M412 Practice Problems for Final Exam

1. Solve the PDE

$$\begin{aligned} u_t + t^3 u_x &= u \\ u(t,0) &= t, \quad t > 0 \\ u(0,x) &= 1 - e^{-x}, \quad x > 0. \end{aligned}$$

2. Solve the PDE

$$\begin{aligned} u_{tt} &= c^2 u_{xx}; \quad x > 0, t > 0 \\ u(0,x) &= f(x); \quad x > 0 \\ u_t(0,x) &= g(x); \quad x > 0 \\ u_x(t,0) &= t; \quad t > 0. \end{aligned}$$

3. Solve the PDE

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u(x,0) = 0, &\quad u(x,2) = 0 \\ u(0,y) = 0, &\quad u(1,y) = 2. \end{aligned}$$

4. Solve the PDE

$$\begin{aligned} u_t &= u_{xx} + e^{-t} \sin 3\pi x \\ u(t,0) &= 0, \quad u(t,1) = 0 \\ u(0,x) &= \sin \pi x. \end{aligned}$$

5. For the PDE in Problem 4, find an equilbrium solution and show that it matches the limit as $t \to \infty$ of your solution to Problem 4.

6. For the PDE

$$\begin{aligned} u_t &= u_{xx} + t \sin x \\ u_x(t,0) &= -1 \\ u_x(t,\pi) &= 0 \\ u(0,x) &= \cos x, \end{aligned}$$

find the total energy

$$\int_0^\pi u(t,x)dx.$$

7. Use separation of variables to show that solutions to the quarter-plane problem

$$\begin{aligned} u_t &= u_{xx}; \quad t > 0, x > 0 \\ u_x(t,0) &= 0 \\ |u(t,+\infty)| &\text{ bounded} \\ u(0,x) &= f(x) \end{aligned}$$

can be written in the form

$$u(t,x) = \int_0^\infty C(\omega) e^{-\omega^2 t} \cos \omega x d\omega,$$

for some appropriate constant $C(\omega)$.

8. Use the method of Fourier tranforms to solve the first order equation

$$\begin{aligned} u_t &= u_x \\ u(0,x) &= f(x). \end{aligned}$$

9. [This question appeared on Exam 3.] Use Fourier's Theorem to prove that if a function $f(x)$ is piecewise smooth on an interval $[0,L]$, then the Fourier cosine series for $f(x)$ converges for all $x \in (0,L)$ to

$$\begin{aligned} &(i): f(x) \text{ if } f \text{ is continuous at the point } x \\ &(ii): \frac{1}{2}(f(x^-) + f(x^+)) \text{ if } f \text{ has a jump discontinuity at the point } x \end{aligned}$$

What does the Fourier cosine series converge to at the endpoints $x = 0$ and $x = L$?

10. We have seen in the homework that if a function $f(x)$ is piecewise smooth on an interval $[0,L]$, then the Fourier sine series for $f(x)$ converges for all $x \in (0,L)$ to

$$\begin{aligned} &(i): f(x) \text{ if } f \text{ is continuous at the point } x \\ &(ii): \frac{1}{2}(f(x^-) + f(x^+)) \text{ if } f \text{ has a jump discontinuity at the point } x. \end{aligned}$$

Use this and Problem 9 to prove that if $f(x)$ is continuous on $[0,L]$ and $f'(x)$ is piecewise smooth on the same interval, then the Fourier cosine series for $f(x)$ can be differentiated term by term.

Solutions

1. For $x \geq \frac{t^4}{4}$, we have

$$\begin{aligned} \frac{dx}{dt} &= t^3; \quad x(0) = x_0 \Rightarrow x(t) = \frac{t^4}{4} + x_0 \\ \frac{du}{dt} &= u; \quad u(0) = 1 - e^{-x_0} \Rightarrow u(t) = (1 - e^{-x_0})e^t, \end{aligned}$$

from which we conclude

$$u(t,x) = (1 - e^{-(x - \frac{t^4}{4})})e^t.$$

For $x \leq \frac{t^4}{4}$, we have

$$\begin{aligned} \frac{dx}{dt} &= t^3; \quad x(t_0) = 0 \Rightarrow x(t) = \frac{t^4}{4} - \frac{t_0^4}{4}, \\ \frac{du}{dt} &= u; \quad u(t_0) = t_0 \Rightarrow u(t) = t_0 e^{t - t_0}, \end{aligned}$$

from which we conclude

$$u(t,x) = (t^4 - 4x)^{1/4} e^{t - (t^4 - 4x)^{1/4}}.$$

Combining these,

$$u(t,x) = \begin{cases} (t^4 - 4x)^{1/4} e^{t-(t^4-4x)^{1/4}}, & x \le \frac{t^4}{4} \\ (1 - e^{-(x-\frac{t^4}{4})})e^t, & x \ge \frac{t^4}{4}. \end{cases}$$

