\big) \geqq (6/a_-^2).$$

We point out that the above estimate is not assumed sharp and should be compared with the more natural estimates of [H1–2], [ZH]. We content ourselves with such estimates here, since it is necessarily the degenerate side ($x \geqq 0$) that will dictate the analysis. Indeed, we find that nothing is lost from this rough estimate in our final estimate on $v(t, x)$.

For derivative estimates in the case $y \leqq x \leqq 0$, we begin with $|x - y| \leqq Kt$, and the bounded-$|\lambda|$ estimates

$$\partial_x G_\lambda(x, y) = \mathbf{O}_a(1)\mu_1^- e^{\mu_1^-(x-y)} + \mathbf{O}(e^{-\eta|x|})e^{\mu_1^-(x-y)}.$$

The first term of $\partial_x G_\lambda(x, y)$ can be analyzed as in [ZH], and yields an estimate of the form $\mathbf{O}(t^{-1})e^{-\frac{(x-y-a_-t)^2}{Mt}}$. The second term of $\partial_x G_\lambda(x, y)$ is of the same form as that of $G_\lambda(x, y)$ and hence yields the same estimate.

For $x \leqq y \leqq 0$, and $|x - y| \leqq Kt$, we have

$$G_\lambda(x, y) = \mathbf{O}_a(1)e^{\mu_2^-(x-y)} + \mathbf{O}(1)e^{\mu_2^- x - \mu_1^- y}.$$

Since $\mathrm{Re}\,\mu_2^- > 0$ in a ball around the origin, we may take d_0 sufficiently small so that

$\operatorname{Re}\mu_2^-|_{\Gamma_d} > \varepsilon > 0$, for some $\varepsilon > 0$. Taking then the contour Γ_d, we have exponential decay in both t and $|x-y|$, which for $|x-y| \leqq Kt$ yields an estimate of $\mathbf{O}(t^{-1/2})e^{-\frac{(x-y-a_-t)^2}{Mt}}$ on the first term—the $t^{-1/2}$ term indicative of $t \to 0$ blow up, as we now have exponential decay in time. Similarly, for the second term of $G_\lambda(x,y)$, we may employ the essential spectrum, Γ_e, as our contour. For $|x-y| \geqq a_-t$, we again have exponential decay in $|x-y|$ and t, and arrive at the same estimate as above. For $|x-y| \leqq a_-t$, we observe that $\operatorname{Re}\mu_2^- > \operatorname{Re}\sqrt{\lambda}$ in a neighborhood of the origin so that by taking the contour described through $\sqrt{\lambda} = (x-y)/2t + ik$ to the right of Γ_d, we have an estimate of

$$\mathbf{O}(t^{-1})\mathbf{O}(e^{-\eta|x|})e^{-\frac{(x-y)^2}{Mt}} I_{\{|x-y| \leqq a_-t\}}.$$

Derivative estimates are similar.

For $x \leqq 0 \leqq y$, we have the bounded $|\lambda|$ estimate

$$G_\lambda(x,y) = \mathbf{O}(e^{-\eta|x|})\mathbf{O}_1(|y|)e^{-\mu_1^- x - \sqrt{\lambda} y}.$$

Taking the contour defined through

$$\sqrt{\lambda(k)} = \frac{y}{2t} + ik,$$

and proceeding as in a Fourier transform analysis of the heat equation, we immediately obtain the claimed estimate. In this case, the derivative estimate is precisely the same.

In the case $y \leqq 0 \leqq x$ and $|\lambda| \leqq M_s$, we have

$$G_\lambda(x,y) = \mathbf{O}_1(|x|^{-1})e^{-\sqrt{\lambda}x - \mu_1^- y}.$$

We cannot now proceed as in [H1], where a Taylor expansion around $\lambda = 0$ was found for the exponent,

$$h(\lambda) = \lambda t - \sqrt{\lambda}x - \frac{a_- - \sqrt{a_-^2 + 4\lambda}}{2} y,$$

and analyzed for a minimizing real λ. Indeed, the $a_+ \to 0$ limit indicates that a contour passing through $\lambda = 0$ (the essential spectrum boundary for example) would be optimal. Rather, we proceed by considering two regions of interest, $|y| \leqq a_-t$ and $|y| \geqq a_-t$. For $|y| \leqq a_-t$, we expect the kernel to have reached the origin, and employ a heat-equation scaled contour. On the other hand, for $|y| \geqq a_-t$, we expect the kernel to be to the left of the origin, propagating to the right with speed a_-. For $|y| \geqq a_-t$, we take the contour defined through

$$\mu_1^-(\lambda(k)) = \frac{x - y - a_-t}{2t} + ik.$$

Proceeding as in [HZ1], [ZH] in an ε-ball around the origin, we observe that interior to such a ball, $\operatorname{Re}(-\sqrt{\lambda}) \leqq \operatorname{Re}(\mu_1^-)$, and we obtain the estimate as there of

$$\mathbf{O}(t^{-1/2})\mathbf{O}_1(|x|^{-1})e^{-\frac{(x-y-a_-t)^2}{Mt}} I_{\{|y| \geqq a_-t\}}.$$

Alternatively, for $|y| \leqq a_- t$, we take the contour defined through

$$\sqrt{\lambda(k)} = \frac{x}{2t} + ik,$$

which yields an estimate as in the case $y \leqq x \leqq 0$ by

$$\mathbf{O}_1(|x|^{-1})e^{-\frac{x^2}{Mt}}\left(t + \frac{y}{a_-} - \frac{3}{2a_-^3}\frac{x^2}{t^2}y\right)^{-1} I_{\{|y| \leqq a_- t\}}.$$

Finally, for $|x| \leqq 1$ we take the contour defined through

$$\sqrt{\lambda(k)} = \frac{1}{2t} + ik,$$

(to the right of Γ_d). Derivative estimates are altered in this case only by additional x-decay, $\mathbf{O}(|x|^{-1})$.

In the case $0 \leqq y \leqq x$, our bounded $|\lambda|$ estimate on $G_\lambda(x, y)$ takes the form

$$G_\lambda(x, y) = \left[\mathbf{O}(\lambda^{-1/2}) \wedge \left(\mathbf{O}_1(|y|) + \mathbf{O}(\log \lambda)\right)\right]\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\sqrt{\lambda}(x-y)},$$

which we analyze along the contour $\sqrt{\lambda} = (x - y)/2t + ik$ in order to arrive at an estimate of the form

$$G(t, x; y) = \left[\mathbf{O}(t^{-1/2}) \wedge \left(\mathbf{O}(t^{-1})\mathbf{O}_1(|y|) + \mathbf{O}(t^{-1}\log t)\right)\right]\mathbf{O}_1(|x|^{-1})\mathbf{O}(|y|)e^{-\frac{(x-y)^2}{Mt}}.$$

Similarly for derivatives, we have

$$G_x(t, x; y) = \mathbf{O}(t^{-1})\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(t^{-1}\log t)\mathbf{O}_1(|x|^{-2})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}}.$$

The case $0 \leqq x \leqq y$ is similar. □

5. Estimates on the perturbation

In this section we will prove Theorem 1.2 by virtue of a lemma similar to Lemma 1.5 of [ZH] which will provide a direct means of employing (1.4) to obtain estimates on v. Such analyses have proven lengthy in the past, and we will be as brief as would seem prudent by omitting computations that have appeared in referenced work. In particular, we render in full detail only those computations germane to changes arising from the analysis of the degenerate case, and those which lead directly to the terms of Theorem 1.2. We have

Lemma 5.1. *Let C_1 and C_2 be constants and let $h_0(x), h(t, x) \geqq 0$ satisfy the relations*

$$\int_{-\infty}^{+\infty} |G_x(t, x; y)| \left|\int^{y} h_0(\xi)\, d\xi\, dy \leqq C_1 h(t, x)$$

or

$$\int_{-\infty}^{+\infty} \left| \int^{y} G_x(t,x;\xi)\,d\xi \right| h_0(y)\,dy \leqq C_1 h(t,x)$$

and

$$\int_0^t \int_{-\infty}^{+\infty} G_x(t-s,x;y) h(s,y)^2\,dy\,ds \leqq C_1 h(t,x),$$

for all $t > 0, x \in \mathbb{R}$, *and where* $\int^x$ *can be chosen either as* $\int_{-\infty}^{x}$ *or* $\int_{x}^{+\infty}$ *for each* x. *If then* $|v(0,x)| \leqq \zeta_0 h_0(x)$ *for some* ζ_0 *sufficiently small, then* $|v(t,x)| \leqq C_2 \zeta_0 h(t,x)$ *for all* $t > 0$, $x \in \mathbb{R}$.

Proof. The proof of Lemma 5.1 is exactly that of Lemma 2.1 of [H2]. □

The following two lemmas, proved in [HZ1] compactify the analysis.

Lemma 5.2. *Let* $f(y) \geqq 0$ *be a nonincreasing function on* $\mathbb{R}_+$, *with* $f(0) \leqq C$ *for some constant* C. *Then for* $a, z, r > 0$ *fixed, we have*

$$\int_0^{+\infty} a^{1/r} e^{-a(z-y)^r} f(y)\,dy \leqq \tilde{C}(\omega)[f(z/\omega) + e^{-a\gamma z^r}],$$

with $0 < \gamma < 1$, $\omega > 1$ *both as close to* 1 *as we like. Moreover, if* $f(\cdot)$ *is integrable, we obtain the estimate*

$$\int_0^{+\infty} a^{1/r} e^{-a(z-y)^r} f(y)\,dy \leqq \tilde{C}(\omega)[a^{1/r} \wedge f(z/\omega) + (1 + a^{-1/r})^{-1} e^{-a\gamma z^r}],$$

where $\wedge$ *represents minimum.*

Lemma 5.3. *For* $z \geqq 0$, $y \leqq 0$, $a > 0$, $r > 0$, *and* $\gamma > 1$, *we have the tail estimate*

$$\int_{-\infty}^{y} e^{-a(z-\xi)^r\,d\xi} \leqq C(r) a^{-1/r} e^{\frac{a}{\gamma}(z-y)^r}.$$

The following lemma often dictates behavior on the degenerate side.

