THE MASLOV AND MORSE INDICES FOR SCHRÖDINGER OPERATORS ON $[0, 1]$

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Abstract. Assuming a symmetric potential and separated self-adjoint boundary conditions, we relate the Maslov and Morse indices for Schrödinger operators on $[0, 1]$. We find that the Morse index can be computed in terms of the Maslov index and two associated matrix eigenvalue problems. This provides an efficient way to compute the Morse index for such operators.

1. Introduction

We consider eigenvalue problems

$$\begin{aligned} Hy &:= -y'' + V(x)y = \lambda y \\ \alpha_1 y(0) &+ \alpha_2 y'(0) = 0 \\ \beta_1 y(1) &+ \beta_2 y'(1) = 0, \end{aligned} \tag{1.1}$$

where $y \in \mathbb{R}^n$, $V \in C([0, 1])$ is a symmetric matrix in $\mathbb{R}^{n\times n}$, and α_1, α_2, β_1, and β_2 are real-valued $n \times n$ matrices such that

$$\operatorname{rank} \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} = n; \qquad \operatorname{rank} \begin{bmatrix}\beta_1 & \beta_2\end{bmatrix} = n, \tag{1.2}$$

$$\alpha_1\alpha_2^t - \alpha_2\alpha_1^t = 0_{n\times n}; \qquad \beta_1\beta_2^t - \beta_2\beta_1^t = 0_{n\times n}, \tag{1.3}$$

where we use superscript t to denote matrix transpose, anticipating the use of superscript T to denote transpose in a complex Hilbert space described below. If (1.2)–(1.3) hold then without loss of generality we can take

$$\begin{aligned} \alpha_1\alpha_1^t + \alpha_2\alpha_2^t = I, \\ \beta_1\beta_1^t + \beta_2\beta_2^t = I \end{aligned} \tag{1.4}$$

(see, for example, [37, page 108]).

In particular, we are interested in counting the number of negative eigenvalues for H (i.e., the Morse index). We proceed by relating the Morse index to the Maslov index, which is described in Section 2. In essence, we'll find that the Morse index can be computed in terms of the Maslov index, and that while the Maslov index is less elementary than the Morse index, it's relatively straightforward to compute in the current setting.

The Maslov index has its origins in the work of V. P. Maslov [41] and subsequent development by V. I. Arnol'd [2]. It has now been studied extensively, both as a fundamental

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geometric quantity [6, 17, 22, 44, 46] and as a tool for counting the number of eigenvalues on specified intervals [7, 9, 12, 13, 14, 15, 19, 21, 30, 31, 33]. In this latter context, there has been a strong resurgence of interest following the analysis by Deng and Jones (i.e., [19]) for multidimensional domains. Our aim in the current analysis is to rigorously develop a relationship between the Maslov index and the Morse index in the relatively simple setting of (1.1), and to take advantage of this setting to compute the Maslov index directly for example cases so that these properties can be illustrated and illuminated. Our approach is adapted from [15, 19],

As a starting point, we define what we will mean by a *Lagrangian subspace*.

Definition 1.1. We say $\ell \subset \mathbb{R}^{2n}$ is a Lagrangian subspace if ℓ has dimension n and

$$(Jx, y)_{\mathbb{R}^{2n}} = 0,$$

for all $x, y \in \ell$. Here, $(\cdot, \cdot)_{\mathbb{R}^{2n}}$ denotes Euclidean inner product on $\mathbb{R}^{2n}$, and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

with I_n the $n \times n$ identity matrix. We sometimes adopt standard notation for symplectic forms, $\omega(x, y) = (Jx, y)_{\mathbb{R}^{2n}}$.

A simple example, important for intuition, is the case $n = 1$, for which $(Jx, y)_{\mathbb{R}^2} = 0$ if and only if x and y are linearly dependent. In this case, we see that any line through the origin is a Lagrangian subspace of $\mathbb{R}^2$. As a foreshadowing of further discussion, we note that each such Lagrangian subspace can be identified with precisely two points on the unit circle S^1.

More generally, any Lagrangian subspace of $\mathbb{R}^{2n}$ can be spanned by a choice of n linearly independent vectors in $\mathbb{R}^{2n}$. We will generally find it convenient to collect these n vectors as the columns of a $2n \times n$ matrix $\mathbf{X}$, which we will refer to as a *frame* for ℓ.

Lagrangian subspaces arise naturally in the current setting if we consider the shooting problem in which we evolve forward the family of solutions of (1.1) that satisfy only the left boundary condition (i.e., the condition at 0). In this setting, it will be natural to view (1.1) as a first order system with $p = y$, $q = y'$, and $\mathbf{p} = \binom{p}{q}$. We obtain

$$\frac{d\mathbf{p}}{dx} = \mathbb{A}(x; \lambda)\mathbf{p}, \tag{1.5}$$

where

$$\mathbb{A}(x; \lambda) = \begin{pmatrix} 0 & I_n \\ -\lambda I_n + V & 0 \end{pmatrix}.$$

Let $\{\mathbf{p}_j(x)\}_{j=1}^n = \{\binom{p_j(x)}{q_j(x)}\}_{j=1}^n$ denote any collection of n linearly independent vectors in $\mathbb{R}^{2n}$ satisfying the left boundary conditions

$$\alpha_1 p_j(0) + \alpha_2 q_j(0) = 0 \quad \forall j \in \{1, 2, \dots, n\},$$

and evolving according to (1.5). For example, using (1.3) we can take the vectors $\{p_j(0)\}_{j=1}^n$ to be the columns of α_2^t, and likewise the vectors $\{q_j(0)\}_{j=1}^n$ to be the columns of $-\alpha_1^t$. We denote by $X(x)$ the $n \times n$ matrix obtained by taking each $p_j(x)$ as a column, and we denote by $Z(x)$ the $n \times n$ matrix obtained by taking each $q_j(x)$ as a column. We will verify in Theorem 3.2 that the $2n \times n$ matrix $\mathbf{X} := \binom{X}{Z}$ is the frame for a Lagrangian subspace that we will denote $\ell(x,\lambda)$. Notice that $\ell(x,\lambda)$ varies as x and λ vary, and in particular if we choose any path Γ in the x-λ plane we can consider the evolution of ℓ along this path.

