Texas Hold'em Project

M442, Spring 2022

Due Tues. May 3

1 Overview

The word *poker* (the etymology of which continues to be a subject of debate) refers to a collection of cardgames in which players compare ranked hands in competition for a *pot* of money that typically grows through betting (i.e., claims made on it) as the hand proceeds. All serious versions of poker have the following property in common: while it is quite easy to learn the basic rules of the game, it is extremely difficult to develop a strategy that might be described as optimal. In this project our goal will be to understand some of the ways in which mathematics—especially game theory, combinatorics, and conditional expected value—can be applied to the analysis of poker. Ultimately, our focus will be on the game Texas Hold'em, often referred to in these notes as just Hold'em[1], but we will also study some academic versions of poker that will help us better understand the mathematics involved. In practice, Hold'em comes in three varieties: no-limit (the one most often televised), pot limit, and limit. We will consider limit Hold'em, which is the version played in most cardrooms. We should note at the outset that strategies in no-limit Hold'em differ significantly from those of limit Hold'em, so in particular the players in most televised games will not play according to the strategy developed in this project. For convenience—and following from our class discussion of game theory—the players under consideration will be designated Rose and Colin. If there is only one player under discussion, we will typically refer to her as Rose.

To be clear, we will not develop a complete strategy for limit Hold'em in this project. In particular, we will say very little about two important aspects of Hold'em: (1) psychological issues such as reading body language or faking tells; and (2) putting a player on a hand (i.e., systematically thinking through what a player is probably holding). In addition we won't say nearly enough about the more subtle aspects of strategy: for example, we won't discuss situations in which Rose should make an unprofitable call after a check-raise[2] to avoid giving her opponents the impression that she can be raised out of a pot. We also won't say much about the following important issue: if Rose has a strong hand, she needs to understand the odds that her opponents have strong hands that are second best to hers. That is, Rose wants to have the best hand, but she wants her opponents to *think* they have the best hands, because then they will put lots of money in her pot. For a more complete view of such non-mathematical aspects of limit Hold'em I recommend our reference [2] as the indispensible starting point.

[1] We know Texas owns it.

[2] A strategy we'll discuss during the project.

In mathematics we have a Fundamental Theorem of Algebra and a Fundamental Theorem of Calculus, and so it seems only reasonable to end this overview with David Sklansky's Fundamental Theorem of Poker (from [6], p. 17):

> Every time you play a hand differently from the way you would have played it if you could see all your opponents' cards, they gain; and every time you play your hand the same way you would have played it if you could see all their cards, they lose. Conversely, every time opponents play their hands differently from the way they would have if they could see all your cards, you gain; and every time they play their hands the same way they would have played if they could see all your cards, you lose.

2 The Rank of Poker Hands

While different versions of poker can involve different hand rankings, the hierarchy we'll discuss in this section is standard in many games, including Hold'em. Poker is generally played with a standard deck of fifty-two cards, and a poker hand typically consists of five of these cards. The number of possible hands is

$$\binom{52}{5} = 2,598,960.$$

In Table 2, we list the various poker hands, the number of ways they can be obtained, and then the probability of getting the hand, which is simply the number of ways it can be obtained divided by $\binom{52}{5}$. The less probable a hand is, the higher we rank it, so a straight flush beats four of a kind, which in turn beats a full house etc. In standard versions of poker no preference is assigned to any particular suit,[3] and so ties are possible. For example, if all five ranks of a club flush are the same as all five ranks of a heart flush the hands tie and the pot is split. If the ranks are not all the same, the flush with the highest top card wins, and if the top cards tie the flush with the highest second card wins etc. (Having given this example, I should probably mention that in Hold'em two players cannot flush in different suits—a flush requires three cards of the suit to be on the board.) As another example, if Player A has $A\heartsuit A\spadesuit 9\heartsuit 8\spadesuit 2\diamondsuit$ (a pair of aces with a nine kicker) and Player B has $A\diamondsuit A\clubsuit K\clubsuit 7\clubsuit 4\heartsuit$ (a pair of aces with a king kicker), then Player B wins.

3 The Rules of Hold'em

Though several different versions of Hold'em are played, I'll only describe one, a common form of limit Hold'em typically referred to as 4/8 for its betting structure. The number of players in any game of Hold'em can vary, but ten players is often considered a full table, so this is the number we'll assume for our general description. A dealer is supplied in most cardrooms, so a button is typically placed in front of the player currently in the dealer's

[3] As opposed, for example, to bridge, in which case suits are ordered alphebetically from worst to best; i.e., in ascending order, clubs, diamonds, hearts, spades.

Hand	Number possible	Probability
Straight flush	40	.000015
Four of a kind	624	.000240
Full house	3744	.001441
Flush	5108	.001965
Straight	10200	.003925
Trip	54912	.021128
Two pairs	123552	.047539
One pair	1098240	.422569
Bupkis	1302540	.501177

Table 1: Standard rank of poker hands.

position, and that player is referred to as the button. (The button moves around the table from hand to hand, moving one player to the previous button's left after each hand.) With ten people at the table, an ante structure would be inefficient, so the Hold'em pot is initiated with a double-blind structure. (Blind bets, like antes, are bets made before any cards have been seen.) The player to the button's immediate left is called the small blind, and she begins building the pot by putting in the required $2.00. The player to the small blind's immediate left is called the big blind, and he puts $4.00 into the pot. The dealer now deals two cards to each player, face down, beginning with the small blind. These are referred to as the players' pocket cards. (The cards are dealt in the traditional manner: one card to each player, followed by a second card to each player; prior to dealing, the dealer will typically *burn* one or more cards by putting them in the *muck* pile where folded (mucked) hands will go.) A round of betting follows, beginning with the player to the immediate left of the big blind, who is said to be *under the gun*. (The rationale here is that the big blind has started the betting, and so the betting continues from him.) Each player has three options, call, raise, or fold, and all raises must be in increments of $4.00. For example, the player to the immediate left of the big blind can do exactly one of the following: (1) fold by putting her cards into the muck pile; (2) call the big blind's bet by putting $4.00 into the pot, or (3) raise the big blind's bet by putting $8.00 into the pot. She cannot, however, raise $1.00, $2.00, $3.00, or any amount over $4.00. The round of betting continues until all raises are called (or all players except one have folded), and then the dealer spreads the flop. A player is not allowed to raise his own bet, and in most cardrooms each round of betting is limited to one bet and three raises. If no one has raised the original big blind bet, the small blind can call for $2.00 (since she already put $2.00 in) and the big blind can call without putting more money in the pot. If no one has raised before the big blind acts, the dealer will give the big blind an option to raise. Typically, the big blind will either raise or say something like, "Big enough." The flop is a set of three community cards dealt face up in the middle of the table. Following the flop, there is another round of betting at the $4.00 level, this one starting with the first active player to the button's left. (In Hold'em, most players will have folded their first two cards.) In this round there is no forced initial bet, so the first player can either check or bet (she can't raise, because there is no active bet to raise, and she certainly won't fold, because she can continue in the hand without putting more money in the pot). Once a player has bet, the remaining players (those acting *behind* the bettor) can fold, call, or raise. Next, the fourth community card (the *turn* card, also referred to as *fourth street*) is dealt, and betting begins at the higher \$8.00 level, again with the first active player to the dealer's left. Finally, the fifth community card (the *river* card, also referred to as *fifth street*) is dealt, and there is a final round of betting at the \$8.00 level, precisely the same as the round following the turn. Once the fourth round of betting is complete the active players reveal their hands, and the best hand takes the pot. In the event of a tie the pot is split. A casino or cardroom makes its money either by taking a *rake* out of each pot (something like 10% of the pot, up to a maximum of the lower bet limit), or charging a sitting fee per hour. Any such charge clearly affects a player's expected value, but we won't take this into account in our analysis.

A player's hand in Hold'em is the best five-card hand she can make from the seven cards available to her: her two pocket cards and the five board (or community) cards. She can use any combination of these, so in particular she can play two, one or neither of her pocket cards. Since she has seven cards to build her hand from her probability of getting each of the five-card poker hands listed above (except bupkis; i.e., a high card hand) will be higher than in Table 2. The appropriate Hold'em probabilities are given in Table 3. While trivial, this is a phenomenally important observation that beginning Hold'em players often overlook. Notice that while the weakest possible pair (a pair of twos) is a reasonably strong holding in a five-card hand (it wins about half the time against a single opponent), it is a very weak holding in a seven-card hand (it wins only about 17% of the time). *One of the biggest mistakes beginning Hold'em players make is that they play too many hands that simply won't hold up in a seven-card game.*

Hand	Number possible	Probability
Straight flush	41,584	.000311
Four of a kind	224,848	.001681
Full house	3,473,184	.025961
Flush	4,047,644	.030255
Straight	6,180,020	.046194
Trip	6,461,620	.048299
Two pair	31,433,400	.234955
One pair	58,627,800	.438225
Bupkis	23,294,460	.174119

Table 2: Hold'em probabilities.

In reading this table, we note that the number of possible seven card deals is

$$\binom{52}{7} = 133,784,560,$$

so the probabilities are obtained by dividing the number possible for each hand by this number. Also, we should observe that some authors distinguish a *trip* from a *set* as follows: a trip describes three-of-a-kind *with a pair on the board*, while a set describes three-of-a-kind *with a pair in the pocket*. Our choice will be to use trip for any three-of-a-kind, but certainly there is an enormous difference between these situations.

3.1 Odds and Probabilities

Since most references on Hold'em work with odds rather than probabilities (in particular, with odds *against* an event occurring), we'll briefly recall how to translate from one description to the other.[4] If the probability that some event occurs is $\frac{2}{5}$ then the odds against it are 3-to-2, typically written $3 : 2$; that is, for every two times the event occurs it fails to occur three times. In general, if the probability of the event occurring is $\frac{a}{b}$ then the odds against the event are $(b - a) : a$. To go the other way, if the odds are $c : d$ against an event occurring, the probability that it will occur is $\frac{d}{c+d}$.