Lemma 5.4. *For* $M > 0$, $t > 0$, *and* $\alpha \in \mathbb{R}$, *we have*

$$\int_0^{+\infty} (1+y)^{\alpha} e^{-\frac{y^2}{Mt}}\,dy \leqq C_{\alpha} \begin{cases} 1, & \alpha < -1, \\ \log(2+\sqrt{t}), & \alpha = -1, \\ (1+t)^{\frac{1}{2}+\frac{\alpha}{2}}, & \alpha > -1. \end{cases}$$

Proof. The integral in Lemma 5.4 is clearly bounded over any finite interval, so that we need only consider

$$\int_1^{+\infty} y^{\alpha} e^{-\frac{y^2}{Mt}}\,dy.$$

Here, we make the substitution $\xi = y^{-1}\sqrt{t}$, which leads to

$$\int_0^{\sqrt{t}} \frac{t^{\frac{\alpha}{2}+\frac{1}{2}}}{\xi^{2+\alpha}} e^{-\frac{1}{M\xi^2}}\, d\xi,$$

from which the claim is clear. □

Proof of Theorem 1.2. In order to employ Lemma 5.1 we need an ansatz $h(t,x)$ for the behavior of $v(t,x)$, which for convenience we will write as $h^+(t,x)$ for $x \geqq 0$ and $h^-(t,x)$ for $x \leqq 0$. We take

$$h^-(t,x) := e^{-\frac{\eta}{2}|x|} D_-(t) + d(x - a_- t),$$

η as in Theorem 1.1, and

$$h^+(t,x) := (1+|x|)^{-1} D_-(t) e^{-\frac{x^2}{2Mt}} + (1+|x|)^{-2} D_+(t) e^{-\frac{x^2}{2Mt}} + d_+(t,x),$$

where we recall that

$$D_-(t) := \begin{cases} (1+t)^{\frac{1-r}{2}}, & 1 < r < 3, \\ (1+t)^{-1}\log(2+t), & r = 3, \\ (1+t)^{-1}, & r > 3, \end{cases}$$

$D_+(t) = \log(2+t) D_-(t)$, and

$$d_+(t,x) := \begin{cases} (1+x)^{-r} \wedge t^{-1}(1+x)^{2-r}, & 1 < r < 2, \\ (1+x)^{-2}\log(2+x) \wedge t^{-1}, & r = 2, \\ (1+t)^{-1/2}(1+x)^{1-r} \wedge t^{-1}(1+x)^{2-r}, & r > 2. \end{cases}$$

In both cases we take $h_0(x) = d(x) \in \Delta_r$. It is useful to observe that for

$$(1+x)^{-1}\log(2+t) \leqq C, \quad (1+|x|)^{-2} D_+(t) e^{-\frac{x^2}{2Mt}} \leqq C(1+|x|)^{-1} D_-(t) e^{-\frac{x^2}{2Mt}},$$

while for $(1+x)^{-1}\log(2+t) \geqq C$, $d_+(t,x)$ may be subsumed into the remaining terms, explaining why $d_+(t,x)$ not contain the $\log t$ error.

Linear analysis for $x \leqq 0$. The linear analysis will consist of integrals of the form

$$\int_{-\infty}^{+\infty} G_x(t,x;y) \int^{y} v_0(\xi)\, d\xi\, dy,$$

as suggested in Lemma 5.1. In the case $|x-y| \geqq Kt$, this analysis is unchanged from that of previous work (see [H1–3], [ZH]) and yields estimates bounded by $h^{\pm}(t,x)$ for $x \gtreqless 0$. The sectorial nature of L insures that we have no difficulty as $t \to 0$, as for example, in the case of equations of odd order (see [HZ1]). We shall find, as in previous analyses, that though the nonlinear integrals are often more tedious to study, the size of $v(t,x)$ is determined by the linear terms.

For $|x-y| \leqq Kt$, we begin in the case $y \leqq x \leqq 0$, for which we have

$$G_x(t,x;y) = \mathbf{O}(t^{-1})e^{-\frac{(x-y-a_-t)^2}{Mt}} + \mathbf{O}(t^{-1/2})\mathbf{O}(e^{-\eta|x|})e^{-\frac{(x-y-a_-t)^2}{Mt}} + \mathbf{O}(e^{-\eta|x|})(|x-y-a_-t|+1)^{-1}I_{\{|x-y|\leqq a_-t\}}.$$

The first two terms are exactly those of the Lax analysis of [H1–2] and yield an estimate by (Lemma 5.2) of

$$C[t^{-1/2}D(x-a_-t) + e^{-\eta|x|}D(x-a_-t)].$$

As in [H1–2] we observe that $D(x-a_-t) \sim (x-a_-t)^{1-r}$ is not integrable in x for $1 < r \leqq 2$. Integrating by parts, however, we obtain the alternate estimate

$$\int_{-\infty}^{+\infty}\int^{y} G_x(t,x;\xi)\,d\xi v_0(y)\,dy \leqq C[e^{-\eta|x|}\log(2+t) + d(x-a_-t)].$$

Observing that $t^{-1/2}D(x-a_-t) \sim t^{1/2-r}$, we note that for $t^{-r/2} \leqq Ce^{-\frac{\eta}{2}|x|}$, we have

$$t^{-1/2}D(x-a_-t) \leqq Ce^{-\frac{\eta}{2}|x|}D_-(t),$$

while for $t^{-r/2} \geqq Ce^{-\frac{\eta}{2}|x|}$, we have $e^{-\eta|x|}\log t \leqq (1/C)e^{-\frac{\eta}{2}|x|}t^{-r/2}\log t$.

For $|x| \geqq \varepsilon t$, any $\varepsilon > 0$, the third term decays at exponential rate in both t and x. For $|x| \leqq \varepsilon t$, we compute

$$e^{-\eta|x|}\int_{x-a_-t}^{x}(|x-y-a_-t|+1)^{-1}(1-y)^{1-r}\,dy.$$

On $y \in \left[x - \frac{a_-}{2}t, x\right]$, we observe that $(|x-y-a_-t|+1) \geqq t/2$, so that

$$e^{-\eta|x|}\int_{x-\frac{1}{2}a_-t}^{x}(|x-y-a_-t|+1)^{-1}(1-y)^{1-r}\,dy$$
$$\leqq Ce^{-\eta|x|}\begin{cases}(1+t)^{-1}\left(1-x+\frac{1}{2}a_-t\right)^{2-r}, & 1<r<2,\\ (1+t)^{-1}\log\left(1-x+\frac{1}{2}a_-t\right), & r=2,\\ C(1+t)^{-1}, & r>2,\end{cases}$$

which is bounded by $e^{-\frac{\eta}{2}|x|}D_-(t)$. For $y \in \left[x-\frac{a_-}{2}t, x-a_-t\right]$, $(1-y) \geqq \left(1-x+\frac{1}{2}a_-t\right)$ so that

$$\int_{x-a_-t}^{x-\frac{1}{2}a_-t}(|x-y-a_-t|+1)^{-1}(1-y)^{1-r}\,dy$$
$$\leqq Ce^{-\eta|x|}\left(1-x+\frac{1}{2}a_-t\right)^{1-r}\log(2+t) \leqq Ce^{-\eta|x|}D_-(t).$$

A similar estimate follows for $x \leqq y \leqq 0$.

The linear analysis for $x \leqq 0$ is determined by its degenerate portion: $x \leqq 0 \leqq y$. Here, we have ($|x-y| \leqq Kt$)

$$G_x(t,x;y) = \mathbf{O}(t^{-1})\mathbf{O}(e^{-\eta|x|})\mathbf{O}_1(|y|)e^{-\frac{y^2}{Mt}}.$$

Extending the integration to $+\infty$, we consider integrals of the form

$$t^{-1}e^{-\eta|x|}\int_0^{+\infty}(1+y)e^{-\frac{y^2}{Mt}}(1+y)^{1-r}\,dy,$$

which according to Lemma 5.4 are bounded by

$$t^{-1}e^{-\eta|x|}\int_0^{+\infty}(1+y)^{2-r}e^{-\frac{y^2}{Mt}}\,dy \leqq Ce^{-\eta|x|}\begin{cases}(1+t)^{1/2-r/2}, & 1<r<3,\\ (1+t)^{-1}\log(2+t), & r=3,\\ (1+t)^{-1}, & r>3,\end{cases}$$

which may be taken as the derivation of $D_-(t)$.

Linear analysis for $x \geqq 0$. We turn now to the analysis for $x \geqq 0$, which we begin with the case $y \leqq 0 \leqq x$, where we have

$$\begin{aligned}G_x(t,x;y) &= \mathbf{O}(t^{-1/2})\mathbf{O}_1(|x|^{-2})e^{-\frac{(x-y-a_-t)^2}{Mt}}I_{\{|y|\geqq a_-t\}}\\ &\quad+\mathbf{O}_1(|x|^{-2})\left(t+\frac{y}{a_-}-\frac{3}{2a_-^3}\frac{x^2}{t^2}y\right)^{-1}e^{-\frac{x^2}{Mt}}I_{\{\{|y|\leqq a_-t\}\cap\{x\geqq 1\}\}}\\ &\quad+\left(t+\frac{y}{a_-}-\frac{3}{2a_-^3t}y\right)^{-1}I_{\{\{|y|\leqq a_-t\}\cap\{x\leqq 1\}\}}.\end{aligned}$$

Over the first term of $G_x(t,x;y)$, we have an estimate by

$$Ct^{-1/2}(1+x)^{-2}\int_{-\infty}^{-a_-t}e^{-\frac{(x-y-a_-t)^2}{Mt}}(1-y)^{1-r}\,dy \leqq C(1+x)^{-2}(1+a_-t)^{1-r}e^{-\frac{x^2}{2Mt}}.$$

For the second, we observe that for $|x| \geqq \varepsilon t$, some $0<\varepsilon \ll a_-$ we have exponential decay in both x and t, and for $|x| \leqq \varepsilon t$, we compute

$$(1+x)^{-2}e^{-\frac{x^2}{Mt}}\int_{-a_-t}^{0}\left(t+\frac{y}{a_-}-\frac{3}{2a_-^3}\frac{x^2}{t^2}y\right)^{-1}(1-y)^{1-r}\,dy.$$

For $y\in[-(a_-/2)t,0]$, $\left(t+y/a_- - \left(3/(2a_-^3)(x^2/t^2)y\right)\right) \geqq t/2$, so that we have

$$\int_{-\frac{a_-}{2}t}^{0}\left(t+\frac{y}{a_-}-\frac{3}{2a_-^3}\frac{x^2}{t^2}y\right)^{-1}(1-y)^{1-r}\,dy \overset{r\neq 2}{\leqq} C(1+t)^{-1}\left[\left(1+\frac{a_-}{2}t\right)^{2-r}\vee 1\right],$$

where $\vee$ represents maximum, and for $r = 2 \log t$ growth is obtained. For $y \in \left[-a_- t, -\frac{a_-}{2} t\right]$, we have

$$\int_{-a_- t}^{-\frac{a_-}{2} t} \left(t + \frac{y}{a_-} - \frac{3}{2a_-^3} \frac{x^2}{t^2} y\right)^{-1} (1-y)^{1-r} \, dy$$
$$\leqq C\left(1 + \frac{a_-}{2} t\right)^{1-r} \log(2+t).$$

