M641 Assignment 8, due Wed. Oct. 31[1]

1. [10 pts] Determine whether or not

$$f(\vec{x}) = \text{sgn }(x_1)$$

is weakly differentiable on $U = B^o(0,1) \subset \mathbb{R}^n$, and justify your claim. Show that f belongs to a space with 0 regularity, but that there are $W^{k,p}(U)$ spaces with negative negative regularity that f does not belong to. Does this contradict Sobolev embeddings?

2. [10 pts] Suppose $U \subset \mathbb{R}^n$ is open and bounded and ∂U is C^1. Show that if $u \in W^{k,p}(U)$ with $k - \frac{n}{p} > 0$ then if $\{\phi_j\}_{j=1}^{\infty} \subset C^\infty(\bar{U})$ approximates u in $W^{k,p}(U)$ (i.e., $\phi_j \to u$ in $W^{k,p}(U)$) it will approach the continuous version of u in the supremum norm.

3. [10 pts] Show that if $u \in H^1([a,b])$ and $\phi \in C^1([a,b])$ then integration by parts is valid:

$$\int_a^b u\phi' dx = u\phi\Big|_a^b - \int_a^b u'\phi dx$$

Be clear about the meaning of $u(a)$ and $u(b)$, and the roll of the previous problem.

4. [10 pts] Answer the following:

a. Verify that the trigonometric functions

$$\{\frac{1}{\sqrt{2}}\} \cup \{\sin(nx), \cos(nx)\}_{n=1}^{\infty}$$

are orthonormal relative to the inner product

$$\langle u, v \rangle := \frac{1}{\pi} \int_0^{2\pi} u(x)v(x)dx.$$

Given any function $f \in L^2([0,2\pi])$ write down the least squares approximation of f in terms of these functions.

b. Is the family of trigonometric functions

$$\{\frac{1}{\sqrt{2}}\} \cup \{\sin(nx), \cos(nx)\}_{n=1}^{\infty}$$

orthonormal in the inner product space $H^1([0,2\pi])$. If not, how can you make it orthonormal?

Note. For Part (b) use the unscaled inner product for $H^1([0,2\pi])$.

5. [10 pts] (**Keener Problem 2.2.7.**) Show that the trigonometric functions form a complete orthogonal basis for any Sobolev space $H^n([0,2\pi])$ whose functions $f(x)$ satisfy $f(0) = f(2\pi)$. What is the difference between k terms of an L^2 and an H^1 Fourier approximation of some C^1 function? Illustrate the difference with a specific example of your choosing.

[1] Happy Halloween!