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SHARP POINTWISE BOUNDS FOR PERTURBED VISCOUS SHOCK WAVES

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Abstract. Refining previous work of Zumbrun, Mascia–Zumbrun, Raoofi, Howard–Zumbrun and Howard–Raoofi, we derive sharp pointwise bounds on behavior of perturbed viscous shock profiles for large-amplitude Lax or overcompressive type shocks and physical viscosity. These extend well-known results of Liu obtained by somewhat different techniques for small-amplitude Lax type shocks and artificial viscosity, completing a program initiated by Zumbrun and Howard. As pointed out by Liu, the key to obtaining sharp bounds is to take account of cancellation associated with the property that the solution decays faster along characteristic than in other directions. Thus, we must here estimate characteristic derivatives for the entire nonlinear perturbation, rather than judicially chosen parts as in the work of Raoofi and Howard–Raoofi, a requirement that greatly complicates the analysis.

Keywords: Hyperbolic–Parabolic conservation laws; traveling waves; asymptotic behavior and stability; Lax and overcompressive shocks; Evans function.

1. Introduction

Three important themes in the work of Tai-Ping Liu are stability of shock waves, pointwise bounds, and the interaction of hyperbolic and parabolic (viscous) effects.

These came together in the landmark paper [27], a tour de force in which he established sharp pointwise bounds on the asymptotic behavior of a perturbed viscous shock profile, for small-amplitude, Lax type shocks and artificial viscosity. A long-standing program of the authors, set out in [43], has been to extend to large-amplitudes, general type profiles, and physical (partially parabolic) viscosities results obtained by Liu and others for small amplitude Lax type profiles and artificial viscosity. For various results in this direction, see, e.g., [42, 34, 37, 16, 15] In this paper, we achieve a definitive result recovering the full bounds of [27] for general Lax or overcompressive type shocks and physical or artificial viscosity, thus completing the program of [43]. The analysis involves an interesting blend of the spectral techniques introduced by the authors to deal with large amplitudes, described in Secs. 2 and 3, a delicate type of cancellation estimate introduced by Liu [26, 27] in the small-amplitude context, described in Sec. 4, and sharp convolution estimates of the type developed in [12–16], taking account of transversality but not cancellation of two interacting signals.

Consider a (possibly) large-amplitude *viscous shock profile*, or traveling-wave solution

$$\bar{u}(x-st); \quad \lim_{x\to\pm\infty} \bar{u}(x) = u_\pm, \tag{1.1}$$

of systems of partially or fully parabolic conservation laws

$$u_t + F(u)_x = (B(u)u_x)_x, \tag{1.2}$$

$x \in \mathbb{R}$, u, $F \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times n}$, where

$$u = \begin{pmatrix} u^I \\ u^{II} \end{pmatrix}, \quad F = \begin{pmatrix} F^I \\ F^{II} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \tag{1.3}$$

$u^I \in \mathbb{R}^{n-r}$, $u^{II} \in \mathbb{R}^r$, r some positive integer, possibly n (full regularization), and

$$\operatorname{Re}\sigma(b_2) \geq \theta > 0.$$

Here and elsewhere, σ denotes spectrum of a matrix or other linear operator. Working in a coordinate system moving along with the shock, we may without loss of generality consider a standing profile $\bar{u}(x)$, $s = 0$.

Following [41], we assume that, by some invertible change of coordinates $u \to w(u)$, followed if necessary by multiplication on the left by a nonsingular matrix function $S(w)$, Eqs. (1.2) may be written in the *quasilinear, partially symmetric hyperbolic-parabolic form*

$$\tilde{A}^0 w_t + \tilde{A} w_x = (\tilde{B} w_x)_x + G, \quad w = \begin{pmatrix} w^I \\ w^{II} \end{pmatrix}, \tag{1.4}$$

$w^I \in \mathbb{R}^{n-r}$, $w^{II} \in \mathbb{R}^r$, $x \in \mathbb{R}$, $t \in \mathbb{R}_+$, where, defining $w_\pm := w(u_\pm)$:

(A1) $\tilde{A}(w_\pm)$, $\tilde{A}_{11}$, $\tilde{A}^0$ are symmetric, $\tilde{A}^0 > 0$.

(A2) Dissipativity: no eigenvector of $dF(u_\pm)$ lies in the kernel of $B(u_\pm)$. (Equivalently, no eigenvector of $\tilde{A}(\tilde{A}^0)^{-1}(w_\pm)$ lies in the kernel of $\tilde{B}(\tilde{A}^0)^{-1}(w_\pm)$.)

(A3) $\tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b} \end{pmatrix}$, $\tilde{G} = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix}$, with $Re\tilde{b}(w) \geq \theta$ for some $\theta > 0$, for all w, and $\tilde{g}(w_x, w_x) = \mathbf{O}(|w_x|^2)$. Here, the coefficients of (1.4) may be expressed in terms of the original equation (1.2), the coordinate change $u \to w(u)$, and the approximate symmetrizer $S(w)$, as

$$\begin{aligned} \tilde{A}^0 &:= S(w)(\partial u/\partial w), \quad \tilde{A} := S(w)dF(u(w))(\partial u/\partial w), \\ \tilde{B} &:= S(w)B(u(w))(\partial u/\partial w), \quad G = -(dSw_x)B(u(w))(\partial u/\partial w)w_x. \end{aligned} \tag{1.5}$$

Alternatively, we assume, simply,

(B1) Strict parabolicity: $n = r$, or, equivalently, $\Re\sigma(B) > 0$.

Along with the above structural assumptions, we make the technical hypotheses:

(H0) F, B, w, $S \in C^{10}$.

(H1) The eigenvalues of $\tilde{A}_* := \tilde{A}_{11}(\tilde{A}^0_{11})^{-1}$ are (i) distinct from 0; (ii) of common sign; and (iii) of constant multiplicity with respect to u.

(H2) The eigenvalues of $dF(u_\pm)$ are real, distinct, and nonzero.

(H3) Nearby $\bar{u}$, the set of all solutions of (1.1)–(1.2) connecting the same values $u_\pm$ forms a smooth manifold $\{\bar{u}^\delta\}$, $\delta \in \mathcal{U} \subset \mathbb{R}^\ell$, $\bar{u}^0 = \bar{u}$.

Remark 1.1. Structural assumptions (A1)–(A3) [alternatively, (B1)] and technical hypotheses (H0)–(H2) admit such physical systems as the compressible Navier–Stokes equations, the equations of magnetohydrodymics, and Slemrod's model for van der Waal gas dynamics [40]. Moreover, existence of waves $\bar{u}$ satisfying (H3) has been established in each of these cases.

Definition 1.2. An ideal shock

$$u(x,t) = \begin{cases} u_-, & x < st, \\ u_+, & x > st, \end{cases} \tag{1.6}$$

is classified as *undercompressive*, *Lax*, or *overcompressive* type according as $i - n$ is less than, equal to, or greater than 1, where i, denoting the sum of the dimensions i_- and i_+ of the center-unstable subspace of $df(u_-)$ and the center-stable subspace of $df(u_+)$, represents the total number of characteristics incoming to the shock.

A viscous profile (1.1) is classified as *pure undercompressive* type if the associated ideal shock is undercompressive and $\ell = 1$, *pure Lax* type if the corresponding ideal shock is Lax type and $\ell = i - n$, and *pure overcompressive* type if the corresponding ideal shock is overcompressive and $\ell = i - n$, ℓ as in (H3). Otherwise it is classified as *mixed under-overcompressive* type; see [43].

Pure Lax type profiles are the most common type, and the only type arising in standard gas dynamics, while pure over- and undercompressive type profiles arise in magnetohydrodynamics (MHD) and phase-transitional models. In this paper, we restrict to the case of pure Lax or overcompressive type shocks.

Finally, we assume that the profile satisfies a linearized stability criterion based on the *Evans function*. As described, e.g., in [1, 6, 10, 20, 21, 34, 42, 43], the Evans function $D(\lambda)$, a Wronskian constructed from solutions of the associated eigenvalue equation, serves as a characteristic function for the linear operator L that arises upon linearization of (1.2) about $\bar{u}(x)$. More precisely, away from essential spectrum, zeroes of the Evans function correspond in location and multiplicity with eigenvalues of L [1, 10, 43]. It was shown in [43, 34], respectively for the strictly parabolic and real viscosity cases, that $L^1 \cap L^p \to L^p$ linearized orbital stability of the profile, $p > 1$, is equivalent to the Evans function condition

($\mathcal{D}$) There exist precisely ℓ zeroes of $D(\cdot)$ in the nonstable half-plane Re $\lambda \geq 0$, necessarily at $\lambda = 0$, where ℓ as in (H3) is the dimension of the manifold connecting u_- and u_+.

Under assumptions (A0)–(A3) [alternatively, (B1)] and (H0)–(H3), condition ($\mathcal{D}$) is equivalent to (i) *strong spectral stability*, $\sigma(L) \subset \{\text{Re}\ \lambda \leq 0\} \cup \{0\}$, (ii) *hyperbolic stability* of the associated ideal shock, and (iii) *transversality* of $\bar{u}$ as a solution of the connection problem in the associated traveling-wave ODE, where hyperbolic stability is defined for Lax and undercompressive shocks by the Lopatinski condition of [28–30, 7] and for Lax and overcompressive shocks by the analogous long-wave stability condition ($\mathcal{D}$ii), below; see [43, 33, 38, 41] for further explanation.

Remark 1.3. Stability criterion ($\mathcal{D}$) has been shown to hold for general small amplitude Lax shocks [8, 11, 19, 23, 24, 35, 36], and for large-amplitude shocks in such cases as Lax type waves arising in isentropic Navier–Stokes equations for the gamma-law gas as $\gamma \to 1$ [35], and undercompressive shocks arising in Slemrod's model for van der Waal gas dynamics [40] (see [39] for Slemrod's model). On the other hand, it has been shown to fail for certain large-amplitude and or nonclassical type shocks [10, 9, 40, 38]. More generally, condition ($\mathcal{D}$) can be readily checked by numerical calculation [2–5, 18, 17].

$$\text{Setting } A_\pm := dF(u_\pm),\ \Gamma_\pm := d^2F(u_\pm),\ \text{and } B_\pm := B(u_\pm),\ \text{denote by}$$
$$a_1^- < a_2^- < \cdots < a_n^- \text{ and } a_1^+ < a_2^+ < \cdots < a_n^+ \tag{1.7}$$

the eigenvalues of A_- and A_+, and $l_j^\pm$, $r_j^\pm$ left and right eigenvectors associated with each $a_j^\pm$, normalized so that $\left(l_j^T r_k\right)_\pm = \delta_k^j$, where δ_k^j is the Kronecker delta function, returning 1 for $j = k$ and 0 for $j \neq k$. Define scalar diffusion coefficients

$$\beta_j^\pm := \left(l_j^T B r_j\right)_\pm \tag{1.8}$$

and scalar coupling coefficients

$$\gamma_j^\pm := \left(l_j^T \Gamma(r_j, r_j)\right)_\pm. \tag{1.9}$$

Under this notation, hyperbolic stability (a consequence of the assumed $(\mathcal{D})$) of a Lax or overcompressive shock profile $\bar{u}$ is the condition:

($\mathcal{D}$ii) The set $\{r_j^\pm : a_j^\pm \gtrless 0\} \cup \{\int_{-\infty}^{+\infty} \frac{\partial \bar{u}^\delta}{\partial \delta_i} dx; i = 1, \dots, \ell\}$ forms a basis for $\mathbb{R}^n$, with $\int_{-\infty}^{+\infty} \frac{\partial \bar{u}^\delta}{\partial \delta_i} dx$ computed at $\delta = 0$.

Remark 1.4. For Lax profiles, ($\mathcal{D}$ii) reduces to the Liu–Majda condition

$$\det\left(\{r_j^\pm : a_j^\pm \gtrless 0\},\ (u_+ - u_-)\right) \neq 0.$$

Following [25, 27], define for a given mass m_j^- the scalar diffusion waves $\varphi_j^-(x,t;m_j^-)$ as (self-similar) solutions of the Burgers equations

$$\varphi_{j,t}^- + a_j^- \varphi_{j,x}^- - \beta_j^- \varphi_{j,xx}^- = -\gamma_j^- \left((\varphi_j^-)^2\right)_x \tag{1.10}$$

with point-source initial data

$$\varphi_j^-(x,-1) = m_j^- \delta_0(x), \tag{1.11}$$

and similarly for $\varphi_j^+(x,t;m_j^+)$. Given a collection of masses $m_j^\pm$ prescribed on outgoing characteristic modes $a_j^- < 0$ and $a_j^+ > 0$, define

$$\varphi(x,t) = \sum_{a_j^- < 0} \varphi_j^-(x,t;m_j^-) r_j^- + \sum_{a_j^+ > 0} \varphi_j^+(x,t;m_j^+) r_j^+. \tag{1.12}$$

In the setting described above, we will determine estimates on perturbed viscous shock profiles in terms of ϕ and a refined collection of template functions (terminology following [43]; notation following [27]).

Definition 1.5 (Template functions.). Let

$$\begin{aligned}
\psi_1(x,t) &= \sum_{a_k^\pm \gtrless 0} (1 + |x - a_k^\pm t| + t^{1/2})^{-3/2},\\
\psi_2(x,t) &= (1+|x|)^{-1/2}(1+|x|+t)^{-1/2}(1+|x|+t^{1/2})^{-1/2}\chi,\\
\psi_3(x,t) &= (1+|x|+t)^{-1}(1+|x|)^{-1}\chi,\\
\psi_4(x,t) &= (1+|x|+t)^{-7/4}\chi,
\end{aligned} \tag{1.13}$$

where χ denotes an indicator function on $x \in [a_1^- t, a_n^+ t]$, and also

$$\begin{aligned}
\psi_1^{j,\pm}(x,t) &= \left(1 + |x - a_j^\pm t| + t^{1/2}\right)^{-3/2},\\
\bar{\psi}_1^{j,\pm}(x,t) &= \psi_1(x,t) - \psi_1^{j,\pm}.
\end{aligned}$$

The goal of our analysis is to establish the following sharp pointwise description of asymptotic behavior, generalizing bounds obtained by Liu [27] for small-amplitude Lax type profiles with artificial viscosity $B = I$.

Theorem 1.6. *Assume* (*A*1)–(*A*3) [*alternatively,* (*B*1)], (*H*0)–(*H*3) *and* ($\mathcal{D}$) *hold, and* $\bar{u}$ *is a pure Lax or overcompressive shock profile. Assume also that* $\tilde{u}$ *solves* (1.2) *with initial data* $\tilde{u}_0$ *and that, for initial perturbation* $u_0 := \tilde{u}_0 - \bar{u}$, *we have*

$|u_0|_{H^5} \le E_0$ and $|u_0(x)|$, $|\partial_x u_0(x)|$, and $|\partial_x^2 u_0(x)| \le E_0(1+|x|)^{-\frac{3}{2}}$, for E_0 sufficiently small. Then, the solution $\tilde u$ continues globally in time, with $\tilde u - \bar u \in L^1 \cap H^5$. Moreover, there exist a choice of $|m_j^{\pm}|$, $|\delta_*| \le CE_0$ and a function $\delta(t)$ (determined, respectively, by (3.1) and (3.12)), such that, for $v := \tilde u - \bar u^{\delta_*} - \varphi - \frac{\partial \bar u^{\delta}}{\partial \delta}(\delta_*)\delta$,

$$\begin{aligned} |v(x,t)| &\le CE_0[\psi_1+\psi_2](x,t),\\ |\partial_x v(x,t)| &\le CE_0\big[t^{-1/2}(\psi_1+\psi_2)+\psi_3+\psi_4\big](x,t),\\ |\delta(t)| &\le CE_0(1+t)^{-\frac{1}{2}},\\ |\dot\delta(t)| &\le CE_0(1+t)^{-1}, \end{aligned} \tag{1.14}$$

and, for all $a_j^{\pm} \gtrless 0$,

$$|(\partial_t + a_j^{\pm}\partial_x)v(x,t)| \le CE_0\big[t^{-1}(1+t)^{1/4}\psi_1^{j,\pm} + t^{-1/2}\big(\bar\psi_1^{j,\pm}+\psi_2\big)+\psi_3+\psi_4\big](x,t), \tag{1.15}$$

for some constant C (independent of x, t and E_0).

Remark 1.7. By Taylor's theorem,

$$\partial_x^k\left(\bar u^{\delta_*+\delta(t)} - \bar u^{\delta_*} - \frac{\partial \bar u^{\delta}}{\partial \delta}\delta(t)\right) = \mathbf{O}(|\delta(t)|^2 e^{-k|x|}), \tag{1.16}$$

for $k = 0$ [respectively, $k = 1$], since $\frac{\partial \bar u^{\delta}}{\partial \delta}$ and its derivatives decay at exponential rate (see Lemma 2.1). Since the right-hand side of (1.16) for $k = 0$ [respectively, $k = 1$] is smaller than the right-hand side of (1.14)(i) [respectively, (ii)], we may conclude that

$$\tilde v := u - \bar u^{\delta_*+\delta(t)} - \phi$$

decays at the same rate as v, i.e. u is well approximated by $\bar u^{\delta_*+\delta(t)} + \phi$: a dynamically changing nearby profile with the same endstates $u_{\pm}$, superposed with outgoing diffusion waves. This is a slight refinement of the asymptotic description $u \sim \bar u^{\delta_*} + \phi$ introduced by Liu [25, 27].

Remark 1.8. In contrast to the modulus bounds given here, the estimates of [27] are stated in each separate characteristic field, giving the further information that along characteristic rays the perturbation v lies mainly in the associated characteristic direction. The difference is rather subtle, however, and so we have opted for simplicity of exposition to omit this level of detail. The additional information can easily be recovered at the expense of additional bookkeeping; see Remark 4.4 for further discussion.

Liu's analysis involved essentially two main ingredients, which were strongly coupled. The first was to obtain approximate Green function bounds taking advantage of the weakly coupled nature of the equations in the small-amplitude case to approximate by a superposition of solutions of scalar conservation laws; the second, by delicate pointwise interaction estimates between the approximate Green kernel and various algebraically decaying source terms, to close a nonlinear iteration and obtain the result. The "coupling" we mention refers to the fact that the Green function estimates blow up as amplitude goes to zero and the shock becomes more and more characteristic, whereas the source terms decay with amplitude; thus, the two effects must be delicately balanced to close the iteration and achieve a correct result. In the large-amplitude case, bounded away from the characteristic limit, the issues are somewhat different; namely, we do not have this "characteristic coupling" problem, but on the other hand we cannot as in [27] obtain approximate Green function bounds by asymptotic development in the amplitude. Likewise, for physical, partially parabolic viscosity, there are new difficulties associated with regularity and the need to gain derivatives.

These new difficulties have largely been surmounted in the authors' previous work. In particular, (i) sharp Green function bounds have been obtained by Mascia–Zumbrun [33] in great generality using Laplace transform/stationary phase estimates, and (ii) global existence and sharp $H^s \cap L^p$ estimates on the nonlinear residual v have been obtained by Raoofi [37] using the linearized estimates of [34], a key cancellation estimate of Liu, and a nonstandard energy estimate to control higher derivatives. These results are described in more detail in Secs. 2 and 4, and we shall use them freely in our analysis. A major advantage of this "bootstrap" approach is that we need not close a nonlinear iteration, but only carry out a *linear* fixed-point argument to obtain our result. This was used in the previous work [15] to establish"nearly optimal" pointwise bounds in a relatively uncomplicated manner. As pointed out in [43, 15], however, to obtain the full bounds of Liu requires estimates not only on v and v_x as in [15], but also *characteristic derivatives* $(\partial_t + a_j^{\pm}\partial_x)v$ as in [27]. To carry out these bounds requires an immense ammount of additional work, with consideration of numerous different cases, and accounts for most of the work of this paper.

Remark 1.9. In the strictly parabolic case, the regularity requirement on the initial perturbation $u = \tilde{u} - \bar{u}$ may be significantly relaxed in Theorem 1.6, from $u \in H^5$ to $u \in C^{0+\alpha}$, $\alpha > 0$, and the bound $E_0(1+|x|)^{-3/2}$ imposed on $|u_0|$ alone, and not derivatives $|\partial_x u_0|$ and $|\partial_x^2 u_0|$, by using parametrix bounds as described in [43, 41, 16, 15] to establish short-time pointwise smoothing estimates. (That these bounds are necessary in the real viscosity case, on the other hand, may be clearly seen in the proof of Proposition 3.3). In the artificial viscosity case $B \equiv$ constant, which includes the case $B \equiv I$ treated by Liu [27], the short-time existence/regularity theory becomes trivial [43, 16], and we require no additional regularity beyond the single pointwise hypothesis $|u_0| \le E_0(1+|x|)^{-3/2}$.

Remark 1.10. Alternatively, we may subsume the strictly parabolic case (B1) under the same analysis used to treat the partially symmetric case (A1)–(A3), substituting for parametrix estimates the more elementary Sobolev estimates of [33, 37].

For, in this case, we may drop the requirement of symmetry of $A_\pm$, which was used only to obtain a skew-symmetric K such that $\Re(KA+B)_\pm > 0$, whereas $K=0$ suffices when B is full rank. Since the A_{11} component is empty, the remaining structural conditions are trivially satisfied upon multiplying 1.2 by a smooth symmetric positive definite $A^0(u)$ such that $\Re A^0 B > 0$ (guaranteed by Lyapunov's theorem). By a more careful accounting, taking advantage of associated simplifications, the results of [41, 37]. with the exception of derivative bounds, may be obtained in the strictly parabolic case under regularity H^1, matching the assumption of Goodman in the classic paper [11] stability with respect to zero-mass perturbations, and the results of [15] (for which are required L^∞ bounds on $\partial_x u$) under regularity H^2. In the present case, for which pointwise derivative bounds are crucial to the argument, we require H^4.

Plan of the paper. In Sec. 2, we recall the results of [33, 15] for our use. In Sec. 3, we give the brief argument establishing Theorem 1.6, subject to certain integral estimates. In Sec. 4, we motivate the analysis to follow by reviewing a key cancellation estimate of Liu [26] upon which our analysis, and that of [27], depends. In Secs. 5 and 7, we carry out the main work of the paper, establishing the deferred integral estimates.

2. Preliminaries

We start by recalling some needed, known results.

2.1. *Profile estimates*

We first recall the profile analysis carried out in [33], generalizing results of [31] in the strictly parabolic case. Profile $\bar{u}(x)$ satisfies the standing-wave ordinary differential equation (ODE)

$$B(\bar{u})\bar{u}' = F(\bar{u}) - F(u_-). \tag{2.1}$$

Considering the block structure of B, this can be written as:

$$F^I(u^I, u^{II}) \equiv F^I(u_-^I, u_-^{II}) \tag{2.2}$$

and

$$b_1(u^I)' + b_2(u^{II})' = F^{II}(u^I, u^{II}) - F^{II}(u_-^I, u_-^{II}). \tag{2.3}$$

Proposition 2.1 [33]. *Given* (H1)–(H3), (2.3) *determines a smooth r-dimensional manifold on which* (2.3) *determines a nondegenerate ODE. Moreover, endstates $u_\pm$ are hyperbolic restpoints of this ODE, i.e. the coefficients of the* *linearized equations about $u_\pm$, written in local coordinates, have no center subspace. In particular, under regularity* $(H0)$,

$$D_x^j D_\delta^i(\bar{u}^\delta(x) - u_\pm) = \mathbf{O}(e^{-\alpha|x|}) \quad \text{as } x \to \pm\infty, \qquad \alpha > 0,\ 0 \le j \le 10,\ i = 0, 1. \tag{2.4}$$

2.2. *Linearized equations and Green distribution bounds*

We next recall some linear theory from [33, 43, 16]. Linearizing (1.2) about $\bar{u}^{\delta_*}(\cdot)$, δ_* to be determined later, gives

$$v_t = Lv := -(Av)_x + (Bv_x)_x, \tag{2.5}$$

with

$$B(x) := B(\bar{u}^{\delta_*}(x)), \quad A(x)v := dF(\bar{u}^{\delta_*}(x))v - dB(\bar{u}^{\delta_*}(x))v\bar{u}_x^{\delta_*}. \tag{2.6}$$

Denoting $A^\pm := A(\pm\infty)$, $B^\pm := B(\pm\infty)$, and considering Lemma 2.1, it follows that

$$|A(x) - A^-| = \mathbf{O}(e^{-\eta|x|}), \quad |B(x) - B^-| = \mathbf{O}(e^{-\eta|x|}) \tag{2.7}$$

as $x \to -\infty$, for some positive η. Similarly for A^+ and B^+, as $x \to +\infty$. Also $|A(x) - A^\pm|$ and $|B(x) - B^\pm|$ are bounded for all x.

Define the (*scalar*) *characteristic speeds* $a_1^\pm < \cdots < a_n^\pm$ (as above) to be the eigenvalues of $A^\pm$, and the left and right (*scalar*) *characteristic modes* $l_j^\pm$, $r_j^\pm$ to be corresponding left and right eigenvectors, respectively (i.e. $A^\pm r_j^\pm = a_j^\pm r_j^\pm$, etc.), normalized so that $l_j^+ \cdot r_k^+ = \delta_k^j$ and $l_j^- \cdot r_k^- = \delta_k^j$. Following Kawashima [22], define associated *effective scalar diffusion rates* $\beta_j^\pm : j = 1, \ldots, n$ by relation

$$\begin{pmatrix} \beta_1^\pm & & 0 \\ & \vdots & \\ 0 & & \beta_n^\pm \end{pmatrix} = \text{diag } L^\pm B^\pm R^\pm, \tag{2.8}$$

where $L^\pm := (l_1^\pm, \ldots, l_n^\pm)^t$, $R^\pm := (r_1^\pm, \ldots, r_n^\pm)$ diagonalize $A^\pm$.

Assume for A and B the block structures:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ B_{21} & B_{22} \end{pmatrix}.$$

Also, let $a_j^*(x)$, $j = 1, \ldots, (n-r)$ denote the eigenvalues of

$$A_* := A_{11} - A_{12}B_{22}^{-1}B_{21},$$

with $l_j^*(x)$, $r_j^*(x) \in \mathbb{R}^{n-r}$ associated left and right eigenvectors, normalized so that $l_j^{*t} r_j \equiv \delta_k^j$. More generally, for an m_j^*-fold eigenvalue, we choose $(n-r) \times m_j^*$ blocks L_j^* and R_j^* of eigenvectors satisfying the *dynamical normalization*

$$L_j^{*t} \partial_x R_j^* \equiv 0,$$

along with the usual static normalization $L_j^{*t} R_j \equiv \delta_k^j I_{m_j^*}$; as shown in [32, Lemma 4.9], this may always be achieved with bounded L_j^*, R_j^*. Associated with L_j^*, R_j^*, define extended, $n \times m_j^*$ blocks

$$\mathcal{L}_j^* := \begin{pmatrix} L_j^* \\ 0 \end{pmatrix}, \quad \mathcal{R}_j^* := \begin{pmatrix} R_j^* \\ -B_{22}^{-1} B_{21} R_j^* \end{pmatrix}.$$

Eigenvalues a_j^* and eigenmodes $\mathcal{L}_j^*$, $\mathcal{R}_j^*$ correspond, respectively, to short-time hyperbolic characteristic speeds and modes of propagation for the reduced, hyperbolic part of degenerate system (1.2).

Define local, $m_j \times m_j$ *dissipation coefficients*

$$\eta_j^*(x) := -L_j^{*t} D_* R_j^*(x), \quad j = 1, \ldots, J \le n - r,$$

where

$$D_*(x) := A_{12} B_{22}^{-1} \big[A_{21} - A_{22} B_{22}^{-1} B_{21} + A_* B_{22}^{-1} B_{21} + B_{22} \partial_x \big(B_{22}^{-1} B_{21} \big) \big]$$

is an effective dissipation analogous to the effective diffusion predicted by formal, Chapman–Enskog expansion in the (dual) relaxation case.

The *Green distribution* (fundamental solution) associated with (2.5) is defined by

$$G(x, t; y) := e^{Lt} \delta_y(x). \tag{2.9}$$

Recalling the standard notation $\operatorname{errfn}(z) := \frac{1}{2\pi} \int_{-\infty}^{z} e^{-\xi^2} d\xi$, we have the following pointwise description.