For $0 \leqq x, y$, we have ($|x-y| \leqq Kt$, $t \geqq T > 1$)

$$G_x(t,x;y) = \mathbf{O}(t^{-1})\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(t^{-1}\log t)\mathbf{O}_1(|x|^{-2})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}}.$$

Hence

$$\left|\int_0^{+\infty} G_x(t,x;y) \int^y v_0(\xi)\, d\xi \, dy\right|$$
$$\leqq C\big(1 + (1+x)^{-1}\log t\big)t^{-1}(1+x)^{-1} \int_0^{+\infty} e^{-\frac{(x-y)^2}{Mt}}(1+y)^{2-r}\, dy.$$

It is convenient to analyze this integral over three regions: (1) $y \in [0, \varepsilon x]$, $0 < \varepsilon \ll 1$, (2) $y \in [\varepsilon x, 2x]$, and (3) $y \in [2x, +\infty)$. For the first, we have

$$Ct^{-1}(1+x)^{-1} \int_0^{\varepsilon x} e^{-\frac{(x-y)^2}{Mt}}(1+y)^{2-r}\, dy$$
$$\leqq Ct^{-1}(1+x)^{-1} e^{-\frac{3x^2}{4Mt}} \int_0^{\varepsilon x} e^{-\frac{(x-y)^2}{8Mt}}(1+y)^{2-r}\, dy.$$

For $r > 3$, we immediately observe that the integrability of $(1+y)^{2-r}$ yields an estimate by

$$Ct^{-1}(1+x)^{-1} e^{-\frac{3x^2}{4Mt}}.$$

For $r \leqq 3$, we integrate $(1+y)^{2-r}$ on $[0, \varepsilon x]$ to get an estimate by

$$Ct^{-1}(1+x)^{-1} e^{-\frac{3x^2}{4Mt}} \begin{cases} \log(1+x), & r = 3, \\ (1+x)^{3-r}, & 1 < r < 3. \end{cases}$$

We recover a bound by $h^+(t,x)$ through the observation that

$$t^{-1}(1+x)^{2-r} e^{-\frac{3x^2}{4Mt}} \leqq Ct^{\frac{1}{2}-\frac{r}{2}}(1+x)^{-1} e^{-\frac{x^2}{2Mt}}.$$

Multiplying now by $1 + (1+x)^{-1}\log t$, we may take this computation as the derivation of the first two terms of $h^+(t,x)$.

On $y \in [\varepsilon x, 2x]$ we have

$$Ct^{-1}(1+x)^{-1}\int_{\varepsilon x}^{2x} e^{-\frac{(x-y)^2}{Mt}}(1+y)^{2-r}\,dy,$$

which we may estimate in two ways. Integrating the kernel yields an estimate by $Ct^{-1/2}(1+x)^{1-r}$, while integrating $(1+y)^{2-r}$ yields an estimate by $Ct^{-1}(1+x)^{2-r}$. We will observe shortly that by integrating by parts we may also obtain an analogous estimate by $(1+x)^{-r}+(1+x)^{-2}e^{-\frac{x^2}{2Mt}}\log(2+x)$, which should be compared with the parts estimate obtained above for $x \leqq 0$. In each case, we have the additional multiplication by $1+(1+x)^{-1}\log t$. For $(1+x)^{-1}\log t \leqq C$, our estimate by $d_+(t,x)$ on these terms is clear; indeed, this computation may be taken as its derivation. For $(1+x)^{-1}\log t \geqq C$, we observe that algebraic t-decay yields exponential x-decay, and thus $(1+x)^{-1}(\log t)t^{-1}(1+x)^{2-r}$ may be subsumed by the first two terms of $h^+(t,x)$.

On $y \in [2x, +\infty)$, we have

$$\begin{aligned} &Ct^{-1}(1+x)^{-1}\int_{2x}^{+\infty} e^{-\frac{(x-y)^2}{Mt}}(1+y)^{2-r}\,dy \\ &\quad \leqq Ct^{-1}(1+x)^{-1}e^{-\frac{3x^2}{4Mt}}\int_{2x}^{+\infty} e^{-\frac{(x-y)^2}{4Mt}}(1+y)^{2-r}\,dy. \end{aligned}$$

In the case that $r > 3$ we have an estimate of

$$Ct^{-1}(1+x)^{-1}e^{-\frac{3x^2}{4Mt}},$$

by the integrability of $(1+y)^{2-r}$. For $r \leqq 3$, we make the change of variable $\xi = y - x$ and consider

$$\int_{x}^{+\infty} e^{-\frac{\xi^2}{4Mt}}(1+\xi+x)^{2-r}\,d\xi.$$

For $2 \leqq r \leqq 3$, $(1+\xi+x)^{2-r} \leqq (1+\xi)^{2-r}$, while for $1 < r \leqq 2$, $(1+\xi+x)^{2-r} \leqq (1+2\xi)^{2-r}$, so that Lemma 5.4 applies to give an estimate of $C(1+x)^{-1}D_-(t)e^{-\frac{x^2}{2Mt}}$. Combining, we have

$$\left|\int_{[\varepsilon x, 2x]^c} G_x(t,x;y)\int^{y} v_0(\xi)\,d\xi\right| \leqq C[(1+x)^{-1}D_-(t)e^{-\frac{x^2}{2Mt}} + (1+x)^{-2}D_+(t)e^{-\frac{x^2}{2Mt}}].$$

We observe that the estimates obtained thus far (on $y \in [\varepsilon x, 2x]$) are not square integrable for $1 < r \leqq 3/2$. In the case $t^{-1/2} \leqq C(1+x)^{-1}$, we may obtain integrability by giving up t-decay; whereas for $(1+x) \geqq \sqrt{t}$ we proceed as in [H1], [H3] through integrating by parts and observing $\int_{\mathbb{R}} G_x\,dy = 0$, so that

$$\int_{-\infty}^{+\infty} G_x(t,x;y)\int^{y} v_0(\xi)\,d\xi\,dy = -\int_{-\infty}^{+\infty}\int^{y} G_x(t,x;\xi)\,d\xi v_0(y)\,dy.$$

For $y \leqq 0$, we have (for $x \geqq 1$)

$$
\begin{aligned}
(5.1)\qquad & \left| \int_{-\infty}^{0} \int_{-\infty}^{y} G_x(t,x;\xi)\, d\xi v_0(y)\, dy \right| \\
&\leqq C \int_{-\infty}^{0} \int_{-\infty}^{y} (1+x)^{-2} \left(t + \frac{\xi}{a_-} - \frac{3}{2a_-^3} \frac{x^2}{t^2} \xi \right)^{-1} e^{-\frac{x^2}{Mt}} I_{\{|\xi| \leqq a_- t\}}\, d\xi v_0(y)\, dy \\
&\quad + C \int_{-\infty}^{0} \int_{-\infty}^{y} t^{-1/2} (1+x)^{-2} e^{-\frac{(x-\xi-a_- t)^2}{Mt}} I_{\{|\xi| \geqq a_- t\}}\, d\xi v_0(y)\, dy \\
&= C \int_{-a_- t}^{0} \int_{-a_- t}^{y} (1+x)^{-2} \left(t + \frac{\xi}{a_-} - \frac{3}{2a_-^3} \frac{x^2}{t^2} \xi \right)^{-1} e^{-\frac{x^2}{Mt}}\, d\xi v_0(y)\, dy \\
&\quad + C \int_{-a_- t}^{0} \int_{-\infty}^{-a_- t} t^{-1/2} (1+x)^{-2} e^{-\frac{(x-\xi-a_- t)^2}{Mt}}\, d\xi v_0(y)\, dy \\
&\quad + C \int_{-\infty}^{-a_- t} \int_{-\infty}^{y} t^{-1/2} (1+x)^{-2} e^{-\frac{(x-\xi-a_- t)^2}{Mt}}\, d\xi v_0(y)\, dy.
\end{aligned}
$$

For the first term on the right-hand side of (5.1), we divide the analysis up as

$$
\begin{aligned}
& \int_{-a_- t}^{-\frac{a_-}{2} t} \int_{-a_- t}^{y} (1+x)^{-2} \left(t + \frac{\xi}{a_-} - \frac{3}{2a_-^3} \frac{x^2}{t^2} \xi \right)^{-1} e^{-\frac{x^2}{Mt}}\, d\xi v_0(y)\, dy \\
&+ \int_{-\frac{a_-}{2} t}^{0} \int_{-a_- t}^{y} (1+x)^{-2} \left(t + \frac{\xi}{a_-} - \frac{3}{2a_-^3} \frac{x^2}{t^2} \xi \right)^{-1} e^{-\frac{x^2}{Mt}}\, d\xi v_0(y)\, dy.
\end{aligned}
$$

For $x \geqq \varepsilon t$, $0 < \varepsilon \ll a_-$, $e^{-\frac{x^2}{Mt}}$ gives exponential decay in x and t. For $x \leqq \varepsilon t$, we begin with the integral over $y \in \left[-a_- t, -\frac{a_-}{2} t\right]$, for which we integrate $\left(t + \frac{\xi}{a_-} - \frac{3}{2a_-^3} \frac{x^2}{t^2} \xi \right)^{-1}$ to get an estimate by

$$
\begin{aligned}
& C \int_{-a_- t}^{-\frac{a_-}{2} t} (1+x)^{-2} e^{-\frac{x^2}{Mt}} \log \left(\frac{t + \frac{y}{a_-} - \frac{3}{a_-^3} \frac{x^2}{t^2} y}{\frac{3x^2}{2a_-^2 t}} \right) v_0(y)\, dy \\
&\qquad \leqq C(1+x)^{-2} e^{-\frac{x^2}{Mt}} (1+t)^{1-r} \log t.
\end{aligned}
$$

For the integral over $y \in \left[-\frac{a_-}{2} t, 0\right]$ we observe that $\left(t + \frac{\xi}{a_-} - \frac{3}{2a_-^3} \frac{x^2}{t^2} \xi \right) \geqq \frac{1}{2} t$, yielding an estimate by $C(1+x)^{-2} e^{-\frac{x^2}{Mt}}$. For the second and third terms of (5.1) we easily obtain an estimate by $(1+x)^{-2} e^{-\frac{x^2}{Mt}}$.

Similarly, for $y \geqq 0$ we have

$$
\int_{0}^{+\infty} \int_{-\infty}^{y} G_x(t,x;\xi)\, d\xi v_0(y) I_{\{y \leqq x\}}\, dy - \int_{0}^{+\infty} \int_{y}^{+\infty} G_x(t,x;\xi)\, d\xi v_0(y) I_{\{y \geqq x\}}\, dy.
$$

Proceeding as above we obtain an estimate (for $(1+x) \geqq \sqrt{t}$) of

$$\begin{cases} (1+x)^{-r}, & 1 < r < 2, \\ (1+x)^{-2}\log(2+x), & r = 2, \end{cases}$$

which yields the final form of $d_+(t,x)$.

