M442, Spring 2022 Practice Problems for the Final

The final exams for M442 Sections 930 and 931 will be held in the usual classroom, Blocker 148, on the following schedule:

- Section 930: Monday, May 9, 3:30 p.m. – 5:30 p.m.

- Section 931: Monday, May 9, 10:30 a.m. – 12:30 p.m.

The exam will cover the following topics: solving ODE in MATLAB, including initial value problems, event location, boundary value problems, and parameter estimation; computing probabilities via counting arguments, including permutations and combinations; conditional probability and computing probability by conditioning, including the multiplication rule and Bayes' Lemma; random variables and expected value; computing expected values by condititioning; game theory, including both pure and mixed strategies.

The exam will consist of two parts: Part 1 will not require MATLAB, while Part 2 will require MATLAB. Students will not be allowed to use notes or references for Part 1, and students must submit Part 1 before beginning Part 2. For Part 2, students can access any notes, references, M-files etc., except no communication between students is allowed.

For some questions, students may be expected to access data files from the course web site.

Office hours the week before the exam will be WRF 1:30 p.m. - 3:00 p.m., both via Zoom and in Blocker 625B. A link for the Zoom meeting will be sent to the class via email.

Practice Problems for Part 1

1. Suppose A and B are events on a sample space S.

a. Show that
$$P(A \cap B^c) = P(A) - P(A \cap B).$$

b. Use (a) to establish

$$P(A^c \cap B^c) = 1 - P(A) - P(B) + P(A \cap B).$$

2. Suppose there are 15 students in a certain class, and they are to be arranged into 5 groups of 3 students each. In how many ways can this arrangement be made?

3. On a multiple choice exam with three possible answers for each of the five questions, what is the probability that a student would get four or more correct answers purely by guessing? It's good practice to work this two different ways, once using a combinatorial argument and once using independence.

4. What is the probability of getting each of the following hands in Texas Hold'em:

a. Straight flush.

b. Full house.

5. Suppose a certain M442 student can solve a regression problem correctly with probability .8, a dimensional analysis problem with probability .7, and a phase plane problem with

probability .25. If a pop quiz is given and there is a 10% chance the problem will be on regression, a 50% chance the problem will be on dimensional analysis, and a 40% chance the problem will be on phase planes, what is the probability the student will be able to solve the problem correctly?

6. Suppose a certain jury is 60% certain that a suspect is guilty, and a new piece of evidence is introduced that shows that both the suspect and whoever committed the crime have a characteristic only shared by 20% of the population. (For example, suppose the suspect has red hair, and a witness observed that whoever committed the crime had red hair.) How certain should the jury be?

7. Suppose we have three cards identical in form except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red, the other black. The three cards are mixed up in a hat, and one card is randomly selected and put down on the ground. If the up side of the chosen card is colored red, what is the probability that the other side is colored black?

8. Show that if X and Y are independent discrete random variables, and $E[|X|]$ and $E[|Y|]$ are both finite, then

$$E[XY] = E[X]E[Y].$$

Explain why you need to assume $E[|X|]$ and $E[|Y|]$ are finite.

9. Suppose a fair coin is flipped until it lands heads on two consecutive flips. What is the expected number of flips required?

10. Find the optimal strategies and game-theoretic value for the matrix game depicted in Figure 1.

Rose/Colin	A	B	C
A	5	2	1
B	4	1	3
C	3	4	3
D	1	6	2

Figure 1: Matrix game for Problem 10.

11. Find the optimal strategies and game-theoretic value for the matrix game depicted in Figure 2.

12. Find the optimal strategies and game-theoretic value for the matrix game depicted in Figure 3.

13. Find the optimal strategies and game-theoretic value for the matrix game depicted in Figure 4.

Rose/Colin	A	B	C
A	2	1	−5
B	10	5	−10
C	1	−3	0

Figure 2: Matrix game for Problem 11.

Rose/Colin	A	B	C
A	−7	0	3
B	4	−1	−2
C	−5	2	1

Figure 3: Matrix game for Problem 12.

14. Find the optimal strategies and game-theoretic value for the matrix game depicted in Figure 5.

Practice Problems for Part 2

1. For a pendulum of length l attached to a pivot that rotates with angular velocity ω along an upright circular frame with radius r (see Figure 6), the equation for the angle θ is

$$\theta'' - \frac{r\omega^2}{l}\cos(\theta - \omega t) + \frac{g}{l}\sin\theta = 0.$$

Fix $r = 1$, $\omega = 1$, $l = 1$, and $g = 9.81$, and initialize this equation with $\theta(0) = 0$ and $\theta'(0) = 0$. (Notice that at $t = 0$ the angle inside the circle will be 0.) Determine the number of times $t > 0$ at which the pendulum will hang straight down ($\theta(t) = 0$) as the pivot moves once around the circle (i.e., as t runs from 0 to 2π). Give the values of t at which these events occur.

2. For a simple pendulum under the influence of gravity and physical air resistance, the pendulum angle $\theta(t)$ satisfies the ODE

$$\theta'' = -\frac{g}{l}\sin\theta - b|\theta'|\theta',$$

Rose/Colin	A	B	C	D
A	1	−1	2	−4
B	2	3	−5	4

Figure 4: Matrix game for Problem 13.

Rose/Colin	A	B	C	D
A	4	−3	2	−4
B	4	−4	4	−2
C	0	1	−3	2
D	−5	2	−7	2
E	3	−2	2	−2

Figure 5: Matrix game for Problem 14.

where b is the coefficient of air resistance. Take $g = 9.81\,\text{m/s}^2$ and $l = .4\,\text{m}$ and find two different values of b so that the following three initial-boundary conditions hold: $\theta(0) = \frac{\pi}{4}$, $\theta'(0) = 0$, and $\theta(1) = 0$.