3.2 A Practical Note on Thinking at the Table

As we will see below the general form for Rose's expected value when calling a bet in Hold'em is

$$E[W_c] = Pq - B(1 - q),$$

where W_c denotes her winnings if she calls (a random variable), P denotes the size of the current pot, B denotes the current bet size, and q denotes the probability that Rose has the winning hand. (In [2] the authors refer to the product Pq as Rose's *pot equity.*) Rose should call (or possibly raise) if $E[W_c] > 0$, which gives

$$q > \frac{B}{P + B}.$$

We will spend quite a bit of time in this project computing values for q in various situations, but here we note a practical matter regarding the calculation of the fraction $\frac{B}{P+B}$. It is probably easiest to count this in terms of number of bets rather than actual dollar amounts. For example, suppose Rose is fourth to act in the first round of betting, and the betting has been:

1. call at 4
2. raise to 8
3. fold
4. Rose to act

Including the small and big blind bets, there is 18 dollars in the pot, and the current bet level is 8. I.e.,

$$\frac{B}{P + B} = \frac{8}{18 + 8} = \frac{8}{26} = \frac{4}{13}.$$

(If Rose's odds of winning are better than $\frac{4}{13}$, Rose should at least call, and possibly she should raise.) Rose can think of the pot as having $P = 4.5$ bets in it, and can regard the betting level as $B = 2$ bets, and compute

$$\frac{B}{P + B} = \frac{2}{6.5} = \frac{4}{13}.$$

[4] We'll work exclusively with probabilities in these notes, so this is purely FYI.

4 Starting Hands

In this section we will analyze the first round of betting, after the two pocket cards have been dealt. First, there are

$$\binom{52}{2} = 1,326$$

possible starting hands, but for practical purposes we don't need to distinguish between hands such as 2♡2♠ and 2♢2♣. (To be precise, it is only at the outset that we don't distinguish between these hands; clearly, differences in suit can become very important once cards begin to appear on the board.) In order to count the number of qualitatively different hands, we first observe that there are 13 different pairs, and then note that once these have been removed we can combine any ace in twenty-four ways (2–K, suited and 2–K, unsuited), any 2 in twenty-two ways (as with the ace except omit aces), etc., down to the queen in either of two ways. This gives

$$13 + (24 + 22 + \cdots + 2) = 13 + 6(26) = 13^2 = 169.$$

Many analyses have been carried out on the relative success rates of starting hands. For example, in [4] the author provides a table of win rates for all 169 possible starting hands (obtained by the simulation of one million hands against randomly selected single opponents). The first ten are given in the table below.

Hand	Winning Percentage
AA	86.1%
KK	74.6%
AKs	68.6%
QQ	68.5%
AKu	67.0%
AQs	64.9%
JJ	64.4%
TT	60.8%
AQu	60.5%
AJs	58.6%

Table 3: Win rates for the top ten starting hands.

Starting hand strategies (SHS's) depend on four primary things: 1. Betting structure of the game; 2. Table position of the player; 3. Game type (disposition of the players); and 4. Bets and raises in the current hand. We will consider each of these in turn.

4.1 Betting structure

As mentioned in the introduction we are focusing in this project on 4/8 limit Hold'em. This is one particular betting structure, and there are many others: (1) There are alternative low-limit structures such as 3/6, 5/10, 15/30, and also higher-limit limit structures; (2)

There are pot-limit structures in which case the maximum bet at any given time is tied to the amount of money in the pot; and (3) no-limit structures in which case any player can bet any amount (limited only by what she has in front of her at the table) at any time. In general, optimal Hold'em strategy will be different for each structure. Since we will only consider 4/8 Hold'em in these notes we won't see these differences in our analysis.

4.2 Table Position

Following Gary Carson's book [1], we'll designate five different locations around the Hold'em table, listed in the following table.

# of chairs left of the button	Position
1&2	The blinds
3&4	Under the gun
5&6	Early position
7&8	Middle position
9&10	Late position (10 is the button)

Table 4: Positions around the Hold'em table.

A late position is better than an early position, because players in late position have more information when placing a bet. We'll see that players in late position can profitably play many more hands than can players in early position.

4.3 Game Type

Clearly, any SHS must take into account the play of other players at the table, and for this it's convenient to consider the following rough characterizations of games.

1. Loose games

We'll refer to a game as *loose* if there are consistently several players in for the flop. To be more precise, let's say a game is loose if at least half the flops involve five or more players.

2. Tight games

We'll refer to a game as *tight* if there are consistently few players in for the flop. To be more precise, let's say a game is tight if $\frac{3}{4}$ of the flops involve only two players.

3. Passive games

We'll refer to a game as *passive* if few hands involve raises. To be more precise, let's say a game is passive if $\frac{1}{5}$ or fewer hands involve a raise.

4. Aggressive games

We'll refer to a game as *aggresive* if hands frequently involve raises. To be more precise, let's say a game is aggressive if $\frac{1}{2}$ or more hands involve a raise.

Clearly, a hand cannot be both loose and tight, but notice that we can have any of the four combinations loose–passive, loose–aggressive, tight–passive, or tight–aggressive. We can also consider finer gradations such as mildly tight or very tight. When we refer to a game as typical–typical we mean a game roughly half way between loose and tight and half-way between passive and aggressive.

4.4 Bets and Raises in the Current Hand

Except for players under the gun (i.e., immediately to the left of the big blind), the SHS will depend on what previous players have done. For example, it's typically a solid statement of strength if the player under the gun raises the big blind, so a player needs a better hand to call this raise than he needs simply to call the big blind. As another (somewhat extreme) example, suppose a player under the gun raises the big blind and the next player re-raises. Anyone at the table who's not packing some fairly convincing heat is in for a world of hurt.

4.5 A Practical SHS

In the next subsection we will begin our mathematical development of an SHS, but first let's consider a broadly applicable practical approach. The web site [3] lists expected values for the 169 different possible starting hands, obtained after 122,031,244 actual games. In Table 5 we arrange all starting hands that had a positive expected value.

244	A	K	Q	J	T	9	8	7	6	5	4	3	2
A	6	4s,12u	4s,12u	4s,12u	4s,12u	4s	4s	4s	4s	4s	4s	4s	
K	X	6	4s,12u	4s,12u	4s,12u	4s	4s						
Q	X	X	6	4s,12u	4s	4s							
J	X	X	X	6	4s	4s							
T	X	X	X	X	6	4s							
9	X	X	X	X	X	6							
8	X	X	X	X	X	X	6						
7	X	X	X	X	X	X	X	6					
6	X	X	X	X	X	X	X	X	6				
5	X	X	X	X	X	X	X	X	X	6			
4	X	X	X	X	X	X	X	X	X	X			
3	X	X	X	X	X	X	X	X	X	X	X		
2	X	X	X	X	X	X	X	X	X	X	X	X	

Table 5: Practical SHS Table.

By symmetry we only need half the table, so I've removed half the entries with X's. (Some authors use one direction for suited cards and the other for unsuited cards, but that strikes me as confusing.) Each intersection with a number in it designates a playable starting hand, and the number specifies the number of such hands possible. For example, any combination of ace-king (*big slick*) should be played (suited or unsuited), and there are 16 such combinations, 4 suited and 12 unsuited. Generally speaking, we can think of this as a reasonable strategy for middle position in a standard game, and we can adjust it according to the situation. It's probably fair to say that the raising hands are the pairs ten and higher, big slick, and any suited *broadway* (any two cards ten or higher of the same suit). For a much more thorough SHS, see [2].

4.6 A Starting Hand Theory

The big blind bet is considered a genuine bet (as oppposed to an ante), so Rose will need to decide whether she should fold, call, or raise. Our point of view will be that Rose will make the decision based on the play of her opponents and the probability that she will be in a strong position after the flop. In particular, we will not yet consider the probability that Rose will ultimately win the hand; rather, we will focus on the probability that the flop will give her a hand she is willing to continue with. In order to understand why we do this, we must be aware that Rose will fold many hands at the flop that would ultimately win. That is, if we computed the probability that Rose has a strong hand after all seven cards have been dealt, we would include many hands that were weak after the flop but saw beneficial draws on the turn and river. While these certainly give Rose winning hands, she would never play them, because her odds *from the flop on* are too small. Our contention, then, is that what matters at this stage is whether or not Rose would play the hand from the flop on. We set

$$\begin{aligned}
P &= \text{Current pot value} \\
B &= \text{Current bet value} \\
R &= \text{event Rose has a strong hand after the flop} \\
R^c &= \text{event Rose has a weak hand after the flop} \\
W_c &= \text{the value of Rose's hand if she calls (a random variable)} \\
W_r &= \text{the value of Rose's hand if she raises (a random variable).}
\end{aligned}$$

First, since the blinds are forced, Rose should not consider herself to have put any money in the pot at the outset, and so her expected value of folding is 0. In fact, since each decision will be independent of the amount of money Rose put into the pot during earlier rounds, *the expected value of a fold will be 0 for all rounds.* For calling, Rose's basic decision equation is

$$E[W_c] = P \cdot P(R) - B \cdot P(R^c).$$

Notice that we assume the players acting after Rose will all fold, which is generally not correct. More precisely, we should condition our expectation on what these players will do. We keep in mind, however, that if players call behind Rose her expected value will increase (more money in the pot), and so this equation penalizes Rose when she is in early position (i.e., her expected value for the same hand will typically be smaller in early position than in late position). In this way, our approximate model captures a correct phenomenon: early position is weaker than late position.

If we expect that N players will call Rose's raise, then Rose's expected value of raising is

$$E[W_r] = (P + NB)P(R) - 2BP(R^c).$$

(Strictly speaking we should condition on the number of raisers Rose gets.) If $E[W_c] > 0$ then Rose should call, and she should raise if $E[W_r] \geq E[W_c]$. (The possibility of equality in this relation suggests my preference for raising when reasonable during pre-flop play.) Our (massively simplified but ultimately fairly useful) raise relation becomes

$$(P + NB)P(R) - 2BP(R^c) \geq PP(R) - BP(R^c) \implies N \geq \frac{P(R^c)}{P(R)}. \tag{1}$$

That is, it is reasonable for Rose to raise if she has reason to believe she will have a number of callers greater than or equal to this ratio $\frac{P(R^c)}{P(R)}$. (This assumes $N \geq 1$; if Rose expects that no one will call her raise then she should certainly raise.) In her consideration, she must keep in mind that *while a raise typically requires several callers, the raise itself will often drive players to fold*. In fact when we discuss flop play we will consider the possibility of Rose's raising expressly for the purpose of *protecting her hand*: that is, raising to alter the pot odds for her opponents so that they will fold and have no chance to draw out on her. While this approach can also be used during pre-flop play, we will limit our discussion of it to flop play where it strikes me as most useful.