Proposition 2.2 [33]. *Under assumptions (A1)–(A3) [alternatively, (B1)], (H0)–(H3), and ($\mathcal{D}$), the Green distribution $G(x, t; y)$ associated with the linearized evolution equations may be decomposed as*

$$G(x, t; y) = H + E + S + R, \tag{2.10}$$

where, for $y \le 0$:

$$\begin{aligned} H(x, t; y) &:= \sum_{j=1}^{J} a_j^{*-1}(x) a_j^*(y) \mathcal{R}_j^*(x) \zeta_j^*(y, t) \delta_{x - \bar{a}_j^* t}(-y) \mathcal{L}_j^{*t}(y) \\ &= \sum_{j=1}^{J} \mathcal{R}_j^*(x) \mathcal{O}(e^{-\eta_0 t}) \delta_{x - \bar{a}_j^* t}(-y) \mathcal{L}_j^{*t}(y), \end{aligned} \tag{2.11}$$

where the averaged convection rates $\bar{a}_j^* = \bar{a}_j^*(x, t)$ in (2.11) denote the time-averages over $[0, t]$ of $a_j^*(x)$ along backward characteristic paths $z_j^* = z_j^*(x, t)$ defined by

$$dz_j^*/dt = a_j^*(z_j^*), \quad z_j^*(t) = x,$$

the dissipation matrix $\zeta_j^* = \zeta_j^*(x, t) \in \mathbb{R}^{m_j^* \times m_j^*}$ is defined by the dissipative flow

$$d\zeta_j^*/dt = -\eta_j^*(z_j^*) \zeta_j^*, \quad \zeta_j^*(0) = I_{m_j},$$

and $\delta_{x-\bar{a}_j^* t}$ *denotes Dirac distribution centered at* $x - \bar{a}_j^* t$.

$$E(x,t;y) := \sum_{j=1}^{\ell} \frac{\partial \bar{u}^{\delta}(x)}{\partial \delta_j}_{|\delta=\delta_*} e_j(y,t), \tag{2.12}$$

$$e_j(y,t) := \sum_{a_k^- > 0} \left(\mathrm{errfn}\left(\frac{y + a_k^- t}{\sqrt{4\beta_k^- t}} \right) - \mathrm{errfn}\left(\frac{y - a_k^- t}{\sqrt{4\beta_k^- t}} \right) \right) l_{jk}^-. \tag{2.13}$$

$$\begin{aligned} &S(x,t;y) \\ &\quad := \chi_{\{t\ge 1\}} \sum_{a_k^- < 0} r_k^- l_k^{-t} \left(4\pi\beta_k^- t\right)^{-1/2} e^{-(x-y-a_k^- t)^2/4\beta_k^- t} \\ &\qquad + \chi_{\{t\ge 1\}} \sum_{a_k^- > 0} r_k^- l_k^{-t} \left(4\pi\beta_k^- t\right)^{-1/2} e^{-(x-y-a_k^- t)^2/4\beta_k^- t} \left(\frac{e^{-x}}{e^x + e^{-x}} \right) \\ &\qquad + \chi_{\{t\ge 1\}} \sum_{a_k^- > 0,\, a_j^- < 0} \left[c_{k,-}^{j,-}\right] r_j^- l_k^{-t} \left(4\pi\bar{\beta}_{jk}^- t\right)^{-1/2} e^{-(x-z_{jk}^-)^2/4\bar{\beta}_{jk}^- t} \left(\frac{e^{-x}}{e^x + e^{-x}} \right) \\ &\qquad + \chi_{\{t\ge 1\}} \sum_{a_k^- > 0,\, a_j^+ > 0} \left[c_{k,-}^{j,+}\right] r_j^+ l_k^{-t} \left(4\pi\bar{\beta}_{jk}^+ t\right)^{-1/2} e^{-(x-z_{jk}^+)^2/4\bar{\beta}_{jk}^+ t} \left(\frac{e^{x}}{e^x + e^{-x}} \right), \end{aligned} \tag{2.14}$$

with

$$z_{jk}^{\pm}(y,t) := a_j^{\pm} \left(t - \frac{|y|}{|a_k^-|} \right) \tag{2.15}$$

and

$$\bar{\beta}_{jk}^{\pm}(x,t;y) := \frac{x^{\pm}}{a_j^{\pm} t} \beta_j^{\pm} + \frac{|y|}{|a_k^- t|} \left(\frac{a_j^{\pm}}{a_k^-} \right)^2 \beta_k^-. \tag{2.16}$$

The remainder R *and its derivatives have the following bounds.*

$$\begin{aligned} &R(x,t;y) \\ &\quad = \mathbf{O}(e^{-\eta(|x-y|+t)}) + \sum_{k=1}^{n} \mathbf{O}((t+1)^{-1/2} e^{-\eta x^+} + e^{-\eta|x|}) t^{-1/2} e^{-(x-y-a_k^- t)^2/Mt} \\ &\qquad + \sum_{a_k^- > 0,\, a_j^- < 0} \chi_{\{|a_k^- t| \ge |y|\}} \mathbf{O}((t+1)^{-1/2} t^{-1/2}) e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt} e^{-\eta x^+} \\ &\qquad + \sum_{a_k^- > 0,\, a_j^+ > 0} \chi_{\{|a_k^- t| \ge |y|\}} \mathbf{O}((t+1)^{-1/2} t^{-1/2}) e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt} e^{-\eta x^-}, \end{aligned} \tag{2.17}$$

$$
\begin{aligned}
&R_y(x,t;y)\\
&\quad = \sum_{j=1}^{J} \mathbf{O}(e^{-\eta t})\delta_{x-\bar a_j^* t}(-y) + \mathbf{O}(e^{-\eta(|x-y|+t)})\\
&\qquad + \sum_{k=1}^{n} \mathbf{O}((t+1)^{-1/2}e^{-\eta x^+} + e^{-\eta|x|} + e^{-\eta|y|})t^{-1}e^{-(x-y-a_k^- t)^2/Mt}\\
&\qquad + \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge|y|\}}\mathbf{O}((t+1)^{-1/2}t^{-1})e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt}e^{-\eta x^+}\\
&\qquad + \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge|y|\}}\mathbf{O}((t+1)^{-1/2}t^{-1})e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt}e^{-\eta x^-},
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
&R_x(x,t;y)\\
&\quad = \sum_{j=1}^{J} \mathbf{O}(e^{-\eta t})\delta_{x-\bar a_j^* t}(-y) + \mathbf{O}(e^{-\eta(|x-y|+t)})\\
&\qquad + \sum_{k=1}^{n} \mathbf{O}((t+1)^{-1}e^{-\eta x^+} + e^{-\eta|x|})t^{-1}(t+1)^{1/2}e^{-(x-y-a_k^- t)^2/Mt}\\
&\qquad + \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge|y|\}}\mathbf{O}((t+1)^{-1/2}t^{-1})e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt}e^{-\eta x^+}\\
&\qquad + \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge|y|\}}\mathbf{O}((t+1)^{-1/2}t^{-1})e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt}e^{-\eta x^-}.
\end{aligned}
\tag{2.19}
$$

Moreover, for $|x-y|/t$ *sufficiently large,* $|G| \le Ce^{-\eta t}e^{-|x-y|^2/Mt)}$ *as in the strictly parabolic case.*

Setting $\tilde G := S + R$, so that $G = H + E + \tilde G$, we have the following useful alternative bounds for $\tilde G$.

Proposition 2.3 [43, 33, 16]. *Under the assumptions of Proposition* 2.2, $\tilde G$ *has the following bounds.*

$$
\begin{aligned}
|\partial_{x,y}^{\alpha}\tilde G(x,t;y)| \le{}& \sum_{j=1}^{J}\sum_{\beta=0}^{\max\{0,|\alpha|-1\}} \mathbf{O}(e^{-\eta t})\partial_y^{\beta}\delta_{x-\bar a_j^* t}(-y) + \mathbf{O}(e^{-\eta(|x-y|+t)})\\
&+ C(t^{-|\alpha|/2} + |\alpha_x|e^{-\eta|x|})\Bigg(\sum_{k=1}^{n} t^{-1/2}e^{-(x-y-a_k^- t)^2/Mt}e^{-\eta x^+}\\
&+ \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge|y|\}}t^{-1/2}e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt}e^{-\eta x^+}\\
&+ \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge|y|\}}t^{-1/2}e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt}e^{-\eta x^-}\Bigg),
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
&|(\partial_t + a_l^- \partial_x)\tilde{G}(x,t;y)| \\
&\quad \le \sum_{j=1}^{J} \mathbf{O}(e^{-\eta t})\delta_{x-\bar{a}_j^* t}(-y) + \mathbf{O}(e^{-\eta(|x-y|+t)}) \\
&\qquad + Ct^{-3/2}\left(e^{-(x-y-a_l^- t)^2/Mt} e^{-\eta x^+} + \sum_{a_k^->0} e^{-(x-a_l^-(t-|y/a_k^-|))^2/Mt} e^{-\eta x^+}\right) \\
&\qquad + Ct^{-1/2}\left(e^{-(x-y-a_l^- t)^2/Mt} e^{-\eta|x|} I_{\{x\ge 0\}}\right. \\
&\qquad \left. + \sum_{a_k^->0} e^{-(x-a_l^-(t-|y/a_k^-|))^2/Mt} e^{-\eta|x|} I_{\{x\ge 0\}}\right) \\
&\qquad + C(t^{-1} + e^{-\eta|x|})\left(\sum_{k\ne l} t^{-1/2} e^{-(x-y-a_k^- t)^2/Mt} e^{-\eta x^+}\right. \\
&\qquad + \sum_{a_k^->0,\, a_j^-<0, j\ne l} \chi_{\{|a_k^- t|\ge|y|\}} t^{-1/2} e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt} e^{-\eta x^+} \\
&\qquad \left. + \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge|y|\}} t^{-1/2} e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt} e^{-\eta x^-}\right),
\end{aligned}
\tag{2.21}
$$

$0 \le |\alpha| \le 2$, for $y \le 0$, and symmetrically for $y \ge 0$, for some η, C, $M > 0$, where $a_j^\pm$ are as in Proposition 2.2, $\beta_k^\pm > 0$, $x^\pm$ denotes the positive/negative part of x, and indicator function $\chi_{\{|a_k^- t|\ge|y|\}}$ is 1 for $|a_k^- t| \ge |y|$ and 0 otherwise. Moreover, all estimates are uniform in the supressed parameter δ_*.

Remark 2.4. In the strictly parabolic case $r = n$, the hyperbolic part H is absent in the decomposition of G given in (2.10).

Remark 2.5. From (2.13), we obtain by straightforward calculation (see [33]) the bounds

$$
\begin{aligned}
|e_j(y,t)| &\le C \sum_{a_k^->0} \left(\mathrm{errfn}\left(\frac{y + a_k^- t}{\sqrt{4\beta_k^- t}}\right) - \mathrm{errfn}\left(\frac{y - a_k^- t}{\sqrt{4\beta_k^- t}}\right)\right), \\
|\partial_t e_j(y,t)| &\le Ct^{-1/2} \sum_{a_k^->0} e^{-|y+a_k^- t|^2/Mt}, \\
|\partial_y e_j(y,t)| &\le Ct^{-1/2} \sum_{a_k^->0} e^{-|y+a_k^- t|^2/Mt}, \\
|\partial_{yt} e_j(y,t)| &\le Ct^{-1} \sum_{a_k^->0} e^{-|y+a_k^- t|^2/Mt}
\end{aligned}
\tag{2.22}
$$

for $y \le 0$, and symmetrically for $y \ge 0$.

2.3. L^p decay

Finally, we recall the $H^s \cap L^p$ result of [37] established using the linearized bounds of [33] and Hausdorff–Young's inequality together with an H^s energy estimate and the cancellation estimate described in Sec. 4, below.

Proposition 2.6 [37, 15]. *Under the conditions of Theorem* 1.6, *there exists a unique, global solution* $\tilde u$ *of* (1.2), $\tilde u \in L^1 \cap H^5$. *Moreover, for* $v := \tilde u - \bar u^{\delta_*} - \varphi - \frac{\partial \bar u^\delta}{\partial \delta}(\delta_*)\delta$, *with* $|m_j^\pm|, |\delta_*| \le CE_0$ *and* $\delta(t)$ *defined as in* (3.1) *and* (3.12),

$$|v(\cdot,t)|_{L^p} \le CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}}, \quad 1 \le p \le \infty, \tag{2.23}$$

and

$$|v(\cdot,t)|_{H^5} \le CE_0(1+t)^{-\frac{1}{2}}, \tag{2.24}$$

for some constant $C > 0$.

3. Nonlinear Analysis: Proof of Theorem 1.6

We now carry out the proof of Theorem 1.6 assuming certain integral estimates to be established in Secs. 5–7. Let $\tilde u$ be the solution of (1.2) guaranteed by Proposition 2.6. Following [25, 27, 37], let $|m_i|$, $|\delta_*| \le CE_0$ be the unique solutions guaranteed by ($\mathcal{D}$ii) and the Implicit Function theorem of

$$\int_{-\infty}^{+\infty} (\tilde u(x,0) - \bar u^{\delta_*})(x)\,dx = \sum_{a_j^- < 0} m_j r_j^- + \sum_{a_j^+ > 0} m_j r_j^+, \tag{3.1}$$

or, equivalently, $\int_{-\infty}^{+\infty} \tilde u(x,0) = \int_{-\infty}^{+\infty} (\bar u^{\delta_*} + \phi)(x)\,dx$. This determines the asymptotic state, by conservation of mass.

Remark 3.1 [25, 43]. In the case of Lax-type shock waves, $\bar u^\delta(x) = \bar u(x+\delta)$, hence $\int \frac{\partial \bar u^\delta}{\partial \delta}\,dx = (u_+ - u_-)$ and δ_* can be explicitly computed as the solution of a linear equation.

Setting $u(x,t) := \tilde u(x,t) - \bar u^{\delta_*}(x)$, use Taylor's expansion around $\bar u^{\delta_*}(x)$ to find

$$u_t + (A(x)u)_x - (B(x)u_x)_x = -(\Gamma(x)(u,u))_x + Q(u,u_x)_x, \tag{3.2}$$

where $\Gamma(x)(u,u) = d^2 f(\bar u^{\delta_*})(u,u) - d^2 B(\bar u^{\delta_*})(u,u)\bar u_x^{\delta_*}$ and

$$Q(u,u_x) = \mathbf{O}(|u||u_x| + |u|^3),$$

with $\Gamma_\pm = \Gamma(\pm\infty)$. Define constant coefficients $b_{ij}^\pm$ and $\Gamma_{ijk}^\pm$ to satisfy

$$\Gamma_\pm\big(r_j^\pm, r_k^\pm\big) = \sum_{i=1}^n \Gamma_{ijk}^\pm r_i^\pm, \quad B_\pm r_j^\pm = \sum_{i=1}^n b_{ij}^\pm r_i^\pm. \tag{3.3}$$

Then, of course, $\beta_i^\pm = b_{ii}^\pm$ and $\gamma_i^\pm := \Gamma_{iii}^\pm$.

Continuing, set $v := u - \varphi - \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t)$, with $\delta(t)$ to be defined later, and assuming $\delta(0) = 0$. Notice that, by our choice of δ_* and diffusion waves $\varphi_i^{\pm}$'s, we have zero initial mass of v, i.e.

$$\int_{-\infty}^{+\infty} v(x,0)dx = 0. \tag{3.4}$$

Replacing u with $v + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t)$ in (3.2) ($\frac{\partial \bar{u}^\delta}{\partial \delta_i}$ computed at $\delta = \delta_*$), and using the fact that $\frac{\partial \bar{u}^\delta}{\partial \delta_i}$ satisfies the linear time independent equation $Lv = 0$, we obtain

$$v_t - Lv = \Phi(x,t) + \mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t)\right)_x + \frac{\partial \bar{u}^\delta}{\partial \delta}\dot{\delta}(t), \tag{3.5}$$

where

$$\begin{aligned}\mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right) = \mathbf{O}\Bigg(&|v|^2 + |\varphi||v| + |v|\left|\frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right| + |\varphi|\left|\frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right| + \left|\frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right|^2 \\ &+ \left|\left(\varphi + v + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right)\left(\varphi + v + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right)_x\right| + \left|\varphi + v + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right|^3\Bigg).\end{aligned} \tag{3.6}$$

Furthermore,

$$\begin{aligned}\mathcal{F}(v, \varphi, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t))_x = \mathbf{O}\Bigg(&\mathcal{F}\left(v, \varphi, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right) \\ &+ \left|\left(v + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right)_x\right|\left|\left(v^{II} + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right)_x\right| \\ &+ \left|v + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right|\left|\left(v^{II} + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right)_{xx}\right|\Bigg),\end{aligned} \tag{3.7}$$

and $\Phi(x,t) := -\varphi_t - (A(x)\varphi)_x + (B(x)\varphi_x)_x - (\Gamma(x)(\varphi,\varphi))_x$. For Φ we write

$$\begin{aligned}\Phi(x,t) &= -(\varphi_t + A\varphi_x - B\varphi_{xx} + \Gamma(\varphi,\varphi)_x) \\ &= -\sum_{a_i^- < 0} \varphi_t^i r_i^- + (A(x)\varphi^i r_i^-)_x - (B(x)\varphi_x^i r_i^-)_x + (\Gamma(x)(\varphi^i r_i^-, \varphi^i r_i^-))_x \\ &\quad - \sum_{a_i^+ > 0} \varphi_t^i r_i^+ + (A(x)\varphi^i r_i^+)_x - (B(x)\varphi_x^i r_i^+)_x + (\Gamma(x)(\varphi^i r_i^+, \varphi^i r_i^+))_x \\ &\quad - \sum_{i \neq j} (\varphi_i \varphi_j \Gamma(x)(r_i^{\pm}, r_j^{\pm}))_x.\end{aligned} \tag{3.8}$$

Let us write a typical term of the first summation ($a_i^- < 0$) in the following form:

$$\begin{aligned}&\varphi_t^i r_i^- + \left(A(x)\varphi^i r_i^-\right)_x - \left(B(x)\varphi_x^i r_i^-\right)_x + \left(\Gamma(x)\left(\varphi^i r_i^-, \varphi^i r_i^-\right)\right)_x \\ &\quad = \left[(A(x) - A^-)\varphi^i r_i^- - (B(x) - B^-)\varphi_x^i r_i^- + (\Gamma(x) - \Gamma^-)\left(\varphi^i r_i^-, \varphi^i r_i^-\right)\right]_x \\ &\qquad + \varphi_t^i r_i^- + \left(\varphi_x^i A^- r_i^-\right) - \left(\varphi_{xx}^i B^- r_i^-\right) + \left((\varphi^i)_x^2 \Gamma^-(r_i^-, r_i^-)\right).\end{aligned} \tag{3.9}$$

Now we use the definition of φ^i in (1.10) and the definition of coefficients b_{ij} and Γ_{ijk} in (3.3) to write the last part of (3.9) in the following form:

$$\begin{aligned} &\varphi^i_t r^-_i + \left(\varphi^i_x A^- r^-_i\right) - \left(\varphi^i_{xx} B^- r^-_i\right) + \left((\varphi^i)^2_x \Gamma^-(r^-_i, r^-_i)\right) \\ &\quad = -\varphi^i_{xx} \sum_{j\neq i} b^-_{ij} r^-_j - (\varphi^i)^2_x \sum_{j\neq i} \Gamma^-_{jii} r^-_j. \end{aligned} \tag{3.10}$$

1 Similar statements hold for $a^+_i > 0$ with minus signs replaced with plus signs.

Applying Duhamel's principle, we obtain from (3.5)

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{+\infty} G(x,t;y) v_0(y) dy \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} G(x,t-s;y) \mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right)_y (y,s) dy\, ds \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} G(x,t-s;y) \Phi(y,s) dy\, ds + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t), \end{aligned} \tag{3.11}$$

where we have used the identity $\int_{-\infty}^{+\infty} G(x,t;y) \frac{\partial \bar{u}^\delta}{\partial \delta_i}(y) dy = e^{Lt} \frac{\partial \bar{u}^\delta}{\partial \delta_i} = \frac{\partial \bar{u}^\delta}{\partial \delta_i}$ and $\delta(0) = 0$. Assuming

$$\begin{aligned} \delta_i(t) &= -\int_{-\infty}^{\infty} e_i(y,t) v_0(y) dy \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} e_i(y,t-s) \mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right)_y (y,s) dy\, ds \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} e_i(x,t-s;y) \Phi(y,s) dy\, ds, \end{aligned} \tag{3.12}$$

and using (3.11), (3.12) and $G = H + E + \tilde{G}$, we obtain:

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{+\infty} (H+\tilde{G})(x,t;y) v_0(y) dy \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} (H+\tilde{G})(x,t-s;y) \mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right)_y (y,s) dy\, ds \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} (H+\tilde{G})(x,t-s;y) \Phi(y,s) dy\, ds, \end{aligned} \tag{3.13}$$

which we can clearly augment with derivative estimates through differentiation on both sides. In addition to $v(x,t)$ and $\delta(t)$, we will keep track in our argument of $v_x(x,t)$, $(v_t + a^\pm_j v_x)$, and $\dot{\delta}(t)$, the latter of which satisfies

$$\begin{aligned} \dot{\delta}_i(t) &= -\int_{-\infty}^{\infty} \partial_t e_i(y,t) v_0(y) dy \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} \partial_t e_i(y,t-s) \mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right)_y (y,s) dy\, ds \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} \partial_t e_i(y,t-s) \Phi(y,s) dy\, ds, \end{aligned} \tag{3.14}$$

where we have taken advantage of the observation, apparent from (2.22), that $e(y,0)=0$.

Define

$$\begin{aligned}\zeta(t) := {} & \sup_{y,0\le s\le t} \frac{|v(y,s)|}{\psi_1(y,s)+\psi_2(y,s)} \\ & + \sup_{y,0\le s\le t} \frac{|v_y(y,s)|}{s^{-1/2}(\psi_1(y,s)+\psi_2(y,s))+\psi_3(y,s)+\psi_4(y,s)} \\ & + \sum_{a_j^-<0,a_j^+>0} \sup_{y,0\le s\le t} \frac{|(\partial_s + a_j^\pm \partial_y)v(y,s)|}{\big(s^{-1}(1+s)^{1/4}\psi_1^{j,\pm} + s^{-1/2}(\bar\psi_1^{j,\pm}+\psi_2)+\psi_3+\psi_4\big)(y,s)} \\ & + \sup_{0\le s\le t} \frac{|\delta(s)|}{(1+s)^{-1/2}} + \sup_{0\le s\le t} \frac{|\dot\delta(s)|}{(1+s)^{-1}}. \end{aligned} \tag{3.15}$$

The desired estimates follow easily using the following integral estimates, to be established in Secs. 5–7.

Lemma 3.2 (Linear estimates I). *If* $\int_{-\infty}^{+\infty} v_0(y)dy = 0$ *and* $|v_0(y)| \le E_0(1+|y|)^{-3/2}$, $E_0>0$, *then*

$$\begin{aligned} \left|\int_{-\infty}^{+\infty} \tilde G(x,t;y)v_0(y)dy\right| &\le CE_0\psi_1(x,t), \\ \left|\int_{-\infty}^{+\infty} \partial_x\tilde G(x,t;y)v_0(y)dy\right| &\le CE_0(t^{-1/2}+e^{-\eta|x|})\psi_1(x,t), \\ \left|\int_{-\infty}^{+\infty} (\partial_t + a_j^\pm\partial_x)\tilde G(x,t;y)v_0(y)dy\right| &\le CE_0\big(t^{-1}\psi_1^{j,\pm}(x,t) \\ &\quad + t^{-1/2}\bar\psi_1^{j,\pm}(x,t) + e^{-\eta|x|}\psi_1(x,t)\big), \\ \left|\int_{-\infty}^{+\infty} e_i(y,t)v_0(y)dy\right| &\le CE_0(1+t)^{-1/2}, \\ \left|\int_{-\infty}^{+\infty} \partial_t e_i(y,t)v_0(y)dy\right| &\le CE_0(1+t)^{-3/2}. \end{aligned} \tag{3.16}$$

Lemma 3.3 (Linear estimates II). *If* $|v_0(x)|$, $|\partial_x v_0(x)|$, $|\partial_x^2 v_0(x)| \le E_0(1+|x|)^{-\frac{3}{2}}$, $E_0>0$, *then, for some* $\theta>0$,

$$\begin{aligned} \int_{-\infty}^{+\infty} H(x,t;y)v_0(y)dy &\le CE_0e^{-\theta t}(1+|x|)^{-\frac{3}{2}} \\ &\le CE_0\psi_1(x,t), \\ \int_{-\infty}^{+\infty} H_x(x,t;y)v_0(y)dy &\le CE_0e^{-\theta t}(1+|x|)^{-\frac{1}{2}} \\ &\le CE_0(1+t)^{-1/2}\psi_1(x,t), \\ \int_{-\infty}^{+\infty} \big(\partial_t + a_j^\pm\partial_x\big)H_x(x,t;y)v_0(y)dy &\le CE_0e^{-\theta t}(1+|x|)^{-\frac{1}{2}} \\ &\le CE_0(1+t)^{-3/4}\psi_1(x,t). \end{aligned} \tag{3.17}$$

Lemma 3.4 (Nonlinear estimates I). *If* $\|v(y,s)\|_{L^\infty} \le CE_0(1+s)^{-3/4}$, $\|v(y,s)\|_{H^4} \le CE_0(1+s)^{-1/2}$, *and* $\zeta(t) < +\infty$ *(see* 3.15*), then*

$$\begin{aligned}
&\left|\int_0^t\int_{-\infty}^{+\infty} \tilde{G}(x,t-s;y)\mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial\delta}\delta\right)_y dy\,ds\right| \le CE_0\zeta(t)[\psi_1(x,t)+\psi_2(x,t)],\\
&\left|\int_0^t\int_{-\infty}^{+\infty} \tilde{G}(x,t-s;y)\Phi(y,s)dy\,ds\right| \le CE_0\psi_1(x,t),\\
&\left|\int_0^t\int_{-\infty}^{+\infty} \tilde{G}_x(x,t-s;y)\mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial\delta}\delta\right)_y dy\,ds\right|\\
&\qquad \le CE_0\zeta(t)\Big[t^{-1/2}(\psi_1(x,t)+\psi_2(x,t))+\psi_3(x,t)+\psi_4(x,t)],\\
&\left|\int_0^t\int_{-\infty}^{+\infty} \tilde{G}_x(x,t-s;y)\Phi(y,s)dy\,ds\right| \le CE_0 t^{-1/2}\psi_1(x,t),\\
&\left|(\partial_t + a_j^\pm\partial_x)\int_0^t\int_{-\infty}^{+\infty} \tilde{G}(x,t-s;y)\mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial\delta}\delta\right)_y dy\,ds\right|\\
&\qquad \le CE_0\zeta(t)\left[t^{-1}(1+t)^{1/4}\psi_1^{j,\pm} + t^{-1/2}(\bar{\psi}_1^{j,\pm}+\psi_2)+\psi_3+\psi_4\right](x,t),\\
&\left|(\partial_t + a_j^\pm\partial_x)\int_0^t\int_{-\infty}^{+\infty} \tilde{G}(x,t-s;y)\Phi(y,s)dy\,ds\right|\\
&\qquad \le CE_0\left[t^{-1}(1+t)^{1/4}\psi_1^{j,\pm}(x,t) + t^{-1/2}(\bar{\psi}_1^{j,\pm}(x,t)+\psi_2(x,t))\right],\\
&\left|\int_0^t\int_{-\infty}^{+\infty} \partial_y e_i(y,t-s)\mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial\delta}\delta\right)_y dy\,ds\right| \le CE_0\zeta(t)(1+t)^{-3/4},\\
&\left|\int_0^t\int_{-\infty}^{+\infty} \partial_{yt} e_i(y,t-s)\mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial\delta}\delta\right)_y dy\,ds\right| \le CE_0\zeta(t)(1+t)^{-1},\\
&\left|\int_0^t\int_{-\infty}^{+\infty} e_i(y,t-s)\Phi(y,s)dy\,ds\right| \le CE_0(1+t)^{-1/2},\\
&\left|\int_0^t\int_{-\infty}^{+\infty} \partial_t e_i(y,t-s)\Phi(y,s)dyds\right| \le CE_0(1+t)^{-1}.
\end{aligned} \tag{3.18}$$

Lemma 3.5 (Nonlinear estimates II). *If* $\|v(y,s)\|_{L^\infty} \le CE_0(1+s)^{-3/4}$, $\|v(y,s)\|_{H^5} \le CE_0(1+s)^{-1/2}$, *and* $\zeta(t) < +\infty$, *then*

$$\begin{aligned}
&\left|\int_0^t\int_{-\infty}^{+\infty} H(x,t-s;y)\mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial\delta}\delta\right)_y dy\,ds\right| \le CE_0\zeta(t)[\psi_1(x,t)+\psi_2(x,t)],\\
&\left|\int_0^t\int_{-\infty}^{+\infty} H(x,t-s;y)\Phi(y,s)dy\,ds\right| \le CE_0\psi_1(x,t),\\
&\left|\int_0^t\int_{-\infty}^{+\infty} H_x(x,t-s;y)\mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial\delta}\delta\right)_y dy\,ds\right|\\
&\qquad \le CE_0\zeta(t)\left[t^{-1/2}(\psi_1(x,t)+\psi_2(x,t))+\psi_3(x,t)+\psi_4(x,t)\right],
\end{aligned}$$

$$\left|\int_0^t\int_{-\infty}^{+\infty} H_x(x,t-s;y)\Phi(y,s)dy\,ds\right| \le CE_0 t^{-1/2}(\psi_1+\psi_2)(x,t),$$
$$\left|(\partial_t + a_j^{\pm}\partial_x)\int_0^t\int_{-\infty}^{+\infty} H(x,t-s;y)\mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial\delta}\delta\right)_y dy\,ds\right|$$
$$\le CE_0\zeta(t)\big[t^{-1}(1+t)^{1/4}\psi_1^{j,\pm} + t^{-1/2}(\bar{\psi}_1^{j,\pm}+\psi_2)+\psi_3+\psi_4\big](x,t),$$
$$\left|(\partial_t + a_j^{\pm}\partial_x)\int_0^t\int_{-\infty}^{+\infty} H(x,t-s;y)\Phi(y,s)dy\,ds\right|$$
$$\le CE_0\big[t^{-1}(1+t)^{1/4}\psi_1^{j,\pm}(x,t) + t^{-1/2}(\bar{\psi}_1^{j,\pm}(x,t)+\psi_2(x,t))\big]. \tag{3.19}$$

Proof of Theorem 1.6. We prove Theorem 1.6 directly from the integral representations (3.12) and (3.13), augmented with similar representations for $\dot{\delta}_i(t)$, $v_x(x,t)$, and $(\partial_t + a_j^{\pm}\partial_x)v(x,t)$, obtained through direct differentiation of (3.12) and (3.13). Recalling the definition of our iteration variable $\zeta(t)$ in (3.15), our goal will be to employ the estimates of Lemmas 3.2–3.5 to establish the inequality

$$\zeta(t) \le CE_0(1+\zeta(t)), \tag{3.20}$$

where E_0 is precisely as in Theorem 1.6. Choosing, then, $E_0 \le \frac{1}{2C}$, and noting that the possibility of $\zeta(t)$ jumping discontinuously from a finite value to an infinite value is precluded by short-time theory (see [15, Remark 3.6]),[a] we will be able to conclude

$$\zeta(t) \le \frac{CE_0}{1-CE_0} \le 1. \tag{3.21}$$

The estimates of Theorem 1.6 follow immediately from (3.21) and (3.15).