3. Data for a predator-prey interaction is stored in the M-file *lvdata.m* (available on the course web site), and in this problem we'll fit this data to the Lotka-Volterra predator-prey model

$$\frac{dy_1}{dt} = ay_1 - by_1y_2$$
$$\frac{dy_2}{dt} = -ry_2 + cy_1y_2.$$

a. Use the central difference derivative approximation to determine values for the parameters a, b, c, and r. Create a stacked plot of your solutions $y_1(t)$ (prey population) and $y_2(t)$ (predator population), each plotted along with the corresponding data.

b. Use nonlinear least squares regression to refine your values of a, b, c, and d from Part (a). Again, create a stacked plot for your model and data, and also compute standard deviation values s_1 and s_2.

4. Repeat the nonlinear fit from Problem 3, taking the initial values $y_1(0)$ and $y_2(0)$ to be parameters. As initial parameter values, use the values obtained from your nonlinear fit for
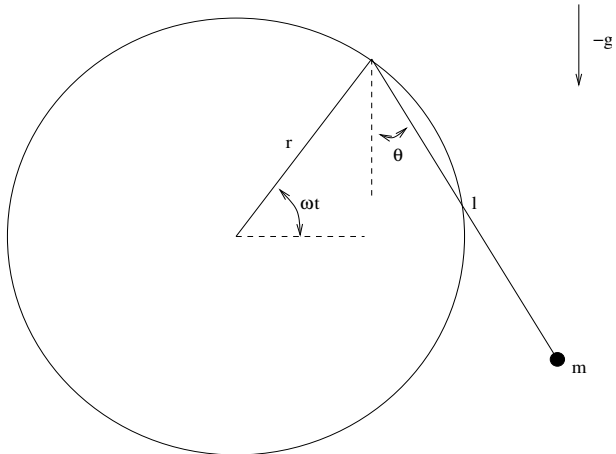
Figure 6: Figure for Problem 1 from Part 2.

Problem 3 along with $y_1(0) = 30$ and $y_2(0) = 4$. Again, create a stacked plot for your model and data, and also compute standard deviation values s_1 and s_2.

Solutions to Problems for Part 1

1. For (a), we write
$$A = (A \cap B) \cup (A \cap B^c),$$
and use the fact that $A \cap B$ and $A \cap B^c$ are mutually exclusive to compute
$$\begin{aligned} P(A) &= P((A \cap B) \cup (A \cap B^c)) \\ &= P(A \cap B) + P(A \cap B^c) \end{aligned}$$

For (b), we replace A in (a) with A^c, to get
$$P(A^c \cap B^c) = P(A^c) - P(A^c \cap B).$$

Clearly $P(A^c) = 1 - P(A)$, and if we again use (a) with the roles of A and B switched, we have
$$P(A^c \cap B) = P(B) - P(A \cap B).$$
Combining these observations, we obtain the conclusion.

2. There are $\binom{15}{3}$ ways to choose the first group, $\binom{12}{3}$ ways to choose the second group etc., then 5! ways to arrange the five groups. In total,
$$\frac{\binom{15}{3}\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}}{5!} = 1401400.$$

3. First, there are 3^5 possible solution arrangements. Clearly, there is only 1 way to answer all five questions correctly. For the number of ways to answer exactly 4 questions correctly,

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we observe that there are $\binom{5}{4} = 5$ ways to select the 4 that will be answered correctly and 2 ways to incorrectly answer the fifth problem. That is,

$$P(\text{at least four correct}) = \frac{1 + 5 \cdot 2}{3^5} = \frac{11}{243}.$$

Alternatively, the probability of getting, say, the first four correct and missing the last one is

$$(\frac{1}{3})^4(\frac{2}{3}).$$

In total, there are five ways in which to get four correct and miss one, and one way in which to get them all. We have

$$P(\text{at least four correct}) = 5(\frac{1}{3})^4(\frac{2}{3}) + (\frac{1}{3})^5 = \frac{11}{243}.$$

4. First, the total number of seven-card deals is

$$\binom{52}{7} = 133,784,560.$$

For (a), it helps to divide the count into two parts, the case of royal flushes and the case of straight flushes with top cards 5 through K. For the first, there are 4 ways to suit the royal flush and $\binom{47}{2}$ ways to select the final two cards, giving in total

$$4 \cdot \binom{47}{2} = 4,324$$

possibilities. Next, there are 9 possible rank arrangements for the remaining straight flushes, and 4 ways to suit each. Of the 47 cards that don't appear in the straight flush, we also have to set aside the card that would create a straight flush with a higher top card. E.g., if the straight flush is clubs with top card king, we have to take out the ace of clubs, which would give a hand that has already been counted in the number of royal flushes. This means that we have

$$9 \cdot 4 \cdot \binom{46}{2} = 37,260.$$

The probability of getting a straight flush is

$$P(\text{straight flush}) = \frac{4,324 + 37,260}{133,784,560} = .000311.$$

For (b), we need to take care with the following three different ways a full house can occur:

