M641 Fall 2012 Assignment 10, due Wed. Nov. 14

1. [10 pts] For Banach spaces X and Y, Keener defines the norm of a linear operator $L : X \to Y$ as

$$\|L\| := \sup_{\|x\|_X \neq 0} \frac{\|Lx\|_Y}{\|x\|_X}.$$

(More precisely, Keener works only with $X = Y = H$, but this is the natural generalization of his definition.) Show first that this defines a proper norm, and also that this defines precisely the same norm as each of the following:

a.

$$\|L\| := \sup_{\|x\|_X \leq 1} \|Lx\|_Y.$$

b.

$$\|L\| := \sup_{\|x\|_X = 1} \|Lx\|_Y.$$

c.

$$\|L\| := \inf_{C \geq 0} \Big\{ C : \|Lx\|_Y \leq C\|x\|_X, \, \forall x \in X \Big\}.$$

2. [10 pts] Show that if X and Y are Banach spaces and $L : X \to Y$ is a linear operator then the following are equivalent:

(i) L is continuous on X (i.e., at every point of X).

(ii) L is continuous at a single point of X.

(iii) L is bounded on X.

3. [10 pts] (**Keener Problem 3.2.1.**) Show that $Tf = f(0)$ is not a bounded linear functional on the space of continuous functions measured with the L^2 norm, but it is a bounded linear functional if measured using the uniform norm.

4. [10 pts] Identify the kernel and range of the integral operator

$$Kf(x) := \int_0^1 \sin\{\pi(x - y)\} f(y) dy$$

for $f \in C([0, 1])$.

5. [10 pts] Consider the operator

$$Lu := u - \int_0^1 u(y) dy$$

on $C([0, 1])$.

a. Compute $\|L\|$ (not just an upper bound).

b. Show that there does *not* exist a function $u \in C([0, 1])$ for which the norm is achieved (i.e., that the supremum in the norm definition is not achieved).