M469 Practice Problems for the Midterm Exam

The midterm exam will be Thursday March 5, 7:00 - 9:00 p.m., in Blocker 220. The midterm will cover linear regression and conversion to linear form, non-dimensionalization, cobwebbing, fixed points and stability of fixed points, maximum sustainable yield, m-cycles and stability of m-cycles, bifurcation theory, delay difference equations including exact solutions and stability of fixed points, linear systems of difference equations including derivations (age-structured population models and Markov models) and exact solutions.

Class will not meet on Friday, March 6.

1. Suppose you have data $\{(x_k, y_k, z_k)\}_{k=1}^N$ that appears consistent with the model

$$z = \frac{a^x y^b}{1 + x^2}.$$

Transform this equation in such a way that you can obtain values for a and b using linear regression. Write down both the error function associated with your transformed equation and the error function you would use to get more precise parameter values with nonlinear regression.

2. Suppose you have data $\{(x_k, y_k, z_k)\}_{k=1}^N$ that appears consistent with the model

$$z = ax + by.$$

Use the method of least squares regression to find a value for b in terms of the data. Your value for a can be specified in terms of your value for b.

3. The modified Newton's method for a root of $h(y) = 0$ near y_0 is

$$y_{t+1} = y_t - \frac{h(y_t)}{h'(y_0)},$$

where we assume $h'(y)$ is continuous in a neighborhood of the root (on an open interval containing the root). Show that if y_0 is sufficiently close to the root then this will converge.

4. Show that if $\hat{y}$ is a fixed point for the difference equation

$$y_{t+1} = f(y_t),$$

such that

$$-1 < f'(\hat{y}) < +1,$$

then $\hat{y}$ is an asymptotically stable fixed point for any iterated recursion

$$y_{t+1} = f^m(y_t),$$

$m = 1, 2, \dots$.

5. Find all fixed points for the difference equation

$$y_{t+1} = \frac{1}{2} y_t (\frac{1}{2} - y_t),$$

and use the method of cobwebbing to determine which limit will be achieved from the starting value $y_0 = -1$.

6. Non-dimensionalize the Beverton-Holt model

$$y_{t+1} = \frac{(1+r)y_t}{1 + \frac{r}{K} y_t},$$

or $r > 0$. Use the derivative method to show that the positive fixed point is always asymptotically stable, and use cobwebbing to show the same thing. Interpret these observations in terms of the original parameters.

7. For the difference equation

$$y_{t+1} = -\frac{r y_t}{1 + y_t^2}, \quad r > 0,$$

find all fixed points and determine the values of r for which each is stable. Also, find all the 2-cycles and determine the values of r for which each is stable. Discuss any bifurcations you observe, and decide whether or not you expect the 2-cycle to bifurcate into a 4-cycle.

8. This problem concerns bifurcations for the non-dimensionalized discrete logistic model

$$y_{t+1} = f(y_t); \quad f(y) = (1+r)y - ry^2.$$

a. The function $f^4(y)$ is graphed along with a 45^o line in Figure 1. Explain the bifurcation that is depicted.

b. The function $f^6(y)$ is graphed along with a 45^o line in Figure 2. Explain the bifurcation that is depicted.

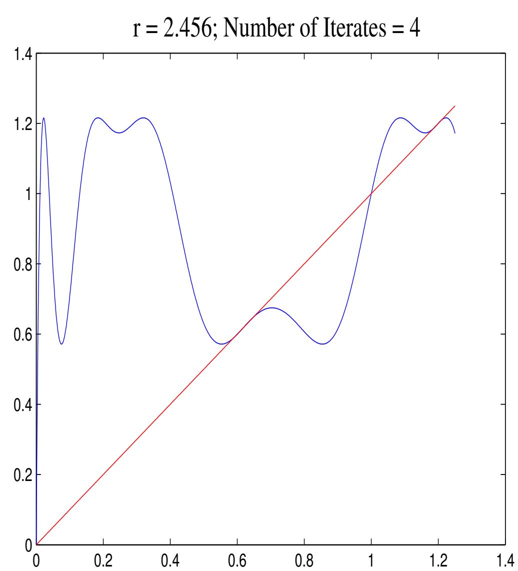

Figure 1: Figure for Problem 8a.

