Single Differential Equations: Parameter Estimation

MATH 469, Texas A&M University

Spring 2020

The Logistic Model

Let's fit the logistic model

$$\frac{dy}{dt} = ry(1 - \frac{y}{K}); \quad y(0) = y_0$$

to the same US population data we used for the discrete logistic model (census numbers for the years 1790-2010, though omitting 2010 for the current calculation).

Recall that we can write the exact solution to this equation as

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}.$$

This is clearly nonlinear in the parameters r and K.

In order to start with a linear fit, notice that we can express our equation as

$$\frac{1}{y}\frac{dy}{dt} = r - \frac{r}{K}y.$$

Derivative Approximation

From the previous slide,

$$\frac{1}{y}\frac{dy}{dt} = r - \frac{r}{K}y.$$

The idea will be to plot values of $\frac{1}{y}\frac{dy}{dt}$ versus values of y, but our data has the form $\{(t_k, y_k)\}_{k=1}^N$, so we don't immediately have values for $\frac{1}{y}\frac{dy}{dt}$.

In order to approximate $y'(t)$ from the data, we recall that if y is differentiable at a value t, then there exists a function $\epsilon(h;t)$ so that

$$y(t+h) = y(t) + y'(t)h + \epsilon(h;t),$$

where

$$\lim_{h\to 0}\frac{\epsilon(h;t)}{h} = 0.$$

Derivative Approximation

From the previous slide,

$$y(t+h) = y(t) + y'(t)h + \epsilon(h;t). \qquad (*)$$

Solving (*) for $y'(t)$, we find

$$y'(t) = \frac{y(t+h) - y(t)}{h} - \frac{\epsilon(h;t)}{h}.$$

For h sufficiently small, $\frac{\epsilon(h;t)}{h}$ will be small, and we will have the forward difference derivative approximation

$$y'(t) \cong \frac{y(t+h) - y(t)}{h}.$$

If y is differentiable to higher orders in t, then we can use a Taylor series to say more about how good this approximation is.

Derivative Approximation

The first three terms in a Taylor expansion for $y(t+h)$ are

$$y(t+h) = y(t) + y'(t)h + \frac{1}{2}y''(t)h^2 + \dots.$$

We can solve again for $y'(t)$ to get

$$y'(t) = \frac{y(t+h) - y(t)}{h} - \frac{1}{2}y''(t)h + \dots,$$

and we see that the error in the forward difference approximation of $y'(t)$ is proportional to h.

Notation. We write

$$f(h) = \mathbf{O}(h)$$

(read: "big O of h") if there exists a constant C so that

$$|f(h)| \leq C|h|,$$

for all h sufficiently small.

Derivative Approximation

With this notation, we can write the forward difference derivative approximation as

$$y'(t) = \frac{y(t+h) - y(t)}{h} + \mathbf{O}(h).$$

We refer to this as a first order derivative approximation.

Proceeding similarly, we can derive higher order derivative approximations. Let's derive a second order approximation, which is what we'll use for this example. For this, we consider higher order Taylor expansions,

$$y(t+h) = y(t) + y'(t)h + \frac{1}{2}y''(t)h^2 + \frac{1}{6}y'''(t)h^3 + \dots$$
$$y(t-h) = y(t) - y'(t)h + \frac{1}{2}y''(t)h^2 - \frac{1}{6}y'''(t)h^3 + \dots$$

Derivative Approximation

If we subtract the second equation from the first, we get

$$y(t+h) - y(t-h) = 2y'(t)h + \frac{1}{3}y'''(t)h^3 + \dots .$$

Solving once again for $y'(t)$, we see that

$$y'(t) = \frac{y(t+h) - y(t-h)}{2h} - \frac{1}{6}y'''(t)h^2 + \dots$$

In this case, the error is $\mathbf{O}(h^2)$, so this is a second-order derivative approximation. This is called the central difference derivative approximation.

Returning to our data $\{(t_k, y_k)\}_{k=1}^N$, if we take 1790 to correspond with $t_1 = 0$, then we have $t_1 = 0$, $t_2 = 10$, $t_3 = 20$, ..., $t_{22} = 210$ (t_{22} corresponds with 2000), with corresponding populations (in millions) $y_1 = 3.93$, $y_2 = 5.31$, $y_3 = 7.24$, $y_4 = 9.64$, ..., $y_{20} = 249.63$, $y_{21} = 281.42$, $y_{22} = 308.75$.

Derivative Approximation

Since our approximation of $y'(t)$ requires both the previous data point and the subsequence data point, we can only obtain approximations at the times $\{t_k\}_{k=2}^{21}$. These approximations are (here, $h = 10$):

$$y'(t_2) \cong \frac{y(t_2+10)-y(t_2-10)}{2\cdot 10} = \frac{y_3-y_1}{20} = \frac{7.24-3.93}{20} = .1655$$

$$y'(t_3) \cong \frac{y_4-y_2}{20} = \frac{9.64-5.31}{20} = .2165$$

$$\vdots$$

$$y'(t_{21}) \cong \frac{y_{22}-y_{20}}{20} = \frac{308.75-249.63}{20} = 2.9560.$$

Let's denote these values $\{y'_k\}_{k=2}^{21}$. I.e., $y'_2 = .1655$, $y'_3 = .2165$, etc.