We'll base our approximation of $P(R)$ on the position Rose will be in after the flop, so the first calculations we'll do regard certain probabilities on hands after the flop. We note at the outset that there are

$$\binom{50}{3} = 19600$$

ways to arrange the flop. (Keep in mind that two cards are in Rose's pocket, and that while 18 other cards have been dealt Rose has no information about them.) We'll study the flop more carefully in Section 5, but it will clarify the following discussion if we jump ahead and say something now about considerations after the flop. Here is a brief list of important considerations:

1. **Overpairs.** An overpair is a pocket pair that is higher than any card in the flop. An overpair is clearly promising and should usually be played.

2. Top Pair. A player has top pair if one of her pocket cards matches the highest ranking card on the flop. Similarly, she can have *middle pair* or *bottom pair*.

3. Pocket Overcard. A pocket overcard is a card in a player's pocket that is higher in rank than any card on the board. Pocket overcards are sometimes worth betting because they can pair a card on the turn or river to give the player top pair.

4. **Board Overcard.** A board overcard is a card on the board that is higher than either of a player's pocket cards. It can be dangerous to bet with an overcard on the board, because it's likely that another player has paired it.

5. **Flush draws.** If the flop has two suited cards there is a good chance someone will be drawing for a flush. If a player has one or two cards in her pocket matching the suit of these cards she should at least consider betting the hand. If she's entirely off suit she should be very careful. If the flop is all one suit then the player is either drawing to a flush or in fairly big trouble.

6. **Straight draws.** There are two very simple (but important) differences between a flush draw (as in number 5) and a straight draw: (1) a flush beats a straight; and (2) if a player has a four flush then there are 9 cards that give her a flush; if she has four cards toward a straight (open-ended; i.e., not a gut shot (a.k.a., an inside straight)) there are 8 cards that give her a straight. So drawing to a straight, she has a smaller chance of making her hand, and the hand she is hoping to make is not as strong as a flush.

4.6.1 Pairs

First, there are 13 ranks that can give a pair, and $\binom{4}{2} = 6$ ways to pair each rank, so there are $13 \cdot 6 = 78$ possible pocket pairs. Consequently, the probability that Rose holds a pair is

$$P(\text{pocket pair}) = \frac{78}{1326} = .058824.$$

This means Rose can expect to get a pocket pair roughly one out of every twenty hands (about 1.5 times per hour in a typical game). She should play these hands wisely. We consider seven basic hands Rose can get after the flop when she holds a pocket pair[5]:

1. four of a kind (quad)
2. full house
3. three of a kind (trip)
4. two pair, pocket pair over singleton
5. two pair, pocket pair under singleton
6. Overpair (the pocket pair is larger than any rank on the flop)
7. Low pair (the pocket pair is smaller than at least one rank on the flop).

In our calculations we will denote the event Rose gets hand j from this list as H_j.

1. Quad. There is only $\binom{2}{2} = 1$ way for the pair to be arranged in the flop, then there are 48 ways to select the third card, so 48 flops in all. We have

$$P(\text{quad}) = \frac{48}{19600} = .002449.$$

2. Full house. Rose has two ways to flop a full house: (A) the flop is a trip (the unfortunate case, because there is a reasonable chance that one of Rose's opponents will have quads); or (B) there is a pair on the board plus one match for her pocket pair. For (A) there are 12 ways to rank the trip and $\binom{4}{3} = 4$ ways to arrange it, so 48 total possibilities. For (B) there are $\binom{2}{1} = 2$ ways to match the pocket pair, 12 ways to rank the remaining pair and $\binom{4}{2} = 6$ ways to arrange it. I.e., 144 possible arrangments. We conclude

$$P(\text{full house}) = \frac{48 + 144}{19600} = .009796.$$

[5]My terminology *low pair* below is not standard, but the natural alternative *underpair* is typically used to describe a situation in which the rank of a pocket pair is below the rank of *every* card on the flop (not just one), and that's not what we mean by a low pair. Also, under certain circumstances it's reasonable to play a pair that is lower in rank than exactly one card on the flop, but I've decided that's one particular hair we won't split.

Even though Case (B) is much better for Rose than Case (A) we will lump them together since there are few circumstances under which Rose wouldn't proceed after flopping a full house.

3. Trip (not a quad or full house). There are $\binom{2}{1} = 2$ ways to trip the pocket pair, then $\binom{12}{2} = 66$ ways to rank the remaining two cards (two different ranks, so they won't pair to make a full house). Finally, we have 4 ways to suit each of these two cards, so that our total hand count is $2 \cdot 66 \cdot 4^2 = 2112$. We conclude

$$P(\text{trip}) = \frac{2112}{19600} = .107755.$$

4. Two pair, pocket pair over singleton. We obtain two pairs when a pair appears in the flop, with one additional card, referred to as the singleton. Since it will often be the case that one of Rose's opponents will have a pocket card that pairs this singleton, we split the case of two pair into two subcases: when the rank of the pocket pair is greater than the rank of the singleton (labeled (4) in our list), and when the rank of the pocket pair is less than the rank of the singleton (labeled (5) in our list). (Equality gives a trip, and has already been considered.) We note that one of Rose's opponents could also have a pocket card that trips the pair that appears in the flop (giving a trip that will most likely beat Rose's two pair), but since there are only two cards left in the deck with that rank, this situation is less likely. Suppose there are N ranks below the rank of the pocket pair. For example, if the pocket pair is jacks, we would have $N = 9$, corresponding with ranks 2 through T. We have N ways to rank the singleton and $\binom{4}{1} = 4$ ways to arrange it. Likewise, we have 11 ways to rank the flop pair (we must avoid both the pocket pair rank and the singleton rank), and $\binom{4}{2} = 6$ ways to arrange it. In total we have

$$\#\text{ pocket pair over singleton } = N \cdot 4 \cdot 11 \cdot 6 = 264N.$$

In this way we conclude

$$P(\text{pocket pair over singleton}) = \frac{264N}{19600} = .013469N.$$

5. Two pair, pocket pair under singleton. First, note that we have $12 - N$ ranks above the rank of the pocket pair. For example, if the pocket pair is jacks we would have $12 - N = 3$, corresponding with ranks Q, K, A. We have $12 - N$ ranks for the singleton, $\binom{4}{1} = 4$ ways to arrange it, 11 ranks for the flop pair and $\binom{4}{2} = 6$ arrangements. In total,

$$\#\text{ pocket pairs under singleton } = (12 - N) \cdot 4 \cdot 11 \cdot 6 = 264(12 - N).$$

(Clearly, the total number of two-pair hands is $264 \cdot 12 = 3168$.[6]) We conclude

$$P(\text{pocket pair under singleton}) = \frac{264(12 - N)}{19600} = .013469(12 - N).$$

6. Overpair. Again, let N denote the number of ranks below the rank of the pocket pair. We have four suits possible for each of these ranks, so in total we have $4N$ cards to choose

[6] Just add the count from this item to the count from Item 4.

the flop from. We conclude that the total number of such flops is $\binom{4N}{3}$. We must keep in mind, however, that this includes two cases already considered: trip flops and flops with a pair. For trip flops, there are N ways to rank the trip and $\binom{4}{3} = 4$ ways to suit it, so $4N$ trip flops that must be removed. For pair flops, there are N ways to rank the second pair, $\binom{4}{2} = 6$ ways to suit it, then $N - 1$ ways to rank the singleton and $\binom{4}{1} = 4$ ways to suit it. In total we have

$$\text{\# overpair flops} = \binom{4N}{3} - 4N - 24N(N-1).$$

We conclude

$$P(\text{overpair flop}) = \frac{\binom{4N}{3} - 4N - 24N(N-1)}{19600}.$$

7. Low pair. Here, we simply note that any hand that does not fall into one of the above situations must be a low pair. There are $48 + 192 + 2112 + 3168 = 5520$ possibilities for hands 1-5, so

$$P(\text{low pair flop}) = \frac{19600 - 5520 - \left[\binom{4N}{3} - 4N - 24N(N-1)\right]}{19600}.$$

In many cases it's appropriate to *buy* the flop (with either a call or a raise) with everything except a pocket pair under the singleton (H_5) and a low pair (H_7). Since there is some reasonable chance that a pair on the board will give someone a trip, we might also omit H_4. In Table 6 we record the probabilities associated with these choices for the thirteen possible pocket pairs. Notice that in the more restrictive case in which a player would prefer to proceed only with the strong hands H_1, H_2, H_3 $P(R) = .120000$ is the same for all hands.

Pair	H_1,H_2,H_3,H_4,H_6	H_1,H_2,H_3,H_6	H_1,H_2,H_3
AA	1.000000	.838367	.120000
KK	.806939	.658776	.120000
QQ	.646531	.511837	.120000
JJ	.515510	.394286	.120000
TT	.410612	.302857	.120000
99	.328571	.234285	.120000
88	.266122	.185306	.120000
77	.220000	.152653	.120000
66	.186939	.133061	.120000
55	.163673	.123265	.120000
44	.146939	.120000	.120000
33	.133469	.120000	.120000
22	.120000	.120000	.120000

Table 6: Pocket pair probabilities.

Referring to our raise relation (1), we observe that if $P(R) \geq .5$ then $\frac{P(R^c)}{P(R)} \leq 1$, and so it is appropriate to raise with only one caller. This suggests that it's often appropriate to raise with AA, KK, QQ, and JJ, even from an early position. (To jump ahead a little, many players will also raise in early position with suited big slick (i.e., with AK suited).) In our examples, we will take this into consideration by at least worrying a little about the possibility that a player raising in early position has one of these hands.