2. We write solutions in the form

$$u(t,x) = F(x-ct) + G(x+ct),$$

where for $x > 0$, we have

$$\begin{aligned} F(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(y)dy \\ G(x) &= \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(y)dy. \end{aligned}$$

This entirely determines the solution for $x - ct > 0$. For $x - ct < 0$, we need to evaluate F at negative numbers. In order to do this, we notice that our final condition gives

$$t = F'(-ct) + G'(ct).$$

Setting $x = -ct$, we find

$$F'(x) = -\frac{x}{c} - G'(-x).$$

We compute, now,

$$\int_0^x F'(y)dy = \int_0^x -\frac{y}{c} - G'(-y)dy \Rightarrow F(x) - F(0) = -\frac{x^2}{2c} + G(-x) - G(0).$$

It's clear from our expressions for F and G that (assuming our solution is continuous) $F(0) = G(0)$, from which we conclude

$$F(x) = -\frac{x^2}{2c} + G(-x).$$

In this we, for $x - ct < 0$,

$$F(x-ct) = -\frac{(x-ct)^2}{2c} + G(ct-x).$$

We have, then

$$u(t,x) = \begin{cases} \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy, & x - ct > 0 \\ -\frac{(x-ct)^2}{2c} + \frac{1}{2}[f(ct-x) + f(x+ct)] + \frac{1}{2c}\int_0^{x+ct} g(y)dy + \frac{1}{2c}\int_0^{ct-x} g(y)dy, & x - ct < 0. \end{cases}$$

3. Since we have a bounded domain, we proceed by separation of variables, letting $u(x,y) = X(x)Y(y)$, for which we find

$$u_{xx} + u_{yy} = 0 \Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0 \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda.$$

Observe here in particular that we have chosen the sign in front of λ so that the variable with both boundary conditions 0 (Y in this case) will have the standard eigenvalue equation, $Y'' + \lambda Y = 0$. We have, $u(x,0) = 0 \Rightarrow Y(0) = 0$, $u(x,2) = 0 \Rightarrow Y(2) = 0$, and $u(0,y) = 0 \Rightarrow X(0) = 0$. We have, then, the two ODE

$$\begin{aligned} Y'' + \lambda Y &= 0; \quad Y(0) = 0, Y(2) = 0 \\ X'' - \lambda X &= 0; \quad X(0) = 0. \end{aligned}$$

For the $Y(y)$ equation, we take $Y(y) = C_1 \cos\sqrt{\lambda}y + C_2 \sin\sqrt{\lambda}y$, and use the boundary conditions to conclude

$$Y_n(y) = \sin\frac{n\pi}{2}y, \quad n = 1, 2, 3....$$

For $X(x)$, we have

$$X(x) = C_3 \cosh\frac{n\pi}{2}x + C_4 \sinh\frac{n\pi}{2}x,$$

for which our boundary condition $X(0) = 0$ determines $C_3 = 0$, eliminating one constant of integration. We finally have our general expansion for $u(x, y)$,

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi}{2}x \sin\frac{n\pi}{2}y.$$

Finally, we employ our last boundary condition, $u(1, y) = 2$ to obtain the Fourier sine series

$$2 = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi}{2} \sin\frac{n\pi}{2}y.$$

We have, then

$$A_n \sinh\frac{n\pi}{2} = \frac{2}{2}\int_0^2 2\sin\frac{n\pi}{2}y dy = -\frac{4}{n\pi}\cos\frac{n\pi}{2}y\Big|_0^2 = -\frac{4}{n\pi}[(-1)^n - 1],$$

where I have explicitly written the fraction $\frac{2}{2}$ as a reminder that it comes from $\frac{2}{H}$. Our solution is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{-\frac{4}{n\pi}[(-1)^n - 1]}{\sinh\frac{n\pi}{2}} \sinh\frac{n\pi x}{2} \sin\frac{n\pi}{2}y.$$