Continuing to view this process as a shooting argument, we can take as our *target* the Lagrangian subspace associated with the boundary condition at $x = 1$. It's clear that if $\ell(1,\lambda)$ intersects this Lagrangian subspace then λ is an eigenvalue of H, and also that the geometric multiplicity of λ corresponds precisely with the dimension of intersection. In order to clarify the nature of this target space, we let $\{\mathbf{p}_j^{(1)}\}_{j=1}^n = \{\binom{p_j^{(1)}}{q_j^{(1)}}\}_{j=1}^n$ denote any collection of n linearly independent (constant) vectors satisfying the right boundary conditions

$$\beta_1 p_j^{(1)} + \beta_2 q_j^{(1)} = 0 \quad \forall j \in \{1,2,\dots,n\}.$$

For example, we see from (1.3) that we can take the vectors $\{p_j^{(1)}\}_{j=1}^n$ to be the columns of β_2^t, and likewise the vectors $\{q_j^{(1)}\}_{j=1}^n$ to be the columns of $-\beta_1^t$. Let X_1 denote the $n \times n$ matrix comprising $\{p_j^{(1)}\}_{j=1}^n$ as its columns, and let Z_1 denote the $n \times n$ matrix comprising $\{q_j^{(1)}\}_{j=1}^n$ as its columns. We see that $\mathbf{X}_1 := \binom{X_1}{Z_1}$ is a frame for the Lagrangian subspace ℓ_1 that can be viewed as our target.

We can now ask the following questions: (1) as $\ell(x,\lambda)$ evolves, for what values of x and λ does it intersect ℓ_1?; (2) what is the dimension of these intersections?; and (3) what is the direction of these intersections? Geometrically, the Maslov index is precisely a count of these intersections, including both multiplicity and direction.

We will find it productive to fix $s_0 > 0$ (taken sufficiently small during the analysis) and $\lambda_\infty > 0$ (taken sufficiently large during the analysis), and to consider the rectangular path

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4,$$

where the paths $\{\Gamma_i\}_{i=1}^4$ are depicted in Figure 1.

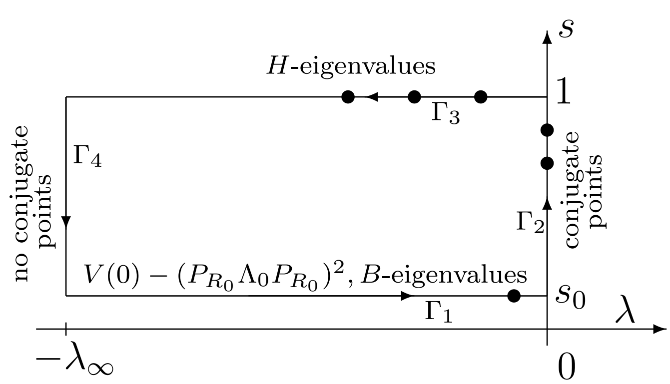

FIGURE 1. Schematic of the path $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$

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As discussed, for example, in [17], the Maslov index enjoys path additivity so that

$$\mathrm{Mas}(\ell,\ell_1;\Gamma)=\mathrm{Mas}(\ell,\ell_1;\Gamma_1)+\mathrm{Mas}(\ell,\ell_1;\Gamma_2)+\mathrm{Mas}(\ell,\ell_1;\Gamma_3)+\mathrm{Mas}(\ell,\ell_1;\Gamma_4).$$

In addition, the Maslov index is homotopy invariant, and it follows immediately that the Maslov index around any closed path will be 0, so that

$$\mathrm{Mas}(\ell,\ell_1;\Gamma)=0.$$

Our analysis is primarily concerned with understanding each of the four quantities

$$\{\mathrm{Mas}(\ell,\ell_1;\Gamma_i)\}_{i=1}^4.$$

As a start, we note that in the setting of eigenvalue problems such as (1.1) it's natural to view the Maslov index along Γ_2 as a distinguished value, and we will designate it the *Principal Maslov Index.* In our setting, this is a readily computable quantity, and we will develop a framework for computing it, and compute values of it in particular cases.

We will show that $\mathrm{Mas}(\ell,\ell_1;\Gamma_3)$ is precisely the Morse index of H that we're trying to compute, and that given any $0<s_0<1$, $\lambda_\infty>0$ can be chosen sufficiently large so that $\mathrm{Mas}(\ell,\ell_1;\Gamma_4)=0$. In the case of Dirichlet boundary conditions we'll find that s_0 can be chosen sufficiently small so that $\mathrm{Mas}(\ell,\ell_1;\Gamma_1)=0$, in which case we get the very simple relationship

$$\mathrm{Mor}(H)=-\mathrm{Mas}(\ell,\ell_1;\Gamma_2). \qquad \text{(Dirichlet case)}$$

More generally, we can have crossings along the bottom shelf (i.e., Γ_1), and in order to efficiently characterize these we'll adapt an elegant theorem from [8] (see also an earlier version in [38]).