- two trips and one extra card

- one trip and two pairs

- one trip, one pair, and two extra cards

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For the first, there are $\binom{13}{2}$ ways to rank the trips, $\binom{4}{3} = 4$ ways to arrange each, and 44 ways to choose the final card, giving

$$\binom{13}{2} \cdot 4^2 \cdot 44 = 54,912.$$

For the second, there are 13 ways to rank the trip, 4 ways to arrange it, $\binom{12}{2}$ ways to rank the pairs, and $\binom{4}{2} = 6$ ways to arrange each, giving in total

$$13 \cdot 4 \cdot \binom{12}{2} \cdot 6^2 = 123,552.$$

For the third, there are 13 ways to rank the trip, and 4 ways to arrange it, then 12 ways to rank the pair and $\binom{4}{2} = 6$ ways to arrange it. There are then $\binom{11}{2}$ ways to rank the remaining two cards without matching the trip or pair or pairing the final two cards, and 4 ways to arrange each, giving

$$13 \cdot 4 \cdot 12 \cdot 6 \cdot \binom{11}{2} \cdot 4^2 = 3,294,720.$$

In total there are

$$54,912 + 123,552 + 3,294,720 = 3,473,184$$

possibilities. We conclude

$$P(\text{full house}) = \frac{3,473,184}{133,784,560} = .025961.$$

5. We define the partition

$$A_1 = \text{event a regression problem is given}$$
$$A_2 = \text{event a dimensional analysis problem is given}$$
$$A_3 = \text{event a phase plane problem is given,}$$

and let B denote the event that the student solves the given problem correctly. Then

$$P(B) = \sum_{j=1}^{3} P(B|A_j)P(A_j)$$
$$= .8 \cdot .1 + .7 \cdot .5 + .25 \cdot .4 = .53.$$

6. For this problem, the relevant events are

$$A = \text{event that the suspect is guilty}$$
$$B = \text{event that the suspect has the characteristic.}$$

We want to compute

$$P(A|B).$$

Since A and A^c form a partition of the sample space, Bayes' Lemma asserts

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{1(.6)}{1(.6) + (.2)(.4)} = .8824.$$

7. One approach to this problem is to proceed almost precisely as for the Monty Hall problem. For this, we begin with the partition

$$A_1 = \text{event the mixed card is drawn}$$
$$A_2 = \text{event the red-red card is drawn}$$
$$A_3 = \text{event the black-black card is drawn.}$$

The conditioning event is

$$B = \text{event the up side of the chosen card is red.}$$

According to Bayes' Lemma, we have

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)}$$
$$= \frac{\left(\frac{1}{2}\right)\frac{1}{3}}{\left(\frac{1}{2}\right)\frac{1}{3} + (1)\frac{1}{3} + (0)\frac{1}{3}} = \frac{1}{3}.$$

Alternatively, we can get by with only defining two events,

$$A = \text{event mixed card is drawn}$$
$$B = \text{event up side of chosen card is red.}$$

Here, by the definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.$$

In this case, the probabilities are slightly more subtle. For $P(A \cap B)$, keep in mind that there is $1/3$ chance of getting the mixed card, and then once you have it, a $1/2$ chance that the red side will be up. For $P(B)$, consider the three cards as six sides, each equally likely to turn up.

8. We compute

$$E[XY] = \sum_{x,y} xy P((X = x) \cap (Y = y))$$
$$= \sum_x \sum_y xy P(X = x)P(Y = y)$$
$$= \sum_x x P(X = x) \sum_y y P(Y = y)$$
$$= E[X]E[Y].$$

8

Here, the second equality is a result of independence. We need to assume $E[|X|]$ and $E[|Y|]$ are finite so that we can rearrange the order of the summation.

9. We let N denote the number of flips required until two consecutive heads occur (a random variable), and let T_k denote the event that the first tail occurs on flip k, $k = 1, 2$, and let T_3 denote the event that the first two flips are heads. (Note that the $\{T_k\}_{k=1}^3$ partition our sample space.) We can now proceed by conditioning,

$$E[N] = \sum_{k=1}^{3} E[N|T_k]P(T_k) = (1 + E[N])\frac{1}{2} + (2 + E[N])\frac{1}{4} + 2\frac{1}{4}.$$

Solving for $E[N]$, we find
$$E[N] = 6.$$

10. In this case we start by applying the minimax method. (See Figure 7.)

Rose/Colin	A	B	C	row mins
A	5	2	1	1
B	4	1	3	1
C	3	4	3	3
D	1	6	2	1
column maxes	5	6	3	

Figure 7: Minimax method for Problem 10.

We conclude that the game is solved by saddle strategy (C,C), and that the value of the game is 3. Note that the other 3s do not correspond with saddle points.

11. We begin by observing that Colin-B dominates Colin-A, and so the problem is reduced to the matrix game given in Figure 8. (Colin-C also dominates Colin-A, but a single dominating column is sufficient to reduce the game.)

Let x denote the probability Colin plays strategy B and $(1 - x)$ the probability he plays strategy C. For Rose's random variables R_A, R_B, and R_C, we have

$$E[R_A] = x - 5(1 - x) = 6x - 5$$
$$E[R_B] = 5x - 10(1 - x) = 15x - 10$$
$$E[R_C] = -3x.$$

These are plotted below using MATLAB, though of course the plots could be done by hand or with a graphing calculator for the exam. (See Figure 9.)