Nonlinear analysis for $x \leqq 0$. We now turn to terms of the form

$$\int_0^t \int_{-\infty}^{+\infty} G_x(t-s,x;y)h(s,y)^2\,dy\,ds,$$

which govern the nonlinear interactions. For $|x-y| \geqq K(t-s)$, we have

$$G_x(t-s,x;y) = \mathbf{O}\big((t-s)^{-1}\big)e^{-\frac{(x-y)^2}{M(t-s)}},$$

as in [H1], [H3], and the small and bounded-time analyses follow as there. In what follows, then, we may take $t \geqq T > 1$.

For $|x-y| \leqq K(t-s),\ y \leqq x \leqq 0$, we have

$$\begin{aligned} G_x(t,x;y) &= \mathbf{O}(t^{-1})e^{-\frac{(x-y-a_-t)^2}{Mt}} + \mathbf{O}(t^{-1/2})\mathbf{O}(e^{-\eta|x|})e^{-\frac{(x-y-a_-t)^2}{Mt}} \\ &\quad + \mathbf{O}(e^{-\eta|x|})(|x-y-a_-t|+1)^{-1}I_{\{|x-y|\leqq a_-t\}}, \end{aligned}$$

which we must integrate against

$$h^-(s,y)^2 \leqq e^{-\eta|y|}D_-(s)^2 + d(y-a_-s)^2.$$

The first two terms of $G_x(t-s,x;y)$ are exactly those of previous analyses, and an estimate by $h^-(t,x)$ follows as there [H1], [H3]. For the third term we first consider

$$\begin{aligned} &\int_0^t \int_{-\infty}^{x} e^{-\eta|x|}\big(|x-y-a_-(t-s)|+1\big)^{-1} I_{\{|x-y|\leqq a_-(t-s)\}} e^{-\eta|y|}D_-(s)^2\,dy\,ds \\ &= \int_0^t \int_{x-a_-(t-s)}^{x} e^{-\eta|x|}\big(|x-y-a_-(t-s)|+1\big)^{-1} e^{-\eta|y|}D_-(s)^2\,dy\,ds \\ &= \int_0^{t/2} \int_{x-a_-(t-s)}^{x} e^{-\eta|x|}\big(|x-y-a_-(t-s)|+1\big)^{-1} e^{-\eta|y|}D_-(s)^2\,dy\,ds \\ &\quad + \int_{t/2}^{t} \int_{x-a_-(t-s)}^{x} e^{-\eta|x|}\big(|x-y-a_-(t-s)|+1\big)^{-1} e^{-\eta|y|}D_-(s)^2\,dy\,ds. \end{aligned}$$

We now divide the analysis as in the linear case, treating separately

$$y \in \Big[x-a_-(t-s), x-\frac{a_-}{2}(t-s)\Big] \quad \text{and} \quad y \in \Big[x-\frac{a_-}{2}(t-s), x\Big].$$

For $s \in [0, t/2]$ and $y \in \left[x - a_-(t-s), x - \frac{a_-}{2}(t-s)\right]$, we observe that

$$e^{-\eta|y|} \leqq e^{-\eta|x - \frac{a_-}{2}(t-s)|} \leqq e^{-\eta|x|} e^{-\eta \frac{a_-}{4} t},$$

exponential decay in x and t. For $s \in [0, t/2]$ and $y \in \left[x - \frac{a_-}{2}(t-s), x\right]$, we observe that

$$\left(|x - y - a_-(t-s)| + 1\right) \geqq \frac{1}{2}(t-s) \geqq \frac{1}{4}t.$$

The estimate follows, then, by integrating $e^{-\eta|y|}$ and $D_-(s)^2$.

For $s \in [t/2, t]$ and $y \in \left[x - a_-(t-s), x - \frac{a_-}{2}(t-s)\right]$, we have

$$\left(|x - y - a_-(t-s)| + 1\right) \geqq \frac{3}{a_-^2}$$

so that

$$\int_{t/2}^{t} \int_{x-a_-(t-s)}^{x-\frac{a_-}{2}(t-s)} e^{-\eta|x|} \left(|x - y - a_-(t-s)| + 1\right)^{-1} e^{-\eta|y|} D_-(s)^2 \, dy \, ds$$
$$\leqq C \int_{t/2}^{t} e^{-\eta|x|} e^{-\frac{\eta}{2}|x - \frac{a_-}{2}(t-s)|} D_-(t)^2 \, ds \leqq C e^{-\eta|x|} D_-(t)^2.$$

For $y \in \left[x - \frac{a_-}{2}(t-s), x\right]$, an interval of width $\frac{a_-}{2}(t-s)$, $\left(|x-y-a_-(t-s)|+1\right) \geqq \frac{1}{2}(t-s)$, so that we have an estimate by

$$C \int_{t/2}^{t} \int_{x-\frac{a_-}{2}t}^{x} e^{-\eta|x|} (t-s)^{-1} e^{-\eta|y|} D_-(t)^2 \, dy \, ds$$
$$= C \int_{t/2}^{t-1/2} e^{-\eta|x|} (t-s)^{-1} D_-(t)^2 \, dy \, ds + C \int_{t-1/2}^{t} e^{-\eta|x|} e^{-\eta|y|} D_-(t)^2 \, dy \, ds$$
$$\leqq C e^{-\eta|x|} D_-(t).$$

The analysis of the second term of $h^-(s,y)^2$, $d(y - a_- s)^2$ is similar. The next analysis is over $x \leqq y \leqq 0$, for which we have

$$G_x(t, x; y) = \mathbf{O}(t^{-1}) e^{-\frac{(x-y-a_-t)^2}{Mt}} + \mathbf{O}(t^{-1/2}) \mathbf{O}(e^{-\eta|x|}) e^{-\frac{(x-y-a_-t)^2}{Mt}}$$
$$+ \mathbf{O}(t^{-1}) \mathbf{O}(e^{-\eta|x|}) e^{-\frac{(x-y)^2}{Mt}} I_{\{|x-y| \leqq a_- t\}}.$$

The first two terms are again exactly as those of previous analyses, and we need only consider the third, over which we begin with

$$\int_0^t \int_x^0 (t-s)^{-1} e^{-\eta|x|} e^{-\frac{(x-y)^2}{M(t-s)}} I_{\{|x-y| \leqq a_-(t-s)\}} e^{-\eta|y|} D_-(s)^2 \, dy \, ds$$

$$= \int_0^t \int_x^{(x+a_-(t-s))\wedge 0} (t-s)^{-1} e^{-\frac{(x-y)^2}{M(t-s)}} e^{-\eta|x|} e^{-\eta|y|} D_-(s)^2 \, dy \, ds.$$

For $|x| \geqq \varepsilon t$, $0 < \varepsilon \ll a_-$, we have exponential decay in both x and t, so we need only consider $|x| \leqq \varepsilon t$. For $s \in [0, t/2]$ $x + a_-(t-s) \geqq x + \frac{a_-}{2} t \geqq \left(\frac{a_-}{2} - \varepsilon\right) t$; hence we have

$$\int_0^{t/2} \int_x^0 (t-s)^{-1} e^{-\frac{(x-y)^2}{M(t-s)}} e^{-\eta|x|} e^{-\eta|y|} D_-(s)^2 \, dy \, ds$$

$$\leqq C t^{-1} e^{-\eta|x|} \int_0^{t/2} D_-(s)^2 \, ds.$$

For $r > 2$, $D_-(s)^2$ is integrable and we have an estimate by $Ct^{-1}e^{-\eta|x|}$. For $1 < r < 2$,

$$\int_0^{t/2} D_-(s)^2 \, ds = \int_0^{t/2} (1+s)^{1-r} \, ds \leqq C(1+t)^{2-r},$$

which gives an estimate by $CD(t)e^{-\eta|x|}$, better than that required.

For $s \in [t, t/2]$ define t^* such that $x + a_-(t - t^*) = 0$. We then have

$$\int_{t/2}^{t^*} \int_x^0 (t-s)^{-1} e^{-\eta|x|} e^{-\frac{(x-y)^2}{M(t-s)}} e^{-\eta|y|} D_-(s)^2 \, dy \, ds$$

$$+ \int_{t^*}^{t} \int_x^{(x+a_-(t-s))} (t-s)^{-1} e^{-\frac{(x-y)^2}{M(t-s)}} e^{-\eta|x|} e^{-\eta|y|} D_-(s)^2 \, dy \, ds.$$

For $s \in [t/2, t^*]$, we have an estimate by

$$\int_{t/2}^{t^*} x(t-s)^{-1} e^{-\eta|x|} D_-(t)^2 \, dy \, ds \leqq C e^{-\eta|x|} D_-(t)^2 x [\log(t/2) - \log(-x/a_-)]$$

$$\leqq C e^{-\frac{\eta}{2}|x|} D_-(t).$$

Similarly, for $s \in [t^*, t]$, we observe that the width of y-integration is $a_-(t-s)$ and the width of x-integration is $|x/a_-|$, giving us the same estimate as above. Again, the analysis of $d(y - a_- s)^2$ is similar.