Theorem 1.2 (Adapted from [8])**.** *Let α_1 and α_2 be as described in (1.2)-(1.3). Then there exist three orthogonal (and mutually orthogonal) projection matrices P_D (the Dirichlet projection), P_N (the Neumann projection), and $P_R=I-P_D-P_N$ (the Robin projection), and an invertible self-adjoint operator Λ acting on the space $P_R\mathbb{R}^n$ such that the boundary condition*

$$\alpha_1 y(0)+\alpha_2 y'(0)=0$$

can be expressed as

$$\begin{aligned}P_D y(0)&=0\\ P_N y'(0)&=0\\ P_R y'(0)&=\Lambda P_R y(0).\end{aligned}$$

Moreover, P_D can be constructed as the projection onto the kernel of α_2 and P_N can be constructed as the projection onto the kernel of α_1. Construction of the operator Λ will be discussed in the following remark. Precisely the same statement holds for β_1 and β_2 for the boundary condition at $x=1$.

Remark 1.3 (Construction of Λ)**.** Let $\mathcal{U}$ denote the unitary matrix

$$\mathcal{U}=-(\alpha_1-i\alpha_2)^{-1}(\alpha_1+i\alpha_2),$$

where the inverse is guaranteed to exist by our assumptions (see Lemma 1.4.7 of [8]). Let $(\mathcal{U}+I)_R$ denote the restriction of $(\mathcal{U}+I)$ to the space $P_R\mathbb{R}^n$, so that $(\mathcal{U}+I)_R$ is invertible. Then

$$\Lambda = -i(\mathcal{U}+I)_R^{-1}(\mathcal{U}-I).$$

It follows that α_2 is invertible on the range of $\alpha_1 P_R$, and $\Lambda = \alpha_2^{-1}\alpha_1 P_R$.

Definition 1.4. Let $(P_{D_0}, P_{N_0}, P_{R_0}, \Lambda_0)$ denote the projection quadruplet associated with our boundary conditions at $x=0$, and let $(P_{D_1}, P_{N_1}, P_{R_1}, \Lambda_1)$ denote the projection quadruplet associated with our boundary conditions at $x=1$. We denote by B the self-adjoint operator obtained by restricting $(P_{R_0}\Lambda_0 P_{R_0} - P_{R_1}\Lambda_1 P_{R_1})$ to the space $(\ker P_{D_0}) \cap (\ker P_{D_1})$.

In Section 3, we will verify the general relationship

$$\operatorname{Mas}(\ell, \ell_1; \Gamma_1) = -\operatorname{Mor}(B) - \operatorname{Mor}(Q(V(0) - (P_{R_0}\Lambda_0 P_{R_0})^2)Q),$$

where Q denotes the projection matrix onto the null space of B.

We see immediately that if $\alpha_2, \beta_2 = 0$ so that α_1, β_1 have full rank, we obtain

$$(P_{D_0}, P_{N_0}, P_{R_0}, \Lambda_0) = (I, 0, 0, 0),$$

and

$$(P_{D_1}, P_{N_1}, P_{R_1}, \Lambda_1) = (I, 0, 0, 0).$$

In this case, $B = 0$, and is restricted to the domain $(\ker P_{D_0}) \cap (\ker P_{D_1}) = \{0\}$. This corresponds with the Dirichlet case mentioned above, for which $\operatorname{Mas}(\ell, \ell_1; \Gamma_1) = 0$. In particular, we have observed that if $(\ker P_{D_0}) \cap (\ker P_{D_1}) = \{0\}$ then $Q \equiv 0$.

On the other extreme, suppose α_2, β_2 both have full rank (the *Neumann-based case*), so that $P_{D_0} = 0$ and $P_{D_1} = 0$, and consequently $(\ker P_{D_0}) \cap (\ker P_{D_1}) = \mathbb{R}^n$. Focusing on the condition at $x = 0$, we notice that this implies $P_{R_0} = I - P_{N_0}$. In this way, $\mathbb{R}^n$ can be decomposed as

$$\mathbb{R}^n = P_{N_0}(\mathbb{R}^n) \oplus P_{R_0}(\mathbb{R}^n),$$

and since P_{N_0} corresponds with projection onto the kernel of α_1 we see that P_{R_0} corresponds with projection onto the range of α_1^t. We conclude that $P_{R_0}\alpha_1^t = \alpha_1^t$. Likewise, since α_1 annihilates $P_{N_0}(\mathbb{R}^n)$ we see that $\alpha_1 P_{R_0} = \alpha_1$. We have, then, using Remark 1.3,

$$P_{R_0}\Lambda_0 P_{R_0} = -P_{R_0}\alpha_2^{-1}\alpha_1 P_{R_0} = -P_{R_0}\alpha_2^{-1}\alpha_1.$$

But according to our condition $\alpha_1\alpha_2^t = \alpha_2\alpha_1^t$, we have $\alpha_2^{-1}\alpha_1 = \alpha_1^t(\alpha_2^t)^{-1}$ so that

$$-P_{R_0}\alpha_2^{-1}\alpha_1 = -\alpha_2^{-1}\alpha_1; \quad \text{i.e., } P_{R_0}\Lambda_0 P_{R_0} = -\alpha_2^{-1}\alpha_1.$$

We conclude that in this case (where α_2, β_2 both have full rank) we have

$$\operatorname{Mas}(\ell, \ell_1; \Gamma_1) = -\operatorname{Mor}(\beta_2^{-1}\beta_1 - \alpha_2^{-1}\alpha_1) - \operatorname{Mor}(Q(V(0) - (\alpha_2^{-1}\alpha_1)^2)Q),$$

where in this case Q is a projection onto the null space of $B = \beta_2^{-1}\beta_1 - \alpha_2^{-1}\alpha_1$.