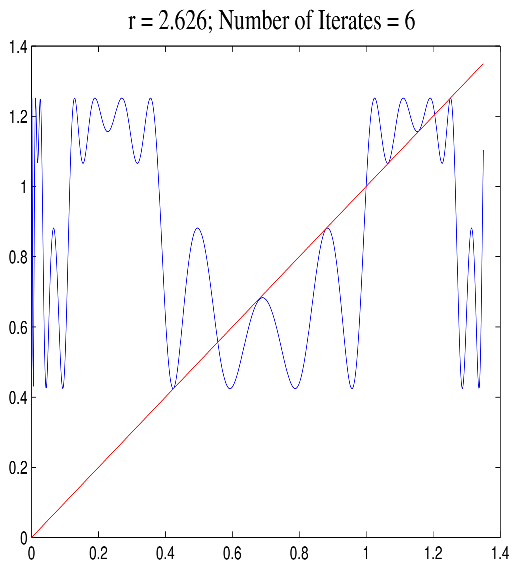

Figure 2: Figure for Problem 8b.

9. Suppose that in the absence of fishermen the population of fish in a certain body of water is modeled by the Gompertz model,

$$y_{t+1} - y_t = -ry_t \ln(\frac{y_t}{K}),$$

and that fishing effort is added as a percentage of population. Write down a difference equation that incorporates this fishing effort, and find the maximum sustainable yield associated with your model. Specify the yield obtained, the equilibrium fish population, and the parameter values for which your results are valid.

10. Solve the delay difference equation

$$y_{t+1} = 2\sqrt{3}y_t - 4y_{t-1},$$

with $y_0 = 2$ and $y_1 = 0$.

11. Find all fixed points for the delay Beverton-Holt model

$$y_{t+1} = \frac{Ry_t}{1 + y_{t-1}}; \quad R = r + 1 > 1,$$

and determine the values R for which each is asymptotically stable.

12. In their paper "Using logistic regression to analyze the sensitivity of PVA models: a comparison of models based on African wild dog models," in Conservation Biology **15** (2001) 1335-1346, the authors Cross and Beissinger model a population of African wild dogs with three age groups: pups, yearlings, and adults. They assume that only yearlings and adults can reproduce, and that while surviving pups and yearlings always move to the next age class, adults can survive back into the adult class. Draw a transition diagram for this situation

and write down a linear system of difference equations that models it. Be sure to write your system in matrix form.

13. Suppose an epidemic is moving through a population of individuals that can be classified as susceptible, infected/infective, and recovered/immune. Assume that during the course of a single time step, measured in some appropriate way (days, weeks, etc.) the following can occur: a susceptible individual can become infected with probability .05, while an infected/infective individual can recover with immunity with probability .1 and can recover without immunity with probability .2. Also, assume an immune individual can lose immunity with probability .01. Draw a transition diagram for this situation and write down a linear system of difference equations that models it. Be sure to write your system in matrix form.

14. Solve the linear system of difference equations

$$\vec{y}_{t+1} = A\vec{y}_t; \quad \vec{y}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 4 & -3 \\ 6 & -5 \end{pmatrix}.$$

Solutions

1. First, we multiply both sides by $1 + x^2$, then we take a natural logarithm of both sides, giving

$$\ln(z(1 + x^2)) = x \ln a + b \ln y.$$

We can now linearly fit $\ln(z(1+x^2))$ as a function of x and $\ln y$, and we obtain values $\ln a$ and b as the parameters. We solve for a by exponentiating. The error for the linear regression is

$$E(\ln a, b) = \sum_{k=1}^{N} \Big(\ln(z_k(1 + x_k^2)) - x_k \ln a - b \ln y_k \Big)^2,$$

while the error for the nonlinear regression is

$$E(a, b) = \sum_{k=1}^{N} (z_k - \frac{a^{x_k} y_k^b}{1 + x_k^2})^2.$$

2. We proceed by minimizing the error

$$E(a, b) = \sum_{k=1}^{N} (z_k - a x_k - b y_k)^2.$$

We have

$$\frac{\partial E}{\partial a} = \sum_{k=1}^{N} 2(z_k - ax_k - by_k)(-x_k) = 0$$
$$\frac{\partial E}{\partial b} = \sum_{k=1}^{N} 2(z_k - ax_k - by_k)(-y_k) = 0,$$

which gives two equations for the two unknowns a and b

$$a\sum_{k=1}^{N} x_k^2 + b\sum_{k=1}^{N} x_k y_k = \sum_{k=1}^{N} x_k z_k$$
$$a\sum_{k=1}^{N} x_k y_k + b\sum_{k=1}^{N} y_k^2 = \sum_{k=1}^{N} y_k z_k.$$

