Blackjack Project

Due Wednesday, Dec. 6

1 Overview

Blackjack, or twenty-one, is certainly one of the best-known games of chance in the world. Even if you've never stepped foot in a casino in your life, you probably know the basic idea: try to get as close to 21 as possible without going over. Though the complete rules of blackjack vary from casino to casino, the game I'll describe is fairly representative of what you're likely to find. I've tried to avoid as much jargon as possible, but several discriptions become prohibitively tedious without such words as "natural," "soft hands," and "hole" cards, so they'll be defined as we go.

2 The rules of play

1. *The number of players.* At most blackjack tables, there is one dealer and from one to six players. The player to the dealer's left receives his cards first, and is said to be on "first base." The final player on the dealer's right is the last to get his cards, and is said to be on "third base."[1]

2. *The deck.* Most casinos continue to use 52 card decks, often two decks or six decks at a time to discourage card counters. However many decks the dealer is using, he will stop roughly half way through and reshuffle.

3. *Betting.* The players make their bets before any cards are dealt.

4. *The deal.* The players and dealer receive two cards each, the dealer's one up and one down, the players' both down. Cards received down are typically referred to as "hole" cards.

5. *Numerical value of the cards.* The numerical value of each face card—jack, queen, and king—is ten. Except for the ace, the numerical value of each other card is simply that card's face value. Critically, the ace can be counted as either a one or an eleven. We refer to a hand with an undetermined ace in it as a "soft" hand and a hand without an undetermined ace in it as a "hard" hand. (For example, a hand with two tens and an ace would be considered "hard" because counting the ace as 11 would bust the hand, so it's determined.)

[1] Try to ignore the fact that this is exactly opposite to a batter's view of the baseball diamond.

6. *Object of play.* The goal of (casino) blackjack is to obtain a total that is greater than the dealer's but does not exceed twenty-one.

7. *Naturals.* If either the player or the dealer receives a total of 21 on his first two cards, the hand is called a *natural.* If a player has a natural and the dealer does not, the player receives $1\frac{1}{2}$ times his original bet from the dealer. If the dealer has a natural and the player does not, the player loses his original bet. (The dealer checks his hand after the deal, and if he has a natural, play stops. The player loses unless he also has a natural.) If both the player and the dealer have naturals, no money changes hands.

8. *The player's draw.* Beginning with the man on first base, the dealer goes around the table asking each player if he would like more cards. The player can take as many cards as he likes, one at a time, face up, until he either "stands" (quits taking cards) or "busts" (goes over 21). If the player busts, he must immediately turn up his hole cards, and he loses his original bet, which is a critical rule in two respects. First, it means that even if the dealer eventually busts, the player has lost—the casino's primary advantage. And second, it means that the remaining players get to see this player's cards prior to making their own decisions.

9. *The dealer's draw.* In almost any casino, the dealer's play will be fixed and his rules of play will be posted at the table. The most common dealer play is as follows: once the players all have their cards, he turns up his single hole card. If his total is 16 or less he *must* draw another card. If his total is 17 or more (necessarily counting his first ace as an eleven[2]), he *must* stand. He continues drawing until his total is 17 or more.

10. *The settlement.* If the player doesn't bust and the dealer does, the player wins even money (the amount he bet). If neither the player nor the dealer busts, the one with with higher total wins an amount equal to the player's original bet. If the dealer and the player both have the same total, no money changes hands.

11. *Splitting pairs.* If a player's initial two cards form a pair (identical, except in suit), he can choose to turn them both face up and play out two separate "twin" hands. His original bet goes on one of the two hands, and an equal amount (usually, non-negotiably) is bet on the other, thus doubling the player's total bet. The player automatically receives a second card face down on each of the split cards and may continue drawing cards face up as usual. One exception to this general rule is that if a player splits aces, he gets exactly one additional card on each hand. If the points in one of his hands sums to 21 in this case, *it does not count as a natural.* (Similarly, if the player has split tens or face cards, an ace will not give him a natural.) Finally, if a player splits a pair and receives a third matching card on one of the hands, he is not permitted to split the new hand.

12. *Doubling down.* After looking at his initial two cards, but before he has drawn, a player may elect to double his bet and draw exactly one more card. A player splitting any pair besides aces can elect to double down on one or both of his twin hands.

[2] Some casinos give their dealers some leeway here, and let them draw on 17 if the 17 includes counting an ace as 11.

3 Definitions

Our approach will be to determine the gaming decision that will optimize the player's expectation at each stage of the game. The following definitions will be critical in our analysis.