We are now prepared to state the main result of our analysis.

Theorem 1.5. *For system* (1.1)*, let* $V \in C([0,1])$ *be a symmetric matrix in* $\mathbb{R}^{n \times n}$*, and let* α_1*,* α_2*,* β_1*, and* β_2 *be as in* (1.2)*-*(1.3)*. In addition, let* Q *denote projection onto the kernel of* B*, and make the non-degeneracy assumption* $0 \notin \sigma(Q(V(0) - (P_{R_0} \Lambda_0 P_{R_0})^2)Q)$*. Then we have*

$$\operatorname{Mor}(H) = -\operatorname{Mas}(\ell, \ell_1; \Gamma_2) + \operatorname{Mor}(B) + \operatorname{Mor}(Q(V(0) - (P_{R_0} \Lambda_0 P_{R_0})^2)Q).$$

Remark 1.6. In the event that $0 \in \sigma(Q(V(0) - (P_{R_0} \Lambda_0 P_{R_0})^2)Q)$, our method still applies, but the resulting expression for $\operatorname{Mor}(H)$ has additional terms that arise from a higher order perturbation expansion.

Remark 1.7. As noted in the lead-in to Theorem 1.5, we have an especially straightforward relation for the Dirichlet case,

$$\operatorname{Mor}(H) = -\operatorname{Mas}(\ell, \ell_1; \Gamma_2).$$

In particular, since B is restricted to the space $(\ker P_{D_0}) \cap (\ker P_{D_1})$, we see that this relation holds if the boundary condition on either side is Dirichlet.

Remark 1.8. Our emphasis on the negative eigenvalues of H (the Morse index) is simply a convention, and we could similarly develop a theorem counting the number of eigenvalues of H below any other fixed real value $\lambda_0 \in \mathbb{R}$. In this case, the number of eigenvalues less than λ_0 would be related to the Maslov index of a path Γ_2^0 with $\lambda = \lambda_0$ fixed, and s going from s_0 to 1 (along with appropriate perturbation terms). This, of course, would allow us to determine the number of eigenvalues of H on any interval $[\lambda_1, \lambda_2] \subset \mathbb{R}$.

Remark 1.9. As we will briefly discuss in Section 3 (see Remark 3.7), the standard Sturm-Liouville oscillation theorem for $n = 1$ (relating the zeros of an eigenfunction to the position of its associated eigenvalue in the sequence of all eigenvalues; e.g, Theorem XIII.7.50 in [20] or Theorem 8.4.5 in [4]) follows in a straightforward manner from Theorem 1.5. In this way, Theorem 1.5 can reasonably be viewed as a generalization of this theory to the current n-dimensional setting. The nature of this generalization is especially elegant in the case that the boundary conditions at $x = 1$ are Dirichlet (see Remark 3.27 in Section 3.4).

We note that there is a long history of such generalizations, including Arnol'd's seminal work with the Maslov index in the 1960's [2]. For a related approach that does not directly refer to the Maslov index, see Chapter 10 in [4]. To the best of our knowledge Theorem 1.5 is the most complete such theorem in the current setting.

The paper is organized as follows. In Section 2 we give a precise definition of the Maslov index, suitable for the current analysis, and summarize some of its properties. In Section 3 we analyze the Maslov index in the setting of (1.1), proving Theorem 1.5, and in Section 4 we discuss several applications intended to illustrate our results.

2. The Maslov index

In this section, we review a definition of the Maslov index appropriate for the current analysis, and outline some of its salient properties. We note that several alternative definitions are available (see, for example, [17]), all with generally the same properties.

Recalling Definition 1.1, we consider the collection of all Lagrangian subspaces of $\mathbb{R}^{2n}$, which we designate the *Lagrangian Grassmannian* and denote $\Lambda(n)$. Let $\Sigma \subset \mathbb{R}$ denote an index interval, and consider any continuous path of Lagrangian subspaces $\Upsilon : \Sigma \to \Lambda(n)$. Given a fixed Lagrangian subspace ℓ_1 (the *target* space, which for us will be associated with data at $x = 1$), we will define the Maslov index $\mathrm{Mas}(\Upsilon, \ell_1; \Sigma)$ associated with intersections of $(\Upsilon)_{t\in\Sigma}$ with ℓ_1.

As a starting point for our construction, which follows particularly [6, 22], we introduce a complex Hilbert space, which we will denote $\mathbb{R}^{2n}_J$. The elements of this space will continue to be real-valued vectors of length $2n$, but we will define multiplication by complex scalars as

$$(\alpha + i\beta)u := \alpha u + \beta J u, \quad u \in \mathbb{R}^{2n}, \alpha + i\beta \in \mathbb{C},$$

and we will define a complex scalar product

$$(u, v)_{\mathbb{R}^{2n}_J} := (u, v)_{\mathbb{R}^{2n}} - i\omega(u, v), \quad u, v \in \mathbb{R}^{2n}$$

(recalling $\omega(u, v) = (Ju, v)_{\mathbb{R}^n}$). It is important to note that, considered as a real vector space, $\mathbb{R}^{2n}_J$ is identical to $\mathbb{R}^{2n}$, and not its complexification $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$. (In fact, $\mathbb{R}^{2n}_J \cong \mathbb{C}^n$ while $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^{2n}$.) However, it is easy to see that $\mathbb{R}^{2n}_J \cong \ell \otimes_{\mathbb{R}} \mathbb{C}$ for any Lagrangian subspace $\ell \in \Lambda(n)$, and we'll take advantage of this correspondence.

