M641 Fall 2012 Midterm Exam Solutions

1. [10 pts] Show that if a linear space $\mathcal{S}$ has dimension n then:

a. Any collection of linearly independent elements of $\mathcal{S}$ must contain n or fewer elements.

b. Any n linearly independent elements of $\mathcal{S}$ forms a basis of $\mathcal{S}$.

Solution. For (a) suppose $\{u_k\}_{k=1}^n$ is a basis of $\mathcal{S}$ and $\{v_j\}_{j=1}^m$ is a linearly independent collection of elements of $\mathcal{S}$ with $m > n$. The calculation in Keener 1.1.1 says precisely that this is a contradition. I.e., we'll show that the set $\{v_j\}_{j=1}^m$ is linearly dependent, so that at least one element can be removed. This process of removal can continue until $n = m$.

We can expand each of the v_j in terms of the $\{u_k\}_{k=1}^n$

$$v_j = \sum_{k=1}^n a_{kj} u_k.$$

Now we try to identify a set of constants $\{c_j\}_{j=1}^m$, not all 0, so that

$$\sum_{j=1}^m c_j v_j = 0.$$

This would require

$$\sum_{j=1}^m c_j \sum_{k=1}^n a_{kj} u_k = 0 \Rightarrow \sum_{k=1}^n \Big(\sum_{j=1}^m c_j a_{kj} \Big) u_k = 0.$$

If the $\{u_k\}_{k=1}^n$ are linearly independent, we must have

$$\sum_{j=1}^m a_{kj} c_j = 0$$

for all $k \in \{1, 2, \dots, n\}$. This is equivalent to the matrix equation

$$A\vec{c} = 0,$$

where $A \in \mathbb{R}^{n \times m}$. Since $m > n$ the null space of A must have dimension at least $m - n$, and so there must exist a non-trivial vector $\vec{c}$, so that $A\vec{c} = 0$. But this means precisely that the set $\{v_j\}_{j=1}^m$ is not linearly independent.

For (b) suppose we have any n linearly independent elements in $\mathcal{S}$, say $\{w_j\}_{j=1}^n$, and suppose this set is not a basis for $\mathcal{S}$. Then there will be an element $s \in \mathcal{S}$ so that s is not a linear combination of the $\{w_j\}_{j=1}^n$. But this means we would have a set of $n+1$ linearly independent elements, which contradicts (a).

2 [10 pts]. Consider the symmetric 5×5 matrix

$$A = \begin{pmatrix} 10 & 1 & -1 & 0 & \frac{\sqrt{39}}{2} \\ 1 & 0 & 1 & -1 & 1 \\ -1 & 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 1 \\ \frac{\sqrt{39}}{2} & 1 & -1 & 1 & 5 \end{pmatrix}.$$

Show that if λ_1 denotes the largest eigenvalue of this matrix then $\lambda_1 \geq 11.5$.

Solution. We know that

$$\lambda_1 = \max_{|\vec{x}|=1} \langle A\vec{x}, \vec{x} \rangle \geq \max_{\substack{|\vec{x}|=1 \\ x_2=x_3=x_4=0}} \langle A\vec{x}, \vec{x} \rangle = \max_{\substack{|\vec{x}|=1 \\ x_2=x_3=x_4=0}} \sum_{i,j \neq 2,3,4} A_{ij} x_i x_j.$$

The right-hand side is precisely the largest eigenvalue of the submatrix

$$A_1 = \begin{pmatrix} 10 & \frac{\sqrt{39}}{2} \\ \frac{\sqrt{39}}{2} & 5 \end{pmatrix}.$$

The eigenvalues of this matrix satisfy

$$(10 - \lambda)(5 - \lambda) - \frac{39}{4} = 0 \Rightarrow \lambda^2 - 15\lambda + \frac{161}{4} = 0.$$

We see that

$$\lambda = \frac{15 \pm \sqrt{225 - 161}}{2} = \frac{15 \pm \sqrt{64}}{2} = \frac{15 \pm 8}{2} = 11.5, 3.5.$$

3 [10 pts]. Answer the following:

a. State the Fredholm Alternative, as discussed in class and in Keener.

b. Prove the following version of the Fredholm Alternative:

For any matrix $A \in \mathbb{C}^{n \times n}$ exactly one of the following holds:

- $A\vec{x} = \vec{b}$ has a unique solution for each $\vec{b} \in \mathbb{C}^n$
- $\mathcal{N}(A^*) \neq \{0\}$

Solution. For (a) the statement from class is as follows:

Suppose $A \in \mathbb{C}^{m \times n}$ and $\vec{b} \in \mathbb{C}^m$. The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b} \in \mathcal{N}(A^*)^\perp$.

For (b), first suppose $A\vec{x} = \vec{b}$ has a solution for each $\vec{b} \in \mathbb{C}^n$. According to the Fredholm Alternative, each of these $\vec{b}$ must be in $\mathcal{N}(A^*)^\perp$, so $\mathcal{N}(A^*)^\perp = \mathbb{C}^n$. Since $\mathbb{C}^n = \mathcal{N}(A^*) \oplus \mathcal{N}(A^*)^\perp$, we conclude that $\mathcal{N}(A^*) = \{0\}$. On the other hand, suppose there exists $\vec{b} \in \mathbb{C}^n$ so that $A\vec{x} = \vec{b}$ does not have a unique solution. If no solution exists for $\vec{b} \in \mathbb{C}^n$ then by the Fredholm Alternative dim $\mathcal{N}(A^*)^\perp < n$ and so $\mathcal{N}(A^*) \neq \{0\}$. On the other hand, if a solution exists but is not unique, then by letting $\vec{x}_1$ and $\vec{x}_2$ denote two solutions we see that $\mathcal{N}(A) \neq \{0\}$. By the Fredholm Alternative,

$$\mathbb{C}^n = \mathcal{R}(A^*) \oplus \mathcal{N}(A),$$

so dim $\mathcal{R}(A^*) < n$. But by the rank-nullity theorem

$$\dim \mathcal{R}(A^*) + \dim \mathcal{N}(A^*) = n,$$

so $\mathcal{N}(A^*) \neq \{0\}$.