The problem specifies that we should eliminate a and solve for b, so we multiply the second equation by

$$\frac{\sum_{k=1}^{N} x_k^2}{\sum_{k=1}^{N} x_k y_k}$$

and subtract the result from the first equation. We find

$$b\Big(\sum_{k=1}^{N} x_k y_k - \frac{\sum_{k=1}^{N} x_k^2 \sum_{k=1}^{N} y_k^2}{\sum_{k=1}^{N} x_k y_k}\Big) = \sum_{k=1}^{N} x_k z_k - \frac{\sum_{k=1}^{N} x_k^2 \sum_{k=1}^{N} y_k z_k}{\sum_{k=1}^{N} x_k y_k},$$

so that

$$b = \frac{\sum_{k=1}^{N} x_k z_k - \frac{\sum_{k=1}^{N} x_k^2 \sum_{k=1}^{N} y_k z_k}{\sum_{k=1}^{N} x_k y_k}}{\sum_{k=1}^{N} x_k y_k - \frac{\sum_{k=1}^{N} x_k^2 \sum_{k=1}^{N} y_k^2}{\sum_{k=1}^{N} x_k y_k}}.$$

We use b to compute a,

$$a = \frac{\sum_{k=1}^{N} y_k z_k - b\sum_{k=1}^{N} y_k^2}{\sum_{k=1}^{N} x_k y_k}.$$

3. We see immediately that since $h(y_r) = 0$, the root y_r is a fixed point of the difference equation

$$y_{t+1} = f(y_t),$$

where

$$f(y) = y - \frac{h(y)}{h'(y_0)}.$$

Since it's a fixed point we're justified in denoting it $\hat{y}$, we know that

$$\lim_{t\to\infty} y_t = \hat{y},$$

if $\hat{y}$ is asymptotically stable. To check of $\hat{y}$ is asymptotically stable, we compute

$$f'(\hat{y}) = 1 - \frac{h'(\hat{y})}{h'(y_0)}.$$

Now since $h'(y)$ is continuous in an interval containing $\hat{y}$, by choosing y_0 sufficiently close to $\hat{y}$ we can make $\frac{h'(\hat{y})}{h'(y_0)}$ as close to 1 as we like, and so ensure $-1 < f'(\hat{y}) < 1$, which implies asymptotic stability.

4. First $\hat{y} = f(\hat{y})$, so

$$f^m(\hat{y}) = f^{m-1}(f(\hat{y})) = f^{m-1}(\hat{y}) = \cdots = f(\hat{y}) = \hat{y},$$

so $\hat{y}$ is certainly a fixed point of the iterated difference equation. To check stability,

$$\frac{d}{dy}f^m(y) = \frac{d}{dy}f(f^{m-1}(y)) = f'(f^{m-1}(y))\frac{d}{dy}f^{m-1}(y) = \cdots = f'(f^{m-1}(y))f'(f^{m-2}(y))\cdots f'(y).$$

We have, then,

$$\frac{d}{dy}f^m(y)\Big|_{y=\hat{y}} = f'(\hat{y})^m.$$

Note that it's also fair to conclude this by simply recognizing that $\hat{y}$ is an m-cycle $\{\hat{y}, \hat{y}, \dots, \hat{y}\}$. The condition for asymptotic stability is

$$-1 < f'(\hat{y})^m < 1,$$

and this is certainly true if $-1 < f'(\hat{y}) < 1$.

5. First, the fixed point equation is

$$\hat{y} = \frac{1}{2}\hat{y}(\frac{1}{2} - \hat{y}) = \frac{1}{4}\hat{y} - \frac{1}{2}\hat{y}^2,$$

so that the fixed points are

$$\frac{3}{4}\hat{y} + \frac{1}{2}\hat{y}^2 = \hat{y}(\frac{3}{4} + \frac{1}{2}\hat{y}) = 0 \Rightarrow \hat{y} = 0, -\frac{3}{2}.$$