It should be at least fairly clear why we consider certain five-card hands worth playing after the flop and others not worth playing. For example, referencing Table 7 in Appendix A we find that the probability of holding a five-card hand better than trip kings is .009211 (i.e., this is the probability of holding trip aces or better). So if there are only two players in the hand, and if each player is willing to play *any* starting hand, then KKK has roughly a .990789 chance of being the best hand after the flop. (This calculation is only rough, because we haven't used all the information we have, that there are three kings the opponent cannot hold in his pocket.) Since this probability is high, KKK is a hand we would be willing to play on the flop. Along these lines we could (though we won't) proceed as follows: let $\{H_j\}_{j=1}^n$ denote a (choice of) partition of the possible flop hands Rose could hold with a certain pocket hand. Clearly, for a pocket pair we could associate H_j with case j above, $j = 1, \dots, 7$. In this way, the probability that Rose has the best hand after the flop is

$$P(R) = \sum_{j=1}^{n} P(R|H_j)P(H_j).$$

The probabilities $P(H_j)$ are precisely what we have just computed, and the probabilities $P(R|H_j)$ can be approximated from Table 7. For example, if Rose holds KK in her pocket, then H_3 = event Rose has trip kings after the flop, $P(H_3) = .107755$, and $P(R|H_3) \approx .990789$. In practice, this sort of calculation is misleading, because it assumes Rose's opponent is willing to play *any* starting hand, whereas most reasonable players will play only about one out of five starting hands. Our approach will be to know the probabilities $P(H_j)$ precisely (these are unaffected by opponent play), but to loosely estimate the $P(R|H_j)$ based on opponent play. We will say quite a bit more about these probabilities in our analysis of flop play in Section 5.

In the following examples our point of view will be as follows: Based on opponent play, Rose will decide which flops she would be willing to play, and this will determine her event R. In particular, Rose will plan to fold if none of these flops hit (unless, of course, the entire table checks around). This means that in the decision equation

$$E[W_c] = P \cdot P(R) - BP(R^c),$$

the subtracted term is entirely justified: Rose will lose precisely an amount B if R^c occurs. The added term is trickier. It is certainly not the case that Rose will certainly earn an amount P if R occurs; in fact, she still has a long way to go before she wins the pot. On the other hand, if she does win the pot she stands to make a good bit *more* than P. (Poker players refer to the odds Rose is getting on this final pot as *implied odds*, a topic that will come up in a more direct way in our analysis of flop play.) Ultimately, we are using $P \cdot P(R)$ as a very simplistic gauge of what Rose hopes to gain if R occurs.

Example 4.1. (Situation 72 from [4].)[7] Rose holds 8♠8♡ in her pocket and is sixth to act

[7]I'll take a number of examples from popular Hold'em literature, mainly to check that our calculations agree with expert advice.

after the big blind. The betting has been as follows:

1. fold
2. call at 4
3. call at 4
4. call at 4
5. call at 4
6. Rose to act.

The pot is $P = 22$ and the bet is $B = 4$. Should Rose fold, call, or raise?

Nothing in the betting so far suggests that anyone at the table has a particularly strong hand, and even though there are still four players to hear from, this suggests we can work with hands H_1, H_2, H_3, H_4, and H_6, which according to Table 6 give $P(R) = .266122$. We have

$$E[W_c] = 22 \cdot .266122 - 4(1 - .266122) = 2.919172,$$

so Rose should certainly (at least) call. In order to raise, Rose would require three or more callers,

$$\frac{P(R^c)}{P(R)} = \frac{1 - .266122}{.266122} = 2.76,$$

and that seems unlikely given that everyone seems content just calling. We conclude that in this case Rose should call but not raise. △

Example 4.2. Rose holds 4♠4♡ in her pocket and is sixth to act after the big blind. The betting has been:

1. call at 4
2. raise to 8
3. fold
4. call at 8
5. fold
6. Rose to act.

The pot is $P = 26$ and the bet is $B = 8$. Should Rose fold, call, or raise?

In this case there has been both a raise and a call, so Rose should think that one or both of those players may have a strong hand. In particular, there is a good chance that another player has a pocket pair, and since just about any pocket pair will beat Rose's fours, she probably doesn't want to play anything except H_1, H_2, and H_3 (and she might want to rule out some full houses from H_2). We have, then, $P(R) = .120000$, and

$$E[W_c] = 26 \cdot .120000 - 8(1 - .120000) = -3.92.$$

Rose should fold. Notice the important lesson we just learned: *It is not always correct to play a pocket pair.* △

Example 4.3. Rose holds the same pocket pair as in the previous example, but this time she is eighth to act after the big blind. The betting has been even more aggressive:

1. raise to 8
2. call at 8
3. call at 8
4. call at 8
5. call at 8
6. call at 8
7. call at 8
8. Rose to act.

The pot is $P = 62$ and the bet is $B = 8$. Should Rose fold, call, or raise?

On one hand, there may be a number of strong hands in this game, but on the other that's a pretty big pot. Assuming again that Rose will only play H_1, H_2, and H_3 we compute

$$E[W_c] = 62 \cdot .12 - 8 \cdot (1 - .12) = .4.$$

Rose should call. Rose requires 8 callers for a raise, and since no one wants to let a pot this big go she might get it, but the odds are certainly against her, so she should probably just call. △

To be complete we need to analyze all 169 qualitatively different flop hands in the same way (so far we've done 13), and we certainly won't do that in this project. However, we will analyze another standard case in the assignments, *unsuited connectors*, and by adding a straightforward suited-hand calculation we will see how to play many suited connectors as well.

5 A Theory of Flop Play

In this section we develop a theory for the round of betting after the flop has been dealt. We note at the outset that to some extent play after the flop is contained in our starting-hand analysis. For example, in Example 4.1 above Rose called the big blind bet under the assumption that she would play hands H_1, H_2, H_3, H_4, and H_6 after the flop. Generally speaking, if one of these hands flops then Rose will call or raise, and if none of these hands flops she will fold. In some sense what we're doing in this section is taking a step back and understanding more precisely which flops Rose should be willing to play when developing her starting hand strategy. On the other hand, Rose must of course be willing to re-evaluate her strategy at any point in the hand, based on how her opponents are playing and how much money is in the pot.

5.1 Drawing Hands and Outs

We typically refer to a card that will significantly improve a player's hand as an *out*. For example, suppose a player holds $8\heartsuit 8\spadesuit$, and the flop is $3\diamondsuit J\diamondsuit 7\spadesuit$. The two remaining eights in the deck would be considered outs for this player. On the other hand, while an ace would improve the hand (to a pair of eights with an ace kicker), we would not consider an ace to be an out. This is why we say *significantly* improve. In order to motivate the theory of outs, we'll consider a simple canonical example.

Example 5.1. Rose holds $J\clubsuit T\clubsuit$ (suited *Sweet Baby James*) in her pocket and the flop is

$$K\diamondsuit 5\clubsuit 7\clubsuit.$$

There are five players remaining after the flop, and Rose is third to act. The betting has been

1. Bet at 4
2. fold
3. Rose to act.

The pot is $P = 24$ and the bet is $B = 4$. Should Rose fold, call, or raise?

Even though no one has shown much strength so far, the only good chance Rose has for winning this hand is a flush. (The king on the flop makes her think simply pairing the ten or jack may not be enough; certainly, anyone with a king in his pocket is going to stay in the hand, and there are still two players to act.) There are two ways for Rose to think about this. First, four clubs are already out, so of the 47 remaining cards in the deck there are nine clubs remaining (these are Rose's outs). This means Rose's probability of getting a club on the turn is

$$P(\text{club on turn}) = \frac{9}{47} = .191489.$$

In the notation of the previous section, with R denoting the event that Rose "wins at the turn" we would write $P(R) = .191489$. On the other hand, Rose could also get a club on the river, so it's useful to know her probability of getting a club on either the turn or the river. In order to compute this, we note that the probability that she *doesn't* get a club on either the turn or the river is

$$\frac{38}{47} \cdot \frac{37}{46} = .650324.$$

Accordingly, the probability that she gets at least one club on the turn or the river is

$$P(\text{club on turn or river}) = 1 - .650324 = .349676.$$

In this case we let R denote the event that Rose gets her flush (and so has a very good chance of winning the hand), and we have $P(R) = .349676$. Let's pause prior to making Rose's decision in this case and write out our general decision equations. △

5.1.1 Decision Equation for Drawing Hands

For drawing hands, my preference is to work with the second probability in Example 5.1, the probability that Rose will draw to the best hand after the river. In order to understand why this might be the more appropriate value to work with, let's first consider the decision equation we would use with the first probability. If we let R denote the event that Rose hits her flush on the turn, then we have

$$E[W_c] = P \cdot P(R) - B \cdot P(R^c),$$

where in the case of Example 5.1 $P(R) = .191489$. Notice, however, that this assumes that Rose will fold her hand if she doesn't make a flush on the turn, and it's certainly not clear that's what she'll do. (Recall that in our evaluation of starting hands, we were fairly certain that Rose should not continue with the hands we threw out; here, that's not the case.) While it is certainly true that Rose should re-evaluate her options after the turn card has been dealt (based on the betting for the remainder of the flop round and the first part of the turn), it's better for her, in my estimation, to consider both rounds at once while making her decision at the flop. For this, of course, we have to modify our decision equation a bit to express the fact that there remains a round of betting after this decision has been made.

To begin, we need to recall that the level of betting increases from 4 to 8 at the turn. In Hold'em games that aren't particularly loose, there are often only two players remaining after the turn (for the round of betting after the river card has been dealt), and so we will assume that if Rose bets on the turn then she will only have one caller. (This makes our model fairly conservative, because it would be better for Rose to have more callers, though we're going to tweak it a little before we're finished. To be fair, as Hold'em has gotten more popular it has attracted more weak players, and so a lot of low-limit games *are* particularly loose.) In this way, her decision equation will be

$$E[W_c] = (P + 8)P(R) - (8 + B)P(R^c),$$

where in this case for Example 5.1 we have $P(R) = .349676$. Here, Rose has put in one bet at the flop (at level B, which depends on whether or not there has been a raise) and one bet at the turn (assumed at level 8). One of Rose's opponents has called her on the turn at level 8. We're not quite finished, though. Note carefully that Rose will certainly fold if she fails to make her flush (unless she opts to bluff), and so she won't lose more than $8 + B$ (ignoring raises etc.; bear with me here). On the other hand, since there still remains a round of betting after the river card has been dealt, she conceivably can make more money than $P + 8$, even with only a single opponent. (These are the *implied odds* mentioned leading into Example 4.1.) The model I suggest is finally

$$E[W_c] = (P + 16)P(R) - (8 + B)P(R^c), \qquad (2)$$

where we have now assumed Rose somehow gets one more bet out of her opponents, either because she has two callers on the turn or because she gets a bet out of her opponent on the river.[8]

Example 5.1 finished. We now finish off Example 5.1 using our drawing-hand flop decision equation (2). We have

$$E[W_c] = (40) \cdot .349676 - 12 \cdot (1 - .349676) = 6.183152,$$

[8]You might be thinking that Rose has to wager another $2B$ on the river, but keep in mind that by that point she knows whether or not she's made her flush, so she only puts this money in the pot if she knows she'll get it back. We're assuming, of course, that a flush is going to win the hand.

and Rose should at least call.