Alternatively, we can shorten this a little by using the Matrix Inversion Theorem. In particular, we start by asserting that $A\vec{x} = \vec{b}$ has a unique solution for each $\vec{b} \in \mathbb{C}^n$ if and only if $\mathcal{N}(A) = \{0\}$ (which ensures that A is invertible). Now for $\mathcal{N}(A) \neq \{0\}$ we simply use the latter part of the previous proof.

4. [10 pts] Answer the following:

a. State Hölder's inequality.

b. Prove the following theorem: Suppose $U \subset \mathbb{R}^n$ is open and $1 < p, q < \infty$, with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} < 1$. Show that if $f \in L^p(U)$ and $g \in L^q(U)$ then

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Solution. For (a): Suppose $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p(U)$ and $v \in L^q(U)$ we have

$$\|uv\|_{L^1} \leq \|u\|_{L^p} \|v\|_{L^q}.$$

For (b),

$$\|fg\|_{L^r}^r = \int_U |f|^r |g|^r d\vec{x}.$$

We use Hölder's inequality with $p_1 = p/r$ and $p_2 = q/r$. This immediately gives

$$\int_U |f|^r |g|^r d\vec{x} \leq \Big(\int_U |f|^p d\vec{x} \Big)^{r/p} \Big(\int_U |g|^q d\vec{x} \Big)^{r/q}.$$

Now taking an r root gives the claim.

5. [10 pts] Find the values $l \in \mathbb{R}$ for which

$$f(x) = \Big| \ln |x| \Big|^l$$

is weakly differentiable on $U = (-1, 1)$.

Solution. First, we verify that $f \in L^1_{loc}(U)$. Notice, in particular, that since we only require *local* integrability there will be no problem at the endstates ± 1. That is, it's sufficient to check when

$$\int_{-1+\delta}^{1-\delta} \Big| \ln |x| \Big|^l dx$$

is finite for any $\delta > 0$. But this is clearly bounded since $\ln |x|$ blows up sub-algebraically as $x \to 0$. If you want to think about this in more detail, consider the integral

$$\int_0^{1-\delta} (-\ln x)^l dx,$$

which is precisely half the integral under consideration. Set $u = -\ln x$ so that $dx = -e^{-u} du$. Then

$$\int_0^{1-\delta} (-\ln x)^l dx = \int_{-\ln(1-\delta)}^{\infty} u^l e^{-u} du.$$

Here, the sub-algebraic nature of $\ln|x|$ becomes the (perhaps more familiar) sub-exponential behavior of u^l. Due to this exponential decay, this last integral is clearly finite.
Next, we need to identify $v \in L^1_{loc}(U)$ so that

$$\int_{-1}^{1} f(x)\phi'(x)dx = -\int_{-1}^{1} v(x)\phi(x)dx$$

for all $\phi \in C_c^\infty(-1,1)$. We compute

$$\begin{aligned}\int_{-1}^{1} |\ln|x||^l \phi'(x)dx &= \lim_{\epsilon\to 0}\int_{\epsilon<|x|<1} |\ln|x||^l \phi'(x)dx \\ &= \lim_{\epsilon\to 0}\left\{|\ln|x||^l\phi(x)\Big|_{-1}^{-\epsilon} + |\ln|x||^l\phi(x)\Big|_{\epsilon}^{1} - \int_{\epsilon<|x|<1} l|\ln|x||^{l-2}\ln|x|\frac{x}{|x|^2}\phi(x)dx\right\}.\end{aligned}$$

For the boundary terms $\phi(-1) = \phi(1) = 0$ by compact support, and

$$\lim_{\epsilon\to 0}\left\{|\ln\epsilon|^l(\phi(-\epsilon) - \phi(\epsilon))\right\} = 0$$

by the continuous differentiability of ϕ. Our candidate for a weak derivative is

$$v(x) = l|\ln|x||^{l-2}\ln|x|\frac{x}{|x|^2} = -l(-\ln|x|)^{l-1}\frac{1}{x}.$$

We need to check when this is in $L^1_{loc}(-1,1)$, so consider

$$\int_{-1+\delta}^{1-\delta} \frac{|\ln|x||^{l-1}}{|x|}dx.$$

Here $\ln|x| < 0$, so for example consider

$$-\int_{0}^{1-\delta} \frac{(\ln x)^{l-1}}{x}dx.$$

We set $y = \ln x \Rightarrow dy = \frac{dx}{x}$ so that we have, for $l \neq 0$,

$$-\int_{-\infty}^{\ln(1-\delta)} y^{l-1}dy = -\frac{y^l}{l}\Big|_{-\infty}^{\ln(1-\delta)},$$

and this is bounded for $l < 0$. For $l = 0$ we have simply $v \equiv 0$, so we conclude that the range of l is

$$l \leq 0.$$