For a matrix U acting on $\mathbb{R}^{2n}_J$, we denote the adjoint by U^{J*} so that

$$(Uu, v)_{\mathbb{R}^{2n}_J} = (u, U^{J*}v)_{\mathbb{R}^{2n}_J},$$

for all $u, v \in \mathbb{R}^{2n}_J$. We denote by $\mathfrak{U}_J$ the space of unitary matrices acting on $\mathbb{R}^{2n}_J$ (i.e., the matrices so that $UU^{J*} = U^{J*}U = I$). In order to clarify the nature of $\mathfrak{U}_J$, we note that we have the identity

$$(Uu, Uv)_{\mathbb{R}^{2n}_J} = (u, v)_{\mathbb{R}^{2n}_J},$$

from which

$$(Uu, Uv)_{\mathbb{R}^{2n}} - i(JUu, Uv)_{\mathbb{R}^{2n}} = (u, v)_{\mathbb{R}^{2n}} - i(Ju, v)_{\mathbb{R}^{2n}}.$$

Equating real parts, we see that U must be unitary as a matrix on $\mathbb{R}^{2n}$, while by equating imaginary parts we see that $UJ = JU$. We have, then,

$$\mathfrak{U}_J = \{U \in \mathbb{R}^{2n\times 2n} \,|\, U^t U = UU^t = I_{2n}, \, UJ = JU\}.$$

In addition, it will be useful to define a matrix U^T satisfying $U^T z := \overline{U^t \overline{z}}$, or $U^T = \tau_1 \circ U^t \circ \tau_1$, where τ_1 is the conjugate operation; that is, if $z = x + Jy$, $x, y \in \ell_1$, then $\tau_1(z) = \overline{z} := x - Jy$. It is also clear that $\tau_1 = 2\Pi_1 - I_{2n}$, where Π_1 is the orthogonal projection onto ℓ_1.

Given our target space ℓ_1, we denote by $\ell_1^\perp$ the Lagrangian subspace perpendicular to ℓ_1 in $\mathbb{R}^{2n}$. I.e., $\ell_1^\perp$ is a Lagrangian subspace, and

$$(u, v)_{\mathbb{R}^{2n}} = 0, \quad \forall u \in \ell_1, v \in \ell_1^\perp.$$

If $\mathbf{X}_{\ell_1}$ is a frame for ℓ_1, then $J\mathbf{X}_{\ell_1}$ is a frame for $\ell_1^\perp$. We can express this as $\ell_1^\perp = J(\ell_1)$, indicating that $\ell_1^\perp$ is the space obtained by mapping all elements of ℓ_1 with J.

For each $s \in \Sigma$ we choose a unitary operator U_s acting on the complex Hilbert space $\mathbb{R}^{2n}_J$ such that $\Upsilon(s) = U_s(\ell_1^\perp)$. This choice is possible by [6, Proposition 1.1]. Indeed, in the current setting, we can associate a canonical frame $\mathbf{X}_{\Upsilon(s)}$ with each $\Upsilon(s)$, as well as a frame $\mathbf{X}_{\ell_1^\perp}$, and find a family of unitary matices satisfying $\mathbf{X}_{\Upsilon(s)} = U_s \mathbf{X}_{\ell_1^\perp}$. (The matrices U_s are not uniquely defined, and in fact we'll find that different choices of U_s can be useful in different settings.)

This relationship provides a natural and productive connection between the elements ℓ of the Lagrangian Grassmannian and elements $U \in \mathfrak{U}_J$. However, the associated unitary matrices are not uniquely specified, and consequently the spectrum of U contains redundant information. For example, in the simple case of $\mathbb{R}^2$ this redundant information corresponds with our previous observation that each element $\ell \in \Lambda(1)$ corresponds with two points on S^1. We overcome this difficulty by defining a new (uniquely specified) unitary matrix W_s in $\mathbb{R}^{2n}_J$ by $W_s = U_s U_s^T$.

We observe that the unitary condition $UJ = JU$ implies U must have the form

$$U = \begin{pmatrix} U_{11} & -U_{21} \\ U_{21} & U_{11} \end{pmatrix} = \begin{pmatrix} U_{11} & 0 \\ 0 & U_{11} \end{pmatrix} + J \begin{pmatrix} U_{21} & 0 \\ 0 & U_{21} \end{pmatrix}.$$

In addition, we have the scaling condition

$$\begin{aligned} U_{11}^t U_{11} + U_{21}^t U_{21} &= I \\ U_{11} U_{11}^t + U_{21} U_{21}^t &= I \\ U_{11}^t U_{21} - U_{21}^t U_{11} &= 0 \\ U_{11} U_{21}^t - U_{21} U_{11}^t &= 0. \end{aligned} \tag{2.1}$$

In this way, there is a natural one-to-one correspondence between matrices $U \in \mathfrak{U}_J$ and the $n \times n$ complex unitary matrices $\tilde{U} = U_{11} + iU_{21}$ (i.e., the $\tilde{U} \in \mathbb{C}^{n\times n}$ so that $\tilde{U}^*\tilde{U} = \tilde{U}\tilde{U}^* = I$).