9

Rose/Colin	B	C
A	1	−5
B	5	−10
C	−3	0

Figure 8: Reduced version of the matrix game for Problem 11.
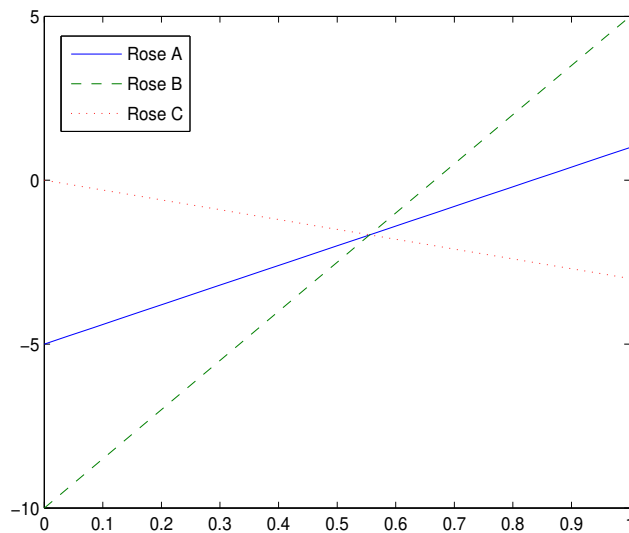

Figure 9: Rose's expected values.

We see that the lines all intersect at the same point. To verify this, we can find the value of x at which $E[R_A] = E[R_C]$ and compare it with the value of x at which $E[R_B] = E[R_C]$. For the former, we have

$$6x - 5 = -3x \implies x = \frac{5}{9},$$

and for the latter

$$15x - 10 = -3x \implies x = \frac{5}{9}.$$

We see that Colin should play Colin-B with probility $\frac{5}{9}$ and Colin-C with probability $\frac{4}{9}$. The value of the game is

$$E[R_C] = -3\left(\frac{5}{9}\right) = -\frac{5}{3}.$$

Next, we need to determine Rose's optimal strategy. For this, we let x denote the probability she plays Rose-A, y the probability she plays Rose-B, and $1 - x - y$ the probability she plays

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Rose-C. If Colin plays optimally, the most she can get is $-\frac{5}{3}$, so she would like to choose x and y to ensure she gets at least this, or equivalently that Colin gets no more than the negative of this. She can accomplish this by choosing x and y so that $E[C_B] = \frac{5}{3}$ and $E[C_C] = \frac{5}{3}$, where $E[C_B]$ and $E[C_C]$ are Colin's expected values

$$E[C_B] = -x - 5y + 3(1 - x - y) = -4x - 8y + 3$$
$$E[C_C] = 5x + 10.$$

This gives us the system of two equations for two unknowns

$$-4x - 8y = -\frac{4}{3}$$
$$5x + 10y = \frac{5}{3}.$$

If we multiply the first of these equations by 5 and the second by 4, we see that they are actually the same equation, so in fact we have only one relation between x and y. If we divide the second equation by 5, we can express this relation as

$$x + 2y = \frac{1}{3}.$$

Since x and y are both non-negative, we must have $x \le \frac{1}{3}$ and $y \le \frac{1}{6}$, but we can take any values x and y satisfying these relations. For example, Rose could take $x = 0$ so that $y = \frac{1}{6}$ and $z = \frac{5}{6}$, meaning that she would play Rose-B with probability $\frac{1}{6}$ and Rose-C with probability $\frac{5}{6}$.

12. First, we can use either an arrow diagram or the minimax method to confirm that this problem does not have a saddle. Next, we let x denote the probability Colin plays strategy A, y the probability he plays strategy B, and $1 - x - y$ the probability he plays strategy C. Then

$$E[R_A] = -7x + 3(1 - x - y) = -10x - 3y + 3$$
$$E[R_B] = 4x - y - 2(1 - x - y) = 6x + y - 2$$
$$E[R_C] = -5x + 2y + (1 - x - y) = -6x + y + 1.$$

Upon setting $E[R_A] = E[R_C]$ and $E[R_B] = E[R_C]$ we obtain a system of two equations for two unknowns:

$$-10x - 3y + 3 = -6x + y + 1 \implies -4x - 4y = -2$$
$$6x + y - 2 = -6x + y + 1 \implies 12x = 3 \Rightarrow x = \frac{1}{4}.$$

Upon substitution of $x = \frac{1}{4}$ back into the first equation, we see that

$$-1 - 4y = -2 \implies y = \frac{1}{4},$$

and consequently

$$z = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}.$$

We conclude that Colin's optimal strategy is:

$$\text{Colin A} : \frac{1}{4}$$
$$\text{Colin B:} \frac{1}{4}$$
$$\text{Colin C:} \frac{1}{2}.$$

The value of the game is

$$E[R_A] = -10\frac{1}{4} - 3\frac{1}{4} + 3 = -\frac{1}{4}.$$

Likewise, if we let x denote the probability that Rose plays strategy A, y the probability she plays strategy B, and $1 - x - y$ the probability she plays strategy C we have

$$
\begin{aligned}
E[C_A] &= 7x - 4y + 5(1 - x - y) = 2x - 9y + 5 \\
E[C_B] &= y - 2(1 - x - y) = 2x + 3y - 2 \\
E[C_C] &= -3x + 2y - (1 - x - y) = -2x + 3y - 1.
\end{aligned}
$$

Upon setting $E[C_A] = E[C_C]$ and $E[C_B] = E[C_C]$, we obtain the system of equations

$$2x - 9y + 5 = -2x + 3y - 1 \implies 4x - 12y = -6$$
$$2x + 3y - 2 = -2x + 3y - 1 \implies 4x = 1 \implies x = \frac{1}{4}.$$

Upon substitution of $x = \frac{1}{4}$ back into the first equation, we see that

$$1 - 12y = -6 \implies y = \frac{7}{12},$$

and consequently

$$z = 1 - \frac{1}{4} - \frac{7}{12} = \frac{1}{6}.$$

We conclude that Rose's optimal strategy is:

$$\text{Rose A:} \frac{1}{4}$$
$$\text{Rose B:} \frac{7}{12}$$
$$\text{Rose C:} \frac{1}{6}.$$

As a check, we compute the value of the game based on Colin's expectations:

$$E[C_A] = 2\frac{1}{4} - 9\frac{7}{12} + 5 = \frac{1}{4},$$

which, as expected, is simply the negative of Rose's expected payoff.

