M412 Assignment 4 Solutions

Two errors in the Practice Problems for Exam 2 have been brought to my attention. In Problem 3, $f(x)$ should be given explicitly as x^2. Also, in the solution to Problem 3, the value of γ should be 2.

1. [10 pts, 5 pts each] Haberman Problem 1.4.1, Parts (f) and (g).

Solutions. For Part (f), the equilibrium solution $\bar{u}(x)$ satisfies

$$\begin{aligned}\bar{u}_{xx} &= -x^2 \\ \bar{u}(0) &= T \\ \bar{u}_x(L) &= 0,\end{aligned}$$

for which

$$\bar{u}(x) = -\frac{1}{12}x^4 + \frac{1}{3}L^3x + T.$$

For Part (g), we have

$$\begin{aligned}\bar{u}_{xx} &= 0 \\ \bar{u}(0) &= T \\ \bar{u}_x(L) + \bar{u}(L) &= 0,\end{aligned}$$

for which

$$\bar{u}(x) = T - \frac{T}{1+L}x.$$

2. [10 pts] Haberman Problem 1.4.5.

Solution. In this case,

$$\begin{aligned}\bar{u}_{xx} &= 0 \\ \bar{u}(0) &= T_1 \\ \bar{u}_x(0) &= T_2 \\ \bar{u}(L) &= T,\end{aligned}$$

where T_1 and T_2 are known and T is to be determined. (The text does not suggestion a notation for the constants labeled T_1 and T_2, so anything is acceptable.) Using only the initial conditions, we find

$$\bar{u}(x) = T_2x + T_1.$$

Setting $\bar{u}(L) = T$, this gives

$$T = T_2L + T_1.$$

3. [10 pts, 5 pts each] Haberman Problem 1.4.7, Parts (a) and (c).

Solution. For Part (a), the equilibrium solution satisfies

$$\begin{aligned}\bar{u}_{xx} &= -1 \\ \bar{u}_x(0) &= 1 \\ \bar{u}_x(L) &= \beta.\end{aligned}$$

Integrating $\bar{u}(x)$ once, we have

$$\bar{u}_x(x) = -x + C_1,$$

for which our two conditions determine first $C_1 = 1$ and second

$$\beta = 1 - L.$$

In order to find the solution, we must integrate a second time

$$\bar{u}(x) = -\frac{1}{2}x^2 + x + C_2.$$

In order to determine the constant C_2, we integrate the full equation,

$$\int_0^L u_t dx = \int_0^L u_{xx} dx + \int_0^L dx = u_x(t, L) - u_x(t, 0) + L = \beta - 1 + L = 0.$$

We have, then,

$$\frac{d}{dt}\int_0^L u dx = 0 \Rightarrow \int_0^L u(t, x) dx = \int_0^L u(0, x) dx = \int_0^L f(x) dx.$$

Since this remains true for all solutions, there must hold

$$\int_0^L \bar{u}(x) dx = \int_0^L f(x) dx.$$

Finally,

$$\int_0^L \bar{u}(x) dx = \int_0^L -\frac{1}{2}x^2 + x + C_2 dx = -\frac{1}{6}L^3 + \frac{1}{2}L^2 + C_2 L,$$

so that

$$C_2 = \frac{1}{L}\int_0^L f(x) dx + \frac{1}{6}L^2 - \frac{1}{2}L.$$

We conclude

$$\bar{u}(x) = -\frac{1}{2}x^2 + x + \frac{1}{L}\int_0^L f(x) dx + \frac{1}{6}L^2 - \frac{1}{2}L.$$

The physical interpretation is that the internal heat production must match the flow caused by a different amount of heat flowing into the bar than out.

For (c), equilibrium solutions satisfy

$$\begin{aligned} \bar{u}_{xx} &= \beta - x \\ \bar{u}_x(0) &= 0 \\ \bar{u}_x(L) &= 0, \end{aligned}$$

from which we deduce

$$\beta = \frac{1}{2}L$$

and

$$\bar{u}_x(x) = \frac{1}{2}Lx - \frac{1}{2}x^2 \Rightarrow \bar{u}(x) = \frac{1}{4}Lx^2 - \frac{1}{6}x^3 + C_2.$$

Proceeding as in Part (a), we find

$$\int_0^L f(x)dx = \int_0^L \frac{1}{4}Lx^2 - \frac{1}{6}x^3 + C_2 dx \Rightarrow C_2 = \frac{1}{L}\int_0^L f(x)dx - \frac{1}{12}L^3 + \frac{1}{24}L^3,$$

and finally

$$\bar{u}(x) = \frac{1}{4}Lx^2 - \frac{1}{6}x^3 + \frac{1}{L}\int_0^L f(x)dx - \frac{1}{24}L^3.$$

4. [10 pts] Haberman Problem 1.4.10.