We next consider the case $x \leqq 0 \leqq y$, for which we have the Green's function estimate

$$G_x(t, x; y) = \mathbf{O}(t^{-1}) \mathbf{O}(e^{-\eta|x|}) \mathbf{O}_1(|y|) e^{-\frac{y^2}{Mt}}.$$

For $y \geqq 0$, we must now integrate against $h^+(s,y)^2$, which is bounded by the terms

$$e^{-\frac{y^2}{Ms}}(1+y)^{-2}D_-(s)^2 + e^{-\frac{y^2}{Ms}}(1+y)^{-4}D_+(s)^2 + d_+(s,y)^2.$$

Beginning with $e^{-\frac{y^2}{Ms}}(1+y)^{-2}D_-(s)^2$, we consider

$$\begin{aligned}
&\int_0^t \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|}(1+y) e^{-\frac{y^2}{M(t-s)}} e^{-\frac{y^2}{Ms}}(1+y)^{-2} D_-(s)^2\, dy\, ds \\
&\quad = \int_0^{t/2} \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|}(1+y) e^{-\frac{y^2}{M(t-s)}} e^{-\frac{y^2}{Ms}}(1+y)^{-2} D_-(s)^2\, dy\, ds \\
&\qquad + \int_{t/2}^{t} \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|}(1+y) e^{-\frac{y^2}{M(t-s)}} e^{-\frac{y^2}{Ms}}(1+y)^{-2} D_-(s)^2\, dy\, ds.
\end{aligned}$$

For $s \in [0, t/2]$ we have t^{-1} decay and by virtue of Lemma 5.4 an estimate of

$$Ct^{-1} e^{-\eta|x|} \int_0^{t/2} \log(2+s) D_-(s)^2\, ds \leqq Ce^{-\eta|x|} D_-(t)^2 \log t \leqq Ce^{-\eta|x|} D_-(t).$$

For $s \in [t/2, t]$ we divide the integration further to get $(t \geqq T > 1)$

$$\begin{aligned}
&\int_{t/2}^{t-1} \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|}(1+y)^{-1} e^{-\frac{y^2}{M(t-s)}} e^{-\frac{y^2}{Ms}} D_-(s)^2\, dy\, ds \\
&\qquad + \int_{t-1}^{t} \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|}(1+y)^{-1} e^{-\frac{y^2}{M(t-s)}} e^{-\frac{y^2}{Ms}} D_-(s)^2\, dy\, ds \\
&\quad \leqq CD_-(t)^2 e^{-\eta|x|} \log(2+t) \int_{t/2}^{t-1} (t-s)^{-1}\, ds + CD_-(t)^2 e^{-\eta|x|} \int_{t-1}^{t} (t-s)^{-1/2}\, ds \\
&\quad \leqq Ce^{-\eta|x|} D_-(t).
\end{aligned}$$

The analysis clearly proceeds similarly for integration against $e^{-\frac{y^2}{Ms}}(1+y)^{-4}D_+(s)^2$.

We also consider, for $1 < r < 2$,

$$\begin{aligned}
&\int_0^t \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|}(1+y) e^{-\frac{y^2}{M(t-s)}} [(1+y)^{-r} \wedge s^{-1}(1+y)^{2-r}]^2\, dy\, ds \\
&\quad = \int_0^{t/2} \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|}(1+y) e^{-\frac{y^2}{M(t-s)}} [(1+y)^{-r} \wedge s^{-1}(1+y)^{2-r}]^2\, dy\, ds \\
&\qquad + \int_{t/2}^{t} \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|}(1+y) e^{-\frac{y^2}{M(t-s)}} [(1+y)^{-r} \wedge s^{-1}(1+y)^{2-r}]^2\, dy\, ds,
\end{aligned}$$

and for $r > 2$

$$\int_0^t \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|} (1+y) e^{-\frac{y^2}{M(t-s)}} [(1+s)^{-1/2}(1+y)^{1-r} \wedge s^{-1}(1+y)^{2-r}]^2 \, dy\, ds$$

$$= \int_0^{t/2} \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|} (1+y) e^{-\frac{y^2}{M(t-s)}} [(1+s)^{-1/2}(1+y)^{1-r} \wedge s^{-1}(1+y)^{2-r}]^2 \, dy\, ds$$

$$+ \int_{t/2}^t \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|} (1+y) e^{-\frac{y^2}{M(t-s)}} [(1+s)^{-1/2}(1+y)^{1-r} \wedge s^{-1}(1+y)^{2-r}]^2 \, dy\, ds,$$

where we will use the estimate

$$(1+y)^{-r} \wedge s^{-1}(1+y)^{2-r} \leqq (1+y)^{-r\alpha} \cdot s^{-(1-\alpha)}(1+y)^{(2-r)(1-\alpha)}, \quad 0 \leqq 1 \leqq \alpha.$$

For $s \in [0, t/2]$ and $r > 5/2$ we have

$$\int_0^{t/2} \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|} (1+y) e^{-\frac{y^2}{M(t-s)}} (1+y)^{3-2r} (1+s)^{-3/2} \, dy\, ds$$

$$\leqq Ct^{-1} e^{-\eta|x|},$$

by the integrability of $(1+y)^{4-2r}$, $r > 5/2$ and of $(1+s)^{-3/2}$. Similarly, for $2 < r \leqq 5/2$, we have

$$\int_0^{t/2} \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|} e^{-\frac{y^2}{M(t-s)}} (1+y)^{3-2r} (1+s)^{-1} \, dy\, ds$$

$$\leqq Ct^{-1} \log(2+t) e^{-\eta|x|},$$

by the integrability of $(1+y)^{3-2r}$, $r > 2$, while for $3/2 < r < 2$, Lemma 5.4 provides an estimate by

$$\int_0^{t/2} \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|} e^{-\frac{y^2}{M(t-s)}} (1+y)^{3-2r} (1+s)^{-1} \, dy\, ds$$

$$\leqq Ct^{-1} e^{-\eta|x|} \int_0^{t/2} (1+t)^{2-r} (1+s)^{-1} \, ds \leqq Ct^{1-r} \log(2+t) e^{-\eta|x|}.$$

For $1 < r < 3/2$, Lemma 5.4 applies to give an estimate of

$$\int_0^{t/2} \int_0^{+\infty} (t-s)^{-1} e^{-\eta|x|} e^{-\frac{y^2}{M(t-s)}} (1+y)^{2-2r} (1+s)^{-1/2} \, dy\, ds$$

$$\leqq Ct^{-1} e^{-\eta|x|} \int_0^{t/2} (1+t)^{\frac{3}{2}-r} (1+s)^{-1/2} \, ds \leqq Ct^{1-r} e^{-\eta|x|}$$

which is better than required, as is the special case $r = 2$.

For $s \in [t/2, t]$ and $r > 2$ we have

$$\int_{t/2}^{t} \int_{0}^{+\infty} (t-s)^{-1} e^{-\eta|x|} e^{-\frac{y^2}{M(t-s)}} (1+y)^{4-2r} (1+s)^{-3/2}\, dy\, ds$$

$$\leqq C(1+t)^{-3/2} e^{-\eta|x|} \int_{t/2}^{t} (t-s)^{-1/2}\, ds \leqq C(1+t)^{-1} e^{-\eta|x|},$$

with a similar estimate for $r = 2$. For $3/2 < r < 2$ we have an estimate by

$$\int_{t/2}^{t} \int_{0}^{+\infty} (t-s)^{-1} e^{-\eta|x|} e^{-\frac{y^2}{M(t-s)}} (1+s)^{-1} (1+y)^{3-2r}\, dy\, ds$$

$$\leqq C(1+t)^{-1} e^{-\eta|x|} \int_{t/2}^{t} (t-s)^{-1/2}\, ds \leqq C(1+t)^{-1/2} e^{-\eta|x|}.$$

For $1 < r \leqq 3/2$ we have by virtue of Lemma 5.4

$$\int_{t/2}^{t} \int_{0}^{+\infty} (t-s)^{-1} e^{-\eta|x|} e^{-\frac{y^2}{M(t-s)}} (1+s)^{-1} (1+y)^{3-2r}\, dy\, ds$$

$$\leqq Ct^{-1} e^{-\eta|x|} \int_{t/2}^{t} (t-s)^{-1+\frac{1}{2}+\frac{3-2r}{2}}\, ds \leqq Ct^{1-r} e^{-\eta|x|}.$$

Nonlinear analysis for $x \geqq 0$. We now turn to the nonlinear analysis for $x \geqq 0$, beginning in the case $y \leqq 0 \leqq x$, for which we have the Green's function estimate

$$\begin{aligned} G_x(t,x;y) &= \mathbf{O}(t^{-1/2}) \mathbf{O}_1(|x|^{-2}) e^{-\frac{(x-y-a_- t)^2}{Mt}} I_{\{|y| \geqq a_- t\}} \\ &\quad + \mathbf{O}_1(|x|^{-2}) \left(t + \frac{y}{a_-} - \frac{3x^2}{2a_-^3 t^2} y \right)^{-1} e^{-\frac{x^2}{Mt}} I_{\{\{|y| \leqq a_- t\} \cap \{x \geqq 1\}\}} \\ &\quad + \mathbf{O}(1) \left(t + \frac{y}{a_-} - \frac{3}{2a_-^3 t} y \right)^{-1} I_{\{\{|y| \leqq a_- t\} \cap \{x \leqq 1\}\}}. \end{aligned}$$

Integrating against $h^-(s,y)^2$ we have first

$$\int_{0}^{t} \int_{-\infty}^{0} (t-s)^{-1/2} (1+x)^{-2} e^{-\frac{(x-y-a_-(t-s))^2}{M(t-s)}} I_{\{|y| \geqq a_-(t-s)\}} e^{-\eta|y|} D_-(s)^2\, dy\, ds$$

$$= \int_{0}^{t} \int_{-\infty}^{-a_-(t-s)} (t-s)^{-1/2} (1+x)^{-2} e^{-\frac{(x-y-a_-(t-s))^2}{M(t-s)}} e^{-\eta|y|} D_-(s)^2\, dy\, ds$$

$$\leqq C(1+x)^{-2} e^{-\frac{x^2}{Mt}} \int_{0}^{t} e^{-\eta a_-(t-s)} D_-(s)^2\, ds \leqq C(1+x)^{-2} e^{-\frac{x^2}{Mt}} D_-(t)^2,$$

better than the estimate required.

For $|y| \leqq a_-(t-s)$, $x \geqq 1$, we have

$$\int_0^t \int_{-a_-(t-s)}^0 (1+x)^{-2}\left(t-s+\frac{y}{a_-}-\frac{3x^2}{2a_-^3(t-s)^2}y\right)^{-1} e^{-\frac{x^2}{M(t-s)}}e^{-\eta|y|}D_-(s)^2\,dy\,ds$$

$$= \int_0^{t/2} \int_{-a_-(t-s)}^0 (1+x)^{-2}\left(t-s+\frac{y}{a_-}-\frac{3x^2}{2a_-^3(t-s)^2}y\right)^{-1} e^{-\frac{x^2}{M(t-s)}}e^{-\eta|y|}D_-(s)^2\,dy\,ds$$

$$+ \int_{t/2}^t \int_{-a_-(t-s)}^0 (1+x)^{-2}\left(t-s+\frac{y}{a_-}-\frac{3x^2}{2a_-^3(t-s)^2}y\right)^{-1} e^{-\frac{x^2}{M(t-s)}}e^{-\eta|y|}D_-(s)^2\,dy\,ds.$$

We divide the integral over $s \in [0, t/2]$ up as

$$\int_0^{t/2} \int_{-a_-(t-s)}^{-\frac{a_-}{2}(t-s)} (1+x)^{-2}\left(t-s+\frac{y}{a_-}-\frac{3x^2}{2a_-^3(t-s)^2}y\right)^{-1} e^{-\frac{x^2}{M(t-s)}}e^{-\eta|y|}D_-(s)^2\,dy\,ds$$

$$\int_0^{t/2} \int_{-\frac{a_-}{2}(t-s)}^0 (1+x)^{-2}\left(t-s+\frac{y}{a_-}-\frac{3x^2}{2a_-^3(t-s)^2}y\right)^{-1} e^{-\frac{x^2}{M(t-s)}}e^{-\eta|y|}D_-(s)^2\,dy\,ds.$$