4. Due to the non-homogeneous term, we must proceed here by eigenfunction expansion. First, we construct eigenfunctions, $X_n(x)$, for the homogeneous problem. Substituting $u(t, x) = T(t)X(x)$ into $u_t = u_{xx}$, and considering our boundary conditions, we determine

$$X'' + \lambda X = 0; \quad X(0) = 0, X(1) = 0,$$

for which we have $X_n(x) = \sin n\pi x$. We now look for a solution as an expansion of these eigenfunctions

$$u(t, x) = \sum_{n=1}^{\infty} c_n(t) \sin n\pi x.$$

Substituting this expansion back into the full non-homogeneous equation, we find

$$\sum_{n=1}^{\infty} \Big(c_n'(t) + n^2\pi^2 c_n(t)\Big) \sin n\pi x = e^{-t}\sin 3\pi x.$$

The key observation we make here is that this is simply a Fourier sine series with fancy constants, $B_n = c_n'(t) - n^2\pi^2 c_n(t)$. Consequently, we have

$$c_n'(t) + n^2\pi^2 c_n(t) = 2\int_0^1 e^{-t}\sin(3\pi x)\sin(n\pi x)dx = \begin{cases} e^{-t}, & n = 3 \\ 0, & n \neq 3. \end{cases}$$

For initial conditions, we take our initial data

$$u(0,x) = \sin \pi x \Rightarrow \sin \pi x = \sum_{n=1}^{\infty} c_n(0) \sin n\pi x,$$

for which

$$c_n(0) = 2\int_0^1 \sin(\pi x)\sin(n\pi x)dx = \begin{cases} 1, & n = 1 \\ 0, & n \neq 1. \end{cases}$$

We have now an ODE to solve for each $n = 1, 2, 3...$, but we observe that if both $c_n'(t) + n^2\pi^2 c_n(t)$ and $c_n(0)$ are 0, the $c_n(t) \equiv 0$. In this case, the only two expansion coefficients that are not identically 0 are $c_1(t)$ and $c_3(t)$. For $c_1(t)$, we have

$$c_1' + \pi^2 c_1 = 0; \quad c_1(0) = 1 \Rightarrow c_1(t) = e^{-\pi^2 t}.$$

For $c_3(t)$, we have

$$c_3' + 9\pi^2 c_3 = e^{-t}; \quad c_3(0) = 0,$$

which we solve by the integrating factor method. (Recall that for a general linear first order equation $y'(t) + p(t)y(t) = g(t)$, the integrating factor is $e^{\int p(t)dt}$, where the constant of integration can be dropped.) In this case, the integrating factor is simply $e^{9\pi^2 t}$, and we have

$$(e^{9\pi^2 t} c_3)' = e^{9\pi^2 t} e^{-t} \Rightarrow e^{9\pi^2 t} c_3(t) = \frac{1}{9\pi^2 - 1} e^{-t(1-9\pi^2)} + C.$$

According to our intial condition $c_3(0) = 0$, we have

$$C = \frac{1}{1 - 9\pi^2}.$$

We conclude that

$$c_3(t) = \frac{1}{1 - 9\pi^2}(e^{-9\pi^2 t} - e^{-t}),$$

with then

$$u(t,x) = e^{-\pi^2 t}\sin(\pi x) + \frac{1}{1 - 9\pi^2}(e^{-9\pi^2 t} - e^{-t})\sin(3\pi x).$$

5. Our equilibrium equation for $\bar{u}(x)$ is

$$\begin{aligned} \bar{u}_{xx} &= 0 \\ \bar{u}(0) &= 0 \\ \bar{u}(1) &= 0, \end{aligned}$$

which is solved by

$$\bar{u}(x) \equiv 0.$$

Taking a limit as $t \to \infty$ of our solution to Problem 4, we see that they agree.