Last, we need to determine whether or not Rose should raise. If we consider only the possibility of her raising on the flop round (even though our model regards two rounds), we obtain the usual raise relation $N \geq P(R^c)/P(R)$, which in this case becomes

$$N \geq \frac{1 - .349676}{.349676} = 1.859790,$$

so that Rose needs two guaranteed callers to raise. (See Section 5.1.2 for an alternative view of raising.) Since one player has already folded there are only three players aside from Rose in the hand, so this is a borderline case. I would suggest she raise. △

See the assignments for another example.

5.1.2 Raising to Protect a Hand

The raising calculation we used in Example 5.1 determines *raising for value*, but a poker player will often raise to *protect her hand*. The idea is simple: the more opponents Rose has, the better the odds are that one of her opponents will hit one of his outs and beat her (even if she hits one of her outs). She raises to make her opponent's odds such that he will fold. We investigate this in the assignments.

5.1.3 Slowplaying

Slowplaying is a technique essentially opposite to raising to protect a hand. We say that Rose is slowplaying her hand if she plays it as if it's not as strong as it actually is; for example, if she checks and calls with a very strong hand we say that she is slowplaying. We saw in our discussion of raising to protect a hand that a bet or raise can scare off an opponent (induce a fold), and if Rose has a very strong holding she wants to keep as many players in the pot as possible. Notice carefully that since Rose should only slowplay with a very strong hand she will raise to protect her hand much more often than she will slowplay to build a pot. This brings us to a fairly good rule of thumb about flop play: *When in doubt between calling and raising on the flop you should usually raise.*[9] The calculations involved with an analysis of slowplaying are the same as those for an analysis of raising to protect a hand, which will be considered in the assignments.

5.1.4 Raising for a Free Card

In certain cases raising can be used to get a *free* card (as it's usually called), or more precisely a card at half price. To understand how this works, we need to first take one step back and notice that players will often check around to the player who has shown the most strength. The idea is simple: if Rose acts earlier than Colin and bets, then if Colin is strong he will likely raise her, and she will have to pay two bets to see the next card. If Rose checks then Colin will bet and Rose will be able to call with a single bet. So if Rose has a

[9]There is an important exception: under certain circumstances you should check *with the intention of raising if someone bets.* This is called check-raising, and we will analyze it in more detail in our discussion of river play.

medium-strength holding and she suspects Colin has a strong holding, she will likely check to him.

Now suppose Colin has a drawing hand, something like a four-flush, and is in last position. There is one bet to him on the flop. He raises to show strength (eight dollars total), hoping that everyone will check to him on the turn. If everyone checks to him on the turn he has the option to check as well and see the river card for free. So if one of his outs hits he will bet, and if none of his outs hits he will check. Notice that if he had not raised on the flop he would have paid four dollars on the flop and (probably) eight more dollars to answer a bet on the turn—so twelve dollars total. If his free card play works he has bought the eight-dollar river card at half price.

5.1.5 Outs Equation

The analysis we carried out in Example 5.1 is easily extended to any drawing hand, and all that changes is the number of outs. For example, suppose Rose holds $J\spadesuit T\diamondsuit$ (unsuited Sweet Baby James) and the flop is

$$8\spadesuit 5\heartsuit 9\clubsuit.$$

Rose has flopped an open-ended straight draw (and not much else, though technically the $T\diamondsuit$ and $J\spadesuit$ are overcards), and so she has eight outs (instead of the nine for a flush). To be general, let's denote the number of outs by x, and compute the probability that Rose gets at least one of her outs in two cards. The probability that Rose does not see one of her outs on either the turn or the river is

$$\frac{47-x}{47} \cdot \frac{46-x}{46},$$

and so the probability that Rose gets at least one of her outs is

$$P(\text{Rose gets an out on turn or river}) = 1 - \frac{47-x}{47} \cdot \frac{46-x}{46}.$$

For the case of an open-ended straight draw (eight outs) this probability is .314524.

5.1.6 Partial Outs

So far we have assumed that if Rose gets one of her outs then she will almost certainly win the hand. In practice, of course, there will often be cards that may or may not win the hand for Rose. For example, suppose Rose holds $A\diamondsuit K\diamondsuit$ in her pocket and the flop is

$$5\diamondsuit T\diamondsuit T\spadesuit.$$

Certainly any diamond is an out for Rose, but she also has a fairly good shot at the pot if either an ace or a king falls. (There is also the possibility of a runner-runner straight; for this, see the next subsection.) Rose must be careful, however, because a diamond is clearly a much better out than either an ace or a king. For a situation like this, we can separate the outs into two categories $x = 9$ outs and $y = 6$ *partial* outs. The probability that Rose gets one of her outs is given by the usual outs equation, as is the probability that Rose gets one of her partial outs (with x replaced by y). This double-counts cases in which Rose gets one of each, and we want to subtract these from her partial outs probability. (If Rose gets both an out and a partial out, then she has an out.[10] For example, if the turn and river are $6\diamondsuit A\spadesuit$ Rose has a flush, and she won't even play the partial out.) The probability that Rose gets one of each is

$$P(\text{Rose gets an out and a partial out}) = 2\frac{x}{47} \cdot \frac{y}{46}.$$

Rose's outs equations become

$$\begin{aligned} P(\text{out}) &= 1 - \frac{47-x}{47} \cdot \frac{46-x}{46} \\ P(\text{partial out}) &= 1 - \frac{47-y}{47} \cdot \frac{46-y}{46} - 2\frac{x}{47} \cdot \frac{y}{46}. \end{aligned}$$

In our example we have $x = 9$ and $y = 6$, so

$$\begin{aligned} P(\text{out}) &= .349676 \\ P(\text{partial out}) &= .191489. \end{aligned}$$

We now decompose $P(R)$ in terms of whether she gets an out or a partial out. For this example, let's suppose Rose estimates that she will certainly win with a flush but that she has only a 50% chance of winning with top pair. If Rose has a single opponent, she will have

$$\begin{aligned} P(R) &= P(R|\text{out})P(\text{out}) + P(R|\text{partial out})P(\text{partial out}) \\ &= 1 \cdot .349676 + .5 \cdot .191489 = .445421. \end{aligned}$$

(Notice that if Rose doesn't feel comfortable estimating her opponents' probabilities she can fall back on Table 8, for which her probability of winning with a pair of kings is roughly .578157 while her probability of winning with a pair of aces is roughly .612345. If she feels she has no information whatsoever about her opponents, it's reasonable to simply average these two values—unless she wants to subdivide into three outs categories.[11])

Let's notice an important thing about outs and partial outs: While outs play well against multiple opponents, partial outs quickly devalue as the number of opponents increases. In order to understand this, suppose Rose has three opponents instead of just one in the previous example. If the probability that her top pair holds up against one opponent is .5, then the probability that it holds up against three opponents is roughly $.5^3 = .125$ (only rough, because the events aren't strictly independent).

5.1.7 Runner-runner Draws

So far, and with good reason, I've been ignoring the so-called runner-runner draws. In the example of the previous subsection Rose has another draw we briefly alluded to, a jack and then a queen, which would give her broadway (an ace-high straight), or perhaps even a royal

[10] Yes, okay, sometimes she has an uber-out, but what matters is that she thinks she has something that will almost certainly win.

[11] For the moment, at least, she doesn't.

flush. In cases like this, when a player requires a card on both the turn and the river, we refer to the draw as a runner-runner draw. Let's compute the probability that Rose makes this draw. She can either draw the jack first and then the queen, or the queen first and then the jack. This gives

$$P(\text{broadway}) = 2\frac{3}{47}\cdot\frac{3}{46} = .008326,$$

close to 1%. Some runner-runner draws are more likely than this (see the assignments), and while it's rarely correct to call with *only* a runner-runner draw it is often profitable to call with an arrangement of draws that *includes* a runner-runner draw. For example, Rose might have an overcard, a runner-runner straight draw, and a runner-runner flush draw. Each of these is weak by itself, but taken together they might give her enough odds to call if the pot is large enough.

5.1.8 Consolidated Outs[12]

For simplicity and uniformity some Hold'em theorists like to organize drawing hands entirely in terms of outs, which means they need to assign a number of outs to partial outs and runner-runner hands. In order to see how this works, let's see how many outs the runner-runner draw described in the previous section would be worth; i.e., we compute the number of outs that correspond with the probability .008326; precisely, we solve the following equation for x,

$$.008326 = 1 - \frac{47-x}{47}\cdot\frac{46-x}{46}.$$

Since it will be useful to have a general formula, let's replace .008326 with p, in which case we find

$$x^2 - (47+46)x + 47\cdot 46p = 0,$$

giving

$$x = \frac{93 \pm \sqrt{93^2 - 4\cdot 47\cdot 46p}}{2}.$$

Notice that if we take addition we will have more than 46.5 outs, which is clearly nonsense, so we take subtraction. In our case $p = .008326$ and so

$$x = .193962.$$

That is, we might reasonably count this runner-runner possibility as roughly .2 outs.[13]

Clearly, we can proceed similarly with partial outs. For example, in the case discussed in Section 5.1.6 we have $p = .5\cdot .191489 = .095745$, and so the number of outs would be

$$x = \frac{93 - \sqrt{93^2 - 4\cdot 47\cdot 46\cdot .095745}}{2} = 2.28.$$

In [2] the authors suggest counting each overcard (the aces and kings in our example) as half an out, so they would use 3 outs, where our analysis suggests using 2.28 outs.

[12]Our project won't use this calculation, but it might be useful for Internet readers.