In this way, the matrix $W_s = U_s U_s^T$ has a natural corresponding matrix $\tilde{W}_s = \tilde{U}_s \tilde{U}_s^T$, where $\tilde{U}^T z = \overline{\tilde{U}^* \bar{z}} = \tilde{U} z$. Ultimately, we will define the Maslov index in terms of $\tilde{W}_s$.

The following properties of the matrices W_s and $\tilde{W}_s$ can be found in [6, Lemma 1.3] or [22, Proposition 2.44].

Lemma 2.1. *If ℓ_1 is a real Lagrangian subspace in $\mathbb{R}^{2n}$, $\Upsilon \colon \Sigma = [a,b] \to \Lambda(n)$ is a continuous path, Π_s and Π_{ℓ_1} are the orthogonal projections onto $\Upsilon(s)$ and ℓ_1 respectively, and U_s is the unitary operator on $\mathbb{R}^{2n}_J$ such that $\Upsilon(s) = U_s(\ell_1^\perp)$, then*

(i) *$W_s = (I_{\mathbb{R}^{2n}_J} - 2\Pi_s)(2\Pi_{\ell_1} - I_{\mathbb{R}^{2n}_J})$;*
(ii) *$\ker(W_s + I_{\mathbb{R}^{2n}_J})$ is isomorphic to $(\Upsilon(s) \cap \ell_1) \oplus J(\Upsilon(s) \cap \ell_1) \cong (\Upsilon(s) \cap \ell_1) \otimes_{\mathbb{R}} \mathbb{C}$;*
(iii) *$\dim_{\mathbb{R}}(\Upsilon(s) \cap \ell_1) = \dim \ker(\tilde{W}_s + I)$.*

Following [6, 22, 44], we define the Maslov index of $\{\Upsilon(s)\}_{s\in\Sigma}$, with target ℓ_1, as the spectral flow of the operator family $\{\tilde{W}_s\}_{s\in\Sigma}$ through -1; that is, as the net count (including multiplicity) of the eigenvalues of $\tilde{W}_s$ crossing the point -1 counterclockwise on the unit circle minus the number of eigenvalues crossing -1 clockwise as the parameter s changes. Specifically, let us choose a partition $a = s_0 < s_1 < \cdots < s_n = b$ of $\Sigma = [a,b]$ and numbers $\epsilon_j \in (0,\pi)$ so that $\ker\left(\tilde{W}_s - e^{i(\pi\pm\epsilon_j)} I\right) = \{0\}$, that is, $e^{i(\pi\pm\epsilon_j)} \in \mathbb{C} \setminus \sigma(\tilde{W}_s)$, for $s_{j-1} < s < s_j$

and $j = 1, \ldots, n$. For each $j = 1, \ldots, n$ and any $s \in [s_{j-1}, s_j]$ there are only finitely many values $\theta \in [0, \epsilon_j]$ for which $e^{i(\pi+\theta)} \in \sigma(\tilde{W}_s)$.

Fix some $j \in \{1, 2, \ldots, n\}$ and consider the value

$$k(s, \epsilon_j) := \sum_{0 \le \theta < \epsilon_j} \dim \ker \big(\tilde{W}_s - e^{i(\pi+\theta)} I \big). \tag{2.2}$$

for $s_{j-1} \le s \le s_j$. This is precisely the sum, along with geometric multiplicity, of the number of eigenvalues of $\tilde{W}_s$ that lie on the arc

$$A_j := \{ e^{is} : s \in [\pi, \pi + \epsilon_j) \}.$$

The stipulation that $e^{i(\pi \pm \epsilon_j)} \in \mathbb{C} \setminus \sigma(\tilde{W}_s)$, for $s_{j-1} < s < s_j$ asserts that no eigenvalue can enter A_j in the clockwise direction or exit in the counterclockwise direction during the interval $s_{j-1} < s < s_j$. In this way, we see that $k(s_j, \epsilon_j) - k(s_{j-1}, \epsilon_j)$ is a count of the number of eigenvalues that entered A_j in the counterclockwise direction minus the number that left in the clockwise direction during the interval (s_{j-1}, s_j).

In dealing with the concatenation of paths, it's particularly important to understand this quantity if an eigenvalue resides at -1 at either $s = s_{j-1}$ or $s = s_j$. If an eigenvalue moving in the counterclockwise direction arrives at -1 at $s = s_j$, then we increment the difference foward. On the other hand, suppose an eigenvalue resides at -1 at $s = s_{j-1}$ and moves in the counterclockwise direction. There is no change, and so we do not increment the difference.

We are ready to define the Maslov index.

Definition 2.2. Let ℓ_1 be a fixed Lagrangian subspace in a real Hilbert space $\mathbb{R}^{2n}$ and let $\Upsilon \colon \Sigma = [a, b] \to \Lambda(n)$ be a continuous path in the Lagrangian–Grassmannian. The Maslov index $\operatorname{Mas}(\Upsilon, \ell_1; \Sigma)$ is defined by

$$\operatorname{Mas}(\Upsilon, \ell_1; \Sigma) = \sum_{j=1}^{n} (k(s_j, \epsilon_j) - k(s_{j-1}, \epsilon_j)). \tag{2.3}$$

We refer to [22, Theorem 3.6] for a list of basic properties of the Maslov index; in particular, as mentioned in our introduction, the Maslov index is a homotopy invariant and is additive under catenation of paths.