6. Integrating the full equation, we have

$$\int_0^\pi u_t dx = \int_0^\pi u_{xx} dx + \int_0^\pi t \sin x dx \Rightarrow \frac{d}{dt}\int_0^\pi u(t,x)dx = u_x(t,\pi) - u_x(t,0) - t\cos x\Big|_0^\pi.$$

It follows that

$$\frac{d}{dt}\int_0^\pi u(t,x)dx = 1 + 2t.$$

Integrating,

$$\int_0^{\pi} u(t,x)dx = t + t^2 + C.$$

In order to find C, we use $u(0,x) = \cos x$ to compute

$$\int_0^{\pi} \cos x dx = C \Rightarrow C = 0.$$

We conclude

$$\int_0^{\pi} u(t,x)dx = t + t^2.$$

7. Separate variables with $u(t,x) = T(t)X(x)$, and set

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda,$$

from which we have the eigenvalue problem

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= 0 \\ X(+\infty) & \text{ bounded.} \end{aligned}$$

In this case, all $\lambda \geq 0$ are eigenvalues, with associated eigenfunctions

$$X_\lambda(x) = \cos \sqrt{\lambda} x.$$

Since the eigenvalues are continuous, we integrate rather than summing, obtaining a general solution of the form

$$u(t,x) = \int_0^{\infty} A(\lambda) e^{-\lambda t} \cos \sqrt{\lambda} x d\lambda.$$

Finally, set $\omega = \sqrt{\lambda}$ to get

$$u(t,x) = \int_0^{\infty} A(\omega^2) e^{-\omega^2 t} \cos \omega x 2\omega d\omega.$$

The stated result follows from the choice

$$C(\omega) = 2\omega A(\omega^2).$$

8. Taking the Fourier transform of this equation, we have

$$\begin{aligned} \hat{u}_t &= -i\omega \hat{u} \\ \hat{u}(t,\omega) &= \hat{f}(\omega) e^{-i\omega t}. \end{aligned}$$

Inverting, we compute

$$u(t,x) = \int_{-\infty}^{+\infty} e^{-i\omega x} \hat{f}(\omega) e^{-i\omega t} d\omega = \int_{-\infty}^{+\infty} e^{-i\omega(x+t)} \hat{f}(\omega) d\omega,$$

where this last expression is the inverse transform of $\hat{f}$, evaluated at $x + t$. That is,

$$u(t,x) = f(x+t).$$

9. Since $f(x)$ is only defined on the interval $[0, L]$, we are free to extend it in any way we like to the full interval $[-L, L]$, where Fourier's theorem is valid. We extend it as an even function, so that the extension $f_E(x)$ is defined by

$$f_E(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L \le x \le 0. \end{cases}$$

If $f(x)$ is piecewise smooth on $[0, L]$, then $f_E(x)$ is piecewise smooth on $[-L, L]$, and Fourier's Theorem states that f_E definitely has a convergent Fourier series,

$$f_E(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L}.$$

We now compute A_0, A_n, and B_n, keeping in mind that $f_E(x)$ is an even function. We have

$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^{+L} f_E(x) dx = \frac{1}{L} \int_0^L f_E(x) dx \\ A_n &= \frac{1}{L} \int_{-L}^{+L} f_E(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L f_E(x) \cos \frac{n\pi x}{L} dx \\ B_n &= \frac{1}{L} \int_{-L}^{+L} f_E(x) \sin \frac{n\pi x}{L} dx = 0. \end{aligned}$$

In this way, we see that the series for $f_E(x)$ is a Fourier cosine series that converges on $[-L, L]$. If it converges on $[-L, L]$, it must converge on $[0, L]$, and since $f(x)$ and $f_E(x)$ agree there, it converges to $f(x)$.

Last, since $f_E(x)$ is an even extension, we have

$$\begin{aligned} \lim_{x \to 0^-} f_E(x) &= \lim_{x \to 0^+} f_E(x) \\ \lim_{x \to L^-} f_E(x) &= \lim_{x \to -L^+} f_E(x), \end{aligned}$$

so that the Fourier cosine series of $f(x)$ converges at $x = 0$ to

$$\lim_{x \to 0^+} f(x),$$

and at $x = L$ to

$$\lim_{x \to L^-} f(x).$$

10. First, under these assumptions, $f(x)$ has a convergent Fourier cosine series (by Problem 9),

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}.$$

Moreover, $f'(x)$ has a convergent sine series

$$f'(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L},$$

with

$$\begin{aligned} B_n &= \frac{2}{L} \int_0^L f'(x) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \Big[f(x) \sin \frac{n\pi x}{L} \Big|_0^L - \frac{n\pi}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \Big] \\ &= -\frac{n\pi}{L} A_n, \end{aligned}$$

which gives precisely the series that arises by differentiating the Fourier cosine series of $f(x)$ term by term.