On the integral over $y \in \left[-a_-(t-s), -\frac{a_-}{2}(t-s)\right]$ we observe that for $x \geqq \varepsilon t$, $e^{-\frac{x^2}{M(t-s)}}$ decays at exponential rate in both x and t. For $x \leqq \varepsilon t$ we integrate $\left(t-s+\frac{y}{a_-}-\frac{3x^2}{2a_-^3(t-s)^2}y\right)^{-1}$ to obtain an estimate by

$$\int_0^{t/2} (1+x)^{-2}e^{-\frac{x^2}{Mt}}\left(\frac{1}{a_-}-\frac{3x^2}{2a_-^3(t-s)^2}\right)$$

$$\times\left[\log\left(\frac{a_-}{2}(t-s)+\frac{3x^2}{4a_-^2(t-s)}\right)-\log\left(\frac{3x^2}{2a_-^3(t-s)}\right)\right]e^{-\eta\frac{a_-}{2}(t-s)}D_-(s)^2\,ds$$

$$\leqq C(1+x)^{-2}e^{-\frac{x^2}{Mt}}e^{-\eta\frac{a_-}{4}t}\log(t) \leqq C(1+x)^{-2}e^{-\frac{x^2}{Mt}}D_+(t).$$

On the integral over $y \in \left[-\frac{a_-}{2}(t-s), 0\right]$, we observe that

$$\left(t-s+\frac{y}{a_-}-\frac{3x^2}{2a_-^3(t-s)^2}y\right) \geqq \frac{1}{2}(t-s) \geqq \frac{1}{4}t,$$

so that we have an estimate by

$$Ct^{-1}(1+x)^{-2}e^{-\frac{x^2}{Mt}}\int_0^{t/2} D_-(s)^2\,ds \leqq C(1+x)^{-2}e^{-\frac{x^2}{Mt}}D_+(t).$$

On the integral over $s \in [t/2, t]$ we have

$$\int_{t/2}^{t} \int_{-a_-(t-s)}^{0} (1+x)^{-2} \left(t - s + \frac{y}{a_-} - \frac{3x^2}{2a_-^3 (t-s)^2} y \right)^{-1} e^{-\frac{x^2}{M(t-s)}} e^{-\eta |y|} D_-(s)^2 \, dy \, ds$$

$$= \int_{t/2}^{t} \int_{-a_-(t-s)}^{-\frac{a_-}{2}(t-s)} (1+x)^{-2} \left(t - s + \frac{y}{a_-} - \frac{3x^2}{2a_-^3 (t-s)^2} y \right)^{-1} e^{-\frac{x^2}{M(t-s)}} e^{-\eta |y|} D_-(s)^2 \, dy \, ds$$

$$+ \int_{t/2}^{t} \int_{-\frac{a_-}{2}(t-s)}^{0} (1+x)^{-2} \left(t - s + \frac{y}{a_-} - \frac{3x^2}{2a_-^3 (t-s)^2} y \right)^{-1} e^{-\frac{x^2}{M(t-s)}} e^{-\eta |y|} D_-(s)^2 \, dy \, ds.$$

For $y \in \left[-\frac{a_-}{2}(t-s), 0 \right]$, $\left(t - s + \frac{y}{a_-} - \frac{3x^2}{2a_-^3 (t-s)^2} y \right) \geqq \frac{1}{2}(t-s)$, so that we have an estimate by

$$C \int_{t/2}^{t-1} \int_{-\frac{a_-}{2}(t-s)}^{0} (1+x)^{-2} (t-s)^{-1} e^{-\frac{x^2}{M(t-s)}} e^{-\eta |y|} D_-(s)^2 \, dy \, ds$$

$$+ C \int_{t-1}^{t} \int_{-\frac{a_-}{2}(t-s)}^{0} (1+x)^{-2} (t-s)^{-1} e^{-\frac{x^2}{M(t-s)}} e^{-\eta |y|} D_-(s)^2 \, dy \, ds$$

$$\leqq C(1+x)^{-2} D_-(t)^2 e^{-\frac{x^2}{Mt}} \int_{t/2}^{t-1} (t-s)^{-1} \, ds + C(1+x)^{-2} D_-(t)^2 e^{-\frac{x^2}{Mt}} \int_{t-1}^{t} ds$$

$$\leqq C(1+x)^{-2} e^{-\frac{x^2}{Mt}} D_-(t).$$

For $y \in \left[-a_-(t-s), -\frac{a_-}{2}(t-s) \right]$, if $x^2/t \geqq \varepsilon > 0$, then

$$\left(t - s + \frac{y}{a_-} - \frac{3x^2}{2a_-^3 (t-s)^2} y \right) \geqq \frac{3\varepsilon}{a_-^3},$$

and we have

$$\int_{t/2}^{t} \int_{-a_-(t-s)}^{-\frac{a_-}{2}(t-s)} (1+x)^{-2} \left(t - s + \frac{y}{a_-} - \frac{3x^2}{2a_-^3 (t-s)^2} y \right)^{-1} e^{-\frac{x^2}{M(t-s)}} e^{-\eta |y|} D_-(s)^2 \, dy \, ds$$

$$\leqq C(1+x)^{-2} e^{-\frac{x^2}{Mt}} D_-(t)^2 \int_{t/2}^{t} e^{-\eta \frac{a_-}{2}(t-s)} \, ds \leqq C(1+x)^{-2} e^{-\frac{x^2}{Mt}} D_-(t).$$

On the other hand, if $\frac{x^2}{t} \leqq \varepsilon$, $\left(\frac{1}{a_-} - \frac{3x^2}{2a_-^3 (t-s)^2} \right) \geqq \frac{1}{a_-} - \frac{3\varepsilon}{a_-^3 t} > 0$, and we have

$$\int_{t/2}^{t}\int_{-a_-(t-s)}^{-\frac{a_-}{2}(t-s)} (1+x)^{-2}\left(t-s+\frac{y}{a_-}-\frac{3x^2}{2a_-^3(t-s)^2}y\right)^{-1} e^{-\frac{x^2}{M(t-s)}}e^{-\eta|y|}D_-(s)^2\,dy\,ds$$

$$\leqq C(1+x)^{-2}D_-(t)^2 e^{-\frac{x^2}{Mt}}\int_{t/2}^{t}\left[\log\left(\frac{\frac{a_-}{2}(t-s)+\frac{3x^2}{4a_-^2(t-s)}}{\frac{3x^2}{2a_-^3(t-s)}}\right)\right]e^{-\eta\frac{a_-}{2}(t-s)}$$

$$\leqq C(1+x)^{-2}D_-(t)e^{-\frac{x^2}{Mt}}.$$

Integrals of $d(y-a_-s)^2$ are similar, as is the case with $x\leqq 1$.

For $x, y\geqq 0$ we have

$$G_x(t,x;y)=\mathbf{O}(t^{-1})\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}}+\mathbf{O}(t^{-1}\log t)\mathbf{O}_1(|x|^{-2})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}}$$

and

$$h^+(s,y)^2\leqq (1+y)^{-2}e^{-\frac{y^2}{Ms}}D_-(s)^2+(1+y)^{-4}e^{-\frac{y^2}{Ms}}D_+(s)^2+d_+(s,y)^2.$$

Proceeding with integrals of the form

$$\int_0^t\int_0^{+\infty} G_x(t-s,x;y)h^+(s,y)^2\,dy\,ds,$$

we begin with $(1+y)^{-2}e^{-\frac{y^2}{Ms}}D_-(s)^2$

$$\int_0^t\int_0^{+\infty}(t-s)^{-1}(1+x)^{-1}e^{-\frac{(x-y)^2}{M(t-s)}}(1+y)^{-1}e^{-\frac{y^2}{Ms}}D_-(s)^2\,dy\,ds$$

$$=\int_0^{t/2}\int_0^{+\infty}(t-s)^{-1}(1+x)^{-1}e^{-\frac{(x-y)^2}{M(t-s)}}(1+y)^{-1}e^{-\frac{y^2}{Ms}}D_-(s)^2\,dy\,ds$$

$$+\int_{t/2}^{t}\int_0^{+\infty}(t-s)^{-1}(1+x)^{-1}e^{-\frac{(x-y)^2}{M(t-s)}}(1+y)^{-1}e^{-\frac{y^2}{Ms}}D_-(s)^2\,dy\,ds.$$

We observe that the combination $e^{-\frac{(x-y)^2}{M(t-s)}}e^{-\frac{y^2}{Ms}}$ gives kernel decay of strength $e^{-\frac{x^2}{Mt}}$. (It is a simple observation that this combination is largest at $y=\frac{s}{t}x$, for which it is constant in s.) For $s\in[0,t/2]$ we integrate $e^{-\frac{y^2}{4s}}(1+y)^{-1}$ according to Lemma 5.4 to arrive at an estimate by

$$Ct^{-1}(1+x)^{-1}e^{-\frac{x^2}{2Mt}}\int_0^{t/2}D_-(s)^2\log(2+s)\,ds\leqq C(1+x)^{-1}e^{-\frac{x^2}{2Mt}}D_-(t),$$

while for $s \in [t/2, t]$ we divide the integration further into

$$\int_{t/2}^{t-1}\int_0^{+\infty}(t-s)^{-1}(1+x)^{-1}e^{-\frac{(x-y)^2}{M(t-s)}}(1+y)^{-1}e^{-\frac{y^2}{Ms}}D_-(s)^2\,dy\,ds$$

$$+\int_{t-1}^{t}\int_0^{+\infty}(t-s)^{-1}(1+x)^{-1}e^{-\frac{(x-y)^2}{M(t-s)}}(1+y)^{-1}e^{-\frac{y^2}{Ms}}D_-(s)^2\,dy\,ds$$

$$\leqq C(1+x)^{-1}e^{-\frac{x^2}{2Mt}}D_-(t)^2\int_{t/2}^{t-1}(t-s)^{-1}\log(2+s)\,ds$$

$$+C(1+x)^{-1}e^{-\frac{x^2}{2Mt}}D_-(t)^2\int_{t-1}^{t}(t-s)^{-1/2}\,ds \leqq C(1+x)^{-1}e^{-\frac{x^2}{2Mt}}D_-(t).$$

Integration against the second term of $G_x(t,x;y)$ clearly proceeds similarly and yields an additional estimate by $C(1+x)^{-2}e^{-\frac{x^2}{2Mt}}$.