1. W is a random variable representing the player's winnings (or losses).

2. x represents the player's total at the time of his decision.

3. D represents the value of the dealer's up card.

4. T represents the dealer's total at the end of the game.

5. J represents the total obtained by a player after drawing one card to whatever he currently holds.

6. $E_S^x[W] = E[W|\text{Player stands with } x]$ represents the expectation of a player who stands on x.

7. $E_D^x[W] = E[W|\text{Player draws exactly one card to } x]$ represents the expectation of a player who draws exactly one more card on x.

8. $M(D)$ represents the minimum standing number. A player holding $x \geq M$ should stand. A player holding $x < M$ should draw.

Notice that E_S^x and E_D^x are conditional expectations, conditioned on the player's having total x and using a particular strategy. This superscript/subscript notation will save a lot of writing. Also, notice that if $E_D^x[W] - E_S^x[W] > 0$, the player should draw. If $E_D^x[W] - E_S^x[W] < 0$, the player should stand. (If these two expectations are the same, the player can either stand or draw with equal expectation of success.) Finally, the D in $E_D^x[W]$ stands for *draw*, and is unrelated to the value D representing the dealer's up card.

At first glance, it might seem possible that while one draw will hurt our chances, multiple draws could improve them. For example, it might appear plausible that we could have $E_D^{12} - E_S^{12} < 0$, but $E_D^{15} - E_S^{12} > 0$. For hard hands, however, we observe that this is not possible. The odds of drawing (in this case) from a 12 to a 13, 14, or 15 are precisely those of drawing from 15 to 22, 23, and 24 (all clearly losing hands). In this way, we create more ways of losing without creating any additional ways of winning, so $E_D^{15} < E_D^{12}$. (One caveat here: In the event that $E_D^x - E_S^x = 0$, and for x small enough so that we cannot possibly bust, we *can* draw to a better position. See the discussion in Section 4.1 below.)

4 Expectations

For simplicity, we will assume that the player bets one dollar, keeping in mind that the optimal strategy for a one dollar bet is exactly the same as the optimal strategy for any other bet (in the sense of expected value). First, we develop a representation for the expected value of a player who chooses to stand on total $x < 21$. (If $x \geq 21$, the player's optimal strategy is clear.) We have,

$$\begin{aligned} E_S^x[W] &= 1 \cdot P(T > 21) + 1 \cdot P(T < x) - 1 \cdot P(x < T \leq 21) \\ &= P(T > 21) + P(T < x) - \Big(1 - P(\{T \leq x\} \cup \{T > 21\})\Big) \\ &= P(T > 21) + P(T < x) - 1 + P(T \leq x) + P(T > 21), \end{aligned}$$

which we will record in the final form

$$E_S^x[W] = 2P(T > 21) + 2P(T < x) + P(T = x) - 1. \tag{1}$$

Similarly, we find a convenient expression for the expectation of a player who draws exactly one card on total x. For this calculation, we will condition on the random variable J, the total a player obtains after exactly one draw:

$$E_D^x[W] = E_D^x[W|J < 17]P(J < 17) + \sum_{j=17}^{21} E_D^x[W|J = j]P(J = j) + E_D^x[W|J > 21]P(J > 21).$$

Each of these conditional expectations is straightforward to compute. We have (keeping in mind that we consider a one dollar bet),

$$\begin{aligned} E_D^x[W|J < 17] &= 1 \cdot P(T > 21) - 1 \cdot P(T \leq 21), \\ E_D^x[W|J = j] &= 1 \cdot P(T > 21) + 1 \cdot P(T < j) - 1 \cdot P(j < T \leq 21), \\ E_D^x[W|J > 21] &= -1. \end{aligned}$$

Combining, we conclude (rearranging terms)

$$\begin{aligned} E_D^x[W] =& P(J < 17)\Big(2P(T > 21) - 1\Big) - P(J > 21) \\ &+ \sum_{j=17}^{21} P(J = j)\Big(P(T > 21) + P(T < j) - P(j < T \leq 21)\Big). \end{aligned} \tag{2}$$

Proceeding similarly, we develop our *decision equation*,

$$\begin{aligned} E_D^x[W] - E_S^x[W] =& -2P(T < x) - P(T = x) - 2P(T > 21)P(J > 21) \\ &+ \sum_{t=17}^{21} P(T = t)\Big(2P(t < J \leq 21) + P(J = t)\Big). \end{aligned} \tag{3}$$

Clearly, the decision equation requires some justification. Justifying this equation will be the first assignment of the project.