Solution. Assuming $c = \rho = 1$ (that is, that we have not arrived at this form of the problem through some scaling of the independent variables), the total thermal energy is

$$\text{total thermal energy } = \int_0^L u(t,x)dx.$$

Integrating the full equation, we have

$$\int_0^L u_t(t,x)dx = \int_0^L u_{xx}(t,x)dx + \int_0^L 4dx = u_x(t,L) - u_x(t,0) + 4L = 1 + 4L.$$

In this case,

$$\frac{d}{dt}\int_0^L u(t,x)dx = 1 + 4L \Rightarrow \int_0^L u(t,x)dx = (1+4L)t + C.$$

Evaluating at $t = 0$, we conclude

$$\int_0^L u(t,x)dx = (1+4L)t + \int_0^L f(x)dx.$$

5. [10 pts] Haberman Problem 1.4.12. (See Haberman's equation (1.2.11) for precisely what he means by a conservation law.)

Solution. For Part (a), integrate the full equation to obtain

$$\int_0^L u_t dx = k\int_0^L u_{xx}dx = ku_x(t,L) - ku_x(t,0) = \alpha - \beta.$$

By *conservation law*, Haberman means the expression

$$\frac{d}{dt}\int_0^L u(t,x)dx = \alpha - \beta.$$

For Part (b), integrate as in Problem 1.4.10 to get

$$\int_0^L u(t,x)dx = (\alpha - \beta)t + \int_0^L f(x)dx.$$

For Part (c), we see from Part (b) that all solutions will continue to change in time unless

$$\alpha = \beta.$$

In this case, we have the equilibrium equation

$$\begin{aligned}\bar{u}_{xx} &= 0\\ \bar{u}_x(0) &= -\alpha/k\\ \bar{u}_x(L) &= -\alpha/k,\end{aligned}$$

from which

$$\bar{u}_x(x) = -\alpha/k.$$

Integrating again, we have

$$\bar{u}(x) = -\frac{\alpha}{k}x + C,$$

where by conservation

$$\int_0^L -\frac{\alpha}{k}x + C dx = \int_0^L f(x)dx \Rightarrow C = \frac{1}{L}\int_0^L f(x)dx + \frac{\alpha}{2k}L,$$

and finally

$$\bar{u}(x) = -\frac{\alpha}{k}x + \frac{1}{L}\int_0^L f(x)dx + \frac{\alpha}{2k}L.$$

6. [10 pts] For the PDE

$$\begin{aligned}u_t &= u_{xx} + \gamma x - 1\\ u_x(t,0) &= 0\\ u_x(t,1) &= 0\\ u(0,x) &= x^2,\end{aligned}$$

determine the value of γ for which an equilibrium solution exists, and find the equilibrium solution.

Solution. Equilibrium solutions satisfy

$$\begin{aligned}\bar{u}_{xx} &= -\gamma x + 1\\ \bar{u}_x(0) &= 0\\ \bar{u}_x(1) &= 0,\end{aligned}$$

from which we find

$$\bar{u}_x(x) = -\frac{1}{2}\gamma x^2 + x + C_1.$$

The condition $\bar{u}_x(0) = 0$ sets $C_1 = 0$, while the condition $\bar{u}_x(1) = 0$ gives

$$0 = -\frac{1}{2}\gamma + 1 \Rightarrow \gamma = 2.$$

Integrating again, we have

$$\bar{u}(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + C_2.$$

In order to determine C we observe similarly as in previous problems that $\int_0^1 u(t,x)dx$ is constant for all t, so that

$$\int_0^1 -\frac{1}{3}x^3 + \frac{1}{2}x^2 + C_2 dx = \int_0^1 x^2 dx = \frac{1}{3} \Rightarrow C_2 = \frac{1}{3} + \frac{1}{12} - \frac{1}{6} = \frac{1}{4}.$$

We conclude

$$\bar{u}(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{4}.$$

7. [10 pts] Solve the PDE in Problem 6 for all time.

Solution. In order to solve this PDE for all time, we define the new variable $v(t, x) = u(t, x) - \bar{u}(x)$, where $u(t, x)$ solves the original problem (stated in Problem 6), and $\bar{u}(x)$ is the equilibrium solution from Problem 6. Upon substitution of $u(t, x) = v(t, x) + \bar{u}(x)$ into the original problem, we find that $v(t, x)$ solves

$$\begin{aligned} v_t &= v_{xx} \\ v_x(t, 0) &= 0 \\ v_x(t, 1) &= 0 \\ v(0, x) &= x^2 - (-\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{4}) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{4}. \end{aligned}$$

This equation for $v(t, x)$ can be solved by separation of variables, and we find

$$\begin{aligned} v(t, x) =& A_0 + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \cos n\pi x, \\ A_0 =& \int_0^1 \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{4} dx \\ A_n =& 2\int_0^1 (\frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{4}) \cos n\pi x. \end{aligned}$$

For A_0,

$$A_0 = \frac{1}{12} + \frac{1}{6} - \frac{1}{4} = 0.$$

For A_n, integrate by parts

$$A_n = \frac{2}{n^2\pi^2}(-1)^n + \frac{4}{n^4\pi^4}(1 - (-1)^n) + \frac{2}{n^2\pi^2}(-1)^n.$$

Combining these observations, we conclude

$$\begin{aligned} u(t, x) = & \sum_{n=1}^{\infty} \Big[\frac{4}{n^2\pi^2}(-1)^n + \frac{4}{n^4\pi^4}(1 - (-1)^n)\Big] e^{-n^2\pi^2 t} \cos n\pi x \\ & + - \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{4}. \end{aligned}$$