We next consider

$$\int_0^t\int_0^{+\infty}(t-s)^{-1}(1+x)^{-1}e^{-\frac{(x-y)^2}{M(t-s)}}(1+y)d_+(s,y)^2\,dy\,ds,$$

where we recall

$$d_+(s,y) := \begin{cases}(1+y)^{-r}\wedge s^{-1}(1+y)^{2-r}, & 1<r\leqq 2,\\ (1+y)^{-2}\log(2+y)\wedge s^{-1}(1+y)^{2-r}, & r=2,\\ (1+s)^{-1/2}(1+y)^{1-r}\wedge s^{-1}(1+y)^{2-r}, & r>2.\end{cases}$$

We again divide the y-integration over three intervals: $y\in[0,\varepsilon x]$, $y\in[\varepsilon x, 2x]$, $y\in[2x,+\infty)$. For $r>5/2$ and $s\in[0,t/2]$ we have

$$\int_0^{t/2}\int_0^{\varepsilon x}(t-s)^{-1}(1+x)^{-1}(1+y)e^{-\frac{(x-y)^2}{M(t-s)}}(1+s)^{-3/2}(1+y)^{3-2r}$$

$$\leqq Ct^{-1}(1+x)^{-1}e^{-\frac{x^2}{2Mt}},$$

by the integrability of $(1+s)^{-3/2}$ in s and of $(1+y)^{4-2r}$, $r>5/2$ in y. For $2<r<5/2$, we similarly have an estimate by

$$Ct^{-1}(1+x)^{-1}e^{-\frac{x^2}{2Mt}}\int_0^{t/2}\int_0^{\varepsilon x}(1+s)^{-1}(1+y)^{3-2r}\,dy\,ds$$

$$\leqq Ct^{-1}\log t(1+x)^{-1}e^{-\frac{x^2}{2Mt}},$$

by the integrability of $(1+y)^{3-2r}$, $r>2$. For $3/2<r<2$, we have an estimate by

$$Ct^{-1}(1+x)^{-1}e^{-\frac{x^2}{2Mt}}\int_0^{t/2}\int_0^{\varepsilon x}(1+s)^{-1/2}(1+y)^{2-2r}\,dy\,ds$$

$$\leqq Ct^{-1/2}(1+x)^{-1}e^{-\frac{x^2}{2Mt}},$$

while for $1 < r \leqq 3/2$ we have

$$Ct^{-1}(1+x)^{-1}e^{-\frac{3x^2}{4Mt}}\int_0^{t/2}\int_0^{\varepsilon x}(1+s)^{-1/2}(1+y)^{2-2r}\,dy\,ds$$
$$\leqq Ct^{-1/2}(1+x)^{2-2r}e^{-\frac{3x^2}{4Mt}} \leqq Ct^{1-r}(1+x)^{-1}e^{-\frac{x^2}{2Mt}},$$

with a similar estimate for $r = 2$.

We next consider the interval $y \in [\varepsilon x, 2x]$, for which we begin with $(1 < r < 2)$

$$\int_0^{t/2}\int_{\varepsilon x}^{2x}(t-s)^{-1}(1+x)^{-1}e^{-\frac{(x-y)^2}{M(t-s)}}(1+y)^{1-2r}\,dy\,ds$$
$$\leqq Ct^{-1}(1+x)^{1-2r}\int_0^{t/2}(1+s)^{\frac{1-r}{2}}\,ds \leqq C(1+x)^{1-2r}.$$

Similarly,

$$\int_0^{t/2}\int_{\varepsilon x}^{2x}(t-s)^{-1}(1+x)^{-1}e^{-\frac{(x-y)^2}{M(t-s)}}(1+s)^{-\frac{1+r}{2}}(1+y)^{2-r}\,dy\,ds$$
$$\leqq Ct^{-1}(1+x)^{2-r}.$$

For $r > 2$ we have integrals of the form

$$\int_0^{t/2}\int_{\varepsilon x}^{2x}(t-s)^{-1}e^{-\frac{(x-y)^2}{M(t-s)}}(1+y)[(1+s)^{-1/2}(1+y)^{1-r} \wedge s^{-1}(1+y)^{2-r}]^2\,dy\,ds$$
$$\leqq Ct^{-1}(1+x)^{-1}\int_0^{t/2}\int_{\varepsilon x}^{2x}(1+s)^{-1}(1+y)^{3-2r}\,dy\,ds \leqq Ct^{-1}\log t(1+x)^{3-2r},$$

which for $r > 2$ is bounded by $Ct^{-1}\log t(1+x)^{1-r}$. Also

$$\int_0^{t/2}\int_{\varepsilon x}^{2x}(t-s)^{-1}(1+x)^{-1}e^{-\frac{(x-y)^2}{M(t-s)}}(1+s)^{-3/2}(1+y)^{4-2r}\,dy\,ds \leqq Ct^{-1}(1+x)^{2-r}.$$

For the final term with $s \in [0, t/2]$ we take $y \in [2x, +\infty)$. For $r > 3/2$ the analysis proceeds exactly as for $y \in [0, \varepsilon x]$. For $1 < r \leqq 3/2$ we have

$$\int_0^{t/2}\int_{2x}^{+\infty}(t-s)^{-1}(1+x)^{-1}e^{-\frac{(x-y)^2}{M(t-s)}}(1+s)^{-1/2}(1+y)^{2-2r}\,dy\,ds.$$

We now make the change of variable $\xi = y - x$ and apply Lemma 5.4 in order to obtain an estimate of $Ct^{1-r}(1+x)^{-1}e^{-\frac{x^2}{2Mt}}$.

We close the proof with estimates over the interval $s \in [t/2, t]$. As the analysis is similar to the case $s \in [0, t/2]$, we provide only a few crucial cases. For $y \in [0, \varepsilon x]$ and $1 < r \leqq 3/2$ we have

$$\int_{t/2}^{t} \int_{0}^{\varepsilon x} (t-s)^{-1}(1+x)^{-1} e^{-\frac{(x-y)^2}{M(t-s)}} (1+s)^{-1}(1+y)^{3-2r}\, dy\, ds$$

$$\leqq Ct^{-1}(1+x)^{2-2r} e^{-\frac{3x^2}{4Mt}} \int_{t/2}^{t} \int_{0}^{\varepsilon x} (t-s)^{-1} e^{-\frac{(x-y)^2}{4M(t-s)}}\, dy\, ds$$

$$\leqq Ct^{-1}(1+x)^{2-2r} e^{-\frac{3x^2}{4Mt}} \int_{t/2}^{t} (t-s)^{-1/2}\, ds \leqq C(1+t)^{-1/2}(1+x)^{2-2r} e^{-\frac{3x^2}{4Mt}}$$

$$\leqq C(1+t)^{1-r}(1+x)^{-1} e^{-\frac{x^2}{2Mt}} \leqq C(1+x)^{-1} e^{-\frac{x^2}{2Mt}} D_-(t).$$

Similarly, for $y \in [2x, +\infty)$ and $1 < r \leqq 3/2$ we have

$$\int_{t/2}^{t} \int_{2x}^{+\infty} (t-s)^{-1}(1+x)^{-1} e^{-\frac{(x-y)^2}{M(t-s)}} (1+s)^{-1}(1+y)^{3-2r}\, dy\, ds,$$

for which we make the change of variable $\xi = y - x$ and employ Lemma 5.4 to obtain an estimate by

$$Ct^{-1}(1+x)^{-1} e^{-\frac{x^2}{2Mt}} \int_{t/2}^{t} (t-s)^{1-r}\, ds \leqq Ct^{1-r}(1+x)^{-1} e^{-\frac{x^2}{2Mt}}.$$

Finally, the terms over $y \in [\varepsilon x, 2x]$ proceed as

$$\int_{t/2}^{t} \int_{\varepsilon x}^{2x} (t-s)^{-1}(1+x)^{-1} e^{-\frac{(x-y)^2}{M(t-s)}} (1+s)^{-1-\frac{r}{2}}(1+y)^{3-r}\, dy\, ds$$

$$\leqq C(1+t)^{-1/2-r/2}(1+x)^{2-r} \leqq C(1+t)^{-1/2-r/2}(1+x)^{2-r}.$$

Integrals of $\mathbf{O}\big((t-s)\log(t-s)\big)\mathbf{O}_1(|x|^{-2})\mathbf{O}_1(|y|)$ against $(1+y)^{-2} e^{-\frac{y^2}{Ms}} D_-(s)^2$ and $d_+(s,y)^2$ proceed similarly, yielding estimates altered from those above by $\log t(1+x)^{-1}$. Integration of $G_x(t,x;y)$ against $(1+y)^{-4} e^{-\frac{y^2}{Ms}} D_+(s)^2$ yields an estimate which may be subsumed into those above $\big(h^+(t,x)\big)$.

6. Further work

Local tracking. Recent advances in the study of stability for viscous shock waves have shown that local tracking rather than asymptotic tracking typically leads to a more detailed understanding of the underlying dynamics [HZ1–2], [ZH]. Indeed, for undercompressive viscous shock waves, whose asymptotic location cannot generally be determined by conservation of mass, local tracking is essential. Such an analysis is hindered, however, in the case of degenerate viscous shock waves by the branch point of the Evans function at $\lambda = 0$. In the case of Lax and undercompressive waves, the Evans function is analytic at $\lambda = 0$ and thus contour integrals over $G_\lambda(x, y)$ in which the contour Γ lies to the left of $\lambda = 0$ may be evaluated as residues (of a meromorphic function), which are precisely projections onto the eigenspace associated to $\bar{u}_x(x)$. In the case that $\lambda = 0$ is a branch point of the Evans function, such an analysis clearly fails. It would appear, however, that we can approximate the eigenspace associated to $\bar{u}_x(x)$ with functions obtained as $\lambda \to 0$ limits in the preceding analysis.

Nonconstant viscosity and higher order degeneracy. Extending these methods to nonconstant viscosity $b(\cdot)$ and higher order degeneracies depends upon developing estimates in the vein of Lemma 3.1 in those cases. The principle difficulty here lies in gaining a sufficiently accurate description of the transition in behavior as λ goes to zero. For first order degeneracy we have taken advantage of the observation that while solutions to the unintegrated equation, $v_{xx} - \big(a(x)v\big)_x = \lambda v$, scale differently for $\lambda \neq 0$ ($v \sim e^{\int a(s)/2\,ds}$) and $\lambda = 0$ ($v \sim e^{\int a(s)\,ds}$), solutions of the integrated equation, $v_{xx} - a(x)v_x = \lambda v$, both scale as $v \sim \bar{u}(x) - u_+ \sim \sqrt{\bar{u}_x} \sim e^{\int a(s)/2\,ds}$. For degeneracies of higher order, say k, this discrepancy in scaling increases: (integrated equations) for $\lambda \neq 0$, $v \sim \sqrt{\bar{u}_x} \sim e^{\int a(s)/2\,ds}$, while for $\lambda = 0$, $v \sim \bar{u}(x) - u_+ \sim (\bar{u}_x)^{1/(k+1)} \sim e^{\int a(s)/(k+1)}$. Estimates of the required accuracy can be obtained in these cases through λ-averaged scalings. For example, for $k = 2$ we may derive estimates similar to those of Lemma 3.1 through looking for solutions of the form

$$v(x) = \frac{\big(\bar{u}(x) - u_+\big)^{3/2}}{\gamma\lambda^{1/4} + \bar{u}(x) - u_+},$$

where $\gamma = \sqrt{-8/f'''(u_+)}$.