We begin with a careful consideration of the integral represention for $v(t,x)$, (3.13), which we will separate for clarity into terms that arise in the case of strict parabolicity — involving $\tilde{G}$ — and the additional terms arising from the relaxation of strict parabolicity — involving H. For the terms involving $\tilde{G}$, we estimate

$$\left|\int_{-\infty}^{+\infty}\tilde{G}(x,t;y)v_0(y)dy\right| + \left|\int_0^t\int_{-\infty}^{+\infty}\tilde{G}(x,t-s;y)\mathcal{F}\left(\varphi,v,\frac{\partial\bar{u}^\delta}{\partial\delta}\delta\right)_y(y,s)dy\,ds\right|$$
$$+\left|\int_0^t\int_{-\infty}^{+\infty}\tilde{G}(x,t-s;y)\Phi(y,s)dy\,ds\right|$$
$$\le CE_0\psi_1(x,t) + CE_0\zeta(t)[\psi_1(x,t)+\psi_2(x,t)] + CE_0\psi_1(x,t),$$

where the estimates follow respectively from the first estimate of Lemma 3.3, the first estimate of Lemma 3.4, and the second estimate of Lemma 3.4 (though the

[a]Note also that the pointwise bounds of [15] directly imply that ζ is finite for each given t.

first and last estimates are the same, we write both for clarity). For the terms in (3.13) involving H, we have similarly

$$\begin{aligned}&\left|\int_{-\infty}^{+\infty} H(x,t;y)v_0(y)dy\right| + \left|\int_0^t\int_{-\infty}^{+\infty} H(x,t-s;y)\mathcal{F}\left(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta\right)_y (y,s)dy\,ds\right|\\ &\quad + \left|\int_0^t\int_{-\infty}^{+\infty} H(x,t-s;y)\Phi(y,s)dy\,ds\right|\\ &\quad \le CE_0\psi_2(x,t) + CE_0\zeta(t)[\psi_1(x,t)+\psi_2(x,t)] + CE_0\psi_1(x,t),\end{aligned}$$

for which the estimates follow respectively from the first estimate of Lemma 3.3, the first estimate of Lemma 3.5, and the second estimate of Lemma 3.5.

$$|v(x,t)| \le CE_0[\psi_1(x,t)+\psi_2(x,t)] + CE_0\zeta(t)[\psi_1(x,t)+\psi_2(x,t)],$$

which can be rearranged as

$$\frac{|v(t,x)|}{\psi_1(x,t)+\psi_2(x,t)} = CE_0(1+\zeta(t)).$$

Keeping in mind that $\zeta(t)$ is a nondecreasing function of t, we have at last

$$\sup_{y,0\le s\le t} \frac{|v(y,s)|}{\psi_1(y,s)+\psi_2(y,s)} \le CE_0(1+\zeta(t)). \tag{3.22}$$

Proceeding similarly for each of the expressions $\delta_i(t)$, $\dot{\delta}_i(t)$, $v_x(x,t)$, and $(\partial_t + a_j^\pm\partial_x)v(x,t)$, we can bound each summand in the definition of $\zeta(t)$ in precisely the same way by $CE_0(1+\zeta(t))$. We conclude the sought estimate

$$\zeta(t) \le CE_0(1+\zeta(t)).$$

As discussed in (3.20) and (3.21), we conclude $\zeta(t) \le 1$, and from this the estimates of Theorem 1.6. □

4. Liu's Cancellation Estimate

Before carrying out the deferred integral estimates, we revisit a key estimate of Liu [26] that at once determines the ultimate rate of decay and motivates the analysis to follow. Consider the illustrative convolution

$$\begin{aligned}u(x,t) &= \int_0^t\int_{-\infty}^{+\infty} g(x-y,t-s)(K(y-s,s)^2)_y\,dy\,ds\\ &= \int_0^t\int_{-\infty}^{+\infty} g_y(x-y,t-s)K(y-s,s)^2\,dy\,ds,\end{aligned} \tag{4.1}$$

$g(x,t) = K(x,t) = (4\pi t)^{-\frac{1}{2}}e^{\frac{-x^2}{4t}}$, similar to quadratic interaction integrals arising through the integration of scattering terms against diffusion waves (see Remark 4.3).

If we replace the integrands in (4.1) by their absolute values, we obtain (see [16]) the sharp estimate

$$|u(x,t)| \leq C(g(x,4t) + g(x-t,4t)) + C\chi_{\{\sqrt{t} \leq x \leq t-\sqrt{t}\}}(x^{-\frac{1}{2}}(t-x)^{-\frac{1}{2}}). \tag{4.2}$$

By taking account of cancellation, however, we may obtain the following stronger bound (also sharp) pointed out by Liu [26]. We follow the notation of Raoofi [37], and also the proof, based on integration by parts in the characteristic direction, which abstracts from the more concrete calculation of Liu the central ideals that will be of use here.

Proposition 4.1 [37, 26]. *For $u(x,t)$ defined in* (4.1) *and $t \geq 1$,*

$$|u(x,t)| \leq Ct^{-\frac{1}{4}}(g(x,8t) + g(x-t,8t)) + C\chi_{\{\sqrt{t} \leq x \leq t-\sqrt{t}\}}(t^{-\frac{1}{2}}(t-x)^{-1} + x^{-\frac{3}{2}}), \tag{4.3}$$

where χ stands for the indicator function, and C is a constant independent of t and x.

The same result holds if K^2 in (4.1) *is replaced with K_x.*

Proof. As $K^2 \sim K_x$, the proof will be stated only for K^2. It is straightforward to verify that the same argument works for K_x at every step. We first state a simple lemma.

Lemma 4.2. *If $0 \leq s \leq \sqrt{t}$, then $e^{\frac{-(x\pm s)^2}{4t}} \leq Ce^{\frac{-x^2}{8t}}$ with C independent of t, s and x.*

Proof. The statement of the lemma is equivalent to

$$\frac{-(x \pm s)^2}{4t} \leq \frac{-x^2}{8t} + D$$

for some D, which (after some calculation) in its turn is equivalent to $(x \pm 2s)^2 - 2s^2 \geq -8Dt$, which holds for $D > \frac{1}{4}$, since $s^2 < t$. □

The argument relies on the following simple properties of the heat kernel g.

$$\int_{-\infty}^{+\infty} g(x-y,t)g(y,t')dy = g(x,t+t') \tag{4.4}$$

$$|g_x(x,t)| \leq Ct^{-\frac{1}{2}}g(x,2t) \tag{4.5}$$

$$|g_t(x,t)| \leq Ct^{-1}g(x,2t) \tag{4.6}$$

$$|g(x,t)| \leq C\,t^{-\frac{1}{2}}. \tag{4.7}$$

Rewriting (4.1), we have

$$\begin{aligned} u(x,t) &= \int_0^t \int_{-\infty}^{+\infty} g_x(x-y,t-s)g(y-s,s)^2\,dy\,ds \\ &= \int_0^{\sqrt{t}} \int_{-\infty}^{+\infty} g_x(x-y,t-s)g(y-s,s)^2\,dy\,ds \\ &\quad + \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} g_x(x-y,t-s)g(y-s,s)^2\,dy\,ds \\ &\quad + \int_{t-\sqrt{t}}^{t} \int_{-\infty}^{+\infty} g_x(x-y,t-s)g(y-s,s)^2\,dy\,ds \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{4.8}$$

(I) and (III) are easy to estimate:

$$|I| = \left| \int_0^{\sqrt{t}} \int_{-\infty}^{+\infty} g_y(x-y,t-s)g(y-s,s)^2\,dy\,ds \right|.$$

By (4.5) and (4.7), the above is less than or equal to

$$C \int_0^{\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} g(x-y,2(t-s))\, g(y-s,2s)\,dy\,ds$$

which, by (4.4), is less than or equal to

$$C \int_0^{\sqrt{t}} (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} g(x-s,2t)\,ds.$$

Now, using Lemma 4.2, the above is

$$\begin{aligned} &\le C\,g(x,4t) \int_0^{\sqrt{t}} (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}}\,ds \\ &\le Ct^{-\frac{1}{2}} g(x,4t) \int_0^{\sqrt{t}} s^{-\frac{1}{2}}\,ds \\ &\le Ct^{-\frac{1}{4}} g(x,4t). \end{aligned}$$

Part (III) in (4.8) can be handled similarly.

The more difficult part is Part (II) of (4.8). We make a change of variable $z := y - s$ to obtain

$$II = \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} g_x(x-y,t-s)g(y-s,s)^2\,dy\,ds \tag{4.9}$$

$$= \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} g_x(x-z-s,t-s)g(z,s)^2\,dz\,ds. \tag{4.10}$$

In order to estimate (II), let us write $g = g(x,\tau)$. Then we write

$$\begin{aligned} g_x(x-z-s,t-s)g(z,s)^2 = &-(g(x-z-s,t-s)g(z,s)^2)_s \\ &- g_\tau(x-z-s,t-s)g(z,s)^2 \\ &+ g(x-z-s,t-s)(g^2)_\tau(z,s). \end{aligned} \tag{4.11}$$

Once again we replace $z+s$ with y and proceed to estimate (II) piece by piece.

The first part of (4.12) can be estimated as follows.

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (g(x-y,t-s)g(y-s,s)^2)_s \, dy \, ds \tag{4.12}$$

$$= \int_{-\infty}^{+\infty} g(x-y,\sqrt{t})\, g(y-t+\sqrt{t}, t-\sqrt{t})^2 \, dy \tag{4.13}$$

$$- \int_{-\infty}^{+\infty} g(x-y,t-\sqrt{t})\, g(y-\sqrt{t},\sqrt{t})^2 \, dy. \tag{4.14}$$

Using (4.4) and (4.7), it follows that

$$\int_{-\infty}^{+\infty} g(x-y,\sqrt{t})\, g(y-t+\sqrt{t}, t-\sqrt{t})^2 \, dy \le Ct^{-\frac{1}{2}} g(x-t+\sqrt{t}, t),$$

and

$$\int_{-\infty}^{+\infty} g(x-y,t-\sqrt{t})\, g(y-\sqrt{t},\sqrt{t})^2 \, dy \le Ct^{-\frac{1}{4}} g(x-\sqrt{t}, t),$$

but by Lemma 4.2

$$\begin{aligned} g(x-\sqrt{t},t) &\le g(x,2t) \\ g(x-t+\sqrt{t},t) &\le g(x-t,2t). \end{aligned}$$

These terms fit in the right-hand side of (4.3).

For the other parts in (4.12), we use (4.6) to obtain

$$\begin{aligned} &\left| \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} g_\tau(x-y,t-s)g(y-s,s)^2 \, dy \, ds \right| \\ &\quad \le \int_{\sqrt{t}}^{t-\sqrt{t}} s^{-\frac{1}{2}}(t-s)^{-1} g(x-s,2t) ds \\ &\quad = \int_{\sqrt{t}}^{\frac{t}{2}} s^{-\frac{1}{2}}(t-s)^{-1} g(x-s,2t) ds \\ &\qquad + \int_{\frac{t}{2}}^{t-\sqrt{t}} s^{-\frac{1}{2}}(t-s)^{-1} g(x-s,2t) ds \\ &\quad =: \mathcal{A} + \mathcal{B}. \end{aligned} \tag{4.15}$$

If $x \le \sqrt{t}$, then

$$\begin{aligned}\mathcal{A} &\le C\,t^{-1} g(x-\sqrt{t},2t)\int_{\sqrt{t}}^{\frac{t}{2}} s^{-\frac{1}{2}}\,ds\\ &\le C\,t^{-\frac{1}{2}}\,g(x-\sqrt{t},2t)\\ &\le Ct^{-\frac{1}{2}}g(x,4t).\end{aligned} \tag{4.16}$$

Similarly when $x \ge t-\sqrt{t}$, we obtain $\mathcal{A} \le Ct^{-\frac{1}{2}}g(x-t,4t)$.

Now for $\sqrt{t} \le x \le t-\sqrt{t}$ we have

$$\begin{aligned}\mathcal{A} &= \int_{\sqrt{t}}^{\frac{t}{2}} s^{-\frac{1}{2}}(t-s)^{-1}g(x-s,2t)ds\\ &\le Ct^{-1}\int_{\sqrt{t}}^{\frac{t}{2}} s^{-\frac{1}{2}}g(x-s,2t)ds\\ &= Ct^{-1}\int_{\sqrt{t}}^{\frac{x}{2}} s^{-\frac{1}{2}}g(x-s,2t)ds + t^{-1}\int_{\frac{x}{2}}^{\frac{t}{2}} s^{-\frac{1}{2}}g(x-s,2t)ds\\ &\le Ct^{-1}g\Big(\frac{x}{2},2t\Big)\int_{\sqrt{t}}^{\frac{x}{2}} s^{-\frac{1}{2}}ds + t^{-1}x^{-\frac{1}{2}}\int_{\frac{x}{2}}^{\frac{t}{2}} g(x-s,2t)ds\\ &\le C(t^{-\frac{1}{2}}g(x,4t)+t^{-1}x^{-\frac{1}{2}})\end{aligned} \tag{4.17}$$

also acceptable, as $t^{-1}x^{-\frac{1}{2}} \le x^{-\frac{3}{2}}$ for $\sqrt{t} \le x \le t-\sqrt{t}$.

Part $\mathcal{B}$ in (4.15) can be estimated similarly. We carry it out briefly only for $\frac{t}{2} \le x \le t-\sqrt{t}$. Let $\xi := t-\sqrt{t}-x$. Then

$$\begin{aligned}\mathcal{B} &\le Ct^{-\frac{1}{2}}\left(\int_{\frac{t}{2}}^{x+\xi} + \int_{x+\xi}^{t-\sqrt{t}}\right)(t-s)^{-1}g(x-s,2t)ds\\ &\le Ct^{-\frac{1}{2}}(t-x)^{-1} + Ct^{-\frac{1}{4}}g\left(\tfrac{t-\sqrt{t}-x}{2},2t\right),\end{aligned} \tag{4.18}$$

to which we apply Lemma 4.2.

There remains the last part of (4.11), i.e.

$$\left|\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty} g(x-y,t-s)(g^2)_\tau(y-s,s)dy\,ds\right|, \tag{4.19}$$

which can easily be shown to be less than or equal to

$$C\int_{\sqrt{t}}^{t-\sqrt{t}} s^{-\frac{3}{2}}g(x-s,2t)ds. \tag{4.20}$$

If $x \le \sqrt{t}$, then (4.20) is of the order

$$\begin{aligned}&C\,g(x-\sqrt{t},2t)\int_{\sqrt{t}}^{t-\sqrt{t}} s^{-\frac{3}{2}}\,ds\\ &\quad\le C\,t^{-\frac{1}{4}}\,g(x-\sqrt{t},2t)\\ &\quad\le C\,t^{-\frac{1}{4}}\,g(x,4t).\end{aligned}$$

For $x \ge t-\sqrt{t}$ we use a similar method.

For $\sqrt{t} \le x \le t-\sqrt{t}$, we use a similar methods to what we used for the previous cases to get

$$(4.20) \le C(t^{-\frac{1}{4}}g(x,4t) + x^{-\frac{3}{2}}).$$

This completes our proof. □

Remark 4.3. Replacing $K(y-at,t)$ by $\kappa(y,t)$ in (4.1), and $g(x-y,t)$ by $\mathcal{G}(x,t;y)$ we obtain bounds similar to (4.3) (with appropriate modifications due to different speeds) provided κ and $\mathcal{G}$ satisfy

$$|\mathcal{G}(x,t;y)| \le Cg(x-y-at,\beta t), \tag{4.21}$$
$$|\mathcal{G}_y(x,t;y)| \le Ct^{-\frac{1}{2}}g(x-y-at,2\beta t), \tag{4.22}$$
$$|\mathcal{G}_t(x,t;y)| \le Ct^{-1}g(x-y-at,2\beta t), \tag{4.23}$$
$$|\kappa(x,t)| \le Cg(x-bt,\beta t), \tag{4.24}$$
$$|\kappa_y(x,t)| \le Ct^{-\frac{1}{2}}g(x-bt,2\beta t), \tag{4.25}$$
$$|\kappa_t(x,t;y)| \le Ct^{-1}g(x-bt,2\beta t), \tag{4.26}$$

for some $a \ne b$, and some constants $C, \beta > 0$, as hold in particular for κ a single diffusion wave and $\mathcal{G}$ a single component of the scattering term S in the decomposition $G = E + H + S + R$. Comparing with (1.14), we see that these are the rate-determing terms.

More generally [43], we may expect analogous cancellation whenever κ and $\mathcal{G}$ have the property that they decay faster along characteristic directions $\partial_s + a_j^{\pm}\partial_y$ [respectively, $\partial_t + a_j^{\pm}\partial_x$] than ∂_y or ∂_s [respectively, ∂_x or ∂_t]. This is the underlying principle guiding our analysis, and that of [27].

Remark 4.4. Reviewing (4.3) more carefully, we see that the signal u is slightly more concentrated near the characteristic direction $dx/dt = 0$ of the propagator g than in the direction $dx/dt = 1$ of the source K_y^2, decaying as $(x + t^{\frac{1}{2}})^{-\frac{3}{2}}$ rather than $t^{-\frac{1}{2}}(|t-x| + t^{\frac{1}{4}})^{-1}$. Systematic, characteristic-by-characteristic bookkeeping taking account of this difference leads to the slightly refined bounds of [27]; these sum, of course, to the simpler modulus bounds presented here.

5. Linear Integral Estimates

It remains to establish the deferred integral estimates used in Sec. 3. We begin, in this section, with the linear integral estimates of Lemmas 3.2 and 3.3.

Proof of Lemma 3.2. The first, fourth, and fifth estimates of Lemma 3.2 have been established in [15]. The second and third follow in a similar fashion from the estimates of Proposition 2.3. □

Proof of Lemma 3.3. Looking at (2.11), we notice that in order to estimate $\int_{-\infty}^{+\infty} H(x,t;y)v_0(y)dy$ it suffices to estimate

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathcal{R}_j^*(x)\mathcal{O}(e^{-\eta_0 t})\delta_{x-\bar{a}_j^* t}(-y)\mathcal{L}_j^{*t}(y)v_0(y)dy &\le CE_0 e^{-\eta_0 t} v_0(\bar{a}_j^* t - x)\\ &\le CE_0 e^{-\eta_0 t}(1+|\bar{a}_j^* t - x|)^{-\frac{3}{2}}\\ &\le CE_0 e^{-\eta_0 t}(1+|x|)^{-\frac{3}{2}}(1+|\bar{a}_j^* t|)^{\frac{3}{2}}\\ &\le CE_0 e^{-\frac{\eta_0 t}{2}}(1+|x|)^{-\frac{3}{2}}. \end{aligned} \tag{5.1}$$

Here we used the crude inequality

$$\frac{1}{1+|a+b|} \le \frac{1+|b|}{1+|a|} \tag{5.2}$$

and the fact that $\bar{a}_j^*$ $\mathcal{R}_j^*$ and $\mathcal{L}_j^{*t}$ are bounded. Observing that $e^{-\frac{eta_0 t}{4}} \le C(N)(1+t)^{-N}$ for any N, and $(1+|x-at|) \le C(1+t)(1+|x|)$, we can bound the righthand side of (5.1) in turn by $CE_0 e^{-\frac{\eta_0 t}{4}}\psi_1(x,t)$, giving (3.17)(i). Estimates (3.17)(ii) and (3.17)(iii) are obtained similarly. □

6. Nonlinear Integral Estimates I

In this section, we carry out the main work of the paper, establishing the nonlinear integral estimates of Proposition 3.4.

Proof of Lemma 3.4. Under the assumption that $\zeta(t)$ is bounded, we have the estimates

$$\begin{aligned}|v(x,t)| &\le \zeta(t)[\psi_1+\psi_2](x,t),\\ |\partial_x v(x,t)| &\le \zeta(t)\big[t^{-1/2}(\psi_1+\psi_2)+\psi_3+\psi_4\big](x,t)\\ |(\partial_t + a_j^{\pm}\partial_x)v(x,t)| &\le \zeta(t)\big[t^{-1}(1+t)^{1/4}\psi_1^{j,\pm} + t^{-1/2}|(\bar{\psi}_1^{j,\pm}+\psi_2)+\psi_3+\psi_4\big](x,t).\end{aligned}$$

In the analysis that follows, we will omit $\zeta(t)$ from our estimates in most cases and focus only on the terms with a given form, the *template*.

We observe at the outset that in proving estimates of form $\psi_1(x,t)$, we will frequently make use of the inequality

$$(1+t)^{-3/4}e^{-\frac{(x-a_j^{\pm}t)^2}{Lt}} \le C(1+|x-a_j^{\pm}t|+t^{-1/2})^{-3/2}. \tag{6.1}$$

In the case $|x-a_k^{\pm}t| \le K\sqrt{t}$, for some constant K, this inequality is immediate. On the other hand, for $|x-a_k^{\pm}t| \ge K\sqrt{t}$, we observe

$$\begin{aligned}(1+t)^{-3/4}e^{-\frac{(x-a_j^{\pm}t)^2}{Lt}} &= (1+t)^{-3/4}|x-a_j^{\pm}t|^{-3/2}t^{3/4}\frac{|x-a_j^{\pm}t|^{3/2}}{t^{3/4}}e^{-\frac{(x-a_j^{\pm}t)^2}{Lt}}\\ &\le C_1(1+t)^{-3/4}t^{3/4}|x-a_j^{\pm}t|^{-3/2}e^{-\frac{(x-a_j^{\pm}t)^2}{2Lt}}\\ &\le C(1+|x-a_j^{\pm}t|+t^{1/2})^{-3/2},\end{aligned}$$

where the seeming blow-up as $|x - a_j^{\pm} t| \to 0$ is controlled by the size of $\sqrt{t}$, which must be smaller than $|x - a_j^{\pm} t|$.

Proof of (3.18(i)). For the first estimate in Lemma 3.4, we begin by considering the nonlinearities $(v(y,s)^2)_y$ and $(v(y,s)v_y(y,s))_y$. Here and in the remaining cases, the analyses of the convection, reflection, and transmission contributions to the Green's kernel $\partial_y \tilde{G}(x,t;y)$ are similar, and we provide full details only for the case of convection. We observe that the contribution

$$\sum_{j=1}^{J} \sum_{\beta=0}^{\max\{0,|\alpha|-1\}} \mathbf{O}(e^{-\eta t}) \partial_y^{\beta} \delta_{x - \bar{a}_j^* t}(-y)$$

is similar to, though less singular than, terms arising in $\partial_y H(x,t;y)$ (see (2.11)), and can be analyzed as in the proof of Lemma 3.5. Finally, we remark that the contribution

$$\mathbf{O}(e^{-\eta(|x-y|+t)})$$

has no effect on the iteration.

(3.18(i)), term one. We first consider integration of our convecting Green's kernel against the nonlinearity $s^{-1/2}(1+s)^{-1/4}\psi_1(y,s)$. In this case, we have integrals of the form

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1 + |y - a_k^- s| + s^{1/2})^{-3/2} s^{-1/2} (1+s)^{-1/4} dy\, ds, \tag{6.2}$$

with a similar integral for $y \geq 0$. Proceeding as in [15], we write

$$x - y - a_j^-(t-s) = (x - a_j^-(t-s) - a_k^- s) - (y - a_k^- s), \tag{6.3}$$

from which we have the estimate

$$\begin{aligned} &e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1 + |y - a_k^- s| + s^{1/2})^{-\gamma} \\ &\quad \leq C \Big[e^{-\epsilon \frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} (1 + |y - a_k^- s| + s^{1/2})^{-\gamma} \\ &\qquad + e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} \big(1 + |y - a_k^- s| + |x - a_j^-(t-s) - a_k^- s| + s^{1/2}\big)^{-\gamma} \Big], \end{aligned} \tag{6.4}$$

where $\gamma = 3/2$. (Here and in future cases, we will state estimates in terms of a parameter γ for general applicability.) For the first estimate in (6.4), we have integrals

$$\begin{aligned} &\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\epsilon \frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} \\ &\qquad \times (1 + |y - a_k^- s| + s^{1/2})^{-3/2} s^{-1/2} (1+s)^{-1/4} dy\, ds. \end{aligned} \tag{6.5}$$

We have three cases to consider: $a_k^- < 0 < a_j^-$, $a_k^- \leq a_j^- < 0$, and $a_j^- < a_k^- < 0$. We will proceed by analyzing the second of these in detail and observing that no qualitatively new calculations arise in the remaining two. For this second case and for $|x| \geq |a_k^-|t$, we write

$$x - a_j^-(t-s) - a_k^- s = (x - a_k^- t) - (a_j^- - a_k^-)(t-s), \tag{6.6}$$

for which we observe that there is no cancellation between summands. We immediately obtain an estimate on (6.5) by

$$\begin{aligned} &C_1 t^{-1} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_0^{t/2} (1+s^{1/2})^{-1/2} s^{-1/2} (1+s)^{-1/4} ds \\ &\quad + C_2 (1+t^{1/2})^{-3/2} t^{-1/2} (1+t)^{-1/4} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_{t/2}^t (t-s)^{-1/2} ds \\ &\leq C(1+t)^{-1} \ln(e+t) e^{-\frac{(x-a_k^- t)^2}{Lt}}, \end{aligned} \tag{6.7}$$

which is sufficient by (6.1). For $|x| \leq |a_j^-|t$, we write

$$x - a_j^-(t-s) - a_k^- s = (x - a_j^- t) - (a_k^- - a_j^-)s, \tag{6.8}$$

for which we observe that there is no cancellation between summands, and proceeding as in (6.7), we obtain an estimate by

$$C(1+t)^{-1} \ln(e+t) e^{-\frac{(x-a_j^- t)^2}{Lt}},$$

which again is sufficient. We remark that this concludes the analysis in the case $a_j^- = a_k^-$, so from here on we may take $a_k^- < a_j^-$ and the case $|a_j^-|t \leq |x| \leq |a_k^-|t$. In this case, and for $s \in [0, t/2]$, we observe through (6.8) the inequality

$$\begin{aligned} &e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} (1 + |y - a_k^- s| + s^{1/2})^{-3/2} (1+s)^{-\gamma} \\ &\quad \leq C\Big[e^{-\frac{(x-a_j^- t)^2}{Lt}} (1 + |y - a_k^- s| + s^{1/2})^{-3/2} (1+s)^{-\gamma} \\ &\qquad + e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} (1 + |y - a_k^- s| + |x - a_j^- t|^{1/2})^{-3/2} (1 + |x - a_j^- t|)^{-\gamma} \Big], \end{aligned} \tag{6.9}$$

with $\gamma = 3/4$. For the first estimate in (6.9), we proceed similarly as in (6.7), while for the second, we have, upon integration of the ϵ-kernel in y, an estimate by

$$\begin{aligned} &C_1 t^{-1/2} (1 + |x - a_j^- t|^{1/2})^{-3/2} (1 + |x - a_j^- t|)^{-3/4} \\ &\quad \times \int_0^{t/2} s^{-1/2} (1+s)^{1/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} ds \leq C(1 + |x - a_j^- t|)^{-3/2}, \end{aligned}$$

where we have used in this last inequality that $a_j^- \neq a_k^-$. We observe that this is clearly sufficient in the case $|x - a_j^- t| \geq \sqrt{t}$, whereas in the case $|x - a_j^- t| \leq \sqrt{t}$, decay of kernel type $\exp((x - a_j^- t)^2/(Lt))$ is immediate, for L sufficiently large, and so we only require decay at rate $t^{-3/4}$, which is straightforward. For $s \in [t/2, t]$, we observe through (6.6) the inequality

$$\begin{aligned}(t-s)^{-\gamma} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} \\ \le C\Big[|x-a_k^- t|^{-\gamma} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} + (t-s)^{-\gamma} e^{-\frac{(x-a_j^- t)^2}{Lt}}\Big], \quad (6.10)\end{aligned}$$

where $\gamma = 1$. For the second estimate in (6.10), we proceed similarly as in (6.7), while for the first we have, upon integration in y of the nonlinearity, an estimate by

$$\begin{aligned}C_2(1+t^{1/2})^{-1/2} t^{-1/2}(1+t)^{-1/4}|x-a_k^- t|^{-1} \int_{t/2}^{t} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} ds \\ \le C(1+t)^{-1/2}|x-a_k^- t|^{-1}.\end{aligned}$$