13. Let y denote the probability Rose plays strategy A and $1 - y$ the probability she plays strategy B. Colin's expectations are:

$$\begin{aligned}
E[C_A] &= -y - 2(1 - y) = y - 2 \\
E[C_B] &= y - 3(1 - y) = 4y - 3 \\
E[C_C] &= -2y + 5(1 - y) = -7y + 5 \\
E[C_D] &= 4y - 4(1 - y) = 8y - 4.
\end{aligned}$$

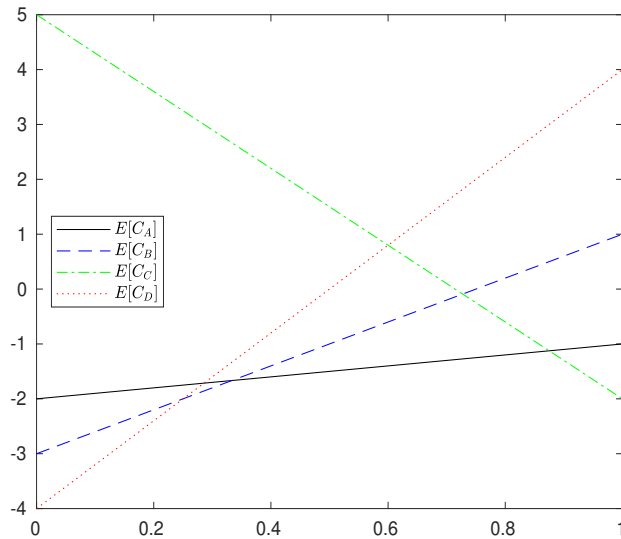A MATLAB plot of these expectations is given below in Figure 10.

Figure 10: Colin's expected values for Problem 13.

We see that Rose minimizes Colin's expectation by choosing y as the point of intersection $E[C_C] = E[C_D]$. If Rose plays this strategy, then Colin will lose more than he should unless he plays only C and D, so we can solve this game by solving the reduced game given in Figure 11.

Rose/Colin	A	B
C	2	-4
D	-5	4

Figure 11: Reduced matrix game for Problem 13.

According to the method of oddments, we conclude the optimal strategies are:

$$\text{Rose A:} \frac{3}{5}$$
$$\text{Rose B:} \frac{2}{5}$$
$$\text{Colin C:} \frac{8}{15}$$
$$\text{Colin D:} \frac{7}{15}.$$

The value of the game is

$$2 \cdot \frac{8}{15} - 4 \cdot \frac{7}{15} = -\frac{4}{5}.$$

14. First, we observe that Colin-C dominates Colin-A, and we drop Colin-A. (See Figure 12.) Next, Rose-E dominates Rose-A, so we drop Rose-A. (See Figure 13.) Finally, Colin-B dominates Colin-D, so we eliminate Colin-D. (See Figure 14.)

Rose/Colin	B	C	D
A	-3	2	-4
B	-4	4	-2
C	1	-3	2
D	2	-7	2
E	-2	2	-2

Figure 12: Reduced matrix game for Problem 14.

Now let x denote the probability Colin plays strategy B and $1-x$ the probability he plays strategy C. Rose's expected value are:

$$
\begin{aligned}
E[R_B] &= -4x + 4(1-x) = -8x + 4 \\
E[R_C] &= x - 3(1-x) = 4x - 3 \\
E[R_D] &= 2x - 7(1-x) = 9x - 7 \\
E[R_E] &= -2x + 2(1-x) = -4x + 2.
\end{aligned}
$$

These are ploted below. (See Figure 15.) We see that the game is solved in terms of Rose's strategies C and E, so we consider the 2×2 subgame given in Figure 16.

Rose/Colin	B	C	D
B	−4	4	−2
C	1	−3	2
D	2	−7	2
E	2	−2	2

Figure 13: Further reduced matrix game for Problem 14.

Rose/Colin	B	C
B	−4	4
C	1	−3
D	2	−7
E	2	−2

Figure 14: Yet further reduced matrix game for Problem 14.

We solve this game by the method of oddments to find the optimal strategies are:

$$\text{Rose C: } \frac{1}{2}$$
$$\text{Rose E: } \frac{1}{2}$$
$$\text{Colin B: } \frac{5}{8}$$
$$\text{Colin C: } \frac{3}{8}.$$

The value of the game is

$$\frac{5}{8} - 3 \cdot \frac{3}{8} = -\frac{1}{2}.$$

Solutions to Part 2

1. This problem is solved using event location with the M-file *rpendulum.m*.
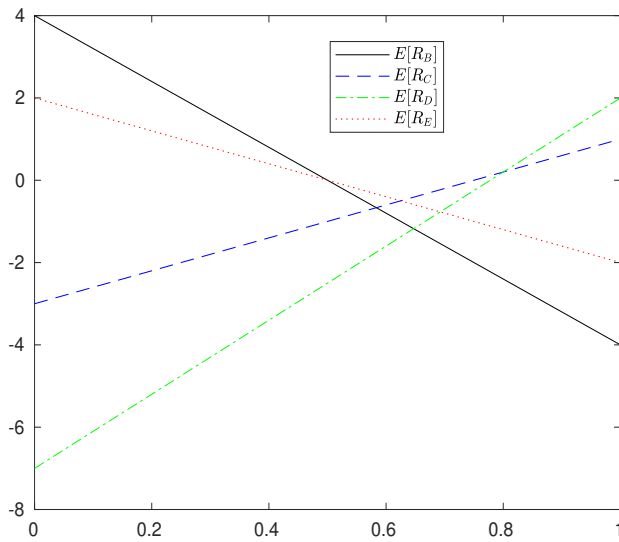
Figure 15: Rose's expected values for Problem 13.

