M412 Exam 1 Practice Problems (not to be turned in)

In addition to these problems, you should of course consider all previously assigned homework problems to be good practice.

Separation of variables

1. Establish the following integral identities:

$$1. \int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} 0, & n \neq m \\ L, & n = m \end{cases}, \quad n, m = 1, 2, ...$$

$$2. \int_{-L}^{L} \cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = 0, \quad m, n = 1, 2, ...$$

2. Use the method of separation of variables to solve the heat equation

$$\begin{aligned} u_t &= u_{xx} \\ u(t,0) &= u(t,1) = 0 \\ u(0,x) &= \sin(2\pi x) - 3\sin(6\pi x). \end{aligned}$$

3. Use the method of separation of variables to solve the heat equation

$$\begin{aligned} u_t &= u_{xx} \\ u_x(t,0) &= u_x(t,1) = 0 \\ u(0,x) &= \begin{cases} 0, & 0 < x \leq \frac{1}{2} \\ 2x, & \frac{1}{2} < x < 1 \end{cases}. \end{aligned}$$

4. Use the method of separation of variables to solve the heat equation

$$\begin{aligned} u_t &= u_{xx} \\ u(t,0) &= u_x(t,1) = 0 \\ u(0,x) &= -3. \end{aligned}$$

5. Use the method of separation of variables to solve the heat equation

$$\begin{aligned} u_t &= u_{xx} \\ u(t,-1) &= u(t,1); \quad u_x(t,-1) = u_x(t,1) \\ u(0,x) &= 3x - 2. \end{aligned}$$

Method of characteristics

6. Solve the partial differential equation

$$\begin{aligned} u_t + uu_x &= 0 \\ u(0,x) &= x. \end{aligned}$$

7. Haberman 12.2.5 (b).

8. Haberman 12.2.5 (c).

The wave equation

9. Haberman 12.4.1.

10. Haberman 12.4.7.

PDE systems

11. Solve the system of PDE

$$\begin{aligned} u_{1_t} &= u_{1_x} + u_{2_x}; \quad u_1(0,x) = f(x) \\ u_{2_t} &= 4u_{1_x} + u_{2_x}; \quad u_2(0,x) = g(x). \end{aligned}$$

Derivations

For the derivations, you will only need to be able to repeat what we did in class.

Solutions.

Solutions to the Haberman problems are in the back of Haberman.

2.

$$u(t,x) = e^{-4\pi^2 t} \sin 2\pi x - 3e^{-36\pi^2 t} \sin 6\pi x.$$

3.

$$u(t,x) = \frac{3}{4} + \sum_{n=1}^{\infty} \Big\{ -\frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{4}{n^2\pi^2} [(-1)^n - \cos \frac{n\pi}{2}] \Big\} e^{-n^2\pi^2 t} \cos n\pi x.$$

4.

$$u(t,x) = -\sum_{n=1}^{\infty} \frac{12}{(2n-1)\pi} e^{-\frac{(2n-1)^2\pi^2}{4} t} \sin \frac{2n-1}{2} \pi x.$$

5.

$$u(t,x) = -2 + \sum_{n=1}^{\infty} \frac{6}{n\pi} (-1)^{n+1} e^{-n^2\pi^2 t} \sin n\pi x.$$

6.

$$u(t,x) = \frac{x}{1+t}.$$

11.

$$\begin{aligned} u_1(t,x) &= \frac{1}{2} f(x-t) - \frac{1}{4} g(x-t) + \frac{1}{2} f(x+3t) + \frac{1}{4} g(x+3t) \\ u_2(t,x) &= -2\Big[\frac{1}{2} f(x-t) - \frac{1}{4} g(x-t)\Big] + 2\Big[\frac{1}{2} f(x+3t) + \frac{1}{4} g(x+3t)\Big]. \end{aligned}$$