For the cobwebbing, we can plot $f(y) = \frac{1}{4}y - \frac{1}{2}y^2 = y(\frac{1}{4} - \frac{1}{2}y)$ by noticing that it's a parabola opening downward with x-intercepts at $y = 0$ and $y = \frac{1}{2}$, and therefore has a maximum value at (the midpoint) $\frac{1}{4}$ of $\frac{1}{4}(\frac{1}{4} - \frac{1}{2}\frac{1}{4}) = \frac{1}{4}\frac{1}{8} = \frac{1}{32}$. We find that for $y_0 = -1$ (see the figure)

$$\lim_{t\to\infty} y_t = 0.$$

6. First, we non-dimensionalize by setting $Y_t = \frac{y_t}{A}$ to get

$$AY_{t+1} = \frac{(1+r)AY_t}{1 + \frac{r}{K}AY_t}.$$

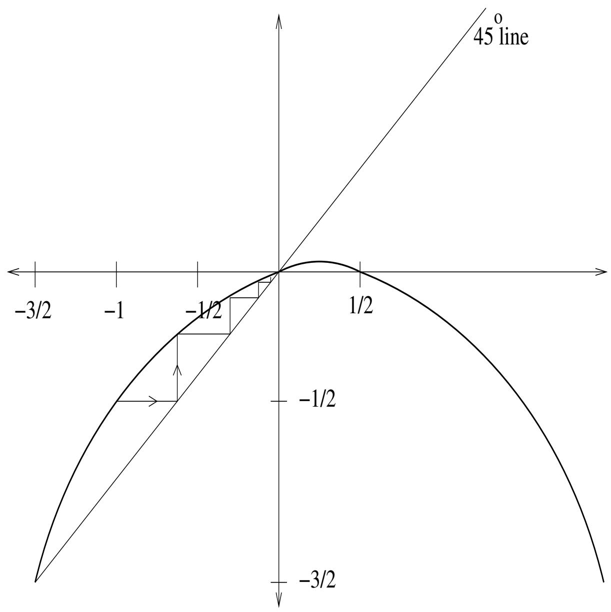

Figure 3: Figure for Problem 5.

It's natural to choose $A = \frac{K}{r}$, though this certainly isn't the only possibility. It's also natural to set $R = 1 + r$, so that we have

$$Y_{t+1} = \frac{RY_t}{1 + Y_t}.$$

The fixed points are solutions of

$$\hat{Y} = \frac{R\hat{Y}}{1 + \hat{Y}} \Rightarrow \hat{Y} = 0, R - 1.$$

For the derivative method, we set

$$f(Y) = \frac{RY}{1 + Y}$$

and compute

$$f'(Y) = \frac{R}{(1 + Y)^2} \Rightarrow f'(R - 1) = \frac{1}{R},$$

and since $r > 0$ this is always between 0 and 1. The cobwebbing diagram is depicted in Figure 4.

The fixed point $\hat{Y} = R - 1$ corresponds with $\hat{y} = K$ in the original variables, so we find that the carrying capacity is always a stable fixed point for the Beverton-Holt model (for $r > 0$).

7. First, the fixed points satisfy

$$\hat{y} = -\frac{r\hat{y}}{1 + \hat{y}^2},$$

so that $\hat{y} = 0$ is the only fixed point. In order to investigate the stability of $\hat{y} = 0$, we set

$$f(y) = -\frac{ry}{1 + y^2} \Rightarrow f'(y) = \frac{r(y^2 - 1)}{(1 + y^2)^2}.$$

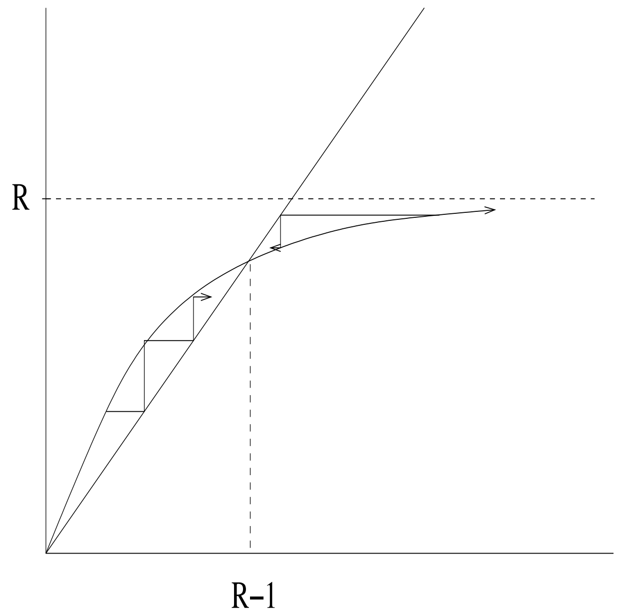

Figure 4: Figure for Problem 6.