[13]Since a draw like this is extremely well disguised, and so might win a large pot, the authors of [2] suggest it might be worth a bit more, say half an out.

5.1.9 The Role of the Board in Counting Outs

For clarity we've been looking at examples in which the board (the three flop cards at this point) is fairly benign. We should note, however, that the way Rose counts her outs may be quite dependent on both her pocket and the board. For example, notice that there is an enormous difference between a case in which Rose has $2\clubsuit 7\clubsuit$ in her pocket with a flop $6\diamondsuit J\clubsuit 9\clubsuit$ and a case in which Rose has $6\diamondsuit 7\clubsuit$ in her pocket with a flop $2\clubsuit J\clubsuit 9\clubsuit$. In the former she has the usual 9 flush outs (plus a very small runner-runner possibility), while in the latter case she must recognize that anyone with even a single club has a flush draw, and moreover that anyone with a club higher than 7 has a better flush draw than she has. While the former hand is almost always worth playing the latter hand is only worth playing if there are few players remaining in the hand so that there is some reasonable chance that no one else has a club.

5.2 Made Hands and Ins

Our thinking changes considerably when we have a made hand. Again, let's motivate the main considerations with an example. I'll take this one from [2] (p. 34).

Example 5.2. Rose holds $9\heartsuit 2\spadesuit$ in her pocket and the flop is

$$9\clubsuit 8\clubsuit 3\heartsuit,$$

giving Rose top pair. There are four players remaining after the flop, and Rose is second to act. The betting has been

1. Bet at 4
2. Rose to act.

The pot is $P = 20$ and the bet is $B = 4$. Should Rose fold, call, or raise?

In this case Rose has a made hand, her pair of nines, which is top pair. The problem with analyzing this hand with outs is that Rose only has two bona fide outs, the remaining nines. (The three remaining twos would be partial outs, counting something like 1.5 outs total.) On the other hand, there are plenty of cards Rose wouldn't mind seeing on the flop: essentially anything nine or smaller that's not a club and doesn't add to the straight-draw possibility of $9\clubsuit 8\clubsuit$. That is, at this point Rose would like to continue with any card that is not an out for one of her opponents. More precisely, there are 29 cards Rose would like to avoid (20 cards ten and higher, 6 clubs 2-7, and 3 non-club sevens). This leaves $47 - 29 = 18$ cards she wouldn't mind seeing, including of course the nines and the non-club twos. We refer to these cards that will keep Rose in the hand as ins, but here's the big difference: in order for Rose to win with this hand she probably needs to see an in on *both* the turn and the river. (Recall that with outs Rose only needed an out on *either* the turn or the river.) The probability that Rose sees an in on both the turn and the river is

$$P(\text{in on turn and river}) = \frac{18}{47} \cdot \frac{17}{46} = .141536.$$

You can probably guess where this is headed, but as in the case of our analysis of drawing hands, let's develop our general decision equation before completing the example.[14] △

5.2.1 Decision Equation for Made Hands

As in the case of drawing hands we'll base our decision equation on two rounds, but this time we need to recognize that Rose may be on the wrong side of the implied odds (poker players would say she's getting *reverse implied odds*[15]). The idea is that one of Rose's opponents may have a drawing hand (for a straight or flush, for example), and in this case Rose will have to pay extra if the draw hits, but she certainly won't get any bets out of her opponent if it fails (her opponent will fold). Consequently, in our decision equation for made hands we will drop off the extra 8 we obtained above with implied odds. We have, then,

$$E[W_c] = (P + 8)P(R) - (8 + B)P(R^c), \tag{3}$$

where in the case of Example 5.2 $P(R) = .141536$.

Example 5.2 finished. We have now

$$E[W_c] = 28 \cdot .141536 - 12(1 - .141536) = -6.33856,$$

and Rose should certainly fold. △

See the assignments for another example.

6 A Theory of Turn Play

You're on your own here. See the assignments.

7 A Theory of River Play

Once the river card has been dealt a player knows exactly what she has, and she can use Table 8 (in the appendix of these notes) to gauge the strength of her hand. For example, if Rose holds a pair of kings the probability that a single opponent has her beat is roughly .421843—the probability that a player has a pair of aces or better. Of course, the problem with this is that it assumes that Rose's opponent has played arbitrary hands (instead of folding the bad ones), and it does not take advantage of the fact that Rose knows five of her opponent's seven cards. In practice, if Rose has paid close attention to the bets her opponent has made, then she should be able to make a reasonable guess as to what he might hold. (This is called *putting her opponent on a hand*, and while it is difficult and often imprecise, it is extremely important: the really good players are often phenomenal at this aspect of

[14] In [2] on pp. 185-189 the authors discuss the same general idea from a slightly different point of view (and with a different hand). In particular, instead of considering which specific cards might give an opponent an out, they envision a generic array of hands four opponents might have: a flush draw, top pair, a gutshot, and two overcards.

[15] No, I'm not making this terminology up, though I have to admit *ins* is mine.

play.[16]) For example, if a player raises before the flop it typically means she has something like AA, KK, QQ, JJ, TT, AKs, AQs, AJs or AKu. Now suppose there is a queen on the flop, and this same opponent raises. This suggests QQ or AQs, with a smaller possibility of hands like AQu or KQs. On the river, Rose should certainly fold anything that doesn't beat a pair of queens, and if the opponent is playing the hand particularly aggressively, she might even fold anything that doesn't beat trip queens. (Meanwhile Rose is paying very close attention to her opponent, making sure he's not simply a moron who raises with just about anything. If he's a moron, she plays solid book poker, uses Table 8 on the river, and wins an insane amount of money.)

Notice, by the way, that it is not a good idea for Rose to start with the five-card board and analyze her opponent's hand in the same way she analyzed the possibilities for her five card hand after the flop. The error here should be clear: the logical inconsistency of it. That is, there are certain hands that might fit this board perfectly, but that no one would have played from the outset. For example, if the board is

$$3\diamondsuit 6\spadesuit T\spadesuit 5\heartsuit K\clubsuit,$$

a pocket $4\heartsuit 7\diamondsuit$ would be great, but almost no one would play such a thing.

Since there are often only two players left at the river, and (more important to us) since the analysis we want to carry out is much simpler for only two players, we will assume the game is head-to-head at the river. For a non-mathematical discussion of a few issues with multiple players at the river, see [2].

7.1 Second position after the river

7.1.1 Colin Bets

First, let's suppose Colin has bet, so Rose can fold, call, or raise.

CALLING IN SECOND POSITION. If Rose calls Colin's bet her expected value is

$$E[W_c] = P \cdot P(R) - B \cdot P(R^c).$$

Notice that these are actually conditional probabilities, conditioned on the event that Colin bets. I.e., we expect Colin to bet from a position of strength (unless he's bluffing), so we have some information from this condition. Also, notice that, due to the possibility of a tie, we should really have three events, R if Rose's hand is the winner, $\bar{R}$ if Colin's hand is the winner, and $\hat{R}$ if the hands tie. This would give

$$E[W_c] = P \cdot P(R) - B \cdot P(\bar{R}) + \frac{P - B}{2} P(\hat{R}),$$

where the final summand corresponds with the case of a tie. In practice, ties are more trouble than they're worth to deal with, and we will leave them out of our calculations.

RAISING IN SECOND POSITION. In order to compute Rose's expected value of raising in second position, we condition on the event that Colin calls her raise,

$$A = \text{event Colin calls Rose's raise.}$$

[16] I am not.

(For this calculation, we assume Colin will not reraise.)

$$E[W_r] = E[W_r|A]P(A) + E[W_r|A^c]P(A^c)$$
$$= \Big[(P+B)P(R|A) - 2BP(R^c|A)\Big]P(A) + P \cdot P(A^c).$$

The important point here is that $P(R|A)$ is typically smaller that $P(R)$, because Colin will only call a raise with a reasonably strong hand. This leads to the following general rule of thumb: *When in doubt between calling and raising on the river you should probably call.*

Example 7.1. Rose holds $J\diamondsuit T\spadesuit$ in her pocket and after the river card has been dealt the board is

$$J\heartsuit 4\spadesuit 8\spadesuit Q\clubsuit T\clubsuit.$$

Rose is head-to-head with Colin, who raised under the gun pre-flop, checked after the flop, and bet after the turn. The pot is $P = 54$, following Colin's bet on the river of $B = 8$. Should Rose fold, call, or raise?

Rose begins by considering the array of hands Colin might have. His raise under the gun was a suggestion of real strength, so Rose supposes first that he has a pocket pair tens or better, big slick unsuited, or any suited broadway (i.e., any two cards ten or better of the same suit). (There is, of course, a presumption here that Rose has been carefully watching her opponents and has reason to believe these are the hands Colin might raise with under the gun.) This reduces Colin's possible hands to 68, which we list below, keeping in mind a few eliminations Rose can make based on what she holds[17]:

$$AA(6), KK(6), QQ(6), JJ(3), TT(3)$$
$$AK(16), AQs(4), AJs(3), ATs(3), KQs(4), KJs(3), KTs(3), QJs(3), QTs(3), JTs(2).$$

His subsequent check after the flop suggests Rose can eliminate any hand with a jack, any hand with two spades, and any hand that overpairs the board. This reduces the possibilities to 33:

$$TT(3), AK(15), AQs(3), ATs(3), KQs(3), KTs(3), QTs(3).$$

His raise after the turn suggests he liked the queen, so presumably he has either a queen in his hand or an open-ended straight draw. This reduces the possibilities to 22:

$$AK(15), AQs(2), KQs(2), KTs(3), QTs(2).$$

Finally, Colin's bet after the river may mean he likes the ten, but it may simply indicate he still feels strong enough after the turn. This reduces the possibilities to 20:

$$AK(15), AQs(2), KQs(2), QTs(1).$$

(Notice that in making his final bet Colin knows there are no flushes, and he's not overly concerned about the straight possibility since it would have to have been runner-runner.) If

[17] While listing these is tedious, it's probably worth doing once since this is a fairly common set of hands for a player to consider strong.

Colin holds $AK(15)$ or $QTs(1)$ then he takes the pot, and if he holds $AQs(2)$ or $KQs(2)$ Rose takes it. This means

$$P(R) = \frac{4}{20} = .2.$$

Rose's expected value of calling is

$$E[W_c] = 54 \cdot .2 - 8 \cdot .8 = 4.4,$$

so Rose should at least call.