Recalling that we are currently working in the case $t \ge |x|/|a_k^-|$, we see that this final estimate is sufficient for the case $|x-a_k^- t| \ge \sqrt{t}$. For the second term in (6.4), we have integrals of the form

$$\begin{aligned}\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} \\ \times (1+|y-a_k^- s| + |x-a_j^-(t-s)-a_k^- s| + s^{1/2})^{-3/2} s^{-1/2}(1+s)^{-1/4} ds. \quad (6.11)\end{aligned}$$

We continue to focus on the case $a_k^- \le a_j^-$. For the cases $|x| \ge |a_k^-|t$ and $|x| \le |a_j^-|t$, respectively, we have no cancellation between the summands in (6.6) and (6.8) and consequently we obtain estimates

$$C(1+t)^{-1/4}[(1+|x-a_k^- t|)^{-3/2} + (1+|x-a_j^- t|)^{-3/2}]. \quad (6.12)$$

For the case $|a_j^-|t \le |x| \le |a_k^-|t$, we divide the analysis into sub-intervals $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we observe through (6.8) the inequality

$$\begin{aligned}&(1+|y-a_k^- s| + |x-a_j^-(t-s)-a_k^- s| + s^{1/2})^{-3/2}(1+s)^{-\gamma} \\ &\le C\big[(1+|x-a_j^- t| + s^{1/2})^{-3/2}(1+s)^{-\gamma} + (1+|y-a_k^- s| \\ &\quad + |x-a_j^-(t-s)-a_k^- s| + |x-a_j^- t|^{1/2})^{-3/2}(1+|x-a_j^- t|)^{-\gamma}\big], \quad (6.13)\end{aligned}$$

for $\gamma = 3/4$. For the first estimate in (6.13), we proceed as in (6.12), while for the second we have, upon integration of the kernel, an estimate on (6.11) by

$$\begin{aligned}C_1 t^{-1/2}(1+|x-a_j^- t|)^{-3/4} \int_0^{t/2} s^{-1/2}(1+s)^{1/2} \\ \times (1+|x-a_j^-(t-s)-a_k^- s| + |x-a_j^- t|^{1/2})^{-3/2} ds \\ \le C(1+t)^{-1/2}(1+|x-a_j^- t|)^{-1}.\end{aligned}$$

For $s \in [t/2, t]$, we observe through (6.6) the inequality

$$\begin{aligned}
&(t-s)^{-1}(1+|x-a_j^-(t-s)-a_k^- s|+s^{1/2})^{-3/2}\\
&\quad \le C\Big[|x-a_k^- t|^{-1}(1+|x-a_j^-(t-s)-a_k^- s|+s^{1/2})^{-3/2}\\
&\qquad + (t-s)^{-1}(1+|x-a_j^-(t-s)-a_k^- s|+|x-a_j^- t|+s^{1/2})^{-3/2}\Big]. \qquad (6.14)
\end{aligned}$$

For the second estimate in (6.14), we proceed similarly as in (6.12), while for the first we have, upon integration of the kernel, an estimate by

$$\begin{aligned}
&C_2|x-a_k^- t|^{-1} t^{-1/2}(1+t)^{-1/4}\\
&\quad \times \int_{t/2}^{t} (t-s)^{1/2}(1+|x-a_j^-(t-s)-a_k^- s|+t^{1/2})^{-3/2} ds\\
&\quad \le C|x-a_k^- t|^{-1}(1+t)^{-1/4}(1+t^{1/2})^{-1/2},
\end{aligned}$$

1 which is sufficient for $t \ge |x|/|a_j^-|$.

(3.18(i)), *term two*. We next consider integration against the nonlinearity $s^{-1/2}(1+s)^{-1/4}\psi_2(y,t)$, for which we have integrals of the form

$$\begin{aligned}
&\int_0^t \int_{-|a_1^-|s}^{0} (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+|y|)^{-1/2}(|y|+s)^{-1/2}\\
&\quad \times (1+|y|+s)^{-3/4}(1+|y|+s^{1/2})^{-1/2} dy\, ds, \qquad (6.15)
\end{aligned}$$

wherein we have observed that for $y \in [-|a_1^-|s, 0]$, s decay yields also decay in y. We first observe an immediate time decay estimate by

$$\begin{aligned}
&C_1 t^{-1}\int_0^{t/2} (1+s)^{-3/4}(1+s^{1/2})^{-1/2} ds\\
&\quad + C_2 t^{-1/2}(1+t)^{-3/4}(1+t^{1/2})^{-1/2}\int_{t/2}^{t} (t-s)^{-1/2} ds\\
&\quad \le C(1+t)^{-1}\ln(e+t). \qquad (6.16)
\end{aligned}$$

In order to determine estimates in space as well, we observe the inequality

$$\begin{aligned}
&e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+|y|)^{-1/2}(|y|+s)^{-1/2}(1+|y|+s)^{-3/4}(1+|y|+s^{1/2})^{-1/2}\\
&\quad \le C\Big[e^{-\epsilon\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}}(1+|y|)^{-1/2}(|y|+s)^{-1/2}\\
&\qquad \times (1+|y|+s)^{-3/4}(1+|y|+s^{1/2})^{-1/2}\\
&\qquad + e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+|x-a_j^-(t-s)|)^{-1/2}(|x-a_j^-(t-s)|+s)^{-1/2}\\
&\qquad \times (1+|x-a_j^-(t-s)|+s)^{-3/4}(1+|x-a_j^-(t-s)|+s^{1/2})^{-1/2}\Big]. \qquad (6.17)
\end{aligned}$$

For the first estimate in (6.17), we have integrals

$$\begin{aligned}
&\int_0^t \int_{-|a_1^-|s}^{0} (t-s)^{-1} e^{-\epsilon\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}}\\
&\quad \times (1+|y|)^{-1/2}(|y|+s)^{-1/2}(1+|y|+s)^{-3/4}(1+|y|+s^{1/2})^{-1/2} dy\, ds. \qquad (6.18)
\end{aligned}$$

We have two cases to consider here, $a_j^- < 0$ and $a_j^- > 0$, of which we focus on the former. In this case, for $|x| \geq |a_j^-|t$, there is no cancellation between $x - a_j^- t$ and $a_j^- s$, and so proceeding as in (6.16), we obtain an estimate by

$$C(1+t)^{-1}\ln(e+t)e^{-\frac{(x-a_j^- t)^2}{Lt}}.$$

For $|x| \leq |a_j^-|t$, we divide the analysis into two cases, $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we observe the inequality

$$\begin{aligned}
&e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}}(|y|+s)^{-1/2}(1+|y|+s)^{-3/4}(1+|y|+s^{1/2})^{-1/2}\\
&\quad\leq C\Big[e^{-\frac{(x-a_j^- t)^2}{Lt}}(|y|+s)^{-1/2}(1+|y|+s)^{-3/4}(1+|y|+s^{1/2})^{-1/2}\\
&\qquad + e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}}(s+|x-a_j^- t|)^{-1/2}(1+s+|x-a_j^- t|)^{-3/4}\\
&\qquad \times (1+s^{1/2}+|x-a_j^- t|^{1/2})^{-1/2}\Big]. \qquad (6.19)
\end{aligned}$$

For the first estimate in (6.19), we proceed similarly as in (6.16), while for the second we have, upon integration of the ϵ kernel, an estimate by

$$\begin{aligned}
&C_1 t^{-1/2}(|x-a_j^- t|)^{-1/2}(1+|x-a_j^- t|)^{-1}\int_0^{t/2} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}}ds\\
&\quad \leq C(|x-a_j^- t|)^{-1/2}(1+|x-a_j^- t|)^{-1},
\end{aligned}$$

where the seeming blow-up as $|x - a_j^- t| \to 0$ can be eliminated by proceeding alternatively for $|x - a_j^- t|$ bounded. For $s \in [t/2, t]$, we have

$$(t-s)^{-\gamma}e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}} \leq C|x|^{-\gamma}e^{-\frac{(x-a_j^-(t-s))^2}{2M(t-s)}}, \qquad (6.20)$$

$\gamma = 1/2$, for which we have an estimate by

$$C_2|x|^{-1/2}t^{-1/2}(1+t)^{-3/4}(1+t^{1/2})^{-1/2}\int_{t/2}^{t} e^{-\frac{(x-a_j^-(t-s))^2}{2M(t-s)}}ds \leq C|x|^{-1/2}(1+t)^{-1},$$

where the seeming blow-up as $|x| \to 0$ can be eliminated by an alternative calculation in the case of $|x|$ bounded. In the current case of $|x| \leq |a_j^-|t$, this final estimate is bounded by $\psi_2(x, t)$. For the second estimate in (6.17), we have integrals

$$\begin{aligned}
&\int_0^t \int_{-|a_1^-|s}^{0}(t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+|x-a_j^-(t-s)|)^{-1/2}\\
&\quad \times (|x-a_j^-(t-s)|+s)^{-1/2}(1+|x-a_j^-(t-s)|+s)^{-3/4}\\
&\quad \times (1+|x-a_j^-(t-s)|+s^{1/2})^{-1/2}\,dy\,ds. \qquad (6.21)
\end{aligned}$$

Focusing again on the case $a_j^- < 0$, we observe that for $|x| \geq |a_j^-|t$, there is no cancellation between $x - a_j^- t$ and $a_j^- s$, and consequently that we have an estimate, upon integration of the kernel, by

$$\begin{aligned} &C_1 t^{-1/2}(1+|x-a_j^- t|)^{-7/4}\int_0^{t/2}(|x-a_j^-(t-s)|+s)^{-1/2}ds \\ &\quad + C_2(1+|x-a_j^- t|)^{-7/4}t^{-1/2}\int_{t/2}^{t}(t-s)^{-1/2}ds \\ &\leq C(1+|x-a_j^- t|)^{-7/4}, \end{aligned} \tag{6.22}$$

which, for $|x - a_j^- t| \geq \sqrt{t}$ decays faster than the claimed estimates. For the case $|x - a_j^- t| \leq \sqrt{t}$, we require only $t^{-3/4}$ decay, which is clear from (6.16). For $|x| \leq |a_j^-|t$, we divide the analysis into cases, $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we observe the inequality

$$\begin{aligned} &(1+|x-a_j^-(t-s)|)^{-1/2}(|x-a_j^-(t-s)|+s)^{-1/2} \\ &\quad \times (1+|x-a_j^-(t-s)|+s)^{-3/4}(1+|x-a_j^-(t-s)|+s^{1/2})^{-1/2} \\ &\leq C\big[(1+|x-a_j^- t|)^{-1/2}(|x-a_j^- t|+s)^{-1/2} \\ &\quad \times (1+|x-a_j^- t|+s)^{-3/4}(1+|x-a_j^- t|+s^{1/2})^{-1/2} \\ &\quad + (1+|x-a_j^-(t-s)|)^{-1/2}(|x-a_j^- t|+s)^{-1/2}(1+|x-a_j^- t|+s)^{-3/4} \\ &\quad \times (1+|x-a_j^- t|^{1/2}+s^{1/2})^{-1/2}\big]. \end{aligned} \tag{6.23}$$

For the first estimate in (6.23), we obtain an estimate, upon integration of the kernel, by

$$\begin{aligned} &C_1 t^{-1/2}(1+|x-a_j^- t|)^{-1}\int_0^{t/2}(|x-a_j^- t|+s)^{-1/2}(1+|x-a_j^- t|+s)^{-3/4}ds \\ &\quad \leq Ct^{-1/2}(1+|x-a_j^- t|)^{-5/4}, \end{aligned}$$

1 while for the second we obtain an estimate by

$$C_1 t^{-1/2}(1+|x-a_j^- t|)^{-3/2}\int_0^{t/2}(|x-a_j^-(t-s)|)^{-1/2}ds \leq C(1+|x-a_j^- t|)^{-3/2}.$$

For $s \in [t/2, t]$, we write

$$\begin{aligned} &(t-s)^{-1/2}(1+|x-a_j^-(t-s)|)^{-1/2} \\ &\quad \leq C\big[|x|^{-1/2}(1+|x-a_j^-(t-s)|)^{-1/2} \\ &\qquad + (t-s)^{-1/2}(1+|x-a_j^-(t-s)|+|x|)^{-1/2}\big]. \end{aligned} \tag{6.24}$$

For the first estimate in (6.24), we obtain an estimate by

$$\begin{aligned} &C_2|x|^{-1/2}t^{-1/2}(1+t)^{-3/4}(1+t^{1/2})^{-1/2}\int_{t/2}^{t}(1+|x-a_j^-(t-s)|)^{-1/2}ds \\ &\quad \leq C\,(1+t)^{-1}|x|^{-1/2}, \end{aligned}$$

which for $|x|$ bounded away from 0 is bounded by $\psi_2(x,t)$, while for the second estimate in (6.14), we obtain an estimate by

$$C_2(1+|x|)^{-1/2}t^{-1/2}(1+t)^{-3/4}(1+t^{1/2})^{-1/2}\int_{t/2}^{t}(t-s)^{-1/2}ds$$
$$\leq C(1+|x|)^{-1/2}(1+t)^{-1}.$$

(3.18(i)), *term three.* We next consider integration against the nonlinearity $(1+s)^{-1}e^{-\eta|y|}$ (which arises, for example, from the term $(\frac{\partial\bar{u}^\delta}{\partial\delta}\delta)^2$), for which, in the case of our convection Green's kernel estimate, we have integrals of the form

$$\int_0^t\int_{-\infty}^0 (t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-1}e^{-\eta|y|}dy\,ds. \tag{6.25}$$

First, we observe a straightforward time decay estimate by

$$C_1t^{-1}\int_0^{t/2}(1+s)^{-1}ds + C_2(1+t)^{-1}\int_{t/2}^{t-1}(t-s)^{-1}ds$$
$$+ C_3(1+t)^{-1}\int_{t-1}^{t}(t-s)^{-1/2}ds \leq C(1+t)^{-1}\ln(e+t). \tag{6.26}$$

In order to determine estimates in space as well, we observe the inequality

$$e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}e^{-\eta|y|}$$
$$\leq \Big[e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}}e^{-\eta|y|} + e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}e^{-\eta_1|x-a_j^-(t-s)|}e^{-\eta_2|y|}\Big]. \tag{6.27}$$

For the first estimate in (6.27), we have integrals

$$\int_0^t\int_{-\infty}^0 (t-s)^{-1}e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-1}e^{-\eta|y|}dy\,ds. \tag{6.28}$$

We have two cases to consider, $a_j^- < 0$ and $a_j^- > 0$, of which we focus on the former. For $|x| \geq |a_j^-|t$, there is no cancellation between $x - a_j^-t$ and a_j^-s, and proceeding almost precisely as in (6.26), we obtain an estimate by

$$C(1+t)^{-1}\ln(e+t)e^{-\frac{(x-a_j^-t)^2}{Lt}}.$$

For $|x| \leq |a_j^-|t$, we divide the analysis into cases, $s \in [0,t/2]$ and $s \in [t/2,t]$. For $s \in [0,t/2]$, we observe the estimate

$$e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-1}$$
$$\leq C\Big[e^{-\frac{(x-a_j^-t)^2}{Lt}}(1+s)^{-1} + e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}}(1+|x-a_j^-t|)^{-1}\Big]. \tag{6.29}$$

For the first estimate in (6.29), we proceed similarly as in (6.26), while for the second we have an estimate by

$$C_1t^{-1}(1+|x-a_j^-t|)^{-1}\int_0^{t/2}e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}}ds \leq C(1+t)^{-1/2}(1+|x-a_j^-t|)^{-1},$$

which is sufficient for $t \geq |x|/|a_j^-|$. For $s \in [t/2, t]$, we have

$$(t-s)^{-1/2} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}} \leq C|x|^{-1/2} e^{-\frac{(x-a_j^-(t-s))^2}{2M(t-s)}},$$

from which we obtain an estimate on (6.28) by

$$C_2|x|^{-1/2}(1+t)^{-1} \int_{t/2}^{t} (t-s)^{-1/2} e^{-\frac{(x-a_j^-(t-s))^2}{2M(t-s)}} ds \leq C|x|^{-1}(1+t)^{-1},$$

for which we have oberved the bound

$$\int_{t/2}^{t} (t-s)^{-1/2} e^{-\frac{(x-a_j^-(t-s))^2}{2M(t-s)}} ds \leq C.$$

The second estimate in (6.27) can be analyzed similarly.

(3.18(i)), *nonlinearity* $v(y,s)\varphi(y,s)$. We next consider integration against the critical nonlinearity $v(y,s)\varphi(y,s)$, which constituted the limiting estimate of the analysis in [15]. Here, we refine the analysis of [15] both through refined estimates on $v(y,s)$ and through the application of the approach of Liu [26, 27] described in Sec. 4, which takes advantage of improved decay for derivatives along the characteristic direction. It is precisely for this analysis that we must keep track of estimates on the characteristic derivatives $(\partial_t + a_j^- \partial_x)v$. We proceed by dividing the integration over s as,

$$\begin{aligned} &\int_0^t \int_{-\infty}^{+\infty} \tilde{G}(x, t-s; y)(v(y,s)\varphi(y,s))_y dy\, ds \\ &\quad = \int_0^{\sqrt{t}} \int_{-\infty}^{+\infty} \tilde{G}(x, t-s; y)(v(y,s)\varphi(y,s))_y dy\, ds \\ &\qquad + \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} \tilde{G}(x, t-s; y)(v(y,s)\varphi(y,s))_y dy\, ds \\ &\qquad + \int_{t-\sqrt{t}}^{t} \int_{-\infty}^{+\infty} \tilde{G}(x, t-s; y)(v(y,s)\varphi(y,s))_y dy\, ds. \end{aligned} \tag{6.30}$$

For the first integral on the right-hand side of (6.30), we integrate by parts in y and use the supremum estimate $\|v(y,s)\|_{L^\infty} \leq C(1+s)^{-3/4}$ (valid for either of our estimates on $v(y,s)$) to obtain integrals of the form,

$$\int_0^{\sqrt{t}} \int_{-\infty}^{0} (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-5/4} e^{-\frac{(y-a_k^- s)^2}{Ms}} dy\, ds,$$

for which we observe the equality

$$e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} e^{-\frac{(y-a_k^- s)^2}{Ms}} = e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} e^{-\frac{t}{Ms(t-s)}\left(y - \frac{xs-(a_j^-+a_k^-)s(t-s)}{t}\right)^2} \tag{6.31}$$

(see [16, Lemma 6]). Integrating over y, we immediately obtain an estimate by

$$Ct^{-1/2}\int_0^{\sqrt{t}}(t-s)^{-1/2}(1+s)^{-3/4}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds.$$

For $|x-a_j^-t|\le K\sqrt{t}$, any fixed K, we have kernel decay $\exp(-(x-a_j^-t)^2/(Lt))$ by boundedness, while for $|x-a_j^-t|\ge K\sqrt{t}$, with K sufficiently large, we have

$$|x-a_j^-(t-s)-a_k^-s| = |(x-a_j^-t)+(a_j^- - a_k^-)s| \ge \left(1-\frac{a_j^- - a_k^-}{K}\right)|x-a_j^-t|, \tag{6.32}$$

and we again have kernel decay $\exp(-(x-a_j^-t)^2/(Lt))$. In either case, we obtain a final estimate by

$$C(1+t)^{-3/4}e^{-\frac{(x-a_j^-t)^2}{Lt}},$$

which is sufficient by (6.1). Similarly, for the third integral in (6.30), we obtain, upon integration in y precisely as above, an estimate by

$$Ct^{-1/2}\int_{t-\sqrt{t}}^{t}(t-s)^{-1/2}(1+s)^{-3/4}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds.$$

Proceeding similarly as above, we obtain an estimate in this case of the form

$$(1+t)^{-3/4}e^{-\frac{(x-a_k^-t)^2}{Lt}}.$$

For the second integral in (6.30), we first consider the case $j=k$, for which, proceedingly similarly as above, we have integrals of the form

$$Ct^{-1/2}\int_{\sqrt{t}}^{t-\sqrt{t}}(t-s)^{-1/2}(1+s)^{-3/4}e^{-\frac{(x-a_j^-t)^2}{Mt}}ds,$$

which has been shown sufficient above. In the case $j\ne k$, the claimed estimate does not follow from such a direct method, and we employ a Liu-type cancellation estimate, as described in Sec. 4, based on integration by parts in the characteristic direction. In order to clarify our analysis, we define the non-convecting variables

$$\begin{aligned} g(x,t;y) &= \tilde{G}^j(x,t,y-a_j^-t)\\ \phi(y,s) &= \varphi_k^-(y+a_k^-s,s)\\ V(y,s) &= v(y+a_k^-s,s), \end{aligned} \tag{6.33}$$

where

$$\tilde{G}^j(x,t;y) = ct^{-1/2}e^{-\frac{(x-y-a_j^-(t-s))^2}{4\beta_j^- t}} \qquad c = r_j^-(l_j^-)^{\mathrm{tr}}/\sqrt{4\pi\beta_j^-}, \tag{6.34}$$

and φ_k^- is as in (1.10). (We will consider corrections to $\tilde{G}^j$ at the end of the analysis.) In this notation, the second integral in (6.30) becomes

$$\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty} g(x,t-s;y+a_j^-(t-s))(V(y-a_k^-s,s)\phi(y-a_k^-s,s))_y dy\,ds. \tag{6.35}$$

Setting $\xi = y + a_j^-(t-s)$, (6.35) becomes

$$\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty} g(x,t-s;\xi)(V(\xi-a_j^-(t-s)-a_k^-s,s)\phi(\xi-a_j^-(t-s)-a_k^-s,s))_\xi d\xi\,ds. \tag{6.36}$$

Denoting by ∂_τ differentiation with respect to the second dependent variable of each function (effectively, a characteristic derivative $\partial_t + a_k^-\partial_x$ on our original variables), we observe the differential relationship

$$\begin{aligned}
&(g(x,t-s;\xi)V(\xi-a_j^-(t-s)-a_k^-s,s)\phi(\xi-a_j^-(t-s)-a_k^-s,s))_s \\
&\quad= -g_\tau(x,t-s;\xi)V(\xi-a_j^-(t-s)-a_k^-s,s)\phi(\xi-a_j^-(t-s)-a_k^-s,s) \\
&\qquad+(a_j^- - a_k^-)g(x,t-s;\xi)(V(\xi-a_j^-(t-s)-a_k^-s,s) \\
&\qquad\times\phi(\xi-a_j^-(t-s)-a_k^-s,s))_\xi \\
&\qquad+g(x,t-s;\xi)(V(\xi-a_j^-(t-s)-a_k^-s,s)\phi(\xi-a_j^-(t-s)-a_k^-s,s))_\tau.
\end{aligned} \tag{6.37}$$

Recalling that the case $j = k$ has already been considered, we can rearrange (6.37) so that the integrand of (6.36) can be written as

$$\begin{aligned}
&g(x,t-s;\xi)(V(\xi-a_j^-(t-s)-a_k^-s,s)\phi(\xi-a_j^-(t-s)-a_k^-s,s))_\xi \\
&\quad=(a_j^- - a_k^-)^{-1}(g(x,t-s;\xi)V(\xi-a_j^-(t-s)-a_k^-s,s) \\
&\qquad\times\phi(\xi-a_j^-(t-s)-a_k^-s,s))_s \\
&\qquad+(a_j^- - a_k^-)^{-1}g_\tau(x,t-s;\xi)V(\xi-a_j^-(t-s)-a_k^-s,s) \\
&\qquad\times\phi(\xi-a_j^-(t-s)-a_k^-s,s) \\
&\qquad-(a_j^- - a_k^-)^{-1}g(x,t-s;\xi)(V(\xi-a_j^-(t-s)-a_k^-s,s) \\
&\qquad\times\phi(\xi-a_j^-(t-s)-a_k^-s,s))_\tau.
\end{aligned} \tag{6.38}$$

For the integration over the first expression on the right-hand side of (6.38), we change the order of integration, and evaluate integration over s to obtain

$$\begin{aligned}
&\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (g(x,t-s;\xi)V(\xi - a_j^-(t-s) - a_k^- s, s) \\
&\quad \times \phi(\xi - a_j^-(t-s) - a_k^- s, s))_s d\xi\, ds \\
&\quad = \int_{-\infty}^{+\infty} g(x,\sqrt{t};\xi)V(\xi - a_j^-\sqrt{t} - a_k^-(t-\sqrt{t}), t-\sqrt{t}) \\
&\quad \times \phi(\xi - a_j^-\sqrt{t} - a_k^-(t-\sqrt{t}), t-\sqrt{t}) d\xi \\
&\quad - \int_{-\infty}^{+\infty} g(x,t-\sqrt{t};\xi)V(\xi - a_j^-(t-\sqrt{t}) - a_k^-\sqrt{t}, \sqrt{t}) \\
&\quad \times \phi(\xi - a_j^-(t-\sqrt{t}) - a_k^-\sqrt{t}, \sqrt{t}) d\xi.
\end{aligned} \tag{6.39}$$

In each integral on the right-hand side of (6.39), we employ the estimate

$$\|V(\cdot,t)\|_{L^\infty} \le C(1+t)^{-3/4}, \tag{6.40}$$

and proceed similarly as in (6.31). For the first, we obtain an estimate by

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (\sqrt{t})^{-1/2} e^{-\frac{(x-\xi)^2}{M\sqrt{t}}} (1 + (t-\sqrt{t}))^{-5/4} e^{-\frac{(\xi - a_j^-\sqrt{t} - a_k^-(t-\sqrt{t})^2}{M(t-\sqrt{t})}} d\xi \\
&\quad \le Ct^{-1/2}(1+t)^{-3/4} e^{-\frac{(x - a_j^-\sqrt{t} - a_k^x - (t-\sqrt{t})^2}{Mt}},
\end{aligned}$$

which gives an estimate by

$$Ct^{-1/2}(1+t)^{-3/4} e^{-\frac{(x-a_k^- t)^2}{Mt}}$$

(see (6.32)). For the second integral on the right-hand side of (6.39), we obtain an estimate by

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (t-\sqrt{t})^{-1/2} e^{-\frac{(x-\xi)^2}{M(t-\sqrt{t})}} (1+\sqrt{t})^{-5/4} e^{-\frac{(\xi - a_j^-(t-\sqrt{t}) - a_k^-\sqrt{t})^2}{M\sqrt{t}}} d\xi \\
&\quad \le Ct^{-1/2}(1+\sqrt{t})^{-3/4} e^{-\frac{(x - a_j^-(t-\sqrt{t}) - a_k^-\sqrt{t})^2}{M\sqrt{t}}},
\end{aligned}$$

which gives an estimate by

$$C(1+t)^{-7/8} e^{-\frac{(x-a_j^- t)^2}{Mt}}.$$

For integration over the second integrand on the right-hand side of (6.38), we have integrals

$$\begin{aligned}\bigg|\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{\mathbb{R}} & g_\tau(x,t-s;\xi)V(\xi-a_j^-(t-s)-a_k^-s,s)\phi(\xi-a_j^-(t-s)-a_k^-s,s)d\xi\,ds\bigg| \\ &\le C_1\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}(t-s)^{-3/2}e^{-\frac{(x-\xi)^2}{M(t-s)}}(1+s)^{-5/4}e^{-\frac{(\xi-a_j^-(t-s)-a_k^-s)^2}{Ms}}d\xi\,ds \\ &\le C_2t^{-1/2}\int_{\sqrt{t}}^{t-\sqrt{t}}(t-s)^{-1}(1+s)^{-3/4}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Ms}}ds, \qquad (6.41)\end{aligned}$$

where for the first inequality in (6.41), we have employed the estimate (6.40), while for the second we have used a calculation similar to (6.31). In this last integral, we have three cases to consider, $a_k^- < 0 < a_j^-$, $a_k < a_j^- < 0$, and $a_j^- < a_k^- < 0$, for which we focus on the second. (We recall that the case $a_k^- = a_j^-$ has already been considered above.) For $|x| \ge |a_k^-|t$, there is no cancellation between summands in (6.6), and we obtain kernel decay,

$$Ct^{-1/2}(1+t)^{-3/4}\ln(e+t)e^{-\frac{(x-a_k^-t)^2}{Lt}},$$

while for $|x| \le |a_j^-|t$, there is no cancellation between summands in (6.8), and we obtain kernel decay

$$Ct^{-1/2}(1+t)^{-3/4}\ln(e+t)e^{-\frac{(x-a_j^-t)^2}{Lt}}.$$

In either case, the seeming blow-up as $t \to 0$ can be eliminated by an alternative analysis in the case of t bounded. For $|a_j^-|t \le |x| \le |a_k^-|t$, we divide the analysis into cases, $s \in [\sqrt{t}, t/2]$ and $s \in [t/2, t-\sqrt{t}]$. For $s \in [\sqrt{t}, t/2]$, we observe through (6.8) the inequality

$$\begin{aligned}(1+s)^{-\gamma}&e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Ms}} \\ &\le C\Big[(1+s)^{-\gamma}e^{-\frac{(x-a_j^-t)^2}{Lt}} + (1+|x-a_j^-t|)^{-\gamma}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Ms}}\Big], \qquad (6.42)\end{aligned}$$

where $\gamma = 3/4$. For the first estimate in (6.42), we proceed as in the case $|x| \le |a_j^-|t$, while for the second we obtain an estimate by

$$\begin{aligned}C_1t^{-3/2}(1+|x-a_j^-t|)^{-3/4}&\int_{\sqrt{t}}^{t/2}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Ms}}ds \\ &\le Ct^{-1/2}(1+t)^{-1/2}(1+|x-a_j^-t|)^{-3/4},\end{aligned}$$

which is sufficient. For $s \in [t/2, t-\sqrt{t}]$, we observe through (6.6) the inequality (6.10) with $\gamma = 1$. For the second estimate in (6.10), we proceed as in the case $|x| \ge |a_k^-|t$, while for the first we have an estimate by

$$\begin{aligned}C_2t^{-1/2}(1+t)^{-3/4}|x-a_k^-t|^{-1}&\int_{t/2}^{t-\sqrt{t}}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Ms}}ds \\ &\le C(1+t)^{-3/4}|x-a_k^-t|^{-1},\end{aligned}$$

which is sufficient for $|x - a_k^- t|$ bounded away from 0. In the case of $|x - a_k^- t|$ bounded, we proceed alternatively.