We have, then,

$$f'(0) = -r,$$

so that $\hat{y} = 0$ is asymptotically stable for

$$-1 < -r < 1 \Rightarrow 1 > r > -1.$$

To find the 2-cycles, we set

$$f(f(y)) = -\frac{rf}{1+f^2} = -\frac{r\frac{-ry}{1+y^2}}{1+(\frac{ry}{1+y^2})^2} = \frac{r^2y(1+y^2)}{(1+y^2)^2+r^2y^2}.$$

The 2-cycle equation becomes

$$y = \frac{r^2y(1+y^2)}{(1+y^2)^2+r^2y^2}.$$

Clearly, the fixed point $\hat{y} = 0$ is a solution, and the other solutions satisfy

$$(1+y^2)^2 + r^2y^2 = r^2 + r^2y^2 \Rightarrow 1 + y^2 = \pm r.$$

Now we've taken $r > 0$, so only the $+r$ choice is possible (for a real-valued fixed point), and we have the two cycle

$$\hat{y}_1 = -\sqrt{r-1}$$
$$\hat{y}_2 = +\sqrt{r-1}.$$

Clearly, this 2-cycle does not emerge until $r > 1$, which is where a bifurcation is expected for the fixed point $\hat{y} = 0$.

Finally, we check stability of the 2-cycle by computing

$$f'(\hat{y}_1)f'(\hat{y}_2) = (\frac{r(r-1)-r}{r^2})^2 = (1-\frac{2}{r})^2.$$

For stability we require $-1 < f'(\hat{y}_1)f'(\hat{y}_2) < 1$, which requires $-1 < 1-\frac{2}{r} < 1$, and so (since the right-hand inequality is true for all $r > 0$)

$$r > 1.$$

Since this 2-cycle is always stable we do not expect a bifurcation into a 4-cycle.

8a. This figure depicts a pitchfork bifurcation as a 2-cycle bifurcates into a 4-cycle.

8b. This figure depicts a tangent bifurcation as a 6-cycle appears.

9. If we incorporate fishing effort as a percentage of population the model becomes

$$y_{t+1} - y_t = -ry_t \ln(\frac{y_t}{K}) - hy_t,$$

and the fixed points satisfy

$$0 = -r\hat{y}\ln(\frac{\hat{y}}{K}) - h\hat{y}.$$

We have a fixed point $\hat{y} = 0$ and a second fixed point

$$\hat{y} = Ke^{-\frac{h}{r}} > 0.$$

The yield at this fixed point is

$$Y(h) = Khe^{-\frac{h}{r}},$$

and we maximize this by computing

$$Y'(h) = Ke^{-\frac{h}{r}}(1-\frac{h}{r}) = 0 \Rightarrow h = r.$$

The equilibrium fish population becomes $\hat{y} = \frac{K}{e}$, and we need to verify that it is stable. To study stability, we set

$$f(y) = y - ry\ln(\frac{y}{K}) - ry,$$

(taking $h = r$) and compute

$$f'(y) = 1 - r\ln(\frac{y}{K}) - 2r \Rightarrow f'(\frac{K}{e}) = 1 - r,$$

so we have stability for

$$-1 < 1 - r < 1 \Rightarrow 2 > r > 0.$$

The yield is

$$Y(r) = \frac{rK}{e}.$$

10. We look for solutions $y_t = a^t$, for which we find

$$a^2 = 2\sqrt{3}a - 4 \Rightarrow a = \frac{2\sqrt{3} \pm \sqrt{12-16}}{2} = \sqrt{3} \pm i.$$

We can write these bases in the polar form $a = re^{i\theta}$, where

$$\begin{aligned} r &= \sqrt{3+1} = 2 \\ \theta &= \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}. \end{aligned}$$

For a_+,

$$a_+ = \sqrt{3} + i = 2e^{i\frac{\pi}{6}} \Rightarrow a_+^t = 2^t e^{i\frac{\pi t}{6}} = 2^t \cos(\frac{\pi t}{6}) + 2^t i \sin(\frac{2\pi}{6}).$$

