M641 Fall 2012 Assignment 2, due Wed. Sept. 12

1 [10 pts]. Let $B^o(0,r)$ denote the open ball of radius r in $\mathbb{R}^n$, and take as definitions

$$|B^o(0,r)| := \int_{B^o(0,r)} 1 d\vec{x}$$
$$|\partial B^o(0,r)| := \int_{\partial B^o(0,r)} 1 dS.$$

a. Show that

$$|B^o(0,r)| = |B^o(0,1)| r^n,$$

and likewise

$$|\partial B^o(0,r)| = |\partial B^o(0,1)| r^{n-1}.$$

b. Show that

$$|\partial B^o(0,r)| = \frac{n}{r} |B^o(0,r)|.$$

c. Show that

$$|\partial B^o(0,1)| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})},$$

where Γ denotes the gamma function

$$\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt.$$

(Hint. Use the integral equality $\int_{\mathbb{R}^n} e^{-|\vec{x}|^2} d\vec{x} = \pi^{n/2}$, and evaluate the integral in polar coordinates.)

d. Show that

$$\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)},$$

so that we find the volumes:

$$|\partial B^o(0,1)| = n\alpha(n),$$

where

$$\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.$$

Note that $|B^o(0,1)| = \alpha(n)$ and this could just as well have been stated for the closed ball $B(0,r)$.

2 [10 pts]. (**Keener Problem 1.2.1.**)

a. Represent the transformation whose matrix representation with respect to the natural basis is

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

relative to the basis $\{(1,1,0)^T,(0,1,1)^T,(1,0,1)^T\}$.

b. The representation of a transformation with respect to the basis

$$\{(1,1,2)^T,(1,2,3)^T,(3,4,1)^T\}$$

is

$$A=\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find the representation of this transformation with respect to the basis

$$\{(1,0,0)^T,(0,1,-1)^T,(0,1,1)^T\}.$$

3. [10 pts] (**Introduction to generalized eigenvectors**) Recall that if λ is an eigenvalue of $A\in\mathbb{R}^{n\times n}$, then its *algebraic multiplicity* is the number of times it repeats as a solution of the characteristic equation $\det(A-\lambda I)=0$ and its *geometric multiplicity* is the number of linearly independent eigenvectors it has. If the algebraic and geometric multiplicities associated with each eigenvalue for a matrix are equal, then the eigenvectors for the matrix form a full basis of $\mathbb{R}^n$ and the matrix can be diagonalized by $P=\begin{pmatrix}\vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n\end{pmatrix}$. If there are eigenvalues of geometric multiplicity differing from their algebraic multiplicity, then we need to consider *generalized eigenvectors*.

Definition. Suppose that for some value λ and some integer $p\geq 1$ there exists a vector $\vec{v}$ so that

$$(A-\lambda I)^p\vec{v}=0 \quad \text{but} \quad (A-\lambda I)^{p-1}\vec{v}\neq 0.$$

Then $\vec{v}$ is said to be a generalized eigenvector for λ of index p.

Find the eigenvalues, eigenvectors and generalized eigenvectors for the matrix

$$A=\begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}.$$

4. [10 pts] (**Representation by Direct Sum**) For a matrix $A\in\mathbb{R}^{n\times n}$, let $\lambda_1,\lambda_2,\dots,\lambda_k$ denote the distinct eigenvalues of A with respective algebraic multiplicities $n_1,n_2,\dots,n_k$, where $k\leq n$ and according to the Fundamental Theorem of Algebra

$$n_1+n_2+\cdots+n_k=n.$$

Let X_j denote the subspace spanned by the generalized eigenvectors for λ_j. (Note that a regular eigenvector is also a generalized eigenvector, and so should be included in this set.)

Theorem. For any matrix $A\in\mathbb{R}^{n\times n}$, and for $j\in\{1,2,\dots,k\}$, the subspace X_j is invariant under A, has dimension n_j, and satisfies

$$(A-\lambda_j I)^{n_j}\vec{v}=0 \quad \forall\vec{v}\in X_j.$$

Moreover, the space $\mathbb{R}^n$ is a direct sum of the spaces $X_1,X_2,\dots,X_k$.

Definition. To say X_j is invariant under A means $\vec{x} \in X_j \Rightarrow A\vec{x} \in X_j$.

Definition. When we say that $\mathbb{R}^n$ is a direct sum of $X_1, X_2, \ldots, X_k$, we mean that for any $\vec{x} \in \mathbb{R}^n$ there exists a unique collection $\vec{x}_1 \in X_1$, $\vec{x}_2 \in X_2$, $\ldots$, $\vec{x}_k \in X_k$ so that

$$\vec{x} = \vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k.$$

For the matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix},$$

from the previous problem, with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ and associated eigenspaces X_1 and X_2, find the unique vectors $\vec{x}_1 \in X_1$ and $\vec{x}_2 \in X_2$ so that

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \vec{x}_1 + \vec{x}_2.$$

5. [10 pts] (**Jordan Canonical Form**) Although some matrices cannot be diagonalized, all matrices can be put into Jordan Canonical form. If a matrix $A \in \mathbb{C}^{n\times n}$ has k distinct eigenvalues $\{\lambda_i\}_{i=1}^k$ with respective multiplicies $\{n_i\}_{i=1}^k$, then the Jordan canonical form of A is

$$J = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_k \end{pmatrix},$$

where each submatrix J_i has the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda_i \end{pmatrix}.$$

We construct our matrix P as follows: let its first n_1 columns comprise the generalized eigenvectors associated with λ_1, its next n_2 columns the generalized eigenvectors associated with λ_2 etc. Then the Jordan form of A is easily computed as

$$J = P^{-1}AP.$$

Verify that this last expression gives the Jordan form for the matrix A from Problem 2, and use this to solve the matrix equation

$$A\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$