The third term in (6.38) can be regarded as the crucial piece, since it is here that we must keep track not only of estimates on $v(y,s)$, but also on estimates of characteristic derivatives on v. We begin by expanding the characteristic derivative,

$$\begin{aligned}
&\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{\mathbb{R}} g(x, t-s;\xi)(V(\xi - a_j^-(t-s) - a_k^- s, s)\phi(\xi - a_j^-(t-s) - a_k^- s, s))_\tau d\xi\, ds \\
&\quad = \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{\mathbb{R}} g(x, t-s;\xi) V(\xi - a_j^-(t-s) - a_k^- s, s) \\
&\quad \times \phi_\tau(\xi - a_j^-(t-s) - a_k^- s, s) d\xi\, ds \\
&\quad + \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{\mathbb{R}} g(x, t-s;\xi) V_\tau(\xi - a_j^-(t-s) - a_k^- s, s) \\
&\quad \times \phi(\xi - a_j^-(t-s) - a_k^- s, s) d\xi\, ds. \qquad (6.43)
\end{aligned}$$

For the first integral on the right-hand side of (6.43), we employ the estimate (6.40) to obtain an estimate by

$$\begin{aligned}
&\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-1/2} e^{-\frac{(x-\xi)^2}{M(t-s)}} (1+s)^{-3/4} (1+s)^{-3/2} e^{-\frac{(\xi - a_j^-(t-s) - a_k^- s)^2}{Ms}} d\xi\, ds \\
&\quad \leq C t^{-1/2} \int_{\sqrt{t}}^{t-\sqrt{t}} (1+s)^{-7/4} e^{-\frac{(x - a_j^-(t-s) - a_k^- s)^2}{Mt}} ds,
\end{aligned}$$

which can be analyzed similarly as was (6.41). For the second integral on the right-hand side of (6.43), we note the following relation, also useful in calculations below,

$$\begin{aligned}
&|V_\tau(\xi - a_j^-(t-s) - a_k^- s, s)| = |(\partial_t + a_k^- \partial_x) v|(\xi - a_j^-(t-s), s) \\
&\quad \leq C s^{-1} s^{1/4} (1 + |\xi - a_j^-(t-s) - a_k^- s| + s^{1/2})^{-3/2} \\
&\qquad + C s^{-1/2} \sum_{a_l^\pm \gtrless 0, l \neq k} (1 + |\xi - a_j^-(t-s) - a_l^- s| + s^{1/2})^{-3/2} \\
&\qquad + C s^{-1/2} (1 + |\xi - a_j^-(t-s)|)^{-1/2} (1 + |\xi - a_j^-(t-s)| + s)^{-1/2} \\
&\qquad \times (1 + |\xi - a_j^-(t-s)| + s^{1/2})^{-1/2} I_{\{a_1^- s \leq (\xi - a_j^-(t-s)) \leq a_n^+ s\}} \\
&\qquad + (1 + |\xi - a_j^-(t-s)|)^{-1} (1+s)^{-1} + (1 + |\xi - a_j^-(t-s)| + s)^{-7/4}. \qquad (6.44)
\end{aligned}$$

For the first estimate in (6.44), we use a supremum norm to obtain integrals

$$\begin{aligned}
&\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-1/2} e^{-\frac{(x-\xi)^2}{M(t-s)}} s^{-1} (1+s)^{-1} e^{-\frac{(\xi - a_j^-(t-s) - a_k^- s)^2}{Ms}} d\xi\, ds \\
&\quad \leq C t^{-1/2} \int_{\sqrt{t}}^{t-\sqrt{t}} s^{-1/2} (1+s)^{-1} e^{-\frac{(x - a_j^-(t-s) - a_k^- s)^2}{Ms}} ds,
\end{aligned}$$

which can be analyzed similarly as was (6.41). For the remaining estimates in (6.44), the critical observation is that when integrated against the convecting diffusion kernel,

$$e^{-\frac{(\xi - a_j^-(t-s) - a_k^- s)^2}{Ms}},$$

they give increased decay in s, from which the claimed estimates can readily be observed. This completes the proof of (3.18)(i).

Proof of (3.18(ii)). Integration against the nonlinearity $\Phi(y,s)$ has been considered in [15], and we need only refine one estimate from that calculation in order to conclude our claim. The critical calculation regards integrals

$$t^{-1/2}\int_{t/2}^{t-\sqrt{t}} (t-s)^{-1}(1+s)^{-1/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds,$$

which was estimated in [15] by the expression,

$$(1+t)^{-3/4}(1+|x-a_k^- t|)^{-1/2}.$$

Here, we observe that alternatively, we may proceed by observing through (6.6) the inequality (6.10) with $\gamma = 1$. For the first estimate in (6.10), we have an estimate by

$$\begin{aligned} &C_2 t^{-1/2}(1+t)^{-1/2}|x-a_k^- t|^{-1}\int_{t/2}^{t-\sqrt{t}} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds \\ &\quad \le C(1+t)^{-1/2}|x-a_k^- t|^{-1}, \end{aligned}$$

while for the second we have an estimate by

$$\begin{aligned} &C_2 t^{-1/2}(1+t)^{-1} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_{t/2}^{t-\sqrt{t}} (t-s)^{-1} ds \\ &\quad \le C(1+t)^{-3/2}\ln(e+t) e^{-\frac{(x-a_k^- t)^2}{Lt}}, \end{aligned}$$

either of which is sufficient. This completes the proof of (3.18(ii)).

Proof of (3.18(iii)–(iv)), x-derivatives. As will be clear from our analysis of characteristic derivatives just below, analysis of x-derivatives is almost precisely the same as the case of characteristic derivatives for which the direction of differentiation matches neither the convection rate of the kernel or the convection rate of the nonlinearity. Since we consider the case of characteristic derivatives in great detail, we will omit the case of x-derivatives.

Proof of (3.18(v)–(vi)), characteristic derivatives. We next develop estimates on characteristic derivatives $(\partial_t + a_l^{\pm}\partial_x)$, $a_l^{\pm} \gtrless 0$, on the nonlinear interaction integrals. These estimates are the primary new contribution of the current analysis.

Observing that $\tilde{G}(x,0;y) = \delta_y(x)I$ by construction, we have, for $a_l^{\pm} \gtrless 0$,

$$
\begin{aligned}
&(\partial_t + a_l^{\pm}\partial_x)\int_0^t\int_{-\infty}^{+\infty}\tilde{G}(x,t-s;y)\left[\Phi(y,s) + \mathcal{F}\left(\varphi, v, \frac{\partial\bar{u}^{\delta}}{\partial\delta}\delta\right)_y(y,s)\right]dy\,ds\\
&\quad = \Phi(x,t) + \mathcal{F}\left(\varphi, v, \frac{\partial\bar{u}^{\delta}}{\partial\delta}\delta\right)_x(x,t)\\
&\qquad + \int_0^t\int_{-\infty}^{+\infty}(\partial_t + a_l^{\pm}\partial_x)\tilde{G}(x,t-s;y)\left[\Phi(y,s) + \mathcal{F}\left(\varphi, v, \frac{\partial\bar{u}^{\delta}}{\partial\delta}\delta\right)_y(y,s)\right]dy\,ds.
\end{aligned}
\tag{6.45}
$$

For $\zeta(t) > 0$ so that

$$
\begin{aligned}
|v(x,t)| &\le \zeta(t)(\psi_1(x,t) + \psi_2(x,t))\\
|\partial_x v(x,t)| &\le \zeta(t)\big[t^{-1/2}(\psi_1(x,t) + |\psi_2(x,t)) + \psi_3(x,t) + \psi_4(x,t)\big],
\end{aligned}
\tag{6.46}
$$

the first two terms on the right-hand side of (6.45) can be seen to be bounded by

$$
\begin{aligned}
&CE_0(1+t)^{-3/4}\sum_{a_j^{\pm}\gtrless 0}\big(1 + |x - a_j^{\pm}t| + t^{1/2}\big)^{-3/2} + \zeta(t)\big[t^{-1}(1+t)^{1/4}\psi_1^{j,\pm}(x,t)\\
&\qquad + t^{-1/2}\big(\bar{\psi}_1^{j,\pm} + \psi_2(x,t)\big) + \psi_3(x,t) + \psi_4(x,t)\big].
\end{aligned}
$$

Proof of (3.18(vi)), nonlinearity $\Phi(y,s)$. The basic elements of our proof are more easily seen in the case of estimate (3.18(vi))—involving the diffusion wave nonlinearity $\Phi(y,s)$—and so our approach will be to establish (3.18(vi)) first and return to (3.18(v)). Proceeding from the right-hand side of (6.45), with nonlinearity $\Phi(y,s)$, we have integrals of the form

$$
\int_0^t\int_{-\infty}^{+\infty}(\partial_t + a_l^{\pm}\partial_x)\tilde{G}(x,t-s;y)\Phi(y,s)dyds. \tag{6.47}
$$

In particuar, we focus on the nonlinearity $(\varphi_k^{-2})_y$, with $\varphi_k^{-}(y,s)$ as defined in (1.10), and the leading order Green's kernel,

$$
\tilde{G}^j(x,t;y) = ct^{-1/2}e^{-\frac{(x-y-a_j^{-}t)^2}{4\beta_j t}},
$$

with $c = r_j^-(l_j^-)^{\mathrm{tr}}/\sqrt{4\pi\beta_j^-}$ (we will discuss corrections to $\tilde{G}^j(x,t;y)$ at the end of the analysis). We divide the analysis into three parts,

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{+\infty} (\partial_t + a_l^- \partial_x)\tilde{G}^j(x,t-s;y)(\varphi_k^-(y,s)^2)_y\,dy\,ds \\
&\quad = \int_0^{\sqrt{t}} \int_{-\infty}^{+\infty} (\partial_t + a_l^- \partial_x)\tilde{G}^j(x,t-s;y)(\varphi_k^-(y,s)^2)_y\,dy\,ds \\
&\qquad + \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (\partial_t + a_l^- \partial_x)\tilde{G}^j(x,t-s;y)(\varphi_k^-(y,s)^2)_y\,dy\,ds \\
&\qquad + \int_{t-\sqrt{t}}^{t} \int_{-\infty}^{+\infty} (\partial_t + a_l^- \partial_x)\tilde{G}^j(x,t-s;y)(\varphi_k^-(y,s)^2)_y\,dy\,ds. \qquad (6.48)
\end{aligned}$$

According to the estimates of Proposition 2.2, the case $j = k$ does not occur (the diffusion waves have been chosen to eliminate precisely this case), leaving three cases to consider, $l = j \neq k$, $l = k \neq j$, and $l \neq j, l \neq k, j \neq k$. We remark that it is in this third case that the characteristic derivative acts similarly as a derivative with respect to x only.

Case 1: $l = j \neq k$. For the case $l = j \neq k$, the characteristic derivative of $\tilde{G}^j$ behaves like a time derivative of the heat kernel, and we have

$$|(\partial_t + a_l^- \partial_x)\tilde{G}^j(x,t-s;y)| \le C(t-s)^{-3/2} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}.$$

For the first estimate in (6.48), upon integration by parts in y, we have integrals of the form

$$\int_0^{\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-2} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-1} e^{-\frac{(y-a_k^- s)^2}{Ms}}\,dy\,ds,$$

for which we observe the equality (6.31). Integration over y leads immediately to an estimate by

$$Ct^{-1/2} \int_0^{\sqrt{t}} (t-s)^{-3/2}(1+s)^{-1/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}\,ds.$$

In the event that $|x - a_j^- t| \le K\sqrt{t}$, we immediately have decay with scaling $\exp((x - a_j^- t)^2/(Lt))$, while for $|x - a_j^- t| \ge K\sqrt{t}$, we have

$$\begin{aligned}
|x - a_j^-(t-s) - a_k^- s| &= |(x - a_j^- t) - (a_k^- - a_j^-)s| \\
&\ge \left(1 - \frac{a_k^- - a_j^-}{K}\right)|x - a_j^- t|, \qquad (6.49)
\end{aligned}$$

for which, with K taken sufficiently large, we again have decay with scaling $\exp((x-a_j^- t)^2/(Lt))$. We have, then, an estimate on this term of the form

$$C_1 t^{-2}(1+t^{1/2})^{-1/2}\int_0^{\sqrt{t}} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}\,ds \le Ct^{-3/2}(1+t^{1/2})^{-1/2}e^{-\frac{(x-a_j^- t)^2}{Lt}}, \tag{6.50}$$

where in obtaining this last inequality we have reserved a small part of the kernel $\exp(-(x-a_j^-(t-s)-a_k^- s)^2/(Mt))$ for integration. Finally, the blow-up as $t\to 0$ can be reduced by putting derivatives on the nonlinearity for small time. For the third integral in (6.48), we integrate the charactericstic derivative by parts to avoid blow-up near $s=t$. Observing the relation

$$(\partial_t + a_j^-\partial_x)\tilde{G}^j(x,t-s;y) = -(\partial_s + a_j^-\partial_y)\tilde{G}^j(x,t-s;y),$$

we have

$$\begin{aligned}
&\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty}(\partial_t + a_l^-\partial_x)\tilde{G}^j(x,t-s;y)(\varphi_k^-(y,s)^2)_y\,dy\,ds\\
&\quad=\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty}(-\partial_s - a_l^-\partial_y)\tilde{G}^j(x,t-s;y)(\varphi_k^-(y,s)^2)_y\,dy\,ds\\
&\quad=\int_{-\infty}^{+\infty}\tilde{G}^j(x,\sqrt{t};y)(\varphi_k^-(y,t-\sqrt{t})^2)_y dy\\
&\qquad-\int_{-\infty}^{+\infty}\tilde{G}^j(x,0;y)(\varphi_k^-(y,t)^2)_y\,dy\\
&\qquad+\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty}\tilde{G}^j(x,t-s;y)(\partial_s + a_l^-\partial_y)(\varphi_k^-(y,s)^2)_y\,dy\,ds.
\end{aligned}\tag{6.51}$$

For the first integral in (6.51), we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty}(\sqrt{t})^{-1/2}e^{-\frac{(x-y-a_j^-\sqrt{t})}{M\sqrt{t}}}(1+(t-\sqrt{t}))^{-3/2}e^{-\frac{(y-a_k^-(t-\sqrt{t}))^2}{M(t-\sqrt{t})}}\,dy\\
&\quad\le Ct^{-1/2}(1+(t-\sqrt{t}))^{-1}e^{-\frac{(x-a_j^-\sqrt{t}-a_k^-(t-\sqrt{t}))^2}{Mt}},
\end{aligned}$$

which gives an estimate by

$$t^{-1/2}(1+t)^{-1}e^{-\frac{(x-a_k^- t)^2}{Lt}},$$

sufficient by an argument similar to (6.49). For the second integral in (6.51), $\tilde{G}^j(x,0;y)$ is a delta function, over which integration yields an estimate by

$$C(1+t)^{-3/2}e^{-\frac{(x-a_k^- t)^2}{Mt}}.$$

For the third integral in (6.51), we have, upon noting that the characteristic derivative is not along the direction of propagation of φ_k^-, integrals of the form

$$\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty}(t-s)^{-1/2}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-3/2}e^{-\frac{(y-a_k^- s)^2}{Ms}}\,dy\,ds$$
$$\leq Ct^{-1/2}\int_{t-\sqrt{t}}^{t}(1+s)^{-3/2}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}\,ds.$$

Proceeding similarly as in (6.49) and the surrounding estimates, we obtain an estimate by

$$C(1+t)^{-3/2}e^{-\frac{(x-a_j^- t)^2}{Lt}},$$

which has been shown sufficient in (6.1). For the second integral in (6.48), we must proceed by taking advantage of increased decay for derivatives along characteristic directions. Recalling our definitions (6.33), we can write this integral in the form

$$\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}g_\tau(x,t-s;y+a_j^-(t-s))(\phi(y-a_k^- s,s)^2)_y\,dy\,ds,$$

which upon the substitution $\xi = y + a_j^-(t-s)$ becomes

$$\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}g_\tau(x,t-s;\xi)(\phi(\xi-a_j^-(t-s)-a_k^- s,s)^2)_\xi\,d\xi\,ds.$$

Proceeding similarly as in (6.37), we write

$$\begin{aligned}(g_\tau(x,t-s;\xi)&\phi(\xi-a_j^-(t-s)-a_k^- s,s)^2)_s\\ &= -g_{\tau\tau}(x,t-s;\xi)\phi(\xi-a_j^-(t-s)-a_k^- s,s)^2\\ &\quad+(a_j^- - a_k^-)g_\tau(x,t-s;\xi)(\phi(\xi-a_j^-(t-s)-a_k^- s,s)^2)_\xi\\ &\quad+g_\tau(x,t-s;\xi)(\phi(\xi-a_j^-(t-s)-a_k^- s,s)^2)_\tau,\end{aligned}\tag{6.52}$$

which can be rearranged as

$$\begin{aligned}g_\tau(x,t-s;\xi)&(\phi(\xi-a_j^-(t-s)-a_k^- s,s)^2)_\xi\\ &=(a_j^- - a_k^-)^{-1}(g_\tau(x,t-s;\xi)\phi(\xi-a_j^-(t-s)-a_k^- s,s)^2)_s\\ &\quad+(a_j^- - a_k^-)^{-1}g_{\tau\tau}(x,t-s;\xi)\phi(\xi-a_j^-(t-s)-a_k^- s,s)^2\\ &\quad-(a_j^- - a_k^-)^{-1}g_\tau(x,t-s;\xi)(\phi(\xi-a_j^-(t-s)-a_k^- s,s)^2)_\tau,\end{aligned}\tag{6.53}$$

where we recall again that the case $j = k$ does not occur here. For integration over the first summand on the right-hand side of (6.53), we exchange order of integration to obtain

$$\begin{aligned}\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}&(g_\tau(x,t-s;\xi)\phi(\xi-a_j^-(t-s)-a_k^-s,s)^2)_s\,d\xi\,ds\\&=\int_{-\infty}^{+\infty}g_\tau(x,\sqrt{t};\xi)\phi(\xi-a_j^-\sqrt{t}-a_k^-(t-\sqrt{t}),t-\sqrt{t})^2d\xi\\&\quad-\int_{-\infty}^{+\infty}g_\tau(x,t-\sqrt{t};\xi)\phi(\xi-a_j^-(t-\sqrt{t})-a_k^-\sqrt{t},\sqrt{t})^2d\xi.\end{aligned}\tag{6.54}$$

For the first integral in (6.54), proceeding similarly as in (6.31), we estimate

$$\begin{aligned}\int_{-\infty}^{+\infty}&(\sqrt{t})^{-3/2}e^{-\frac{(x-\xi)^2}{M\sqrt{t}}}(1+(t-\sqrt{t}))^{-1}e^{-\frac{(\xi-a_j^-\sqrt{t}-a_k^-(t-\sqrt{t}))^2}{M(t-\sqrt{t})}}d\xi\\&\le Ct^{-1/2}(\sqrt{t})^{-1}(1+(t-\sqrt{t}))^{-1/2}e^{-\frac{(x-a_j^-\sqrt{t}-a_k^-(t-\sqrt{t}))^2}{Mt}},\end{aligned}$$

for which we conclude in a manner similar to (6.32) an estimate by

$$Ct^{-1}(1+t)^{-1/2}e^{-\frac{(x-a_k^-t)^2}{Lt}}.$$

Proceeding similarly for the second integral in (6.54), we estimate

$$\begin{aligned}\int_{-\infty}^{+\infty}&(t-\sqrt{t})^{-3/2}e^{-\frac{(x-\xi)^2}{M(t-\sqrt{t})}}(1+\sqrt{t})^{-1}e^{-\frac{(\xi-a_j^-(t-\sqrt{t})-a_k^-\sqrt{t})^2}{M\sqrt{t}}}d\xi\\&\le Ct^{-1/2}(t-\sqrt{t})^{-1}(1+\sqrt{t})^{-1/2}e^{-\frac{(x-a_j^-(t-\sqrt{t})-a_k^-\sqrt{t})^2}{Mt}},\end{aligned}$$

for which we conclude an estimate by

$$Ct^{-3/2}(1+t)^{-1/4}e^{-\frac{(x-a_j^-t)^2}{Lt}}.$$

For integration over the second summand on the right-hand side of (6.53), we estimate

$$\begin{aligned}\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}&(t-s)^{-5/2}e^{-\frac{(x-\xi)^2}{M(t-s)}}(1+s)^{-1}e^{-\frac{(\xi-a_j^-(t-s)-a_k^-s)^2}{Ms}}d\xi\,ds\\&\le Ct^{-1/2}\int_{\sqrt{t}}^{t-\sqrt{t}}(t-s)^{-2}(1+s)^{-1/2}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds.\end{aligned}$$

For this last integral, we have three cases to consider, $a_k^-<0<a_j^-$, $a_k^-<a_j^-<0$, and $a_j^-<a_k^-<0$, of which we focus on the second. In the case $|x|\ge|a_k^-|t$, there is no cancellation between summands in (6.6), and we have an estimate by

$$\begin{aligned}C_1t^{-5/2}(1+\sqrt{t})^{-1/2}e^{-\frac{(x-a_k^-t)^2}{Lt}}&\int_{\sqrt{t}}^{t/2}e^{-\epsilon\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds\\&+C_2t^{-1/2}(\sqrt{t})^{-2}(1+t)^{-1/2}e^{-\frac{(x-a_k^-t)^2}{Lt}}\int_{t/2}^{t-\sqrt{t}}e^{-\epsilon\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds\\\le Ct^{-3/2}e^{-\frac{(x-a_k^-t)^2}{Lt}},&\end{aligned}\tag{6.55}$$

while similarly in the case $|x| \le |a_j^-|t$, there is no cancellation between summands in (6.8), and we obtain an estimate by

$$Ct^{-3/2}e^{-\frac{(x-a_j^- t)^2}{Lt}}.$$

For the case $|a_j^-|t \le |x| \le |a_k^-|t$, we divide the analysis into cases $s \in [\sqrt{t}, t/2]$ and $s \in [t/2, t-\sqrt{t}]$ (we take t large enough so that $\sqrt{t} < t/2$, proceeding alternatively for t bounded, in which we need not establish t decay). For $s \in [\sqrt{t}, t/2]$, we observe through (6.8) the inequality (6.42) with $\gamma = 1/2$. For the first estimate in (6.42), we proceed as in (6.55), while for the second we have an estimate by

$$\begin{aligned} &C_1 t^{-5/2}(1+|x-a_j^- t|)^{-1/2}\int_{\sqrt{t}}^{t/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}\,ds \\ &\quad \le Ct^{-3/2}(1+t)^{-1/2}(1+|x-a_j^- t|)^{-1/2}. \end{aligned}$$

For $s \in [t/2, t-\sqrt{t}]$, we observe through (6.6) the inequality (6.10) with $\gamma = 3/2$. For the second estimate in (6.10), we proceed as in (6.55), while for the first we obtain an estimate by

$$\begin{aligned} &C_2 t^{-1/2}(1+t)^{-1/2}|x-a_k^- t|^{-3/2}\int_{t/2}^{t-\sqrt{t}} (t-s)^{-1/2}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}\,ds \\ &\quad \le Ct^{-1/4}(1+t)^{-1/2}|x-a_k^- t|^{-3/2}, \end{aligned}$$

which is better than the required estimate along the a_k^- directions (since $j \ne k$). For integration over the third summand on the right-hand side of (6.53), we estimate

$$\begin{aligned} &\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty} (t-s)^{-3/2}e^{-\frac{(x-\xi)^2}{M(t-s)}}(1+s)^{-2}e^{-\frac{(\xi-a_j^-(t-s)-a_k^- s)^2}{Ms}}\,d\xi\,ds \\ &\quad \le Ct^{-1/2}\int_{\sqrt{t}}^{t-\sqrt{t}} (t-s)^{-1}(1+s)^{-3/2}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}\,ds, \end{aligned}$$

for which we can proceed almost exactly as in our analysis of the second summand on the right-hand side of (6.53) (the full analysis is omitted).