To check whether or not Rose should raise, we need to consider which hands Colin will call a raise with. With a queen on the board, he has to doubt Rose would raise (unless she's bluffing) with a pair of jacks or worse, so if he's a solid player capable of a tough fold he will likely fold precisely the two hands Rose can beat. In this way, raising would be tragic: Rose would have $P(R|A) = 0$, and so

$$E[W_r] = -16 \cdot \frac{16}{20} + 54\frac{4}{20} = -2,$$

taking her from a positive expectation to a negative expectation. △

7.2 First Position After the River

A player in first position after the river has two choices, check or bet. This situation is quite similar to second position after the river, and we'll omit its expected-value analysis. While the player in first position on the river is at a disadvantage, he can compensate for this somewhat by bluffing (with a hand that misses its draw) or check-raising (with a hand that's a clear winner). We'll investigate each of these possibilities in the assignments, but to prepare for the analysis there, we'll give an example of each here.

7.2.1 Bluffing

Suppose Colin[18] holds $T\clubsuit Q\clubsuit$ in his pocket and after the turn card has been dealt the board is

$$2\clubsuit 7\diamondsuit A\clubsuit K\diamondsuit.$$

Colin has put Rose on a fairly good hand such as trip aces or trip kings. If a club falls on the river then Colin will certainly bet, but at the same time he might notice that if a diamond falls on the river it will look as if he might have made his flush. Should he bet both clubs and diamonds? In general, this is a very tough question, and this is where Colin is well-served by having watched his opponent's reactions during the game: is Rose the sort of player who calls just about any hand on the river, or is Rose capable of making a tough fold even with a good hand and a large pot? In the assignments, we will use game theory to give a mathematician's answer to this question, but let's take a practical look at it here. First, suppose Colin never bluffs, and that, knowing this, Rose will always fold when Colin bets. Then Colin's expected value is

$$E[W_b] = \frac{9}{46}P.$$

[18] Since Rose has consistently been our second-position player, we'll let Colin be our first position player.

I.e., Colin gets the pot if his flush comes in, and otherwise he folds. (Keep in mind that a fold always has zero expectation.) On the other hand, suppose Colin bets any club or diamond, and Rose continues to fold when Colin bets. Then Colin's expected value is

$$E[W_b] = \frac{20}{46}P,$$

a considerable improvement. (Notice that while there are only 9 live clubs there are 11 live diamonds.) Of course, Rose is a very savvy player, so let's suppose she sees what Colin is doing and decides to call all his bets. Colin's expected value becomes

$$E[W_b] = \frac{9}{46}(P+B) - \frac{11}{46}B = \frac{9}{46}P - \frac{2}{46}B,$$

even worse than it was when he was only betting clubs. But Colin also is very savvy, and decides to mix it up, betting exactly half the time when he has a diamond. (For example, Colin might glance down at his watch and if the second hand is on the left side he will bet and if it is on the right side he will check.) If Rose continues to call all of Colin's bets his expected value becomes

$$E[W_b] = \frac{9}{46}(P+B) - .5 \cdot \frac{11}{46}B = \frac{9}{46}P + \frac{4.5}{46}B,$$

better than when he was betting only clubs. Of course, Rose quickly twigs to this, and ... you're starting to get the idea. This is precisely the type of problem game theory was designed to solve.

7.2.2 Check-Raising

Suppose Colin has $A\diamondsuit A\spadesuit$ in his pocket, and after the river card has been dealt the board is

$$2\diamondsuit A\clubsuit 8\spadesuit 7\heartsuit Q\clubsuit.$$

Colin has trip aces, and with no straight possibilities, no flush possibilities, and no full house or quad possibilities, Rose is toast. The only question now is, how can Colin get as much money out of Rose as possible? If he bets and she calls he'll only get one bet out of her, but suppose he checks (suggesting weakness), and she bets. Then he can re-raise and get a second bet out of her. This is a trick called check-raising, and while we will only discuss and analyze it for river play, it's actually a useful tool at all rounds. The question is, under what conditions should Colin check-raise, and this is another question we will answer in the assignments.

8 Assignments

1. **Unsuited connectors.** Consider unsuited-connector starting hands such as $8\heartsuit 9\diamondsuit$, $T\heartsuit J\spadesuit$, etc., with top cards in the rank range 5–J (the other cases must be analyzed in a slightly different, though certainly no more difficult, way). Compute the number of such hands out of the 1,326 possible starting hands and the probability of getting such a hand.

2. **Unsuited connectors at the flop.** For the unsuited connectors described in Item 1, compute the probability of getting each of the following five-card hands after the flop:

1. Straight
2. 8-out straight draw with a pocket card pairing
3. 8-out straight draw without a pocket card pairing
4. 4-out straight draw with a pocket card pairing
5. 4-out straight draw without a pocket card pairing
6. Trip or quad the top card (includes possible full house)
7. Trip or quad the bottom card (includes possible full house)
8. Pair both cards (no trips)
9. Top pair with top card (no straight draw, board pair possible)
10. Low-pair with top card (no straight draw, board pair possible)
11. Top pair with bottom card (no straight draw, board pair possible)
12. Low-pair with bottom card (no straight draw, board pair possible).

Notes. An 8-out straight draw is precisely what it sounds like: a hand for which eight different cards will lead to a straight. For example, if Rose holds 78 and the 8 is paired then the possible 8-out straight draws are 5678, 6789, and 789T (i.e., respective flops 568, 689, and 89T, where of course order doesn't matter). If a pocket card doesn't pair, there are more possibilities: 5678, 6789, 789T, 4_678_T, and 5_789_J (the latter two are sometimes called double gut-shot draws). Likewise, a 4-out straight draw is a hand for which four different cards will lead to a straight. Finally, for (9)-(12) take care when the top card is 5.

2a. **Checking item 2.** To verify some of your calculations for Item 2, compute the number of ways that the top card can be paired, no trip, no quad, and no pair on the bottom card (board pair possible). Check that this is the same as

$$\frac{N_2 + N_4}{2} + N_9 + N_{10},$$

where N_j denotes the number of ways to get hand H_j.
Notes. We divide $N_2 + N_4$ by 2, because those hands include pairing both top and bottom cards. While N_9 and N_{10} both depend on the rank of the pocket connectors, the sum $N_9 + N_{10}$ does not.

3. **Starting-hand play with unsuited connectors.** Use your calculations in Item 2 to answer the following:

Starting Hand 1. Rose holds $T\clubsuit J\diamondsuit$ in her pocket and is sixth to act after the big blind.

The betting has been:

1. fold
2. call at 4
3. call at 4
4. fold
5. call at 4
6. Rose to act.

The pot is at $P = 18$ and the bet is $B = 4$. Should Rose fold, call, or raise?

Starting Hand 2. Rose holds $7\heartsuit 8\spadesuit$ in her pocket and is fourth to act after the big blind. The betting has been:

1. call at 4
2. raise to 8
3. call at 8
4. Rose to act.

The pot is $P = 26$ and the bet is $B = 8$. Should Rose fold, call, or raise?

4. **Flush draws.** This is a very simple calculation that I want to add because it's quite useful: Suppose Rose has any two suited cards in her pocket, and compute the probability she flops (1) a flush and (2) a flush draw (i.e., a four-flush). If you combine these calculations with your work from Item 2 (and overlook some overlapping cases), you can now analyze play with suited connectors.[19]

5. **Play at the flop.** Analyze each of the following situations:

Flop Hand 1. (From p. 272 in [2].) Rose holds $Q\clubsuit 9\clubsuit$ in her pocket and the flop is

$$K\heartsuit T\diamondsuit 4\clubsuit.$$

Six players, including Rose and both blinds, called before the flop. The betting after the flop has been:

1. check
2. bet at 4
3. raise to 8
4. Rose to act.

The pot is $P = 36$ and the bet is $B = 8$. Should Rose fold, call, or raise? (I have in mind your ignoring the runner-runner flush possibility here, though we will reconsider it in Item 7.)

[19] Though it's not part of this project.

Flop Hand 2. Rose holds $Q\heartsuit T\heartsuit$ in her pocket and the flop is

$$T\spadesuit 8\clubsuit 4\heartsuit.$$

Four players, including Rose and both blinds, called before the flop. The betting after the flop has been:

1. bet at 4
2. call at 4
3. Rose to act.

The pot is $P = 24$ and the bet is $B = 4$. Should Rose fold, call, or raise?

6. **Raising at the flop to protect a hand.** Suppose in the flop round the pot is P when the betting gets to Rose and the bet level is $B = 4$ (i.e., someone has bet but no one has raised). Rose would like to decide between calling and raising, and she wants to think about it in the following way: How many outs does Colin (Rose's generic opponent) need in order to call the bet if she only calls, and how many outs does Colin need in order to call the bet if she raises? (Her raising will certainly reduce his pot odds, but the question is, by how much?) Derive relations for Colin's number of outs x in both cases (you will get an inequality relation for x in terms of P), and as an example determine whether or not Rose should raise if the pot is $P = 44$. (We are assuming Rose has already determined that she will at least call, so her cards are not important for your calculation. Notice that when the betting gets to Colin the pot will be $P + 4$ if Rose only calls and $P + 8$ if Rose raises.)

7. **Runner-runner hands.** This problem regards runner-runner hands.

7a. Suppose Rose holds $7\heartsuit 8\diamondsuit$ in her pocket and the flop is

$$6\diamondsuit 2\clubsuit K\spadesuit.$$

Compute the probability that Rose hits her runner-runner straight draw by the river.

7b. Suppose Rose holds $2\spadesuit 6\spadesuit$ in her pocket, and the flop is

$$T\spadesuit 9\heartsuit 3\diamondsuit.$$

Compute the probability that Rose hits her runner-runner flush draw by the river. Is the runner-runner flush possibility enough to salvage Flop Hand 1 from Item 5 from the muck pile?