It will be useful to anticipate some later developments and briefly discuss how the Maslov index applies to the contour Γ described in Figure 1. For this, we'll find it notationally convenient to use the notation $\ell(s, \lambda) = \ell(x, \lambda)|_{x=s}$ (effectively, distinguishing between the independent variable x and the variable endpoint s). For $(s, \lambda) \in \Gamma$, let $\tilde{W}_{s,\lambda}$ denote the unitary complex matrix associated with $\ell(s, \lambda)$ and target ℓ_1. For this discussion, we will use the important fact, verified below, that we have monotonicity in λ in the following sense: *as λ increases (with s fixed), the eigenvalues of $\tilde{W}_{s,\lambda}$ move clockwise around S^1.*

Focusing first on Γ_1 (for which $s = s_0$): as our contour proceeds in the counterclockwise direction the eigenvalues of $\tilde{W}_{s_0,\lambda}$ move clockwise around S^1. In this way, crossings necessarily correspond with eigenvalues of $\tilde{W}_{s_0,\lambda}$ rotating out of some A_j, thus reducing the Maslov index.

Each of these crossings corresponds with a solution to the eigenvalue problem

$$\begin{aligned} H_s y &:= -y'' + V(x)y = \lambda y \\ \alpha_1 y(0) &+ \alpha_2 y'(0) = 0 \\ \beta_1 y(s) &+ \beta_2 y'(s) = 0 \end{aligned} \tag{2.4}$$

(with $s = s_0$.) It's convenient to set $\xi = x/s$ and $u(\xi) = y(x)$ so that u solves the eigenvalue problem

$$\begin{aligned} H(s)u &:= -u'' + s^2 V(s\xi)u = s^2 \lambda u \\ \alpha_1 u(0) &+ \frac{1}{s}\alpha_2 u'(0) = 0 \\ \beta_1 u(1) &+ \frac{1}{s}\beta_2 u'(1) = 0. \end{aligned} \tag{2.5}$$

It's clear that crossings along Γ_1 correspond with the existence of eigenvalues of the operator $H(s_0)$. More precisely, a crossing will occur along Γ_1 at λ, provided ${s_0}^2\lambda$ is an eigenvalue of $H(s_0)$. The number of negative eigenvalues of $H(s_0)$, including multiplicity, is its Morse index, and since each such eigenvalue decreases the Maslov index by its multiplicity we obtain the relation

$$\operatorname{Mas}(\ell, \ell_1; \Gamma_1) = -\operatorname{Mor}(H(s_0)).$$

Remark 2.3. We note for future reference that H_s and $H(s)$ refer to different operators with different domains. To be precise,

$$\begin{aligned} \operatorname{dom}(H_s) &= \Big\{ y \in H^2(0,s) : \alpha_1 y(0) + \alpha_2 y'(0) = 0, \beta_1 y(s) + \beta_2 y'(s) = 0 \Big\} \\ \operatorname{dom}(H(s)) &= \Big\{ u \in H^2(0,1) : \alpha_1 u(0) + \frac{1}{s}\alpha_2 u'(0) = 0;\ \beta_1 u(1) + \frac{1}{s}\beta_2 u'(1) = 0 \Big\} \end{aligned}$$

Of particular importance, $\lambda(s)$ is an eigenvalue of $H(s)$ if and only if $\lambda_s = \lambda(s)/s^2$ is an eigenvalue of H_s.

Suppose we have an intersection at the corner point $(s_0, 0)$, where Γ_1 meets Γ_2. Since the eigenvalues of $\tilde{W}_{s_0,\lambda}$ are moving clockwise around S^1, this must correspond with an eigenvalue of $\tilde{W}_{s_0,\lambda}$ stopping at -1 from the clockwise direction. This eigenvalue does not leave A_j, and so the Maslov index does not increment.

On the other hand, let's consider what happens on Γ_3. In this case, λ will be decreasing (for counterclockwise movement along Γ), so eigenvalues of $\tilde{W}_{s_0,\lambda}$ will move in the counterclockwise direction along S^1. Accordingly, crossings will correspond with eigenvalues moving into some A_j, and so the Maslov index will increase. These crossings correspond with eigenvalues of H (i.e., $H(1)$), and so

$$\operatorname{Mas}(\ell, \ell_1; \Gamma_3) = \operatorname{Mor}(H).$$

Suppose we have an intersection at the corner point $(1, 0)$. By monotonicity in λ, as λ decreases from 0 the the eigenvalues of $\tilde{W}_{s_0,\lambda}$ will move in the counterclockwise direction *into* some A_j. Since these eigenvalues are already in A_j at the start of the time interval, the Maslov index does not change.

Finally, let's consider the contour Γ_2. Aside from the Dirichlet case, we don't necessarily have monotonicity (with respect to s) along Γ_2, but we can still say something about the Maslov index based on eigenvalue curves E_{s_*,λ_*}, which we'll define as continuous paths in the s-λ plane crossing through (s_*,λ_*) and along which λ is an eigenvalue of $H(s)$. Suppose such a curve crosses Γ_2 at some point $(s_*,0)$. If it bends upward, we can consider a small box local to the intersection, so that the path exits this box through its top shelf. As with our discussion of Γ_3 this will correspond with an increase in the Maslov index, and so by homotopy invariance the crossing at $(s_*,0)$ will correspond with a decrease in the Maslov index. Likewise, if the path crossing $(s_*,0)$ bends downward the crossing will correspond with an increase in the Maslov index.

3. Application to the Schrödinger Equation

We now focus on the eigenvalue problem (1.1), and especially the first-order form (1.5). Throughout our analysis, we will make use of the following remark concerning the matrices used in defining our boundary conditions.