Rose/Colin	B	C
C	1	−3
E	−2	2

Figure 16: Final reduced matrix game for Problem 14.

```
function rpendulum
%RPENDULUM: MATLAB function M-file to solve the
%rotating pendulum problem.
%
g = 9.81;
%DEFINE THE RIGHT-HAND SIDE
rhs = @(t,y) [y(2); -g*sin(y(1))+cos(y(1)-t)];
options = odeset('Events',@subevents,'RelTol',1e-12,'AbsTol',1e-12);
[t,y,te,ye,ie] = ode45(rhs,[0 2*pi],[0; 0],options);
[te ye ie]
%
function [lookfor stop direction] = subevents(t,y)
lookfor = y(1);
stop = 0;
direction = 0;
```

There are six times:

$$1.5749$$
$$3.1319$$
$$3.1755$$
$$4.6228$$
$$5.8952$$
$$6.1269$$

Regarding the options statement, if this system is solved without refined accuracy the second and third instances of verticality (i.e., $t = 3.1319$ and $t = 3.1755$) are missed.

2. First, in order to find two different values of b, the strategy is to find one solution for which the pendulum swings across and back, and one solution for which it doesn't swing across. The first of these will correspond with the smaller value of b, so we begin with an initial guess of 0. This is carried out in *pendbvp6.m*, which creates Figure 17.

```
%PENDBVP6: MATLAB script M-file that uses bvp4c
%to find the coefficient of air resistance b for a pendulum
%initiated with theta(0)=pi/4, theta'(0)=0, so that theta(1)=0.
%
g=9.81; l=.4; bguess = 0;
rhs = @(t,y,b) [y(2);-(g/l)*sin(y(1))-b*y(2)*abs(y(2))];
bc = @(yleft,yright,b) [yleft(1)-pi/4;yright(1);yleft(2)];
%
initial_partition = linspace(0,1,25);
initial_solution = @(t) [(pi/4)*cos(3*pi*t/2); -(3*pi^2/8)*sin(3*pi*t/2)];
%initial_solution = @(t) [(pi/4)*cos((pi/2)*t); -(pi^2/8)*sin((pi/2)*t)];
solinit = bvpinit(initial_partition,initial_solution,bguess);
sol = bvp4c(rhs,bc,solinit)
fprintf('b = %5.4f\n',sol.parameters)
plot(sol.x,sol.y(1,:))
title('Pendulum angle as a function of time t','interpreter','latex','FontSize',16)
xlabel('Time $t$','interpreter','latex','FontSize',14)
ylabel('Pendulum angle $\theta$','interpreter','latex','FontSize',14)
```

The coefficient of air resistance in this case is $b = .4147$. In order to find a second value of b, we run *pendbvp6.m* again, changing the initial guess to 5 and using the initial solution commented out in the previous run. In this case, we obtain Figure 18, and $b = 10.9531$.

3. For Part (a), we can express the Lotka-Volterra system as

$$\frac{1}{y_1}\frac{dy_1}{dt} = a - by_2$$
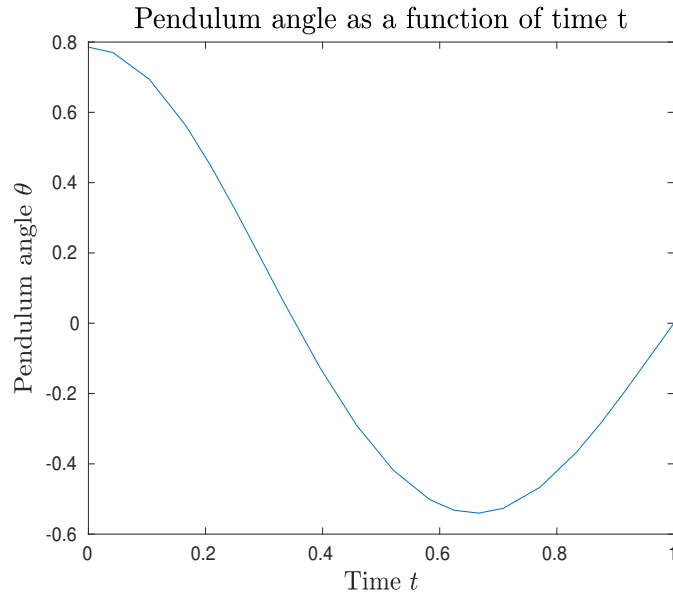$$\frac{1}{y_2}\frac{dy_2}{dt} = -r + cy_1.$$

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Figure 17: Nonlinearly damped pendulum.

