M641 Fall 2012, Assignment 3, due Wed. Sept. 19

1 [10 pts]. Prove that if x, y, z are in an inner product space $\mathcal{S}$ and $\alpha \in \mathbb{C}$ then:
(i)

$$\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

(ii)

$$\langle \sum_{i=1}^{n} x_i, y \rangle = \sum_{i=1}^{n} \langle x_i, y \rangle$$

(iii)

$$\langle x + y, x + y \rangle = \langle x, x \rangle + 2\text{Re } \langle x, y \rangle + \langle y, y \rangle$$

(iv)

$$\langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$$

(v)

$$x = 0 \text{ if and only if } \langle x, y \rangle = 0 \, \forall y \in \mathcal{S}$$

(vi)

$$x = y \text{ if and only if } \langle x, z \rangle = \langle y, z \rangle \, \forall z \in \mathcal{S}$$

Note. Keep in mind here that the idea is to proceed directly from the properties defining an inner product.

2 [10 pts]. Show that for $A \in \mathbb{C}^{m \times n}$ the induced matrix norm

$$\|A\| := \max_{|\vec{x}|=1} |A\vec{x}|$$

defines a proper norm, and also that with this norm if A and B are square matrices $\|AB\| \leq \|A\| \|B\|$.

3 [10 pts]. (**Keener Problem 1.1.1.**) Prove that every basis in a finite dimensional space has the same number of elements.

4 [10 pts]. (**Keener Problem 1.1.3.**)

a. Verify that in an inner product space

$$\text{Re } \langle x, y \rangle = \frac{1}{4} \Big(\|x + y\|^2 - \|x - y\|^2 \Big).$$

b. Show that in any real inner product space there is at most one inner product which generates the same induced norm.

c. In $\mathbb{R}^n$, with $n > 1$, show that

$$\|x\|_p := \Big(\sum_{k=1}^{n} |x_k|^p \Big)^{1/p}$$

can be induced by an inner product if and only if $p = 2$.

5 [10 pts]. (**Keener Problem 1.1.8.**) Verify that the choice $\gamma = \frac{\langle x, y \rangle}{\|y\|^2}$ minimizes $\|x - \gamma y\|^2$. Show that $|\langle x, y \rangle|^2 = \|x\|^2 \|y\|^2$ if and only if x and y are linearly dependent.