Case 2: $l = k \ne j$. For the case $l = k \ne j$, the characteristic derivative of φ_k^- behaves like a time derivative of the heat kernel, and we have

$$(\partial_t + a_k^-\partial_x)\varphi_k^-(y,s) = \mathbf{O}((1+s)^{-3/2})e^{-\frac{(y-a_k^- s)^2}{Ms}}.$$

In general, our strategy for this case will be to integrate by parts, shifting the characteristic derivative onto φ. For the first integral in (6.48), after integration by parts in y, we estimate

$$\begin{aligned} &\int_0^{\sqrt{t}}\int_{-\infty}^{+\infty} (t-s)^{-3/2}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-1}e^{-\frac{(y-a_k^- s)^2}{Ms}}\,dy\,ds \\ &\quad \le Ct^{-1/2}\int_0^{\sqrt{t}} (t-s)^{-1}(1+s)^{-1/2}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}\,ds, \end{aligned}$$

which is bounded by

$$Ct^{-1/2}(1+t)^{-3/4}e^{-\frac{(x-a_j^- t)^2}{Lt}},$$

sufficient for $j \neq l$ by the argument of (6.1). For the third integral in (6.48), we have

$$\begin{aligned}
&\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty}(\partial_t + a_k^-\partial_x)\tilde{G}^j(x,t;y)\partial_y(\varphi_k^-(y,s)^2)dy\,ds\\
&\quad = -\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty}(\partial_s + a_k^-\partial_y)\tilde{G}^j(x,t;y)\partial_y(\varphi_k^-(y,s)^2)dy\,ds\\
&\quad = -\int_{-\infty}^{+\infty}\tilde{G}^j(x,0;y)\partial_y(\varphi_k^-(y,t))^2 dy\\
&\qquad + \int_{-\infty}^{+\infty}\tilde{G}^j(x,\sqrt{t};y)\partial_y(\varphi_k^-(y,t-\sqrt{t}))^2 dy\\
&\qquad - \int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty}\partial_y\tilde{G}^j(x,t-s;y)(\partial_s + a_k^-\partial_y)((\varphi_k^-)^2)dy\,ds. \qquad (6.56)
\end{aligned}$$

For the first integral in (6.56), we have an estimate by

$$|\varphi_k^-(x,t)\partial_x\varphi_k^-(x,t)| \le C(1+t)^{-3/2}e^{-\frac{(x-a_k^- t)^2}{Lt}},$$

while for the second we estimate

$$\begin{aligned}
&\int_{-\infty}^{+\infty}(\sqrt{t})^{-1/2}e^{-\frac{(x-y-a_j^-\sqrt{t})^2}{M\sqrt{t}}}(1+(t-\sqrt{t}))^{-3/2}e^{-\frac{(y-a_k^-(t-\sqrt{t}))}{M(t-\sqrt{t})^2}}dy\\
&\quad \le Ct^{-1/2}(1+(t-\sqrt{t}))^{-1}e^{-\frac{(x-a_j^-\sqrt{t}-a_k^-(t-\sqrt{t}))^2}{M(t-\sqrt{t})}}, \qquad (6.57)
\end{aligned}$$

which gives an estimate by

$$t^{-1/2}(1+t)^{-1}e^{-\frac{(x-a_k^- t)^2}{Lt}}.$$

For the third integral on the right-hand side of (6.56), we estimate

$$\begin{aligned}
&\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty}(t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-2}e^{-\frac{(y-a_k^- s)^2}{Ms}}dy\,ds\\
&\quad \le Ct^{-1/2}\int_{t-\sqrt{t}}^{t}(t-s)^{-1/2}(1+s)^{-3/2}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}ds,
\end{aligned}$$

which gives an estimate by

$$t^{-1/4}(1+t)^{-3/2}e^{-\frac{(x-a_k^- t)^2}{Lt}}.$$

For the second integral in (6.48), we first integrate by parts, moving the characteristic derivative onto φ^k, and then proceed by a Liu-type characteristic derivative estimate as described in Sec. 4. We have

$$\begin{aligned}
&\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (\partial_t + a_k^- \partial_x) \tilde{G}^j(x,t;y) \partial_y(\varphi^k(y,s)^2) dy\, ds \\
&\quad = -\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (\partial_s + a_k^- \partial_y) \tilde{G}^j(x,t;y) \partial_y((\varphi^k)^2) dy\, ds \\
&\quad = -\int_{-\infty}^{+\infty} \tilde{G}^j(x,\sqrt{t};y) \partial_y(\varphi^k(y,t-\sqrt{t}))^2 dy \\
&\qquad - \int_{-\infty}^{+\infty} \tilde{G}^j_y(x,t-\sqrt{t};y) \varphi^k(y,\sqrt{t})^2 dy \\
&\qquad + \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} \tilde{G}^j(x,t-s;y)(\partial_s + a_k^- \partial_y)(\varphi^k(y,s)^2)_y\, dy\, ds. \qquad (6.58)
\end{aligned}$$

For the first integral on the right-hand side of (6.58), we proceed exactly as in (6.57) to obtain an estimate by

$$t^{-1/2}(1+t)^{-1} e^{-\frac{(x-a_k^- t)^2}{Lt}},$$

while for the second we estimate

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (t-\sqrt{t})^{-1} e^{-\frac{(x-y-a_j^-(t-\sqrt{t}))^2}{M(t-\sqrt{t})}} (1+\sqrt{t})^{-1} e^{-\frac{(y-a_k^-\sqrt{t})^2}{M\sqrt{t}}} dy \\
&\quad \le C t^{-1/2} (t-\sqrt{t})^{-1/2} (1+\sqrt{t})^{-1} (\sqrt{t})^{1/2} e^{-\frac{(x-a_j^-(t-\sqrt{t})-a_k^-\sqrt{t})^2}{Mt}},
\end{aligned}$$

which gives an estimate by

$$t^{-3/4}(1+t)^{-1/2} e^{-\frac{(x-a_j^- t)^2}{Lt}}.$$

An argument similar to that of (6.1) shows that this last estimate is sufficient for $j \neq l$, which is the current setting. For the third integral on the right-hand side of (6.58), we proceed in terms of the non-convecting variables (6.33), for which the integral can be re-written as

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} g(x,t-s;y+a_j^-(t-s))(\partial_\tau(\phi(y-a_k^- s,s)^2))_y\, dy\, ds,$$

where we recall that ∂_τ represents differentiation with respect to the second argument of ϕ. Setting $\xi = y + a_j^-(t-s)$, this last integral becomes

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} g(x,t-s;\xi)\big(\partial_\tau(\phi(\xi - a_j^-(t-s) - a_k^- s, s)^2)\big)_\xi\, d\xi\, ds,$$

for which we write

$$\begin{aligned}
&(g(x,t-s;\xi)\partial_\tau(\phi(\xi - a_j^-(t-s) - a_k^- s, s)^2))_s \\
&\quad = -g_\tau(x,t-s;\xi)\partial_\tau(\phi(\xi - a_j^-(t-s) - a_k^- s, s)^2) \\
&\qquad + (a_j^- - a_k^-)g(x,t-s;\xi)(\partial_\tau(\phi(\xi - a_j^-(t-s) - a_k^- s, s)^2))_\xi \\
&\qquad + g(x,t-s;\xi)\partial_{\tau\tau}(\phi(\xi - a_j^-(t-s) - a_k^- s, s)^2).
\end{aligned} \tag{6.59}$$

For integration over the left-hand side of (6.59), we exchange the order of integration to obtain

$$\begin{aligned}
&\int_{-\infty}^{+\infty} g(x,\sqrt{t};\xi)\partial_\tau(\phi(\xi - a_j^-\sqrt{t} - a_k^-(t-\sqrt{t}), t-\sqrt{t})^2)d\xi \\
&\quad - \int_{-\infty}^{+\infty} g(x,t-\sqrt{t};\xi)\partial_\tau(\phi(\xi - a_j^-(t-\sqrt{t}) - a_k^-\sqrt{t}, \sqrt{t})^2)d\xi.
\end{aligned} \tag{6.60}$$

For the first expression in (6.60), we estimate

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (\sqrt{t})^{-1/2} e^{-\frac{(x-\xi)^2}{M\sqrt{t}}} (1 + (t-\sqrt{t}))^{-2} e^{-\frac{(\xi - a_j^-\sqrt{t} - a_k^-(t-\sqrt{t}))^2}{M(t-\sqrt{t})}} d\xi \\
&\quad \le Ct^{-1/2}(1+t)^{-3/2} e^{-\frac{(x - a_j^-\sqrt{t} - a_k^-(t-\sqrt{t}))^2}{Mt}},
\end{aligned}$$

which gives an estimate by

$$Ct^{-1/2}(1+t)^{-3/2} e^{-\frac{(x-a_k^- t)^2}{Lt}}.$$

For the second integral in (6.60), we estimate

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (t-\sqrt{t})^{-1/2} e^{-\frac{(x-\xi)^2}{M(t-\sqrt{t})}} (1+\sqrt{t})^{-2} e^{-\frac{(\xi - a_j^-(t-\sqrt{t}) - a_k^-\sqrt{t})^2}{M\sqrt{t}}} d\xi \\
&\quad \le Ct^{-1/2}(1+\sqrt{t})^{-3/2} e^{-\frac{(x - a_j^-(t-\sqrt{t}) - a_k^-\sqrt{t})^2}{Mt}},
\end{aligned}$$

which gives an estimate by

$$Ct^{-1/2}(1+t)^{-3/4} e^{-\frac{(x-a_j^- t)^2}{Lt}},$$

sufficient for $l \neq j$. For integration over the first term on the right-hand side of (6.59), we estimate

$$\begin{aligned}
&\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-3/2} e^{-\frac{(x-\xi)^2}{M(t-s)}} (1+s)^{-2} e^{-\frac{(\xi - a_j^-(t-s) - a_k^- s)^2}{Ms}} d\xi\, ds \\
&\quad \le Ct^{-1/2} \int_{\sqrt{t}}^{t-\sqrt{t}} (t-s)^{-1}(1+s)^{-3/2} e^{-\frac{(x - a_j^-(t-s) - a_k^- s)^2}{Mt}} ds.
\end{aligned} \tag{6.61}$$

We have three cases to consider here, $a_k^- < 0 < a_j^-$, $a_k^- < a_j^- < 0$, and $a_j^- < a_k^- < 0$, of which we focus on the second. In the case $|x| \ge |a_k^-|t$, we observe that there is no cancellation between summands in (6.6), and consequently we obtain an estimate by

$$\begin{aligned} &C_1 t^{-3/2} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_{\sqrt{t}}^{t/2} (1+s)^{-3/2} ds \\ &\quad + C_2 t^{-1/2}(1+t)^{-3/2} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_{t/2}^{t-\sqrt{t}} (t-s)^{-1} ds \\ &\quad \leq C t^{-1/2}(1+t)^{-1} e^{-\frac{(x-a_k^- t)^2}{Lt}}. \end{aligned} \tag{6.62}$$

Similarly, for $|x| \leq |a_j^-|t$, there is no cancellation between summands in (6.8), and the same calculation gives an estimate by

$$C t^{-1/2}(1+t)^{-1} e^{-\frac{(x-a_j^- t)^2}{Lt}}.$$

For $|a_j^-|t \leq |x| \leq |a_k^-|t$, we divide the analysis into cases $s \in [\sqrt{t}, t/2]$ and $s \in [t/2, t-\sqrt{t}]$. For $s \in [\sqrt{t}, t/2]$, we observe through (6.8) the inequality (6.42) with $\gamma = 3/2$. For the first estimate in (6.42), we proceed as in (6.62), while for the second, we obtain an estimate by

$$\begin{aligned} &C_1 t^{-3/2}(1+|x-a_j^- t|)^{-3/2} \int_{\sqrt{t}}^{t/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds \\ &\quad \leq C t^{-1/2}(1+t)^{-1/2}(1+|x-a_j^- t|)^{-3/2}, \end{aligned}$$

which is sufficient. For $s \in [t/2, t-\sqrt{t}]$, we observe through (6.6) the estimate (6.10) with $\gamma = 1$. For the second estimate in (6.10), we proceed as in (6.62), while for the first, we have an estimate by

$$\begin{aligned} &C_2 t^{-1/2}|x-a_k^- t|^{-1}(1+t)^{-3/2} \int_{t/2}^{t-\sqrt{t}} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds \\ &\quad \leq C(1+t)^{-3/2}|x-a_k^- t|^{-1}, \end{aligned}$$

which is sufficient since we are in the case $t \geq |x|/|a_k^-|$. For integration over the third term on the right-hand side of (6.59), we estimate

$$\begin{aligned} &\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-1/2} e^{-\frac{(x-\xi)^2}{M(t-s)}} (1+s)^{-3} e^{-\frac{(\xi-a_j^-(t-s)-a_k^- s)^2}{Ms}} d\xi\, ds \\ &\quad \leq C t^{-1/2} \int_{\sqrt{t}}^{t-\sqrt{t}} (1+s)^{-5/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds, \end{aligned} \tag{6.63}$$

which can be analyzed similarly as in the immediately preceeding case.

Case 3: $l \neq j$, $l \neq k$, $k \neq j$. For the case $l \neq j$, $l \neq k$, $k \neq j$, we proceed directly in the first and third integrals of (6.48) and through the method of Liu described in Sec. 4 and elsewhere for the second integral in (6.48). In each case, we obtain precisely the reduced decay estimate for characteristic directions other than a_l^-. This concludes the proof of Estimate (3.18(vi)).

Proof of (3.18(v)), nonlinearity $\mathcal{F}$. The analysis of characteristic derivatives in the case of nonlinearity $\mathcal{F}$ constitutes the single most involved section of the paper.

(3.18(v)), *Nonlinearity* $\varphi(y,s)v(y,s)$. The critical nonlinearity of $\mathcal{F}$ is $\varphi(y,s)v(y,s)$, for which we consider integrals

$$\int_0^t \int_{-\infty}^{+\infty} (\partial_t + a_l^{\pm}\partial_x)\tilde{G}^j(x,t-s;y)(\varphi_k^{\pm}(y,s)v(y,s))_y dy ds. \tag{6.64}$$

Case 1: $l=j$. For the case $l=j$, and for $s\in[0,t/2]$, we integrate by parts in y and employ a supremum norm on $|v|$ to arrive at integrals

$$\begin{aligned}\int_0^{t/2}\int_{-\infty}^{+\infty} &(t-s)^{-2}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-5/4}e^{-\frac{(y-a_k^-s)^2}{Ms}}dy\,ds \\ &\le Ct^{-1/2}\int_0^{t/2}(t-s)^{-3/2}(1+s)^{-3/4}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds. \end{aligned}\tag{6.65}$$

We have three cases to consider, $a_k^- < 0 < a_j^-$, $a_k^- \le a_j^- < 0$, and $a_j^- < a_k^- < 0$, of which we focus on the second. (We observe that for the nonlinearity $\varphi(y,s)v(y,s)$, the case $a_j^- = a_k^-$ arises.) For the case $a_k^- \le a_j^- < 0$, we first observe that in the case $|x|\ge|a_k^-|t$, there is no cancellation between summands in (6.6), and proceeding similarly as in (6.62) we determine an estimate by

$$Ct^{-2}(1+t)^{1/4}e^{-\frac{(x-a_k^-t)^2}{Lt}}.$$

In the case $|x|\le|a_j^-|t$, there is no cancellation between summands in (6.8) and we obtain an estimate by

$$Ct^{-2}(1+t)^{1/4}e^{-\frac{(x-a_j^-t)^2}{Lt}}.$$

We note in particular that the case $a_j^- = a_k^-$ has been accomodated in this analysis. For the critical case $|a_j^-|t\le|x|\le|a_k^-|t$ (now $j\ne k$), we observe through (6.8) the inequality (6.42) with $\gamma=3/4$. For the first estimate in (6.42), we proceed similarly as in (6.62), while for the second, we have an estimate by

$$\begin{aligned}C_1t^{-2}(1+|x-a_j^-t|)^{-3/4}&\int_0^{t/2}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds \\ &\le Ct^{-1}(1+t)^{-1/2}(1+|x-a_j^-t|)^{-3/4},\end{aligned}$$

which is (precisely) sufficient for $t\ge|x|/|a_k^-|$. For $s\in[t/2,t-\sqrt{t}]$, we do not integrate by parts in y, and consequently obtain integrals

$$\begin{aligned}\int_{t/2}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}&(t-s)^{-3/2}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}s^{-1/2}(1+s)^{-5/4}e^{-\frac{(y-a_k^-s)^2}{Ms}}dy\,ds \\ &\le Ct^{-1/2}\int_{t/2}^{t-\sqrt{t}}(t-s)^{-1}(1+s)^{-5/4}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds,\end{aligned}\tag{6.66}$$

the last of which can be analyzed similarly as was (6.65). We note that the form of the nonlinearity in (6.66) arises from a supremum norm on v (in the case that φ is differentiated) and from the observation that the estimates on v_y that do not decay at rate $s^{-5/4}$ have spatial decay different from that of φ so that when multiplied by φ the combination decays at rate $s^{-5/4}$ or better. In the final case, $s \in [t-\sqrt{t}, t]$, the expression $(t-s)^{-3/2}$ is not integrable up to $s = t$, and we must proceed by integrating the characteristic derivative by parts. We have

$$
\begin{aligned}
&\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty} (\partial_t + a_l^- \partial_x)\tilde{G}^j(x, t-s; y)(\varphi_k^-(y,s)v(y,s))_y\, dy\, ds \\
&\quad = \int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty} (-\partial_s - a_l^- \partial_y)\tilde{G}^j(x, t-s; y)(\varphi_k^-(y,s)v(y,s))_y\, dy\, ds \\
&\quad = \int_{-\infty}^{+\infty} \tilde{G}^j(x, 0; y)(\varphi_k^-(y,t)v(y,t))_y\, dy \\
&\qquad - \int_{-\infty}^{+\infty} \tilde{G}^j(x, \sqrt{t}; y)(\varphi_k^-(y, t-\sqrt{t})v(y, t-\sqrt{t}))_y\, dy \\
&\qquad - \int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty} \tilde{G}_y^j(x, t-s; y)(\partial_s + a_l^- \partial_y)(\varphi_k^-(y,s)v(y,s))dy\, ds. \qquad (6.67)
\end{aligned}
$$

For the first integral on the right-hand side of (6.67), we have the immediate estimates

$$
\begin{aligned}
|\varphi_x^k(x,t)v(x,t)| &\le CE_0\zeta(t)(1+t)^{-7/4}e^{-\frac{(x-a_k^- t)^2}{Lt}} \\
|\varphi^k(x,t)v_x(x,t)| &\le CE_0\zeta(t)t^{-1/2}(1+t)^{-5/4}e^{-\frac{(x-a_k^- t)^2}{Lt}},
\end{aligned}
$$

where the second of these follows from the observation above that the combination $\varphi^k(x,t)v_x(x,t)$ decays at a rate faster than a product of the supremum norms. For the second integral on the right-hand side of (6.67), we estimate

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} (\sqrt{t})^{-1/2}e^{-\frac{(x-y-a_j^-\sqrt{t})^2}{M\sqrt{t}}}(t-\sqrt{t})^{-1/2}(1+(t-\sqrt{t}))^{-5/4}e^{-\frac{(y-a_k^-(t-\sqrt{t}))^2}{M(t-\sqrt{t})}}\, dy \\
&\quad \le Ct^{-1/2}(1+t)^{-5/4}e^{-\frac{(x-a_j^-\sqrt{t}-a_k^-(t-\sqrt{t}))^2}{Mt}},
\end{aligned}
$$

which gives an estimate by

$$
Ct^{-1/2}(1+t)^{-5/4}e^{-\frac{(x-a_k^- t)^2}{Lt}}.
$$

For the third integral on the right-hand side of (6.67), we have two nonlinearities to consider. We begin with integrals

$$
\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty} \tilde{G}_y^j(x, t-s; y)v(y,s)(\partial_s + a_j^- \partial_y)\varphi^k(y,s)dy\, ds, \qquad (6.68)
$$

for which we estimate

$$\begin{aligned}\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty}&(t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-7/4}e^{-\frac{(y-a_k^- s)^2}{Ms}}dy\,ds\\&\le Ct^{-1/2}\int_{t-\sqrt{t}}^{t}(t-s)^{-1/2}(1+s)^{-7/4}s^{1/2}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}ds,\end{aligned}$$

which gives an estimate by

$$C(1+t)^{-3/2}e^{-\frac{(x-a_k^- t)^2}{Lt}}.$$

For the second nonlinearity, we have integrals

$$\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty}\tilde{G}_y^j(x,t-s;y)\varphi^k(y,s)(\partial_s+a_j^-\partial_y)v(y,s)dy\,ds. \tag{6.69}$$

Observing again that the combination $\varphi^k(y,s)(\partial_s+a_j^-\partial_y)v(y,s)$ decays at rate $s^{-1}(1+s)^{-3/4}$, we see that we can proceed as in the previous case.

Case 2: $l=k\neq j$. For the case $l=k$, we divide the analysis as in (6.48) into integrals

$$\begin{aligned}\int_0^t\int_{-\infty}^{+\infty}&(\partial_t+a_l^-\partial_x)\tilde{G}^j(x,t-s;y)(\varphi_k^-(y,s)v(y,s))_y\,dy\,ds\\&=\int_0^{\sqrt{t}}\int_{-\infty}^{+\infty}(\partial_t+a_l^-\partial_x)\tilde{G}^j(x,t-s;y)(\varphi_k^-(y,t)v(y,s))_y\,dy\,ds\\&\quad+\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}(\partial_t+a_l^-\partial_x)\tilde{G}^j(x,t-s;y)(\varphi_k^-(y,t)v(y,s))_y\,dy\,ds\\&\quad+\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty}(\partial_t+a_l^-\partial_x)\tilde{G}^j(x,t-s;y)(\varphi_k^-(y,s)v(y,s))_y\,dy\,ds.\end{aligned}\tag{6.70}$$

For the first integral in (6.70), upon integration by parts in y, we estimate

$$\begin{aligned}\int_0^{\sqrt{t}}\int_{-\infty}^{+\infty}&(t-s)^{-3/2}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-5/4}e^{-\frac{(y-a_k^- s)^2}{Ms}}dy\,ds\\&\le Ct^{-1/2}\int_0^{\sqrt{t}}(t-s)^{-1}(1+s)^{-5/4}s^{1/2}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}ds,\end{aligned}$$

which gives an estimate by

$$Ct^{-1/2}(1+t)^{-3/4}e^{-\frac{(x-a_j^- t)^2}{Lt}},$$

sufficient for $l \neq j$. For the third integral in (6.70), we do not integrate by parts in y, and consequently we can estimate

$$\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty}(t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}s^{-1/2}(1+s)^{-5/4}e^{-\frac{(y-a_k^- s)^2}{Ms}}\,dy\,ds$$
$$\leq Ct^{-1/2}\int_{t-\sqrt{t}}^{t}(t-s)^{-1/2}(1+s)^{-5/4}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}\,ds,$$

which gives an estimate by

$$Ct^{-1/2}(1+t)^{-1}e^{-\frac{(x-a_k^- t)^2}{Lt}},$$

which is sufficient. For the second estimate in (6.70), we integrate by parts as in (6.67), moving the characteristic derivative onto the nonlinearity, and, when appropriate, moving the y derivative onto $\tilde{G}^j$. We have

$$\begin{aligned}
&\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}(\partial_t + a_k^-\partial_x)\tilde{G}^j(x,t-s;y)(\varphi_k^-(y,s)v(y,s))_y\,dy\,ds\\
&\quad=\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}(-\partial_s - a_k^-\partial_y)\tilde{G}^j(x,t-s;y)(\varphi_k^-(y,s)v(y,s))_y\,dy\,ds\\
&\quad=-\int_{-\infty}^{+\infty}\tilde{G}^j(x,\sqrt{t};y)(\varphi_k^-(y,t-\sqrt{t})v(y,t-\sqrt{t}))_y\,dy\\
&\qquad-\int_{-\infty}^{+\infty}\tilde{G}^j_y(x,t-\sqrt{t};y)\varphi_k^-(y,\sqrt{t})v(y,\sqrt{t})dy\\
&\qquad-\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}\tilde{G}^j_y(x,t-s;y)(\partial_s + a_k^-\partial_y)(\varphi_k^-(y,s)v(y,s))dy\,ds. \qquad (6.71)
\end{aligned}$$

For the first integral on the right-hand side of (6.71), we proceed precisely as with the second term in (6.67). For the second integral on the right-hand side of (6.71), we estimate

$$\int_{-\infty}^{+\infty}(t-\sqrt{t})^{-1}e^{-\frac{(x-y-a_j^-(t-\sqrt{t}))^2}{M(t-\sqrt{t})}}(1+\sqrt{t})^{-5/4}e^{-\frac{(y-a_k^2\sqrt{t})^2}{M\sqrt{t}}}\,dy$$
$$\leq Ct^{-1}(1+\sqrt{t})^{-3/4}e^{-\frac{(x-a_j^-(t-\sqrt{t})-a_k^-\sqrt{t})^2}{Mt}},$$

which gives an estimate by

$$Ct^{-1}(1+t)^{-3/8}e^{-\frac{(x-a_j^- t)^2}{Lt}},$$

suffificient for $l \neq j$. For the third integral on the right hand side of (6.71), we have two integrals to consider,

$$\begin{aligned}
&\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}\tilde{G}^j_y(x,t-s;y)v(y,s)(\partial_s + a_k^-\partial_y)\varphi_k^-(y,s)dy\,ds\\
&\quad+\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}\tilde{G}^j_y(x,t-s;y)\varphi_k^-(y,s)(\partial_s + a_k^-\partial_y)v(y,s)dy\,ds. \qquad (6.72)
\end{aligned}$$

For the first integral in (6.72), we employ the supremum norm on $|v(y,s)|$ and estimate

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-9/4} e^{-\frac{(y-a_k^- s)^2}{Ms}} \, dy \, ds$$
$$\leq Ct^{-1/2} \int_{\sqrt{t}}^{t-\sqrt{t}} (t-s)^{-1/2} (1+s)^{-7/4} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds. \quad (6.73)$$

We have three cases to consider for this last integral, $a_k^- < 0 < a_j^-$, $a_k^- < a_j^- < 0$, and $a_j^- < a_k^- < 0$, of which we focus on the second (we have already considered the case $j = k$). For $|x| \geq |a_k^-|t$, there is no cancellation between summands in (6.6), and we obtain an estimate by

$$C(1+t)^{-3/2} e^{-\frac{(x-a_k^- t)^2}{Lt}},$$

while for $|x| \leq |a_j^- t|$, there is no cancellation between summands in (6.8) and we similarly obtain an estimate by

$$C(1+t)^{-11/8} e^{-\frac{(x-a_j^- t)^2}{Lt}},$$

sufficient for $l \neq j$. For the critical case $|a_j^-|t \leq |x| \leq |a_k^-|t$, we divide the analysis into cases, $s \in [\sqrt{t}, t/2]$ and $s \in [t/2, t-\sqrt{t}]$. For $s \in [\sqrt{t}, t/2]$ we observe through (6.8) the inequality (6.42) with $\gamma = 3/2$. For the first estimate in (6.42), we proceed as above to obtain an estimate by

$$C(1+t)^{-11/8} e^{-\frac{(x-a_j^- t)^2}{Lt}},$$

while for the second we estimate

$$Ct^{-1}(1+|x-a_j^- t|)^{-3/2} \int_{\sqrt{t}}^{t/2} (1+s)^{-1/4} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds$$
$$\leq C(1+t)^{-5/8}(1+|x-a_j^- t|)^{-3/2},$$

sufficient for $l \neq j$. For $s \in [t/2, t-\sqrt{t}]$, we observe through (6.6) the inequality (6.10) with $\gamma = 1/2$. For the second estimate in (6.10), we immediately obtain an estimate by

$$C(1+t)^{-3/2} e^{-\frac{(x-a_k^- t)^2}{Lt}},$$

precisely as required, while for the first we estimate

$$Ct^{-1/2}(1+t)^{-7/4}|x-a_k^- t|^{-1/2} \int_{t/2}^{t-\sqrt{t}} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds$$
$$\leq C(1+t)^{-7/4}|x-a_k^- t|^{-1/2},$$

which is sufficient since $t \geq |x|/|a_k^-|$. For the second integral in (6.72), we have five estimates on $|(\partial_s + a_k^- \partial_y)v|$ to consider (see (6.44)), beginning with integrals

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} s^{-1}(1+s)^{-1/4} \times (1+|y-a_k^- s| + s^{1/2})^{-3/2} e^{-\frac{(y-a_k^- s)^2}{Ms}} dy\, ds. \tag{6.74}$$

Writing

$$x - y - a_j^-(t-s) = (x - a_j^-(t-s) - a_k^- s) - (y - a_k^- s),$$

we observe the estimate

$$\begin{aligned} &e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+|y-a_k^- s| + s^{1/2})^{-3/2} e^{-\frac{(y-a_k^- s)^2}{Ms}} \\ &\quad \leq C\Big[e^{-\epsilon\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} (1+|y-a_k^- s| + s^{1/2})^{-3/2} e^{-\frac{(y-a_k^- s)^2}{Ms}} \\ &\qquad + e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+|x-a_j^-(t-s)-a_k^- s| + s^{1/2})^{-3/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Ms}} \Big]. \end{aligned} \tag{6.75}$$

For the first estimate in (6.75), we have integrals

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-1} e^{-\epsilon\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} s^{-1}(1+s)^{-1/4} \times e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} (1+|y-a_k^- s| + s^{1/2})^{-3/2} e^{-\frac{(y-a_k^- s)^2}{Ms}} dy\, ds. \tag{6.76}$$

We have three cases to consider, $a_k^- < 0 < a_j^-$, $a_k^- < a_j^- < 0$, and $a_j^- < a_k^- < 0$, of which we focus on the second. For $|x| \geq |a_k^-|t$, we observe that there is no cancellation between summands in (6.6), and consequently that we obtain an estimate by

$$\begin{aligned} &C_1 t^{-1} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_{\sqrt{t}}^{t/2} s^{-1/2}(1+s)^{-1} e^{-\frac{((a_j^- - a_k^-)(t-s))^2}{M(t-s)}} ds \\ &\quad + C_2 t^{-1}(1+t)^{-1} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_{t/2}^{t-\sqrt{t}} (t-s)^{-1/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} \\ &\quad \leq C(1+t)^{-3/2} e^{-\frac{(x-a_k^- t)^2}{Lt}}. \end{aligned}$$

For $|x| \leq |a_j^-|t$, we observe that there is no cancellation between summands in (6.6), and consequently we obtain an estimate by

$$\begin{aligned} &C_1 t^{-1} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_{\sqrt{t}}^{t/2} e^{-\epsilon\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} s^{-1/2}(1+s)^{-1} ds \\ &\quad + C_2 t^{-1}(1+t)^{-1} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_{t/2}^{t-\sqrt{t}} (t-s)^{-1/2} e^{-\epsilon\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} ds \\ &\quad \leq C t^{-1/2}(1+t)^{-3/4} e^{-\frac{(x-a_j^- t)^2}{Lt}}, \end{aligned}$$

which is sufficient for $j \neq l$. For $|a_j^-|t \leq |x| \leq |a_k^-|t$, we observe the inequality

$$\begin{aligned} &e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} s^{-1}(1+s)^{-1/2} \\ &\quad \leq C\Big[e^{-\frac{(x-a_j^- t)^2}{Lt}} e^{-\epsilon\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} s^{-1}(1+s)^{-1/2} \\ &\qquad + e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} |x-a_j^- t|^{-1}(1+|x-a_j^- t|)^{-1/2}\Big]. \end{aligned} \tag{6.77}$$