4.1 Consistency Check of Decision Equation

Let's consider two baseline cases: $x(\text{hard}) < 12$ and $x(\text{soft}) < 17$, for which we know we want to draw. Does our decision equation (3) agree with our intuition? Observe that in each of these cases, the first three addends on the right-hand side of the decision equation are all zero, the first two because $T \geq 17$ by dealer's rules, and the third because, since we cannot bust by drawing with $x(\text{hard}) < 12$ or $x(\text{soft}) < 17$, $P(J > 21) = 0$. All that's left is the sum of probabilites, which must all be greater than 0 by our axioms of probability. Hence, $E_D^x[W] - E_S^x[W] \geq 0$, and we conclude that it's at least as good to draw as to stand, and we should draw. In order to understand this last assertion, consider the case $x(\text{hard}) = 4$, the lowest hard hand possible. In this case, every probability in equation (3) is zero, and we have

$$E_D^x[W] - E_S^x[W] = 0.$$

That is, for $x(\text{hard})$ this small, whether we stand or draw, our only chance of winning is if the dealer busts. Since we cannot bust, however, we clearly want to draw.

5 Assignments

1. Finish our derivation of the decision equation by establishing equation (3).

2. (Player's probabilities) Assuming the case of a perfectly shuffled deck of cards, and no counting (i.e., if a certain card is drawn, it is no less likely that that card will be drawn again, which is almost the case for multiple decks of cards), determine for $12 \leq x(\text{hard}) \leq 17$, the player's probabilities for the decision equation (3):

$$\begin{aligned} &P(J = t), t = 17, ..., 21 \\ &P(t < J \leq 21), t = 17, ..., 21 \\ &P(J > 21). \end{aligned}$$

Keep in mind that each of these probabilities is actually a conditional probability, conditioned on the player's current hand, $x(\text{hard})$. For example, in the case $x(\text{hard}) = 12$, you will need to compute

$$\begin{aligned} &P(J = t | x(\text{hard}) = 12), t = 17, ..., 21 \\ &P(t < J \leq 21 | x(\text{hard}) = 12), t = 17, ..., 21 \\ &P(J > 21 | x(\text{hard}) = 12). \end{aligned}$$

3. Write a MATLAB M-file that simulates a dealer using the rules described in these notes. Notice that dealer's probabilities should be conditioned on the knowledge we have, that the dealer's up card has value D. In fact, whatever decision we make (in the absence of counting) must be based solely on the value of this card. Consequently, each dealer probability in equation (3) should be regarded as a conditional probability. We might want to compute, for example, $P(T = 17 | D = 10)$—the probability that the dealer ends up with 17, given that he has a 10 showing. Observe that this would be an extremely tedious calculation analytically. We would have to consider every possible way the dealer could get from 10 to 17, from multiple draws like a 2 in the hole and draws 2,2,A, to a 7 in the hole and no draw. This is why we proceed in the dealer's case through simulation.

4. Focus for a moment on the probability $P(T = 17|D = 6)$, which you will obtain by simulation using your program from Part 3. Determine the number of hands that must be simulated to ensure that (follow me here) the probability that the error on this probability is larger than .005 is less than .05. More generally, you will be using the same program to estimate either six or seven probabilities at once, depending upon whether or not a natural is possible. Assuming six observations, determine the number of simulations required to ensure that the probability of one or more of the errrors being larger than .005 is less than .05. (I.e., there is a 95% chance that none of the probabilities has an error this large. Notice that it's more or less fair here to assume that the random variables are independent, even though they're clearly not. They would be entirely independent if you considered them to arise from independent simulations, and so doing the whole thing with one simulation can be considered a minor cheat to save time.)

5. Use the number of simulations determined from Part 4 to develop a table of Dealer's probabilities. For calibration, you should compare your results with the following:

For the case $D = 6$, we find:

```
>>dealer21
17: 0.16551
18: 0.10615
19: 0.10629
20: 0.10173
21: 0.097125
21, natural: 0
busted: 0.4232
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For the case $D = 2$, we find

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>>dealer21
17: 0.13991
18: 0.13488
19: 0.12967
20: 0.12394
21: 0..11807
21, natural: 0
busted: 0.35352
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Since these probabilities are approximate (that's the nature of simulation), you will not get exactly the same values, but your values should agree with these within $\pm.005$ (see Part 4).

6. Use equation (3) and the analysis above to develop a basic strategy (no splitting pairs or doubling down) for hard blackjack hands. That is, develop a table of minimum standing numbers $M(D)$ for all possible values of D.