M641 Fall 2012 Assignment 5, due Wednesday Oct. 3

1 [10 pts]. (**Keener Problem 1.3.2.**)

a. Prove an inequality relating the eigenvalues of a symmetric matrix before and after one of its diagonal elements is increased.

b. Use this inequality and the minimax principle to show that the smallest eigenvalue of

$$A = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & -4 \\ 4 & -4 & 3 \end{pmatrix}$$

is smaller than $-1/3$.

Note. Keener means a *real* symmetric matrix. For Part b, he means for you to proceed similarly as with his example starting on p. 21.

2 [10 pts]. (**Keener Problem 1.3.3**.) Use the minimax principle to show that the intermediate eigenvalue λ_2 of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{pmatrix}$$

is not positive.

3 [10 pts]. Solve the following.

A. (**Keener Problem 1.2.5.**) Find a basis for the range and null space of the following matrices:

a.

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 5 \end{pmatrix}.$$

b.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

B. (**Keener Problem 1.4.1.**) Under what conditions do the matrices of (Keener's) Problem 1.2.5 have solutions $A\vec{x} = \vec{b}$? Are they unique?

4 [10 pts]. (**Keener Problem 1.4.3.**) Show that the matrix $A = (a_{ij})$ where $a_{ij} = \langle \phi_i, \phi_j \rangle$ is invertible if and only if the vectors ϕ_i are linearly independent.

5 [10 pts]. (**Keener Problem 1.4.4.**) A square matrix A (with real entries) is positive-definite if $\langle A\vec{x}, \vec{x} \rangle > 0$ for all $\vec{x} \neq 0$. Use the Fredholm Alternative to prove that a positive definite matrix is invertible.

Note. Be clear about the role of the Fredholm Alternative.