For integration over the first estimate in (6.77), we have an estimate by

$$\begin{aligned} &C_1 t^{-1} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_{\sqrt{t}}^{t/2} s^{-1/2}(1+s)^{-1} e^{-\epsilon\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} ds \\ &\quad + C_2 t^{-1}(1+t)^{-1} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_{t/2}^{t-\sqrt{t}} (t-s)^{-1/2} e^{-\epsilon\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} ds \\ &\leq C(1+t)^{-5/4} e^{-\frac{(x-a_j^- t)^2}{Lt}}, \end{aligned}$$

which is sufficient for $l \neq j$. For integration over the second estimate in (6.77), we have an estimate by

$$\begin{aligned} &C_1 t^{-1}|x-a_j^- t|^{-1}(1+|x-a_j^- t|)^{-1/2} \int_{\sqrt{t}}^{t/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} ds \\ &\quad + C_2(1+t)^{-1/2}|x-a_j^- t|^{-1}(1+|x-a_j^- t|)^{-1/2} \\ &\quad \times \int_{t/2}^{t-\sqrt{t}} (t-s)^{-1/2} e^{-\epsilon\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} ds \\ &\leq C t^{-1/2}|x-a_j^- t|^{-1}(1+|x-a_j^- t|)^{-1/2}, \end{aligned}$$

which, along with an alternative estimate in the case $|x-a_j^- t| \leq \sqrt{t}$, is sufficient for $l \neq j$. We remark that the critical observation in this calculation was that since we require less decay along the j-characteristic, we use the same estimate (6.77) for both cases $s \in [\sqrt{t}, t/2]$ and $s \in [t/2, t-\sqrt{t}]$. For the second estimate in (6.75), we have integrals

$$\begin{aligned} &\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} s^{-1}(1+s)^{-1/4} \\ &\quad \times (1+|x-a_j^-(t-s)-a_k^- s|+s^{1/2})^{-3/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Ms}} e^{-\epsilon\frac{(y-a_k^- s)^2}{Ms}} dy\, ds. \end{aligned} \tag{6.78}$$

We have three cases to consider, $a_k^- < 0 < a_j^-$, $a_k^- < a_j^- < 0$, and $a_j^- < a_k^- < 0$, of which we focus on the second. For $|x| \geq |a_k^-|t$, we observe that there is no cancellation between summands in (6.6), and consequently we obtain an estimate by

$$\begin{aligned} &C_1 t^{-1}(1+|x-a_k^- t|)^{-3/2}\int_{\sqrt{t}}^{t/2} s^{-1/2}(1+s)^{-1/4}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}s}}ds \\ &\quad + C_2 t^{-1}(1+t)^{-1/4}(1+|x-a_k^- t|)^{-3/2}\int_{t/2}^{t-\sqrt{t}}(t-s)^{-1/2}ds \\ &\leq C(1+t)^{-3/4}(1+|x-a_k^- t|)^{-3/2}, \end{aligned} \tag{6.79}$$

which is the required estimate since $l = k$. For the case $|x| \leq |a_j^-|t$, we have no cancellation between summands in (6.8) and proceeding as in (6.79) we immediately obtain an estimate by

$$C(1+t)^{-3/4}(1+|x-a_j^- t|)^{-3/2}.$$

For $|a_j^-|t \leq |x| \leq |a_k^-|t$, we divide the analysis into cases $s \in [\sqrt{t}, t/2]$ and $s \in [t/2, t-\sqrt{t}]$. For $s \in [\sqrt{t}, t/2]$, we observe the estimate (6.77). For integration over the first estimate in (6.77), we have an estimate by

$$\begin{aligned} &C_1 t^{-1} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_{\sqrt{t}}^{t/2} s^{-1/2}(1+s)^{-1} e^{-\epsilon\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}s}} ds \\ &\quad \leq C(1+t)^{-5/4} e^{-\frac{(x-a_j^- t)^2}{Lt}}, \end{aligned}$$

while for integration over the second estimate in (6.77), we have an estimate by

$$\begin{aligned} &C_1 t^{-1}|x-a_j^- t|^{-1}(1+|x-a_j^- t|)^{-1/2}\int_{t/2}^{t-\sqrt{t}} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}s}} ds \\ &\quad \leq C(1+t)^{-1/2}|x-a_j^- t|^{-1}(1+|x-a_j^- t|)^{-1/2}, \end{aligned}$$

which, along with an alternative estimate in the case $|x-a_j^- t| \leq \sqrt{t}$, is sufficient for $l \neq j$. For $s \in [t/2, t-\sqrt{t}]$, we compute an estimate directly from (6.78)

$$C_2 t^{-1}(1+t)^{-1}\int_{t/2}^{t-\sqrt{t}}(t-s)^{-1/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}s}} ds \leq C(1+t)^{-7/4},$$

which for $t \geq |x|/|a_k^-|$ gives an estimate by

$$C(1+|x|+t)^{-7/4}.$$

For the remaining estimates in $|(\partial_s + a_l^- \partial_y)v|$, we have decay with a different scaling than in the diffusion wave. For example, we have terms of the form

$$\begin{aligned} &\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}(t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} s^{-1/2}(1+s)^{-1/2} \\ &\quad \times (1+|y-a_m^- s|+s^{1/2})^{-3/2} e^{-\frac{(y-a_k^- s)^2}{\bar{M}s}} dy\, ds, \end{aligned}$$

where $m \neq k$. In such cases, we observe that for y near $a_m^- s$ we have exponential decay in s (which gives exponential decay in $\sqrt{t}$), while for y away from $a_m^- s$, we have integrals of the form

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} s^{-1/2}(1+s)^{-1/2}(1+s)^{-3/2} e^{-\frac{(y-a_k^- s)^2}{Ms}} dy\, ds,$$

which are better than previous cases (see (6.73)).

Case 3: $l \neq k$, $l \neq j$. For the case $l \neq k$, $l \neq j$, we divide the analysis into precisely the same three terms as in (6.70). For the first integral in (6.70), upon integration by parts in y, we estimate

$$\begin{aligned}
&\int_0^{\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-3/2} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-5/4} e^{-\frac{(y-a_k^- s)^2}{Ms}} dy\, ds \\
&\quad \leq C_1 t^{-1/2} \int_0^{\sqrt{t}} (t-s)^{-1}(1+s)^{-3/4} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds \\
&\quad \leq C t^{-1/2}(1+t)^{-7/8} e^{-\frac{(x-a_j^- t)^2}{Lt}},
\end{aligned}$$

which is sufficient for $l \neq j$. For the third integral in (6.70), we estimate

$$\begin{aligned}
&\int_{t-\sqrt{t}}^{t} \int_{-\infty}^{+\infty} (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} s^{-1/2}(1+s)^{-5/4} e^{-\frac{(y-a_k^- s)^2}{Ms}} \\
&\quad \leq C t^{-1/2} \int_{t-\sqrt{t}}^{t} (t-s)^{-1/2}(1+s)^{-5/4} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds \\
&\quad \leq C(1+t)^{-3/2} e^{-\frac{(x-a_k^- t)^2}{Lt}},
\end{aligned}$$

which is sufficient. For the second integral in (6.70), we begin with the case $j = k$, for which we write

$$\begin{aligned}
&\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (\partial_t + a_l^- \partial_x)\tilde{G}^j(x, t-s; y)(\varphi^k(y,s)v(y,s))_y\, dy\, ds \\
&\quad = -\int_{\sqrt{t}}^{t/2} \int_{-\infty}^{+\infty} (\partial_t + a_l^- \partial_x)\tilde{G}_y^j(x, t-s; y)\varphi^k(y,s)v(y,s) dy\, ds \\
&\qquad + \int_{t/2}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (\partial_t + a_l^- \partial_x)\tilde{G}^j(x, t-s; y)(\varphi^k(y,s)v(y,s))_y\, dy\, ds. \qquad (6.80)
\end{aligned}$$

For the first integral on the right-hand side of (6.80), we estimate

$$\begin{aligned}
&\int_{\sqrt{t}}^{t/2} (t-s)^{-3/2} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-5/4} e^{-\frac{(y-a_j^- s)^2}{Ms}} ds \\
&\quad \leq C t^{-1/2} \int_{\sqrt{t}}^{t/2} (t-s)^{-1}(1+s)^{-5/4} s^{1/2} e^{-\frac{(x-a_j^- t)^2}{Mt}} ds \\
&\quad \leq C_1(1+t)^{-5/4} e^{-\frac{(x-a_j^- t)^2}{Mt}} ds,
\end{aligned}$$

which is sufficient for $l \neq j$. Similarly, for the second integral on the right-hand side of (6.80), we estimate

$$\begin{aligned}
&\int_{t/2}^{t-\sqrt{t}} (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} s^{-1/2}(1+s)^{-5/4} e^{-\frac{(y-a_j^- s)^2}{Ms}} ds \\
&\quad \leq C t^{-1/2} \int_{t/2}^{t-\sqrt{t}} (t-s)^{-1/2}(1+s)^{-5/4} e^{-\frac{(x-a_j^- t)^2}{Mt}} ds \\
&\quad \leq C_1 (1+t)^{-5/4} e^{-\frac{(x-a_j^- t)^2}{Mt}} ds,
\end{aligned}$$

which again is sufficient for $l \neq j$. In the case $j \neq k$, we employ the non-convecting variables (6.33), along with

$$g_l(x,t;y+a_j^-(t-s)) := (\partial_t + a_l^- \partial_x)\tilde{G}^j(x,t-s;y), \tag{6.81}$$

for which we can write the second integral in (6.70) as

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} g_l(x,t-s;y+a_j^-(t-s))(\phi(y-a_k^- s,s)V(y-a_k^- s,s))_y \, dy \, ds.$$

Setting $\xi = y + a_j^-(t-s)$, this becomes

$$\begin{aligned}
&\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} g_l(x,t-s;\xi)(\phi(\xi - a_j^-(t-s) - a_k^- s, s) \\
&\quad \times V(\xi - a_j^-(t-s) - a_k^- s, s))_y \, dy \, ds.
\end{aligned} \tag{6.82}$$

Proceeding similarly as in (6.37), we write

$$\begin{aligned}
&(g_l(x,t-s;\xi)\phi(\xi - a_j^-(t-s) - a_k^- s,s)V(\xi - a_j^-(t-s) - a_k^- s,s))_s \\
&\quad = -g_{l\tau}(x,t-s;\xi)\phi(\xi - a_j^-(t-s) - a_k^- s,s)V(\xi - a_j^-(t-s) - a_k^- s,s) \\
&\qquad + (a_j^- - a_k^-)g_l(x,t-s;\xi)(\phi(\xi - a_j^-(t-s) - a_k^- s,s) \\
&\qquad \times V(\xi - a_j^-(t-s) - a_k^- s,s))_\xi \\
&\qquad + g_l(x,t-s;\xi)(\phi(\xi - a_j^-(t-s) - a_k^- s,s)V(\xi - a_j^-(t-s) - a_k^- s,s))_\tau.
\end{aligned} \tag{6.83}$$

For integration over the left-hand side of (6.83), we exchange the order of integration to obtain

$$\begin{aligned}
&\int_{-\infty}^{+\infty} g_l(x,\sqrt{t};\xi)\phi(\xi - a_j^-\sqrt{t} - a_k^-(t-\sqrt{t}), t-\sqrt{t}) \\
&\quad \times V(\xi - a_j^-\sqrt{t} - a_k^-(t-\sqrt{t}), t-\sqrt{t}) d\xi \\
&\quad - \int_{-\infty}^{+\infty} g_l(x,t-\sqrt{t};\xi)\phi(\xi - a_j^-(t-\sqrt{t}) - a_k^-\sqrt{t}, \sqrt{t}) \\
&\quad \times V(\xi - a_j^-(t-\sqrt{t}) - a_k^-\sqrt{t}, \sqrt{t}) d\xi.
\end{aligned} \tag{6.84}$$

For the first integral in (6.84), using the supremum norm of V, we estimate

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (\sqrt{t})^{-1} e^{-\frac{(x-\xi)^2}{M\sqrt{t}}} (1+(t-\sqrt{t}))^{-5/4} e^{-\frac{(\xi - a_j^- \sqrt{t} - a_k^-(t-\sqrt{t}))^2}{M(t-\sqrt{t})}} d\xi \\
&\quad \le C t^{-1/2} (\sqrt{t})^{-1/2} (1+(t-\sqrt{t}))^{-5/4} (t-\sqrt{t})^{1/2} e^{-\frac{(x-a_j^-\sqrt{t}-a_k^-(t-\sqrt{t}))^2}{Mt}} \\
&\quad \le C(1+t)^{-3/2} e^{-\frac{(x-a_k^- t)^2}{Lt}}.
\end{aligned}$$

For the second integral in (6.84), again using the supremum norm of V, we estimate

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (t-\sqrt{t})^{-1} e^{-\frac{(x-\xi)^2}{M(t-\sqrt{t})}} (1+\sqrt{t})^{-5/4} e^{-\frac{(\xi - a_j^-(t-\sqrt{t}) - a_k^- \sqrt{t})}{M(t-\sqrt{t})}} d\xi \\
&\quad \le C t^{-1/2} (t-\sqrt{t})^{-1/2} (1+\sqrt{t})^{-5/4} (\sqrt{t})^{1/2} e^{-\frac{(x-a_j^-(t-\sqrt{t})-a_k^-\sqrt{t})}{Mt}} \\
&\quad \le C(1+t)^{-11/8} e^{-\frac{(x-a_j^- t)}{Lt}},
\end{aligned}$$

which is slightly better than required for $l \neq j$ (we require $t^{-10/8}$). For integration over the first term on the right-hand side of (6.83), we estimate

$$\begin{aligned}
&\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-2} e^{-\frac{(x-\xi)^2}{M(t-s)}} (1+s)^{-5/4} e^{-\frac{(\xi - a_j^-(t-s) - a_k^- s)^2}{Ms}} d\xi\, ds \\
&\quad \le C t^{-1/2} \int_{\sqrt{t}}^{t-\sqrt{t}} (t-s)^{-3/2} (1+s)^{-5/4} s^{1/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds. \qquad (6.85)
\end{aligned}$$

We have three cases to consider, $a_k^- < 0 < a_j^-$, $a_k^- < a_j^- < 0$, and $a_j^- < a_k^- < 0$, of which we focus on the second. For $|x| \ge |a_k^-|t$, there is no cancellation between summands in (6.6), and we can estimate

$$\begin{aligned}
&C_1 t^{-2} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_{\sqrt{t}}^{t/2} (1+s)^{-5/4} s^{1/2} ds \\
&\quad + C_2 (1+t)^{-5/4} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_{t/2}^{t-\sqrt{t}} (t-s)^{-3/2} ds \\
&\quad \le C(1+t)^{-3/2} e^{-\frac{(x-a_k^- t)^2}{Lt}}. \qquad (6.86)
\end{aligned}$$

In the case $|x| \le |a_j^-|t$, we have no cancellation between summands in (6.8) and similarly obtain an estimate by

$$C(1+t)^{-3/2} e^{-\frac{(x-a_k^- t)^2}{Lt}}.$$

For the critical case $|a_j^-|t \le |x| \le |a_k^-|t$, we divide the analysis into subcases $s \in [\sqrt{t}, t/2]$ and $s \in [t/2, t-\sqrt{t}]$. For $s \in [\sqrt{t}, t/2]$, we observe through (6.8) the inequality (6.42) with $\gamma = 3/4$. For the first estimate in (6.42), we proceed similarly as in (6.86), while for the second we estimate

$$C_1 t^{-2}(1+|x-a_j^- t|)^{-3/4}\int_{\sqrt{t}}^{t-\sqrt{t}}(1+s)^{-1/2}s^{1/2}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}ds$$
$$\leq Ct^{-3/2}(1+|x-a_j^- t|)^{-3/4},$$

which is sufficient for $t \geq |x|/|a_k^-|$. For $s \in [t/2, t-\sqrt{t}]$, we observe through (6.6) the inequality (6.10) with $\gamma = 3/2$. For the second estimate in (6.10), we proceed similarly as in (6.86), while for the first we estimate

$$C_2(1+t)^{-5/4}|x-a_k^- t|^{-3/2}\int_{t/2}^{t-\sqrt{t}} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}ds \leq C(1+t)^{-3/4}|x-a_k^- t|^{-3/2}.$$

For the third expression on the right-hand side of (6.83), we estimate

$$\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}(t-s)^{-1}e^{-\frac{(x-\xi)^2}{M(t-s)}}s^{-1}(1+s)^{-1}e^{-\frac{(\xi-a_j^-(t-s)-a_k^- s)^2}{Ms}}d\xi\, ds$$
$$\leq Ct^{-1/2}\int_{\sqrt{t}}^{t-\sqrt{t}}(t-s)^{-1/2}s^{-1/2}(1+s)^{-1}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Ms}}ds, \qquad (6.87)$$

where we have once again observe the increased rate of time decay for the combination

$$(\phi(\xi-a_j^-(t-s)-a_k^- s,s)V(\xi-a_j^-(t-s)-a_k^- s,s))_\tau.$$

We have three cases to consider, $a_k^- < 0 < a_j^-$, $a_k^- < a_j^- < 0$, and $a_j^- < a_k^- < 0$, of which we focus on the second. For $|x| \geq |a_k^-|t$, there is no cancellation between summands in (6.6), and we can estimate

$$C_1 t^{-1}e^{-\frac{(x-a_k^- t)^2}{Lt}}\int_{\sqrt{t}}^{t/2}s^{-1/2}(1+s)^{-1}ds$$
$$+C_2 t^{-1}(1+t)^{-1}e^{-\frac{(x-a_k^- t)^2}{Lt}}\int_{t/2}^{t-\sqrt{t}}(t-s)^{-1/2}ds$$
$$\leq C(1+t)^{-5/4}e^{-\frac{(x-a_k^- t)^2}{Lt}}, \qquad (6.88)$$

which is sufficient for $k \neq k$. For $|x| \leq |a_j^-|t$, there is no cancellation between summands in (6.8) and we obtain an estimate by

$$C(1+t)^{-5/4}e^{-\frac{(x-a_j^- t)^2}{Lt}},$$

For $|a_j^-|t \leq |x| \leq |a_k^-|t$, we divide the analysis into cases, $s \in [\sqrt{t}, t/2]$ and $s \in [t/2, \sqrt{t}]$. For $s \in [\sqrt{t}, t/2]$, we observe the inequality (6.42) with $\gamma = 3/2$. For the first estimate in (6.42), we proceed as in (6.88), while for the second we estimate

$$C_1 t^{-1}(1+|x-a_j^- t|)^{-3/2}\int_{\sqrt{t}}^{t/2} s^{-1/2}(1+s)^{1/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}\,ds \\ \le C(1+t)^{-1/2}(1+|x-a_j^- t|)^{-3/2},$$

which is precisely enough for $l \neq j$. For $s \in [t/2, t-\sqrt{t}]$, we observe through (6.6) the inequality (6.10) with $\gamma = 1/2$. For the second estimate in (6.10), we proceed similarly as in (6.88), while for the first we estimate

$$C_2 t^{-1}(1+t)^{-1}|x-a_k^- t|^{-1/2}\int_{t/2}^{t-\sqrt{t}} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}\,ds \\ \le C(1+t)^{-3/2}|x-a_k^- t|^{-1/2},$$

which is sufficient for $l \neq k$. This ends the proof of (3.18(vi)) for the leading order convection kernels $\tilde{G}^j(x,t;y)$.

(3.18(v)), *Nonlinearity* $(v(y,s)^2)_y$. We next consider integrals

$$\int_0^t \int_{-\infty}^{+\infty} (\partial_t + a_l^{\pm}\partial_x)\tilde{G}^j(x,t-s;y)(v(y,s)^2)_y\,dy\,ds \\ = \int_0^{t/2}\int_{-\infty}^{+\infty} (\partial_t + a_l^{\pm}\partial_x)\tilde{G}^j(x,t-s;y)(v(y,s)^2)_y\,dy\,ds \\ + \int_{t/2}^{t}\int_{-\infty}^{+\infty} (\partial_t + a_l^{\pm}\partial_x)\tilde{G}^j(x,t-s;y)(v(y,s)^2)_y\,dy\,ds, \tag{6.89}$$

for which we have two cases to consider, $l \neq j$ and $l = j$.

Case 1: $l \neq j$. For the case $l \neq j$, we integrate the first integral on the right-hand side of (6.89) by parts in y to obtain integrals

$$\int_0^{t/2}\int_{-\infty}^{+\infty} \partial_y(\partial_t + a_l^{\pm}\partial_x)\tilde{G}^j(x,t-s;y)v(y,s)^2\,dy\,ds.$$

Taking supremum norm on one $v(y,s)$, we have two cases two consider, one for each estimate on v. For the first, we have

$$\int_0^{t/2}\int_{-\infty}^{+\infty} (t-s)^{-3/2} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-3/4}(1+|y-a_k^- s|+s^{1/2})^{-3/2}\,dy\,ds, \tag{6.90}$$

where $a_k^- < 0$. We observe through (6.3) the inequality (6.4) with $\gamma = 3/2$. For the first estimate in (6.4), we have integrals

$$\int_0^{t/2}\int_{-\infty}^{+\infty} (t-s)^{-3/2} e^{-\epsilon\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}} \\ \times (1+s)^{-3/4}(1+|y-a_k^- s|+s^{1/2})^{-3/2}\,dy\,ds. \tag{6.91}$$

We have three cases to consider $a_k^- < 0 < a_j^- s$, $a_k^- \leq a_j^- < 0$, and $a_j^- < a_k^- < 0$, of which we focus on the second. In the case $s \in [0, \sqrt{t}]$, we have

$$e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}} \leq Ce^{-\frac{(x-a_j^- t)^2}{Lt}},$$

from which we obtain an estimate by

$$C_1 t^{-3/2} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_0^{\sqrt{t}} (1+s)^{-1} ds \leq C(1+t)^{-3/2} \ln(1+t) e^{-\frac{(x-a_j^- t)^2}{Lt}},$$

sufficient for $l \neq j$. For $s \in [\sqrt{t}, t/2]$, we begin with the case $|x| \geq |a_k^-|t$, for which there is no cancellation between summands in (6.6), and we obtain an estimate by

$$C_1 t^{-3/2} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_{\sqrt{t}}^{t/2} (1+s)^{-1} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}} ds. \tag{6.92}$$

In the event that $j = k$, we obtain an estimate by

$$C(1+t)^{-3/2} \ln(1+t) e^{-\frac{(x-a_k^- t)^2}{Lt}},$$

which is sufficient for $k = j \neq l$, whereas in the event that $j \neq k$, we obtain an estimate by

$$C(1+t)^{-3/2} e^{-\frac{(x-a_k^- t)^2}{Lt}}.$$

Proceeding similarly in the case $|x| \leq |a_j^-|t$, we obtain an estimate by

$$C(1+t)^{-3/2} \ln(1+t) e^{-\frac{(x-a_j^- t)^2}{Lt}},$$

which is sufficient for $l \neq j$. For $|a_j^-|t \leq |x| \leq |a_k^-|t$, we observe through (6.8) the inequality (6.9) with $\gamma = 3/4$. For the first estimate in (6.9), we proceed as in (6.92), while for the second we estimate

$$C_1 t^{-3/2} (1 + |x - a_j^- t|)^{-1} \int_{\sqrt{t}}^{t/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}} ds \leq Ct^{-1}(1 + |x - a_j^- t|)^{-1},$$

which is sufficient for $t \geq |x|/|a_k^-|$ and $l \neq j$. For the second estimate in (6.4) we have integrals

$$\int_{\sqrt{t}}^{t/2} \int_{-\infty}^{+\infty} (t-s)^{-3/2} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-3/4} \\ \times (1 + |y - a_k^- s| + |x - a_j^-(t-s) - a_k^- s| + s^{1/2})^{-3/2} \, dy \, ds,$$

for which we observe through (6.8) the inequality (6.13) with $\gamma = 3/4$. Integrating the exponential kernel, for the first estimate in (6.13) we obtain an estimate by

$$C_1 t^{-1}(1+|x-a_j^- t|)^{-3/2}\int_{\sqrt{t}}^{t/2}(1+s)^{-3/4}ds \le Ct^{-3/4}(1+|x-a_j^- t|)^{-3/2},$$

while for the second estimate in (6.13), we obtain an estimate by

$$C_1 t^{-1}(1+|x-a_j^- t|)^{-3/4}\int_{\sqrt{t}}^{t/2}(1+|x-a_j^- t|+s^{1/2})^{-3/2}\,ds \le Ct^{-1}(1+|x-a_j^- t|)^{-1}.$$

For the second estimate on $v(y,s)$, (6.90) is replaced by

$$\int_0^{t/2}\int_{-|a_1^-|s}^{0}(t-s)^{-3/2}e^{-\frac{(x-y-a_k^-(t-s))^2}{M(t-s)}}(1+|y|)^{-1/2} \times(1+|y|+s)^{-5/4}(1+|y|+s^{1/2})^{-1/2}\,dy\,ds, \tag{6.93}$$

for which we observe through (6.8) the inequality (6.17). For the first estimate in (6.17), we have integrals

$$\int_0^{t/2}\int_{-|a_1^-|s}^{0}(t-s)^{-3/2}e^{-\epsilon\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}e^{-\frac{(x-a_j^-(t-s))^2}{\bar{M}(t-s)}}(1+|y|)^{-1/2} \times(1+|y|+s)^{-5/4}(1+|y|+s^{1/2})^{-1/2}\,dy\,ds,$$

for which we have two cases to consider, $a_j^- < 0$ and $a_j^- > 0$. For the case $a_j^- > 0$, there is no cancellation between x and $a_j^-(t-s)$ and the claimed estimate can be deduced in straightforward fashion. For the case $a_j^- < 0$, we first consider the subcase $|x| \ge |a_j^-|t$, for which there is no cancellation between summands on the right-hand side of

$$x - a_j^-(t-s) = (x-a_j^- t) + a_j^- s,$$

and we immediately obtain an estimate by

$$C_1 t^{-3/2}e^{-\frac{(x-a_j^- t)^2}{Lt}}\int_0^{t/2}(1+s)^{-1}\,ds \le C(1+t)^{-3/2}\ln(e+t)e^{-\frac{(x-a_j^- t)^2}{Lt}}, \tag{6.94}$$

which is sufficient for $l \ne j$. For $|x| \le |a_j^-|t$, we observe the inequality (6.19). For the first estimate in (6.19), we proceed similarly as in (6.94), while for the second we estimate

$$C_1 t^{-3/2}(1+|x-a_j^- t|)^{-3/2}\int_0^{t/2}(1+s)^{1/2}e^{-\frac{(x-a_j^-(t-s))^2}{\bar{M}(t-s)}} \le C(1+t)^{-1/2}(1+|x-a_j^- t|)^{-3/2},$$

which is sufficient for $l \neq j$. For the second estimate in (6.17), we have integrals

$$\int_0^{t/2}\int_{-|a_1^-|s}^{0}(t-s)^{-3/2}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+|y|+|x-a_j^-(t-s)|)^{-1/2}$$
$$\times(1+|y|+|x-a_j^-(t-s)|+s)^{-5/4}(1+|y|+|x-a_j^-(t-s)|+s^{1/2})^{-1/2}\,dy\,ds,$$

for which we have two cases to consider, $a_j^- < 0$ and $a_j^- > 0$, and as before we need focus only on the former. For $a_j^- > 0$ and $|x| \geq |a_j^-|t$, there is no cancellation between $x - a_j^- t$ and $a_j^- s$, and we immediately obtain an estimate by

$$C_1 t^{-1}(1+|x-a_j^-t|)^{-3/2}\int_0^{t/2}(1+|x-a_j^-(t-s)|)^{-3/4}ds$$
$$\leq C(1+t)^{-3/4}(1+|x-a_j^-t|)^{-3/2}. \tag{6.95}$$