The Linear Fit

Recall that the relation we want to fit is

$$\frac{1}{y}\frac{dy}{dt} = r - \frac{r}{K}y.$$

We have 20 data points:

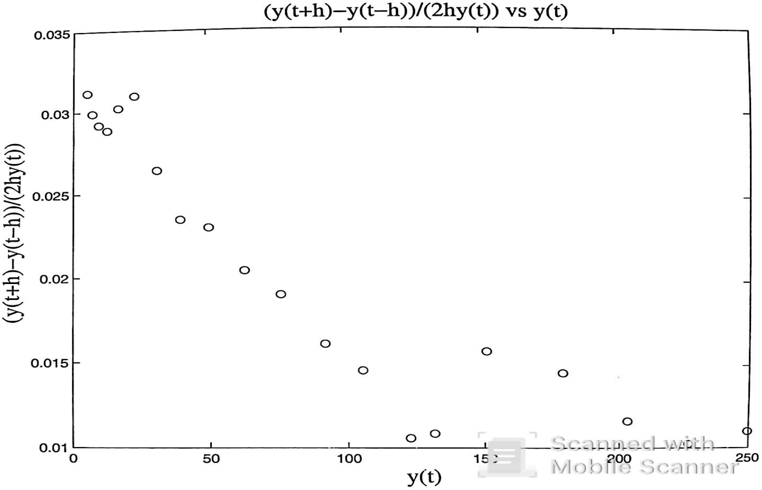
$$(y_2, \frac{1}{y_2}y_2') = (5.31, \frac{1}{5.31}.1655) = (5.31, .0312)$$

$$(y_3, \frac{1}{y_3}y_3') = (7.24, \frac{1}{7.24}.2165) = (7.24, .0299)$$

$$\vdots$$

$$(y_{21}, \frac{1}{y_{21}}y_{21}') = (281.42, \frac{1}{281.42}2.956) = (281.42, .0105).$$

A scatter plot of these points is given on the next slide.



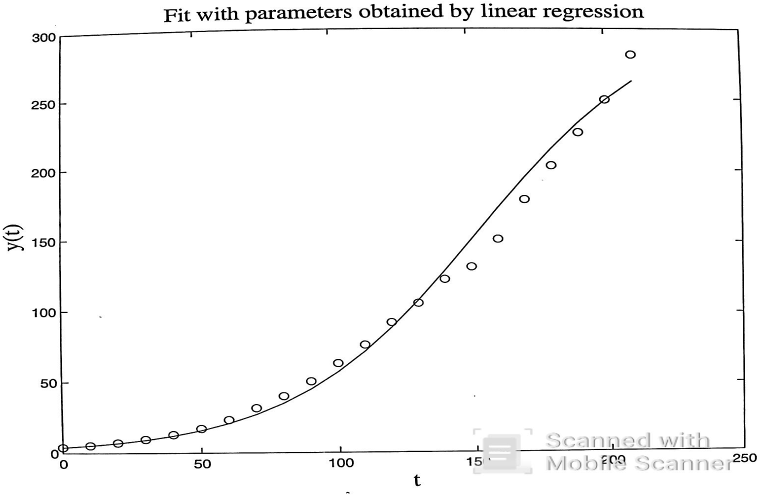
(y(t+h)−y(t−h))/(2hy(t)) vs y(t)
0.035
0.03
0.025
0.02
0.015
0.01
0
50
100
150
200
250
y(t)
(y(t+h)−y(t−h))/(2hy(t))

The Linear Fit

The slope and intercept of the regression line are respectively $-.00009108$ and $.0286$. We see that $r = .0286$ and

$$K = -\frac{.0286}{.00009108} = 314.0097.$$

We can incorporate these values into our model, and check the result against our data. A plot of this fit is given on the next slide.



Fit with parameters obtained by linear regression
300
250
200
150
100
50
0
0
50
100
150
200
250
y(t)
t

The Nonlinear Fit

We can now use the parameter values obtained by linear regression as initial approximations for the parameter values that we'll find with nonlinear regression. For the nonlinear regression, we'll view y_0 as a third parameter, and we'll use the first census count of 3.93 million as an initial approximation.

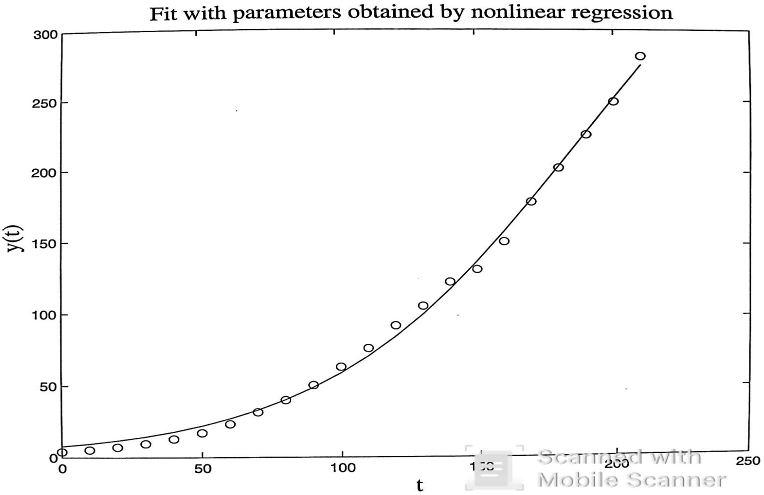
In order to emphasize the dependence of our solution $y(t)$ on the parameter values, let's write

$$y(t; r, K, y_0) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}.$$

For the nonlinear fit, we need to minimize the SSR

$$E(r, K, y_0) = \sum_{k=1}^{22} (y_k - y(t_k; r, K, y_0))^2.$$

We do this with MATLAB, and we find $r = .0215$, $K = 445.9715$, and $y_0 = 7.7463$. A plot of this fit is given on the next slide.


Fit with parameters obtained by nonlinear regression
300
250
200
150
100
50
0
y(t)
0
50
100
150
200
250
t