Since the real and imaginary parts must both be solutions, we can write our general solution as

$$y_t = C_1 2^t \cos(\frac{\pi t}{6}) + C_2 2^t \sin(\frac{\pi t}{6}).$$

Turning now to the initial values, we obtain

$$\begin{aligned} 2 &= C_1 \\ 0 &= 2C_1 \cos(\frac{\pi}{6}) + 2C_2 \sin(\frac{\pi}{6}) = 2C_1 \frac{\sqrt{3}}{2} + 2C_2 \frac{1}{2} = \sqrt{3}C_1 + C_2, \end{aligned}$$

so that $C_1 = 2$ and $C_2 = -2\sqrt{3}$. We conclude

$$y_t = 2^{t+1} \cos(\frac{\pi t}{6}) - \sqrt{3} 2^{t+1} \sin(\frac{\pi t}{6}).$$

11. First, the fixed point equation is

$$\hat{y} = \frac{R\hat{y}}{1+\hat{y}},$$

from which we find $\hat{y} = 0, R-1$. In order to check stability, we set

$$f(y_1, y_2) = \frac{Ry_1}{1+y_2},$$

and we compute

$$\begin{aligned} f_{y_1} &= \frac{R}{1+y_2} \\ f_{y_2} &= -\frac{Ry_1}{(1+y_2)^2}. \end{aligned}$$

For $\hat{y} = 0$,

$$\begin{aligned} f_{y_1}(0,0) &= R \\ f_{y_2}(0,0) &= 0, \end{aligned}$$

and the perturbation equation becomes

$$z_{t+1} = Rz_t,$$

with associated bases (recall: $z_t = a^t$) $a = 0, R$. In this case, the condition for stability is simply

$$-1 < R < 1.$$

(In the original coordinates, this is $-2 < r < 0$.)

For $\hat{y} = R - 1$,

$$\begin{aligned} f_{y_1}(R-1, R-1) &= 1 \\ f_{y_2}(R-1, R-1) &= -\frac{R(R-1)}{R^2} = -(1 - \frac{1}{R}), \end{aligned}$$

and the perturbation equation becomes

$$z_{t+1} = z_t - (1 - \frac{z_{t-1}}{R}),$$

with associated bases solving

$$a^2 - a + (1 - \frac{1}{R}) = 0 \Rightarrow a = \frac{1 \pm \sqrt{1 - 4(1 - \frac{1}{R})}}{2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{4}{R} - 3}.$$

Recalling $R > 1$, we first observe that for $1 < R \leq \frac{4}{3}$ the bases are both real-valued and both between 0 and 1 since the radical is less than 1. For $R > \frac{4}{3}$ we have complex bases

$$a = \frac{1}{2} \pm \frac{i}{2}\sqrt{3 - \frac{4}{R}},$$

and we must check complex modulus

$$|a| = \sqrt{\frac{1}{4} + \frac{1}{4}(3 - \frac{4}{R})} = \sqrt{1 - \frac{1}{R}} < 1,$$

which is true for all $R > 1$, so we conclude that $\hat{y} = R - 1$ is asymptotically stable for all $R > 1$.

12. First, for the transition diagram, see Figure 5.

If we let $\{y_j\}_{j=1}^3$ be as depicted in the transition diagram, we can model the populations with the system

$$\vec{y}_{t+1} = U\vec{y}_t,$$

where

$$U = \begin{pmatrix} 0 & f_{12} & f_{13} \\ s_{21} & 0 & 0 \\ 0 & s_{32} & s_{33} \end{pmatrix}.$$

Here, I've used U since this is an Usher model.