Systems and planar waves. The methods employed here have already been shown extraordinarily robust. They have been successful in the study of scalar conservation laws with high order diffusion and dispersion [HZ1]; in the study of multivariate conservation laws [HoZ1–2], [Z3], [ZS]; and in the study of systems [ZH]. We see no reason why they will not similarly extend, for example, to degenerate solutions of the p-system

$$v_t - u_x = 0,$$

$$u_t - \sigma(v)_x = \mu u_{xx},$$

whose stability has been extensively studied by Matsumura, Mei and Nishihara (see, e.g., [M1–4], [MM1–2], [N]), and to degenerate solutions of the multivariate equation

$$u_t + f(u)_x = \triangle u, \quad t, x, y \in \mathbb{R}_+ \times \mathbb{R}^2,$$

whose stability has recently been analyzed by Nishikawa [Ni].

Acknowledgements

I would like to express my appreciation to Kevin Zumbrun for suggesting the problem and for offering numerous useful suggestions as I worked on it. I am also grateful to

Todd Kapitula for his insight into ODE dynamics near a branch point. Though my basic approach differs considerably from that of Mei, Matsumura, Nishihara and Nishikawa, their work has certainly guided the analysis. Finally, this research was supported in part by the NSF under Grant No. DMS-9804390.

References

[AGJ] *J. Alexander*, *R. Gardner*, and *C. K. R. T. Jones*, A topological invariant arising in the analysis of traveling waves, J. reine angew. Math. **410** (1990), 167–212.

[BF] *F. W. Byron*, *Jr.* and *R. W. Fuller*, Mathematics of classical and quantum physics, Dover publications, New York 1969, 1970.

[C] *W. A. Coppel*, Stability and asymptotic behavior of differential equations, D.C. Heath and Co., Boston 1965.

[F] *M. V. Fedoryuk*, Asymptotic analysis, Springer-Verlag, Berlin-Heidelberg 1993.

[FD] *W. Fickett* and *W. C. Davis*, Detonation, University of California Press, London 1979.

[GZ] *R. Gardner* and *K. Zumbrun*, The Gap Lemma and geometric criteria for instability of viscous shock profiles, CPAM **51** (1998), 789–847.

[G] *J. Goodman*, Nonlinear asymptotic stability of viscous shock profiles for conservation laws, Arch. Rat. Mech. Anal. **95** (1986), 325–344.

[Ha] *S. E. Harris*, Sonic shocks governed by the modified Burgers' equation, Euro. J. Appl. Math. **7** (1996), 201–222.

[HoZ1] *D. Hoff* and *K. Zumbrun*, Asymptotic behavior of multidimensional scalar viscous shock fronts, preprint 1999.

[HoZ2] *D. Hoff* and *K. Zumbrun*, Pointwise Green's function bounds for multi-dimensional scalar shock fronts, preprint 2000.

[H] *E. Hopf*, The partial differential equation $u_t + uu_x = \mu u_{xx}$, CPAM **3** (1950), 201–230.

[H1] *P. Howard*, Pointwise estimates for the stability of a scalar conservation law, Thesis, Indiana University, adv. K. Zumbrun.

[H2] *P. Howard*, Pointwise estimates on the Green's function for a scalar linear convection-diffusion equation, J. Diff. Equ. **155** (1999), 327–367.

[H3] *P. Howard*, Pointwise Green's function approach to stability for scalar conservation laws, CPAM **52** (1999), 1295–1313.

[HZ1] *P. Howard* and *K. Zumbrun*, Pointwise estimates and stability for dispersive-diffusive shock waves, preprint 1999.

[HZ2] *P. Howard* and *K. Zumbrun*, A local tracking scheme for viscous shock waves, preprint 1999.

[Ito] *K. Ito*, Asymptotic decay toward planar rarefaction waves of solutions for viscous conservation laws in several space dimensions, Math. Models Meth. Appl. Sci. **6** (1996), 315–338.

[JGK] *C. K. R. T. Jones*, *R. Gardner*, and *T. Kapitula*, Stability of traveling waves for nonconvex scalar viscous conservation laws, CPAM **46** (1993), no. 4, 505–526.

[KR] *T. Kapitula* and *J. Rubin*, Existence and stability of standing hole solutions to complex Ginzburg-Landau equations, Nonlin. **13** (2000), no. 1, 77–112.

[KS] *T. Kapitula* and *B. Sandstede*, Stability of bright solitary-wave solutions to perturbed nonlinear Schrödinger equations, Physica D **124** (1998), 58–103.

[KM1] *S. Kawashima* and *A. Matsumura*, Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion, Comm. Math. Phys. **101** (1985), 97–127.

[KM2] *S. Kawashima* and *A. Matsumura*, Stability of shock profiles in viscoelasticity with non-convex constitution relations, CPAM **47** (1994), 1547–1569.

[Le] *N. Levinson*, The asymptotic nature of solutions of linear systems of differential equations, Duke Math. J. **15** (1948), 111–126.

[Li1] *T. Li*, On the initiation problem for a combustion model, J. Diff. Equ. **112** (1994), no. 2, 351–373.

[Li2] *T. Li*, Rigorous asymptotic stability of a CJ detonation wave in the limit of small resolved heat release, Combust. Th. Modell. **1** (1997), 259–270.

[Li3] *T. Li*, Stability of strong detonation waves and rates of convergence, Electr. J. Diff. Equ. **1998** (1998), no. 9, 1–17.

[Li4] *T. Li*, Stability and instability of detonation waves, in: Hyperbolic problems: theory, numerics, applications, Seventh International Conference in Zürich, February 1998, Vol. II, 641–650.

[L] *T.-P. Liu*, Nonlinear stability of shock waves for viscous conservation laws, Mem. AMS **56** (1985), no. 328.

[LY] *T.-P. Liu* and *L. A. Ying*, Nonlinear stability of strong detonations for a viscous combustion model, SIAM J. Math. Anal. **26** (1995), 519–528.

[M] *A. Majda*, A qualitative model for dynamics combustion, SIAM J. Appl. Math. **41** (1981), 70–93.

[MN1] *A. Matsumura* and *K. Nishihara*, On the stability of traveling wave solutions of a one-dimensional model system for compressive viscous gas, Japan. J. Appl. Math. **2** (1985), 17–25.

[MN2] *A. Matsumura* and *K. Nishihara*, Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas, Japan. J. Appl. Math. **3** (1986), 1–13.

[MN3] *A. Matsumura* and *K. Nishihara*, Global stability of the rarefaction wave of a one-dimensional model system for compressible viscous gas, Comm. Math. Phys. **144** (1992), 325–335.

[MN4] *A. Matsumura* and *K. Nishihara*, Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex linearity, Comm. Math. Phys. **165** (1994), no. 1, 83–96.

[MN5] *A. Matsumura* and *K. Nishihara*, Global asymptotics toward the rarefaction wave for solutions of viscous p-systems with boundary effect, Quart. Appl. Math. **58** (2000), no. 1, 69–83.

[M1] *M. Mei*, Stability of shock profiles for nonconvex scalar viscous conservation laws, Math. Models Meth. Appl. Sci. **5** (1995), no. 3, 279–296.

[M2] *M. Mei*, Stability of traveling wave solutions for non-convex equations of barotropic viscous gas, Osaka J. Math. **34** (1997), 308–318.

[M3] *M. Mei*, Asymptotic behavior of solutions for degenerate hyperbolic systems of viscous conservation laws with boundary effect, ZAMP **50** (1999), 617–633.

[MeiN] *M. Mei* and *K. Nishihara*, Nonlinear stability of traveling waves for one-dimensional viscoelastic materials with non-convex nonlinearity, Tokyo J. Math. **20** (1997), no. 1, 241–264.

[MM] *R. Miller* and *A. Michel*, Ordinary Differential Equations, Academic Press, 1982.

[N1] *K. Nishihara*, A note on the stability of traveling wave solutions of Burgers equation, Japan. J. Appl. Math. **2** (1985), 27–35.

[N2] *K. Nishihara*, Stability of traveling waves with degenerate shock for system of one-dimensional viscoelastic model, J. Diff. Equ. **120** (1995), 304–318.

[Ni] *M. Nishikawa*, Convergence rate to the traveling wave for viscous conservation laws, Funkcialoj Ekvacioj **41** (1998), 107–132.

[RM] *R. R. Rosales* and *A. Majda*, Weakly nonlinear detonation waves, SIAM J. Appl. Math. **43** (1983), 10086–1118.

[S] *D. Sattinger*, On the stability of waves of nonlinear parabolic systems, Adv. Math. **22** (1976), 312–355.

[SZ] *A. Szepessy* and *K. Zumbrun*, Stability and rarefaction waves in viscous media, Arch. Rat. Mech. Anal. **133** (1996), 249–298.

[T] *L. Truskinovsky*, Transition to detonation in dynamic phase changes, Arch. Rat. Mech. Anal. **125** (1994), 375–397.

[Z1] *K. Zumbrun*, Formation of diffusion waves in a scalar conservation law with convection, Trans. Amer. Math. Soc. **347** (1995), no. 3, 1023–1032.

[Z2] *K. Zumbrun*, Refined wave tracking and nonlinear stability of viscous Lax shocks, preprint 1999.

[Z3] *K. Zumbrun*, Lecture notes, 1999, available on: http://www.math.indiana.edu/home/zumbrun.

[Z4] *K. Zumbrun*, Refined Green's function bounds for stability of viscous shock waves, preprint 2000.

[ZH] *K. Zumbrun* and *P. Howard*, Pointwise semigroup methods and stability of viscous shock waves, Indiana U. Math. J. **47** (1998), no. 3, 741–871.

[ZS] *K. Zumbrun* and *D. Serre*, Viscous and inviscid stability of multi-dimensional planar shock fronts, Indiana Univ. Math. J. **48** (1999), no. 3, 937–992.

Department of Mathematics, Texas A&M University, College Station, TX 77843-3368
e-mail: phoward@math.tamu.edu

Eingegangen 19. Juni 2000