# M412 Practice Problems for Exam 2

Please keep in mind that Exam 2 will be given Wednesday, October 19 7:00–9:00.

## Equilibrium Problems

1. Haberman 1.4.1 Parts (d) and (h).

2. Haberman Problem 1.4.11.

3. For the partial differential equation

$$\begin{aligned} u_t &= u_{xx} + 4x - \gamma \\ u_x(t,0) &= 0 \\ u_x(t,1) &= 0, \\ u(0,x) &= x^2, \end{aligned}$$

find the value of $\gamma$ for which an equilibrium solution exists and then find the equilibrium solution.

4. Solve the PDE from problem 3 for all $t$. (You need not evaluate the integrals that arise for your Fourier coefficients, though note that $A_0$ should be 0 by construction.)

## Laplace's Equation

5. Haberman Problem 2.5.1 (e).

6. Haberman Problem 2.5.5 (a).

7. Haberman Problem 2.5.8 (a).

## The heat equation in two space dimenions

8. Haberman Problem 7.3.1 (c).

9. Haberman Problem 7.3.1 (d).

## The wave equation

10. Solve the partial differential equation

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1, t > 0 \\ u_x(t,0) &= 0 \\ u_x(t,1) &= 0 \\ u(0,x) &= f(x) \\ u_t(0,x) &= g(x). \end{aligned}$$

### Derivation problems

For this exam, you need to be familiar with our derivation of the continuity equation in three dimensions, the heat equation in three dimensions, and the wave equation in one dimension.

11. Haberman 2.5.17.

12. Haberman 2.5.18.

## Solutions

Haberman gives at least partial solutions to problems marked with *. Here are solutions to some of the others.

2. (a)

$$\int_0^L u(t,x)dx = (7-\beta+\frac{1}{2}L^2)t + \int_0^L f(x)dx.$$

(b) $\beta = 7+\frac{1}{2}L^2$,

$$\bar{u}(x) = -\frac{1}{6}x^3 + (7+\frac{1}{2}L^2)x + \frac{1}{L}\int_0^L f(x)dx + \frac{1}{18}L^3 - \frac{1}{2}(7+\frac{1}{2}L^2)L.$$

3. An equilibrium solution exists for $\gamma = 2$,

$$\bar{u}(x) = -\frac{2}{3}x^3 + x^2 + \frac{1}{6}.$$

4.

$$u(t,x) = -\frac{3}{2}x^3 + x^2 + \frac{1}{6} + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t}\cos n\pi x,$$

where

$$A_n = 2\int_0^L (f(x) - (-\frac{2}{3}x^3 + x^2 + \frac{1}{6}))\cos n\pi x dx.$$

9.

$$u(t,x,y) = \sum_{n=1}^{\infty} A_n e^{-k\frac{(n-\frac{1}{2})^2\pi^2}{L^2}t}\sin\frac{(n-\frac{1}{2})\pi}{L}x$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm} e^{-k(\frac{(n-\frac{1}{2})^2\pi^2}{L^2}+\frac{m^2\pi^2}{H^2})t}\sin\frac{(n-\frac{1}{2})\pi x}{L}\cos\frac{m\pi y}{H},$$

where

$$A_n = \frac{2}{LH}\int_0^L \Big(\int_0^H f(x,y)dy\Big)\sin\frac{(n-\frac{1}{2})\pi x}{L}dx$$

and

$$B_{nm} = \frac{4}{LH}\int_0^L \Big(\int_0^H f(x,y)\cos\frac{m\pi y}{H}dy\Big)\sin\frac{(n-\frac{1}{2})\pi x}{L}dx.$$

10.
$$u(t,x) = A_0 + B_0 t + \sum_{n=1}^{\infty} (A_n \cos n\pi t + B_n \sin n\pi t) \cos n\pi x,$$

where

$$\begin{aligned}
A_0 &= \int_0^1 f(x)dx \\
A_n &= 2\int_0^1 f(x) \cos n\pi x dx \\
B_0 &= \int_0^1 g(x)dx \\
B_n &= \frac{2}{n\pi} \int_0^1 g(x) \cos n\pi x dx.
\end{aligned}$$

11. This is precisely the continuity equation that we derived in class.
12. Yes, it's as trivial as it looks.