In the MATLAB M-file *lvlinear1.m*, the values of a and b are obtained by plotting $\frac{1}{y_1}\frac{dy_1}{dt}$ as a function of y_2, and the values of c and r are obtained by plotting $\frac{1}{y_2}\frac{dy_2}{dt}$ as a function of y_1.

```
%LVLINEAR1: MATLAB script M-file to carry out linear parameter
%estimation for the Lotka-Volterra model with the Hudson Bay data
%
lvdata;
%CENTRAL DIFFERENCE FIT
dHdt = (H(3:end)-H(1:end-2))/2;
YH = dHdt./H(2:end-1);
XH = L(2:end-1);
dLdt = (L(3:end)-L(1:end-2))/2;
YL = dLdt./L(2:end-1);
XL = H(2:end-1);
%LINEAR FITS
p1 = polyfit(XH,YH,1);
p2 = polyfit(XL,YL,1);
a = p1(2)
b = -p1(1)
c = p2(1)
r = -p2(2)
%SOLVE AND PLOT
lvrhs = @(t,y) [a*y(1) - b*y(1)*y(2);-r*y(2)+c*y(1)*y(2)];
[t,y]=ode45(lvrhs,[0 20],[30;4]);
subplot(2,1,1)
plot(years,H,'o',t,y(:,1))
```
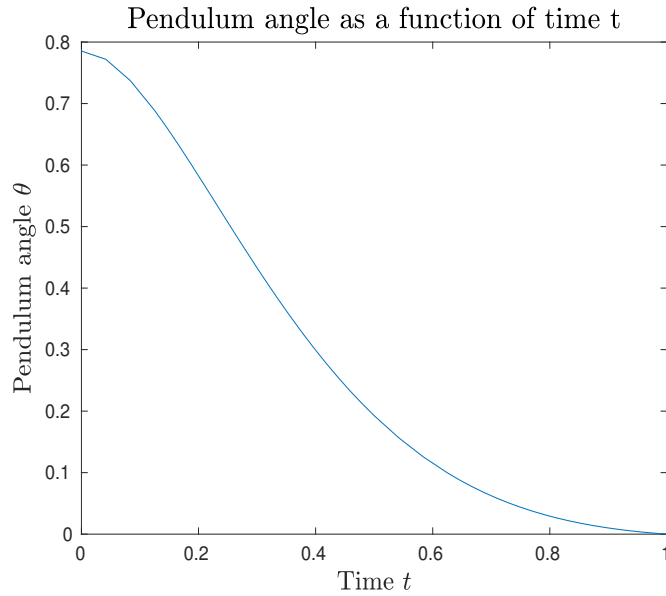
Figure 18: Nonlinearly damped pendulum with stronger damping.

title('Prey Population, model and data','Interpreter','latex','FontSize',16)
subplot(2,1,2)
plot(years,L,'o',t,y(:,2))
title('Predator Population, model and data','Interpreter','latex','FontSize',16)
xlabel('Years','Interpreter','latex','FontSize',14)

We find

$$a = .4732$$
$$b = .0240$$
$$c = .0234$$
$$r = .7646.$$

The figure created by *lvlinear1.m* is depicted in Figure 19.

For (b), we carry out the nonlinear regression calculation with *lvnonlinear.m*.

%LVNONLINEAR: MATLAB script M-file that takes an initial
%approximation of parameter values and carries out nonlinear
%regression to obtain best-fit parameter values for the
%Lotka-Volterra system based on data in lvdata.m.
global years H L w1 w2;
lvdata;
%Weights
w1 = std(H(2:end));
w2 = std(L(2:end));
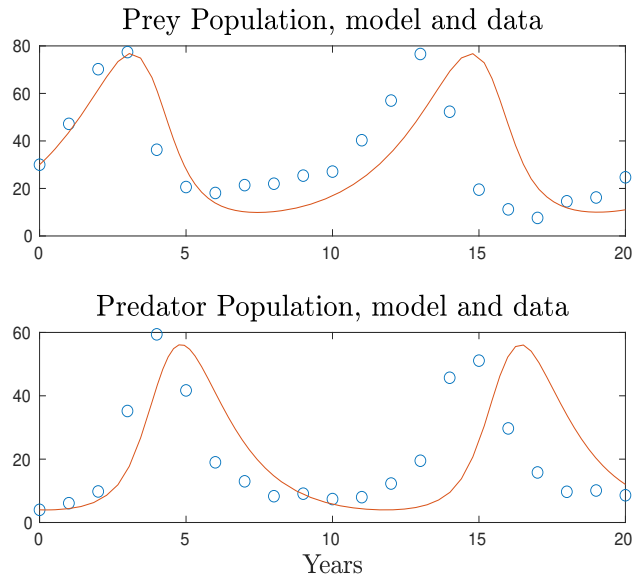guess = [.4732 .0240 .0234 .7646];

Figure 19: Figure for Lotka-Volterra linear fit.

```
[p,error]=fminsearch(@lverr, guess);
a = p(1)
b = p(2)
c = p(3)
r = p(4)
q = 2*(length(H)-1)-length(p);
s = sqrt(error/q)
s1 = w1*s
s2 = w2*s
%
[t,y]=ode45(@lvpe,[0,20],[H(1); L(1)],[],p);
subplot(2,1,1)
plot(t,y(:,1),years,H,'o')
title('Prey population, model and data','Interpreter','latex','FontSize',16)
subplot(2,1,2)
plot(t,y(:,2),years,L,'o')
title('Predator population, model and data','Interpreter','latex','FontSize',16)
xlabel('Years','Interpreter','latex','Fontsize',14)
%
function error = lverr(p)
%LVERR: Function defining error function for
%example with the Lotka-Volterra predator-prey system.
global years H L w1 w2;
[t,y] = ode45(@lvpe,years,[H(1);L(1)],[],p); %Notice that we pass
%a parameter vector
error = norm(y(:,1)-H')^2/w1^2+norm(y(:,2)-L')^2/w2^2;
```

end
%
function value = lvpe(t,y,p)
%LVPE: ODE for example Lotka-Volterra
%predator-pley model paramter
%estimation. p(1) = a, p(2) = b, p(3) = c, p(4) = r.
value = [p(1)*y(1)-p(2)*y(1)*y(2); -p(4)*y(2)+p(3)*y(1)*y(2)];
end