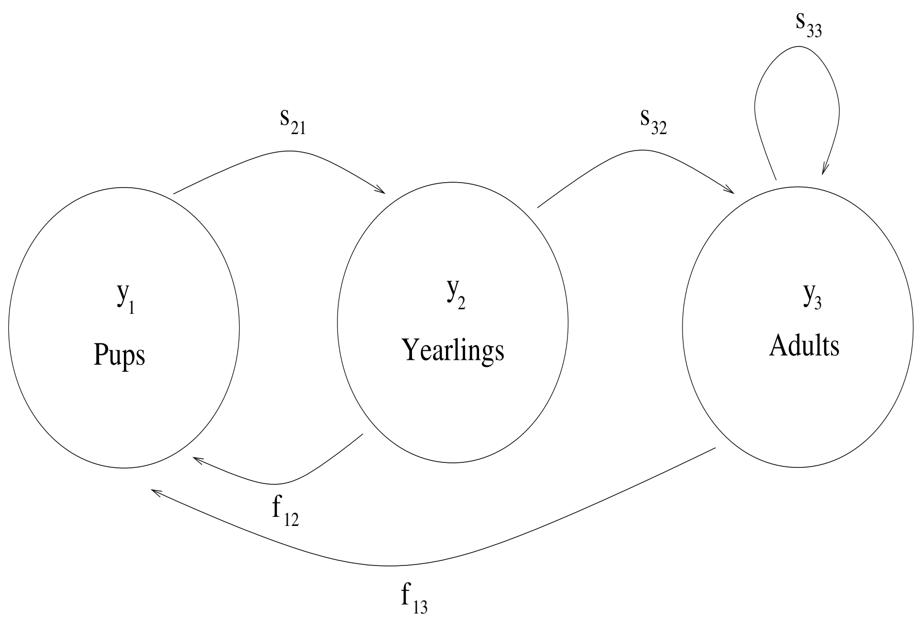

Figure 5: Figure for Problem 12.

13. First, the transition diagram is given in Figure 6.

If we let $\{y_j\}_{j=1}^3$ be as in the transition diagram, denoting here the probability that an individual is in group j, we can model the populations with

$$\vec{y}_{t+1} = M\vec{y}_t,$$

where M denotes the Markov matrix

$$M = \begin{pmatrix} .95 & .2 & .01 \\ .05 & .7 & 0 \\ 0 & .1 & .99 \end{pmatrix}.$$

14. We begin by searching for the eigenvalues of A,

$$\det \begin{pmatrix} 4-\lambda & -3 \\ 6 & -5-\lambda \end{pmatrix} = (4-\lambda)(-5-\lambda) + 18 = \lambda^2 + \lambda - 2$$
$$= (\lambda - 1)(\lambda + 2) = 0.$$

We conclude that the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 1$.

For $\lambda_1 = -2$,

$$\begin{pmatrix} 6 & -3 \\ 6 & -3 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and likewise the eigenvector for $\lambda_1 = 1$ is

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

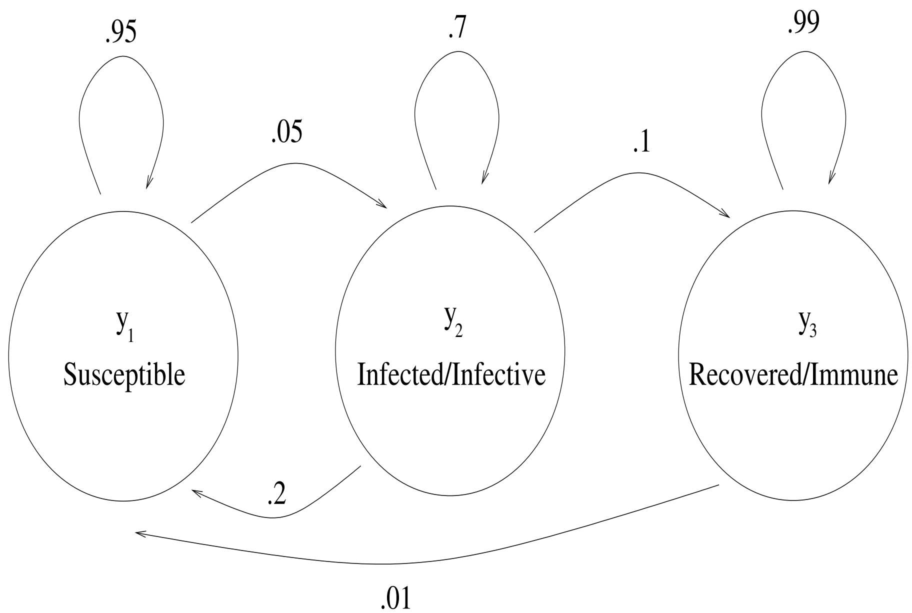

Figure 6: Figure for Problem 13.

In this way, the general solution is

$$\vec{y}_t = C_1(-2)^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 1^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The intial condition gives

$$\begin{aligned} 1 &= C_1 + C_2 \\ 2 &= 2C_1 + C_2. \end{aligned}$$

Subtracting the second equation from the first, we find $C_1 = 1$ so that $C_2 = 0$. We conclude

$$\vec{y}_t = (-2)^t \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$