8. **Play at the turn.** Develop your own decision equations for drawing and made hands at the turn, and use them to analyze each of the following situations:

Turn Hand 1. (From p. 28 of [2].) Rose holds $4\clubsuit 5\clubsuit$ in her pocket and the board after the turn is

$$J\heartsuit T\diamondsuit 6\clubsuit 2\spadesuit,$$

giving Rose a gut-shot straight draw. The game is head-to-head and Rose's opponent has bet 8, bringing the pot to 80. Should Rose fold, call, or raise?

Turn Hand 2. (From p. 291 in [2]) Rose holds $A\heartsuit Q\heartsuit$ in her pocket and the board after the turn is

$$Q\spadesuit T\spadesuit 7\heartsuit 3\clubsuit,$$

giving Rose top pair. There are five players still in the hand, and the betting after the turn is

1. check
2. check
3. bet at 8
4. call at 8
5. Rose to act.

The pot is at $P = 106$ and the bet to Rose is $B = 8$. Should Rose fold, call, or raise?

9. **A mathematician's view of bluffing.** Before analyzing play at the river, let's take a brief look at how bluffing can be analyzed with game theory in the following simplified game: Rose and Colin each place a \$1.00 ante into a pot, and then each is dealt a single card from a limitless deck consisting only of queens, kings, and aces. After looking at his card, Colin must decide either to bet \$2.00 or to fold. If Colin bets, Rose must decide either to fold or call. If Rose calls there is a showdown: aces beat kings and queens, kings beat queens, and no money changes hands in the event of a tie. Find optimal strategies for Rose and Colin, and give straightforward directions for how each of them can follow her or his optimal strategy.

10. **A gambler's view of bluffing.** Item 9 gives a good example of how a mathematician might view bluffing, but a gambler often views it from a slightly different angle. In particular, a gambler will often assume he can read his opponent's hand and will base his game theoretic calculations on this additional information. To see how this works, consider a slight revision of Item 9: Suppose that in the game proposed in Item 9 Colin knows that Rose has a king, and what he wants to determine is how often he should bluff when he is dealt the queen. Find the optimal strategies and the game theoretic value for this situation.

11. **Bluffing on the river.** Poker is sometimes described as *a contest between a made hand and a drawing hand.* In this assigment we will analyze precisely this situation for two players head-to-head after the river card has been dealt. In particular, we'll use game theory to determine when the player with the drawing hand should bluff if he doesn't get his hand. Consider two players, Rose and Colin, Colin in first position. We assume Rose has a made hand (i.e., Colin more or less knows what Rose has) and that Colin has drawn to a hand at the river and obtained either a strong hand or a weak hand. For example, Rose might have bet strong on a flop with an ace, suggesting she has a pair or even trip aces, and Colin might be drawing for a flush. We assume that if Colin gets his strong hand he certainly beats Rose (e.g., his mighty flush beats her measly trip aces), but that if he gets his weak hand he certainly loses (e.g., Rose's trip aces wallop Colin's bupkis). We let Colin have two choices, to check or bet, and we assume that he will only check if he has not made his hand, and that since Rose understands this, his check is tantamount to a fold. (This isn't necessarily the case. In practice, Collin could be check-raising with a strong hand, hoping to get one extra

bet out of Rose if she bets, he raises, and she calls his raise—as opposed to Colin betting and Rose calling.) If Colin bets, Rose has two options: fold or call. (In this scenario it's not reasonable for Rose to raise, because either Colin certainly has her beat with a strong hand or he will simply fold a weak hand. So by raising, she gives herself a way to lose additional money, but no way to gain any additional money.) Determine the optimal strategies for Rose and Colin in terms of the following variables:

$$\begin{aligned} q &= \text{probability Colin gets a strong hand} \\ P &= \text{current value of the pot} \\ B &= \text{current bet level.} \end{aligned}$$

Note. An easy way to make the game zero-sum is to consider Colin and Rose to each have an investment of $\frac{P}{2}$ in the pot.

River Hand 1. Suppose Colin holds $9\diamondsuit T\diamondsuit$ in his pocket and after the turn the board is

$$5\diamondsuit A\spadesuit K\diamondsuit 2\spadesuit.$$

Colin has put Rose on AA, KK, or AK, so that she has him beaten unless he draws to a flush. How often should Colin bluff if he misses his flush? Assume the pot is $P = 80$ and the current bet is $B = 8$. (Notice that since Colin could have a 3 and a 4 in his pocket he can bluff with any card, since he could, at least in theory, have Rose beaten no matter what falls on the river. Though see the next item, which addresses bluffing only with certain cards.)

12. **More bluffing on the river.** In some cases Colin can't reasonably bluff with an arbitrary river card. Revise your bluffing model from Item 11, assuming q is the probability that Colin makes his draw and r is the probability the river card does not make Colin's hand, but could reasonably be used as a bluffing card (and that $q + r$ is not necessarily 1.).

River Hand 2. Suppose Colin holds $A\diamondsuit A\spadesuit$ in his pocket and after the turn the board is

$$7\heartsuit 8\heartsuit Q\spadesuit K\diamondsuit.$$

Colin has put Rose on trip queens (i.e., pocket queens), so he needs trip aces or better to win. How often should Colin bluff when he misses his third ace? Assume the pot is $P = 50$ and the bet is $B = 8$.

13. **Check-raising.** In his book *Hold'Em Poker* (our reference [5]), while discussing play after the river card has been dealt, David Sklansky writes (p. 75, the italics are all Sklansky's): "Taking check raising first, we will assume that your hand cannot lose and that the possibility of a reraise can be neglected. Whether to check, hoping to raise, or come right out betting with this hand depends on three probabilities: the chances you will be called if you bet (assume you won't be raised); the chances your opponent will bet if you check, but will not call your raise; the chances he will bet *and* call your raise. A check raise is profitable if the second figure added to *twice* the third figure exceeds the first figure." Verify that the former actuary (i.e., Sklansky) got his math right. (This problem doesn't involve game theory.)

Appendix A Hand Strength After the Flop

In evaluating the strength of Hold'em hands after the flop, it's useful to add (at least) three categories to the usual five-card poker hands: flush draws (including those that contain pairs), open-ended straight draws (including those that contain pairs), and 4-out straight draws (*excluding* those that contain pairs, since pairs are considered better). In creating Table 7 I placed flush draws and open-ended straight draws above pairs and 4-out straight draws below pairs. This is not a precise arrangement (e.g., I haven't checked that a pair of twos is really better than a 4-out straight draw), but it's certainly good enough for our purposes. The calculations for these probabilities are similar to those for the standard poker hands.

Appendix B Strength of a Final Hold'em Hand

In Table 8 we list the probability of having a certain Hold'em hand or better after all seven cards have been dealt. One of the trickiest things about computing the probabilities for this table is that due to different overlap with straights and flushes the probability of pairing or tripping one rank may differ from the probability of pairing or tripping another. All in all, creating this table did not make for a fun afternoon.[20]

Appendix C Dappers and Schnooks

In the July 27, 2009 issue of *The New Yorker* Malcolm Gladwell writes:

> ...one of the things that happens to us when we become overconfident is that we start to blur the line between the kinds of things that we can control and the kinds of things that we can't. The psychologist Ellen Langer once had subjects engage in a betting game against either a self-assured, well-dressed opponent or a shy and badly dressed opponent (in Langer's delightful phrasing, the 'dapper' or the 'schnook' condition), and she found that her subjects bet far more aggressively when they played against the schnook. They looked at their awkward opponent and thought, I'm better than he is. Yet the game was pure chance: all the players did was draw cards at random from a deck, and see who had the high hand. This is called the "illusion of control": confidence spills over from areas where it may be warranted ("I'm savvier than that schnook") to areas where it isn't warranted at all ("and that means I'm going to draw higher cards").

References

[1] G. Carson, *The complete book of hold'em poker*, Kensington Publishing Corp. 2001.

[2] E. Miller, D. Sklansky, and M. Malmuth, *Small Stakes Hold'em*, Two Plus Two Publishing LLC 2005 (third printing).

[20] Even so, I was too lazy to subdivide two-pair hands into categories, and that certainly should be done.

[3] www.pokerroom.com/poker/poker-school/ev-stats/total-stats-by-card

[4] D. Purdy, *The Illustrated Guide to Texas Hold'em*, Sourcebooks, Inc., 2005.

[5] D. Sklansky, *Hold'em Poker*, Two Plus Two Publishing LLC, 2005, (This is a reprint of the 1997 revision of the original 1976 book, which was the first serious book on hold'em, just when it was starting to become popular.)

[6] D. Sklansky, *The Theory of Poker*, Two Plus Two Publishing LLC, 2007 (ninth printing).

Hands	Probability
Straight flush	.000015
Quad or better	.000255
Full house or better	.001696
Flush or better	.003661
Straight or better	.007586
AAA or better	.009211
KKK or better	.010836
QQQ or better	.012461
JJJ or better	.014086
TTT or better	.015711
999 or better	.017336
888 or better	.018961
777 or better	.020586
666 or better	.022211
555 or better	.023836
444 or better	.025461
333 or better	.027086
222 or better	.028711
Two pairs or better	.076254
Flush draw or better	.119171
Open-ended straight draw or better	.152913
AA or better	.184403
KK or better	.215602
QQ or better	.246509
JJ or better	.277126
TT or better	.307452
99 or better	.337778
88 or better	.368104
77 or better	.398430
66 or better	.428756
55 or better	.459082
44 or better	.489699
33 or better	.520607
22 or better	.551805
4-out straight draw	.603223
Bupkis	.396777

Table 7: Hold'em hand rankings and probabilities after the flop.

Hands	Probability
Straight Flush	.000311
Quad or better	.001992
Full house or better	.027953
Flush or better	.058207
Straight or better	.104401
AAA or better	.108131
KKK or better	.111860
QQQ or better	.115581
JJJ or better	.119296
TTT or better	.123002
999 or better	.126709
888 or better	.130415
777 or better	.134122
666 or better	.137828
555 or better	.141535
444 or better	.145249
333 or better	.148971
222 or better	.152700
Two pairs or better	.387655
AA or better	.421843
KK or better	.456031
QQ or better	.489952
JJ or better	.523607
TT or better	.556996
99 or better	.590429
88 or better	.623862
77 or better	.657295
66 or better	.690728
55 or better	.724117
44 or better	.757772
33 or better	.791693
22 or better	.825881
Bupkis	.174119

Table 8: Probabilities for final Hold'em hands.