For $|x| \leq |a_j^-|t$, we observe the inequality (6.23). For the first estimate in (6.23), we proceed similarly as in (6.95), while for the second we obtain an estimate by

$$C_1 t^{-1}(1+|x-a_j^-t|)^{-3/2}\int_0^{t/2}(1+|x-a_j^-(t-s)|)^{-1/2}\,ds$$
$$\leq C(1+t)^{-1/2}(1+|x-a_j^-t|)^{-3/2},$$

which is sufficient for $l \neq j$. For the case $s \in [t/2, t]$, we need to shift the characteristic derivative onto the nonlinearity. We accomplish this precisely as in (6.51), computing

$$\int_{t/2}^{t}\int_{-\infty}^{+\infty}(\partial_t + a_l^-\partial_x)\tilde{G}^j(x,t-s;y)(v(y,s)^2)_y\,dy\,ds$$
$$= \int_{t/2}^{t}\int_{-\infty}^{+\infty}(-\partial_s - a_l^-\partial_y)\tilde{G}^j(x,t-s;y)(v(y,s)^2)_y\,dy\,ds$$
$$= -\int_{-\infty}^{+\infty}\tilde{G}^j_y(x,t/2;y)v(y,t/2)^2\,dy$$
$$-\int_{-\infty}^{+\infty}\tilde{G}^j(x,0;y)(v(y,t)^2)_y\,dy$$
$$-\int_{t/2}^{t}\int_{-\infty}^{+\infty}\tilde{G}^j_y(x,t-s;y)(\partial_s + a_l^-\partial_y)v(y,s)^2\,dy\,ds, \tag{6.96}$$

where in the first and last integrals on the right-hand side we have additionally integrated by parts in y. For the first integral on the right-hand side of (6.96), and for the first estimate on $v(y,s)$, we have integrals

$$\int_{-\infty}^{+\infty}t^{-1}e^{-\frac{(x-y-a_k^-(t/2))^2}{M(t/2)}}(1+|y-a_k^-(t/2)|+(t/2)^{1/2})^{-3/2}(1+(t/2))^{-3/4}\,dy$$
$$\leq Ct^{-1/2}(1+t)^{-3/2}, \tag{6.97}$$

which is sufficient for $t \geq c|x|$, for any constant $c > 0$. In the case $|x| \geq t/c$, we observe the decomposition

$$x - y - \frac{1}{2}a_j^- t = \left(x - \frac{1}{2}a_j^- t - \frac{1}{2}a_k^- t\right) - \left(y - \frac{1}{2}a_k^- t\right),$$

through which we observe the inequality

$$\begin{aligned} &e^{-\frac{(x-y-a_j^-(t/2))^2}{M(t/2)}}\left(1 + \left|y - \frac{1}{2}a_k^- t\right| + (t/2)^{1/2}\right)^{-3/2} \\ &\quad \leq C\Bigg[e^{-\epsilon\frac{(x-y-a_j^-(t/2))^2}{M(t/2)}} e^{-\frac{(x-a_j^-(t/2)-a_k^-(t/2))^2}{Lt}}\left(1 + \left|y - \frac{1}{2}a_k^- t\right| + (t/2)^{1/2}\right)^{-3/2} \\ &\qquad + e^{-\frac{(x-y-a_j^-(t/2))^2}{M(t/2)}} \\ &\qquad \times \left(1 + \left|y - \frac{1}{2}a_k^- t\right| + \left|x - a_j^-(t/2) - a_k^-(t/2)\right| + (t/2)^{1/2}\right)^{-3/2}\Bigg]. \end{aligned} \tag{6.98}$$

In the event that $|x| \geq t/c$, for c sufficiently small, we have exponential decay in both $|x|$ and t for the first estimate in (6.98), while for the second, we have an estimate by

$$C_1 t^{-1/2}(1+t)^{-3/4}(1 + |x| + t)^{-3/2},$$

which again is sufficient. For the second estimate on $v(y, s)$, we have integrals

$$\begin{aligned} &\int_{-|a_1^-|\frac{t}{2}}^{0} t^{-1} e^{-\frac{(x-y-a_k^-(t/2))^2}{M(t/2)}}(1 + |y|)^{-1/2} \\ &\qquad \times (1 + |y| + (t/2))^{-1/2}(1 + |y| + (t/2)^{1/2})^{-1/2}(1 + (t/2))^{-3/4} dy \\ &\qquad \leq C t^{-1/2}(1+t)^{-3/2}, \end{aligned} \tag{6.99}$$

which is sufficient for $t \geq c|x|$, for any constant $c > 0$. The case $|x| \geq t/c$ can be analyzed similarly as in the immediately preceding case. For the second integral on the right-hand side of (6.96), we observe that $G^j(x, 0; y)$ is a delta function with mass at $x = y$, and consequently, we have an estimate by

$$C|v(x,t)v_x(x,t)| \leq C(1+t)^{-3/4}[t^{-1/2}(\psi_1(x,t) + \psi_2(x,t)) + \psi_3(x,t) + \psi_4(x,t)],$$

which is sufficient. For the third integral on the right-hand side of (6.96), and for the first estimate on $(\partial_s + a_l^- \partial_y)v(y, s)$, we have integrals

$$\begin{aligned} &\int_{t/2}^{t}\int_{-\infty}^{+\infty}(t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} s^{-1}(1+s)^{-1/2}(1 + |y - a_l^- s)| + s^{1/2})^{-3/2}\, dy\, ds \\ &\quad \leq C t^{-1/2}(1+t)^{-1/4}\int_{t/2}^{t}\int_{-\infty}^{+\infty}(t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} s^{-1/2}(1+s)^{-1/4} \\ &\qquad \times (1 + |y - a_l^- s)| + s^{1/2})^{-3/2}\, dy\, ds. \end{aligned} \tag{6.100}$$

This last integral has already been considered in the analysis of (6.2), and we obtain an estimate by

$$Ct^{-1/2}(1+t)^{-1/4}\psi_1(x,t).$$

We proceed in precisely the same manner for the second and third estimates on $(\partial_s + a_l^- \partial_y)v(y,s)$, obtaining an estimate by

$$C\big[t^{-1/2}\big(\bar\psi_1^{l,-}(x,t)+\psi_2(x,t)\big)+\psi_3(x,t)+\psi_4(x,t)\big].$$

For the fourth estimate on $(\partial_s + a_l^- \partial_y)v(y,s)$, we have integrals

$$\int_{t/2}^{t}\int_{-\infty}^{+\infty}(t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-3/4}(1+|y|+s)^{-1}(1+|y|)^{-1}\,dy\,ds. \tag{6.101}$$

In this case, we observe the inequality

$$\begin{aligned} &e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+|y|)^{-1}\\ &\quad\le C\Big[e^{-\epsilon\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}e^{-\frac{(x-a_j^-(t-s))^2}{\bar M(t-s)}}(1+|y|)^{-1}\\ &\qquad + e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+|y|+|x-a_j^-(t-s)|)^{-1}\Big]. \end{aligned} \tag{6.102}$$

For the first estimate in (6.102), we have integrals

$$\begin{aligned} &\int_{t/2}^{t}\int_{-|a_1^-|s}^{0}(t-s)^{-1}e^{-\epsilon\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}e^{-\frac{(x-a_j^-(t-s))^2}{\bar M(t-s)}}\\ &\quad\times(1+s)^{-3/4}(1+|y|+s)^{-1}(1+|y|)^{-1}\,dy\,ds, \end{aligned} \tag{6.103}$$

for which we have two cases to consider, $a_j^- < 0$ and $a_j^- > 0$, and we focus on the former. For $a_j^- < 0$ and additionally $|x| \ge |a_j^-|t$, we have no cancellation between $x - a_j^- t$ and $a_j^- s$, and we obtain an estimate by

$$\begin{aligned} &C_2(1+t)^{-7/4}\ln(e+t)e^{-\frac{(x-a_j^-t)^2}{Lt}}\int_{t/2}^{t-1}(t-s)^{-1}ds\\ &\quad + C_2(1+t)^{-7/4}\ln(e+t)e^{-\frac{(x-a_j^-t)^2}{Lt}}\int_{t-1}^{t}(t-s)^{-1/2}ds\\ &\le C(1+t)^{-7/4}[\ln(e+t)]^2e^{-\frac{(x-a_j^-t)^2}{Lt}}, \end{aligned} \tag{6.104}$$

which is sufficient. For $|x| \le |a_j^-|t$, we observe the inequality (6.20) with $\gamma = 1/2$, for which we estimate

$$\begin{aligned} &C(1+t)^{-7/4}\ln(e+t)(1+|x|)^{-1/2}\int_{t/2}^{t}(t-s)^{-1/2}e^{-\frac{(x-a_j^-(t-s))^2}{\bar M(t-s)}}ds\\ &\le C(1+t)^{-7/4}\ln(e+t)(1+|x|)^{-1/2}, \end{aligned} \tag{6.105}$$

which is sufficient for $t \geq |x|/|a_j^-|$. For the second estimate in (6.102), we have integrals

$$\begin{aligned}\int_{t/2}^{t}\int_{-|a_1^-|s}^{0} &(t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+|y|+|x-a_j^-(t-s)|)^{-1}(1+s)^{-7/4}\,dy\,ds\\ &\leq C(1+t)^{-7/4}\int_{t/2}^{t}(t-s)^{-1/2}(1+|x-a_j^-(t-s)|)^{-1}\,ds\\ &\leq C(1+t)^{-7/4},\end{aligned}$$

which is sufficient for $|x| \leq Kt$, some constant K. For $|x| \geq Kt$, we estimate

$$\begin{aligned}\int_{t/2}^{t}\int_{-|a_1^-|s}^{0} &(t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+|y|+|x-a_j^-(t-s)|)^{-1}(1+s)^{-7/4}\,dy\,ds\\ &\leq C(1+t)^{-7/4}(1+|x|+t)^{-1}\int_{t/2}^{t}(t-s)^{-1/2}\,ds\\ &\leq C(1+t)^{-5/4}(1+|x|+t)^{-1}.\end{aligned}$$

For the final estimate on $(\partial_s + a_l^-\partial_y)v(y,s)$, we have integrals

$$\begin{aligned}\int_{t/2}^{t}\int_{-|a_1^-|s}^{0} &(t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-3/4}(1+|y|+s)^{-7/4}\,dy\,ds\\ &\leq C(1+t)^{-5/2}\int_{t/2}^{t}(t-s)^{-1/2}\,ds\\ &\leq C(1+t)^{-2},\end{aligned} \tag{6.106}$$

which is sufficient for $|x| \leq Kt$. Since $|y|$ is bounded by $|a_1^-|s$, for $|x| \geq Kt$ and K sufficiently large, we have exponential decay in both $|x|$ and t.

Case 2: $l = j$. For the case $l = j$, we have additional decay at rate $(t-s)^{-1/2}$, from which we immediately recover the claimed estimates.

(3.18(v)), *Nonlinearity* $|\frac{\partial \bar{u}^\delta}{\partial \delta}\delta|^2$. We next consider integrals

$$\int_0^t\int_{-\infty}^{+\infty}(\partial_t + a_l^{\pm}\partial_x)\tilde{G}_y^j(x,t-s;y)(1+s)^{-1}e^{-\eta|y|}\,dy\,ds,$$

where in this case additional y-derivatives on the nonlinearity give no additional decay.

Case 1: $l \neq j$. For the case $l \neq j$, we have integrals

$$\begin{aligned}&\int_0^{t-1}\int_{-\infty}^{+\infty}(t-s)^{-3/2}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-1}e^{-\eta|y|}\,dy\,ds\\ &\quad+\int_{t-1}^{t}\int_{-\infty}^{+\infty}(t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-1}e^{-\eta|y|}\,dy\,ds.\end{aligned} \tag{6.107}$$

In either case, we observe the inequality (6.27). For the first estimate in (6.27), we have, upon integration of $e^{-\eta|y|}$, integrals

$$\int_0^{t-1} (t-s)^{-3/2} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-1}\, ds + \int_{t-1}^{t} (t-s)^{-1/2} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-1}\, ds. \tag{6.108}$$

Focusing as in previous cases on the subcase $a_j^- < 0$, we first observe that for $|x| \geq |a_j^-|t$, there is no cancellation between $x - a_j^- t$ and $a_j^- s$, and we consequently have an estimate by

$$\begin{aligned} &Ct^{-3/2} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_0^{t/2} (1+s)^{-1}\, ds \\ &\quad + C_2(1+t)^{-1} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_{t/2}^{t} (t-s)^{-1/2} e^{-\frac{(a_j^- s)^2}{M(t-s)}}\, ds \\ &\leq Ct^{-3/2}[\ln(e+t)] e^{-\frac{(x-a_j^- t)^2}{Lt}}, \end{aligned} \tag{6.109}$$

where in this last inequality, we have observed that for $s \in [t/2, t]$, we have

$$e^{-\frac{(a_j^- s)^2}{M(t-s)}} \leq e^{-\eta_1 t},$$

for $\eta_1 > 0$. For $|x| \leq |a_j^-|t$, we divide the analysis into cases, $s \in [0, t/2]$, $s \in [t/2, t-1]$, and $s \in [t-1, t]$. For $s \in [0, t/2]$, we observe the inequality (6.29). For the second estimate in (6.29), we obtain an estimate by

$$C_1 t^{-3/2}(1 + |x - a_j^- t|)^{-1} \int_0^{t/2} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}}\, ds \leq Ct^{-1}(1 + |x - a_j^- t|)^{-1},$$

which is sufficient for $t \geq |x|/|a_j^-|$ and $l \neq j$. For the first estimate in (6.29), we proceed as in (6.109). For $s \in [t/2, t]$, we observe the inequality (6.20) with $\gamma = 1$, for which we have an estimate by

$$\begin{aligned} &C_2(1+t)^{-1}|x|^{-1} \int_{t/2}^{t-1} (t-s)^{-1/2} e^{-\frac{(x-a_j^-(t-s))^2}{2\bar{M}(t-s)}}\, ds + C_3(1+t)^{-1}|x|^{-1} \int_{t-1}^{t} ds \\ &\quad \leq C(1+t)^{-1}|x|^{-1}. \end{aligned}$$

Case 2: $l = j$. For the case $l = j$, we have additional decay at rate $(t-s)^{-1/2}$, from which we immediately recover the claimed estimates.

This ends the proof of estimate (3.18(v)) for the leading order convection kernel $\tilde{G}^j(x, t; y)$.

(3.18(v)–(vi)), *remainder estimates.* In our proofs of (3.18(v)–(vi)), we have considered only the leading order convection kernel

$$\tilde{G}^j(x,t;y) = ct^{-1/2}e^{-\frac{(x-y-a_j^- t)^2}{4\beta_j t}}.$$

We must also consider the remaining three scattering estimates (terms in $S(x,t;y)$) and additionally the remainder estimates $R(x,t;y)$. Beginning with the remainder estimate, we focus our attention on the nonlinearity $(\varphi^k(y,s)^2)_y$ (analysis of the remaining nonlinearities is similar). We have the decomposition

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{+\infty} (\partial_t + a_l^- \partial_x) R(x,t-s;y)(\varphi^k(y,s)^2)_y \, dy\, ds \\
&\quad = \int_0^{\sqrt{t}} \int_{-\infty}^{+\infty} (\partial_t + a_l^- \partial_x) R(x,t-s;y)(\varphi^k(y,s)^2)_y \, dy\, ds \\
&\qquad + \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (\partial_t + a_l^- \partial_x) R(x,t-s;y)(\varphi^k(y,s)^2)_y \, dy\, ds \\
&\qquad + \int_{t-\sqrt{t}}^{t} \int_{-\infty}^{+\infty} (\partial_t + a_l^- \partial_x) R(x,t-s;y)(\varphi^k(y,s)^2)_y \, dy\, ds, \qquad (6.110)
\end{aligned}$$

where for $y \le 0$ and $a_l^- < 0$,

$$\begin{aligned}
&(\partial_t + a_l^- \partial_x) R(x,t;y) \\
&\quad = \sum_{j=1}^{J} \mathbf{O}(e^{-\eta t})\delta_{x-\bar{a}_j^* t}(-y) + \mathbf{O}(e^{-\eta(|x-y|+t)}) \\
&\qquad + \mathbf{O}((t+1)^{-3/2}e^{-\eta x^+} + e^{-\eta|x|})t^{-1}(t+1)^{1/2}e^{-(x-y-a_l^- t)^2/Mt} \\
&\qquad + \sum_{k\neq l} \mathbf{O}((t+1)^{-1}e^{-\eta x^+} + e^{-\eta|x|})t^{-1}(t+1)^{1/2}e^{-(x-y-a_k^- t)^2/Mt} \\
&\qquad + \chi_{\{|a_k^- t|\ge|y|\}}\mathbf{O}((t+1)^{-1/2}t^{-1})e^{-(x-a_l^-(t-|y/a_k^-|))^2/Mt}e^{-\eta x^+} \\
&\qquad + \sum_{a_k^->0,\, a_j^-<0,\, j\neq l} \chi_{\{|a_k^- t|\ge|y|\}}\mathbf{O}((t+1)^{-1/2}t^{-1})e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt}e^{-\eta x^+} \\
&\qquad + \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge|y|\}}\mathbf{O}((t+1)^{-1/2}t^{-1})e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt}e^{-\eta x^-}.
\end{aligned}$$

(6.111)

In each estimate with decay $t^{-3/2}$ or t^{-2}, we have time decay better than that of characteristic derivatives of $\tilde{G}^j(x,t;y)$, and we can proceed as in the above analyses. For terms $\sum_{j=1}^{J} \mathbf{O}(e^{-\eta t})\delta_{x-\bar{a}_j^* t}(-y)$, we proceed as in the proof of (3.5), in which the interactions arising from our relaxation of strict parabolicity are considered. The only genuinely new term is the exponentially decaying contribution, which has reduced decay in t. Focusing on this term, we integrate by parts in y, observing that in the Lax and overcompressive cases differentiation with respect to y improves

t decay by a factor $t^{-1/2}$ (this is a fundamental point of difference between the Lax and overcompressive cases considered here, and the undercompressive case). We have integrals

$$\begin{aligned}
&e^{-\eta|x|}\int_0^{\sqrt{t}}\int_{-\infty}^{+\infty}(t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-1}e^{-\frac{(y-a_k^-s)^2}{Ms}}dy\,ds\\
&\quad\le Ce^{-\eta|x|}t^{-1}\int_0^{\sqrt{t}}(1+s)^{-1}s^{1/2}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds\\
&\quad\le Ce^{-\eta|x|}t^{-1}e^{-\frac{(x-a_j^-t)^2}{Mt}}\int_0^{\sqrt{t}}(1+s)^{-1}s^{1/2}ds\\
&\quad\le C(1+t)^{-3/4}e^{-\eta|x|}e^{-\frac{(x-a_j^-t)^2}{Mt}},
\end{aligned}\tag{6.112}$$

1 for which we observe that

$$e^{-\eta|x|}e^{-\frac{(x-a_j^-t)^2}{Mt}}\le Ce^{-\eta_1|x|}e^{-\eta_2 t}.$$

For the third integral in (6.110), we estimate

$$\begin{aligned}
&e^{-\eta|x|}\int_{t-\sqrt{t}}^{t}\int_{-\infty}^{+\infty}(t-s)^{-1/2}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-3/2}e^{-\frac{(y-a_k^-s)^2}{Ms}}dy\,ds\\
&\quad\le Ce^{-\eta|x|}t^{-1/2}\int_{t-\sqrt{t}}^{t}(1+s)^{-3/2}s^{1/2}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds\\
&\quad\le Ce^{-\eta|x|}t^{-1/2}(1+t)^{-1}e^{-\frac{(x-a_k^-t)^2}{Lt}}\int_{t-\sqrt{t}}^{t}e^{-\epsilon\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds\\
&\quad\le C(1+t)^{-1}e^{-\eta|x|}e^{-\frac{(x-a_k^-t)^2}{Mt}},
\end{aligned}$$

which is sufficient, precisely as above. For the second integral in (6.110), and for t large enough so that $\sqrt{t}<t/2$, we estimate (integrating by parts in y for $s\in[\sqrt{t},t/2]$)

$$\begin{aligned}
&e^{-\eta|x|}\int_{\sqrt{t}}^{t/2}\int_{-\infty}^{+\infty}(t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-1}e^{-\frac{(y-a_k^-s)^2}{Ms}}dy\,ds\\
&\quad+e^{-\eta|x|}\int_{t/2}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}(t-s)^{-1/2}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-3/2}e^{-\frac{(y-a_k^-s)^2}{Ms}}dy\,ds\\
&\le C_2e^{-\eta|x|}t^{-1/2}\int_{t/2}^{t-\sqrt{t}}(t-s)^{-1/2}(1+s)^{-1}s^{1/2}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds\\
&\quad+C_2e^{-\eta|x|}t^{-1/2}\int_{t/2}^{t-\sqrt{t}}(1+s)^{-3/2}s^{1/2}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds.
\end{aligned}\tag{6.113}$$

3 For $s\in[\sqrt{t},t/2]$, we first observe that for the case $j=k$, we have an estimate by

$$C(1+t)^{-1/2}e^{-\eta|x|}e^{-\frac{(x-a_j^-t)^2}{Mt}},\tag{6.114}$$

which is sufficient, as observed above. For the case $j \neq k$, we recall the inequality (6.42) with $\gamma = 1/2$. For the first estimate in (6.42), we proceed as in (6.114), while for the second we estimate

$$C_1 e^{-\eta|x|} t^{-1}(1+|x-a_j^- t|)^{-1/2} \int_{\sqrt{t}}^{t/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds$$
$$\leq C(1+t)^{-1/2} e^{-\eta|x|}(1+|x-a_j^- t|)^{-1/2}.$$

Observing the estimate

$$e^{-\eta|x|}(1+|x-a_j^- t|)^{-1/2} \leq C e^{-\eta_1|x|}(1+t)^{-1/2},$$

we observe that this estimate is sufficient. For $s \in [t/2, t-\sqrt{t}]$, we first observe that in the event that $j = k$, this last integral in (6.113) provides an estimate by

$$C t^{-1/2} e^{-\eta|x|} e^{-\frac{(x-a_j^- t)^2}{Mt}},$$

while in the event that $j \neq k$, we have an estimate by

$$C(1+t)^{-1} e^{-\eta|x|},$$

either of which is sufficient.

(3.18(v)–(vi)), *full scattering estimates.* In the scattering estimates $S(x,t;y)$ of Proposition 2.2, we have three corrections to $\tilde{G}^j(x,t;y)$, respectively from convection, reflection, and transmission. For convection, the term is

$$\tilde{G}_c^j(x,t;y) = ct^{-1/2} e^{-\frac{(x-y-a_j^- t)^2}{4\beta_j t}} \left(\frac{e^{-x}}{e^x + e^{-x}}\right),$$

$a_j^- > 0$, which satisfies

$$|(\partial_t + a_j^- \partial_x)\tilde{G}_c^j(x,t;y)| \leq C[t^{-3/2} + t^{-1/2} e^{-\eta|x|}] e^{-\frac{(x-y-a_j^- t)^2}{4\beta_j t}}.$$

The estimate with $t^{-3/2}$ decay is precisely as in the case of $\tilde{G}^j$, and can be analyzed similarly. For the estimate with exponential decay in $|x|$, we can proceed precisely as with the exponentially decaying term arising in the analysis of $R(x,t;y)$. The remaining corrections, from the reflection and transmission terms in $S(x,t;y)$ can be analyzed similarly.

In our analysis of $\tilde{G}^j(x,t;y)$, we employed the relation

$$\partial_x \tilde{G}^j(x,t;y) = -\partial_y \tilde{G}^j(x,t;y)$$

(see the argument of (6.51), in which the characteristic derivative $(\partial_t + a_j^- \partial_x)\tilde{G}^j(x,t-s;y)$ is converted into a characteristic derivative in the variables of integration $-(\partial_s + a_j^- \partial_y)\tilde{G}^j(x,t-s;y)$). Designating the scattering term arising from reflection as

$$\tilde{G}_R^{j,k}(x,t;y) = c_R t^{-1/2} e^{-\frac{(x-(a_j^-/a_k^-)y-a_j^- t)^2}{4\bar{\beta}_{jk} t}} \left(\frac{e^{-x}}{e^x + e^{-x}}\right),$$

we observe that the analogous estimate is

$$\partial_x \tilde{G}_R^{j,k}(x,t;y) = -\frac{a_k^-}{a_j^-}\partial_y \tilde{G}_R^{j,k}(x,t;y) + \mathbf{O}(t^{-1/2}e^{-\eta|x|})e^{-\frac{(x-(a_j^-/a_k^-)y-a_j^- t)^2}{4\bar{\beta}_{jk}^- t}},$$

from which we see that the conversion from a characteristic derivative in variables x and t to a characteristic derivative in variables y and s takes the form

$$\begin{aligned}(\partial_t + a_j^-\partial_x)\tilde{G}_R^{j,k}(x,t-s;y) &= -(\partial_s + a_k^-\partial_y)\tilde{G}_R^{j,k}(x,t-s;y) \\ &\quad + \mathbf{O}((t-s)^{-1/2}e^{-\eta|x|})e^{-\frac{(x-(a_j^-/a_k^-)y-a_j^-(t-s))^2}{4\bar{\beta}_{jk}^-(t-s)}}.\end{aligned}$$

In this way, we again have precisely the improved decay required by the analysis, namely

$$(\partial_s + a_k^-\partial_y)\tilde{G}_R^{j,k}(x,t-s;y) = \mathbf{O}((t-s)^{-3/2})e^{-\frac{(x-(a_j^-/a_k^-)y-a_j^-(t-s))^2}{4\bar{\beta}_{jk}^-(t-s)}}.$$

The argument involving transmission terms is entirely similar.

Finally, in each case in which a characteristic derivative in x and t is shifted to one in y and s an exponentially decaying error term arises, such as (from reflection)

$$\mathbf{O}(t^{-1/2}e^{-\eta|x|})e^{-\frac{(x-(a_j^-/a_k^-)y-a_j^- t)^2}{4\bar{\beta}_{jk}^- t}}.$$

In all cases, these terms can be analyzed as were the the exponentially decaying corrections in $(\partial_t + a_j^-\partial_x)R(x,t;y)$ (see (6.112)).

This completes the proof of Lemma 3.4. □

7. Nonlinear Integral Estimates II

Finally, we complete the paper by establishing the nonlinear integral estimates of Proposition 3.5.

Proof of Lemma 3.5. To show (3.19)(i), we need to estimate

$$\begin{aligned}&\left|\int_0^t\int_{-\infty}^{+\infty}\mathcal{R}_j^*(x)\mathcal{O}(e^{-\eta_0(t-s)})\delta_{x-\bar{a}_j^*(t-s)}(-y)\mathcal{L}_j^{*t}(y)\Upsilon(y,s)dy\,ds\right| \\ &\quad \le C\int_0^t (e^{-\eta_0(t-s)})|\Upsilon(-x+\bar{a}_j^*(t-s),s)|ds \qquad (7.1)\end{aligned}$$

for the various sources $\Upsilon(y,s)$ arising in the bounds for $\mathcal{F}$, Φ.

For example, the typical term

$$|v||v_{yy}|(y,s) \le C(1+s)^{-1/2}(\psi_1+\psi_2)(y,s)$$

arising in the bounds for $\mathcal{F}$ leads to sources

$$\begin{aligned}\Upsilon_1(y,s) &= (1+s)^{-1/2}\psi_1(y,s)\\ &= (1+s)^{-1/2}(1+|y-a_i^{\pm}t|)^{-3/2}\end{aligned} \tag{7.2}$$

and

$$\begin{aligned}\Upsilon_2(y,s) &= (1+s)^{-1/2}\psi_2(y,s)\\ &\le (1+s)^{-1/2}(1+|y|)^{-1/2}(1+|y|+t)^{-1/2}(1+|y|+t^{1/2})^{-1/2}.\end{aligned} \tag{7.3}$$

Substituting Υ_1 in (7.1), we obtain

$$\begin{aligned}&\int_0^t e^{-\eta_0(t-s)}(1+s)^{-\frac{1}{2}}(1+|x-\bar{a}_j^*(t-s)-a_i^{\pm}s|)^{-\frac{3}{2}}ds\\ &\quad = \int_0^t e^{-\eta_0(t-s)}(1+s)^{-\frac{1}{2}}(1+|x-a_i^-t-(\bar{a}_j^*-a_i^{\pm})(t-s)|)^{-\frac{3}{2}}ds,\end{aligned} \tag{7.4}$$

which, by (5.2), is smaller than

$$\int_0^t e^{-\eta_0(t-s)}(1+s)^{-\frac{1}{2}}(1+|x-a_i^-t|)^{-\frac{3}{2}}(|1+(\bar{a}_j^*-a_i^-)(t-s)|)^{\frac{3}{2}}ds,$$

which in turn $\big($absorbing $t-s$ powers in $e^{\frac{-\eta_0(t-s)}{2}}\big)$ is smaller than

$$\begin{aligned}(1+|x-a_i^-t|)^{-\frac{3}{2}}\int_0^t e^{-\frac{\eta_0}{2}(t-s)}(1+s)^{-\frac{1}{2}}ds &\le C(1+|x-a_i^-t|)^{-\frac{3}{2}}(1+t)^{-1/2}\\ &= C(1+t)^{-1/2}\psi_1(x,t).\end{aligned} \tag{7.5}$$

Similarly, substituting Υ_2 in (7.1), observing that

$$(1+|y|+t^{1/2})^{-1/2} \sim \min\{(1+|y|)^{-1/2},(1+s)^{-1/4}\},$$

and following the same procedure, we obtain an estimate of

$$\begin{aligned}&C(1+t)^{-1/2}(1+|x|)^{-1/2}(1+|x|+t)^{-1/2}\min\{(1+|x|)^{-1/2},(1+t)^{-1/4}\}\\ &\quad\sim C(1+t)^{-1/2}(1+|x|)^{-1/2}(1+|x|+t)^{-1/2}(1+|x|+t^{1/2})^{-1/2}\\ &\quad\le C(1+t)^{-1/2}(\psi_1+\psi_2)(x,t),\end{aligned} \tag{7.6}$$

where, in the final step, we have used the fact that, for $\chi(x,t)=0$,

$$(1+|x|)^{-1/2}(1+|x|+t)^{-1/2}(1+|x|+t^{1/2})^{-1/2} \sim (1+|x|+t)^{-3/2} \le C\psi_1(x,t).$$

Bounds for other cases follow similarly. □

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