The parameter values are found to be

$$a = .5548$$
$$b = .0284$$
$$c = .0264$$
$$r = .8273,$$

and the standard deviations are

$$s_1 = 5.1103$$
$$s_2 = 3.8790.$$

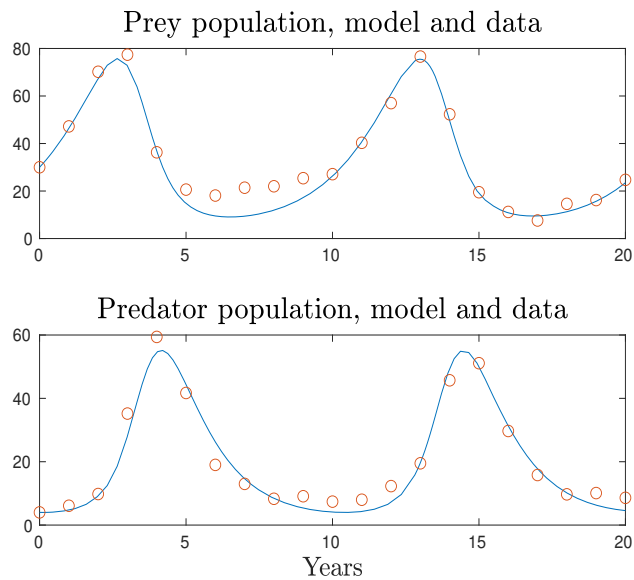The figure created by *lvnonlinear.m* is depicted in Figure 20.

Figure 20: Lotka-Volterra model and data with nonlinear fit.

4. For this one, we require only minor modifications to the file *lvnonlinear.m*. The new file, given below, is *lvnonlinear2.m*.

```
%LVNONLINEAR2: MATLAB script M-file that takes an initial
%approximation of parameter values and carries out nonlinear
%regression to obtain best-fit parameter values for the
%Lotka-Volterra system based on data in lvdata.m. In this
%case, y1(0) and y2(0) are treated as parameters.
global years H L w1 w2;
lvdata;
%Weights
w1 = std(H(1:end));
w2 = std(L(1:end));
guess = [.5548 .0284 .0264 .8273 30 4];
[p,error]=fminsearch(@lverr, guess);
a = p(1)
b = p(2)
c = p(3)
r = p(4)
y01 = p(5)
y02 = p(6)
q = 2*(length(H))-length(p);
s = sqrt(error/q)
s1 = w1*s
s2 = w2*s
%
[t,y]=ode45(@lvpe,[0,20],[y01; y02],[],p(1:4));
subplot(2,1,1)
plot(t,y(:,1),years,H,'o')
title('Prey population, model and data','Interpreter','latex','FontSize',16)
subplot(2,1,2)
plot(t,y(:,2),years,L,'o')
title('Predator population, model and data','Interpreter','latex','FontSize',16)
xlabel('Years','Interpreter','latex','Fontsize',14)
%
function error = lverr(p)
%LVERR: Function defining error function for
%example with the Lotka-Volterra predator-prey system.
global years H L w1 w2;
[t,y] = ode45(@lvpe,years,[p(5);p(6)],[],p(1:4)); %Notice that we pass
%a parameter vector
error = norm(y(:,1)-H')^2/w1^2+norm(y(:,2)-L')^2/w2^2;
end
%
function value = lvpe(t,y,p)
%LVPE: ODE for example Lotka-Volterra
%predator-pley model paramter
%estimation. p(1) = a, p(2) = b, p(3) = c, p(4) = r.
```

value = [p(1)*y(1)-p(2)*y(1)*y(2); -p(4)*y(2)+p(3)*y(1)*y(2)];
end

Notice, in particular, how the parameters are incorporated into the subfunction *lverr*. We find

$$
\begin{aligned}
a &= .4835 \\
b &= .0248 \\
c &= .0272 \\
r &= .9133 \\
y_1(0) &= 35.3836 \\
y_2(0) &= 3.9795 \\
s_1 &= 4.5089 \\
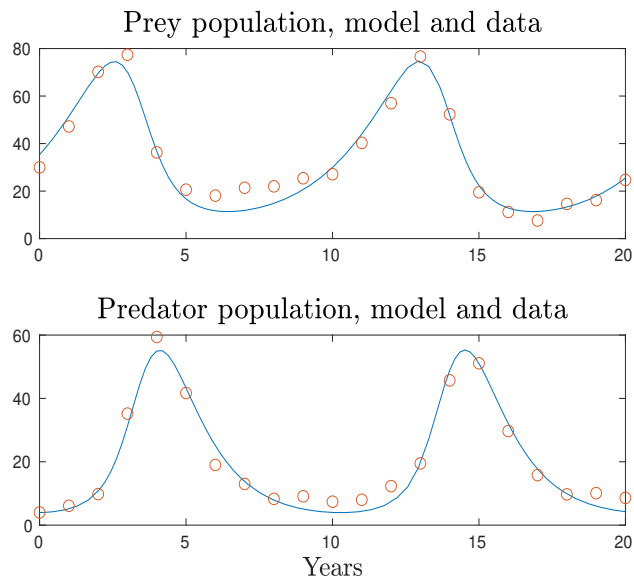s_2 &= 3.5071.
\end{aligned}
$$

The figure generated is given in Figure 21.

Figure 21: Figure for Lotka-Volterra fit.