Remark 3.1. Note that (1.3), (1.4) imply that

$$\begin{bmatrix} \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{bmatrix} \begin{bmatrix} \beta_1^t & \beta_2^t \\ -\beta_2^t & \beta_1^t \end{bmatrix} = I_{2n}, \tag{3.1}$$

which, in turn, implies that

$$\begin{bmatrix} \beta_1^t & \beta_2^t \\ -\beta_2^t & \beta_1^t \end{bmatrix} \begin{bmatrix} \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{bmatrix} = I_{2n}. \tag{3.2}$$

Or,

$$\beta_1^t\beta_2 - \beta_2^t\beta_1 = 0_{n\times n} \tag{3.3}$$

$$\beta_1^t\beta_1 + \beta_2^t\beta_2 = I. \tag{3.4}$$

Similar equalities hold for matrices α_1, α_2.

Following [19], for each $\lambda \in \mathbb{R}$ and $s \in (0,1]$ we define the following set of vector valued functions on $[0,s]$:

$$Y_\lambda = \Big\{ \mathbf{p} \in H^1(0,s) : \mathbf{p} \text{ solves (1.5) and } \alpha_1 p(0) + \alpha_2 q(0) = 0 \Big\}. \tag{3.5}$$

That is, we consider the (n dimensional) solution space to the equation (1.5), defined on $[0,s]$, consisting of the solutions that satisfy the boundary condition at 0.

We define the trace map $\Phi_s^\lambda : Y_\lambda \to \mathbb{R}^{2n}$ by the following formula:

$$\Phi_s^\lambda : \mathbf{p} \mapsto \mathbf{p}(s). \tag{3.6}$$

I.e., for the path of Lagrangian spaces $\ell(s,\lambda)$, we have $\ell(s,\lambda) = \Phi_s^\lambda(Y_\lambda)$.

In what follows, we will use the observation that if $\mathbf{X} = \binom{X}{Z}$ is the frame for a Lagrangian subspace, then

$$X^tZ - Z^tX = 0.$$

To see this, we observe that since $\mathbf{X}$ is the frame of a Lagrangian subspace, each of its columns $\binom{x}{z} \in \mathbb{R}^{2n}$ must satisfy

$$(J\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix})_{\mathbb{R}^{2n}} = 0, \quad \Rightarrow (\begin{pmatrix} -y \\ x \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix})_{\mathbb{R}^{2n}} = 0,$$

from which the identity $X^t Z - Z^t X = 0$ is apparent.

Theorem 3.2. *For all $s \in (0,1]$ and $\lambda \in \mathbb{R}$ the plane $\Phi_s^\lambda(Y_\lambda)$ belongs to the space $\Lambda(n)$ of Lagrangian n-planes in $\mathbb{R}^{2n}$, with the Lagrangian structure $\omega(v_1, v_2) = (Jv_1, v_2)_{\mathbb{R}^{2n}}$.*

Proof. Our target space ℓ_1 can be represented by a $2n \times n$ matrix $\binom{-\beta_2^t}{\beta_1^t}$. Since $-\beta_2\beta_1^t = -\beta_1\beta_2^t$ by (1.3), the symplectic form ω vanishes on ℓ_1. Also, ℓ_1 is n-dimensional (1.2). Hence, ℓ_1 is Lagrangian.

Next, we represent $\Phi_s^\lambda(Y_\lambda)$ as a $2n \times n$ matrix $\binom{X(s,\lambda)}{Z(s,\lambda)}$. Then

$$\begin{aligned}(X^t Z - Z^t X)' &= (X^t)' Z + X^t Z' - (Z^t)' X - Z^t X' &(3.7)\\ &= Z^t Z + X^t (V - \lambda I) X - X^t (V - \lambda I) X - Z^t Z = 0. &(3.8)\end{aligned}$$

But since

$$X^t(0,\lambda) Z(0,\lambda) - Z^t(0,\lambda) X(0,\lambda) = -\alpha_2\alpha_1^t + \alpha_1\alpha_2^t = 0,$$

we see that $X^t Z - Z^t X = 0$. Therefore, the symplectic form ω vanishes on $\Phi_s^\lambda(Y_\lambda)$. And since Y_λ is n-dimensional, $\Phi_s^\lambda(Y_\lambda)$ is also n-dimensional, and, therefore, it is Lagrangian. ☐

At this point, we would like to relate the crossings of the path $\{\Phi_\lambda^s(Y_\lambda)\}$ to eigenvalues of differential operators H_s introduced in (2.5). We remark that $y \in \ker(H_s - \lambda I)$ if and only if the vector valued function $\mathbf{p}$ is a solution of (1.5) on $[0, s]$ that satisfies the boundary conditions $\alpha_1 p(0) + \alpha_2 q(0) = 0$ and $\beta_1 p(s) + \beta_2 q(s) = 0$. In addition, let H_s^D denote the operator H_s with Dirichlet boundary conditions (i.e., $\begin{bmatrix}\beta_1 & \beta_2\end{bmatrix} = \begin{bmatrix} I_n & 0\end{bmatrix}$).

As discussed in Section 2, we proceed by associating each Lagrangian subspace $\ell(s,\lambda)$ with a matrix $U_{s,\lambda} \in \mathfrak{U}_J$. In particular, $U_{s,\lambda}$ should map $\ell_1^\perp$ to $\ell(s,\lambda)$. In terms of frames, this asserts that

$$\mathbf{X}(s,\lambda) = U(s,\lambda)\begin{bmatrix}\beta_1^t \\ \beta_2^t\end{bmatrix},$$

where we will need to scale $\mathbf{X}$ to ensure that $U_{s,\lambda}$ is unitary (see below). According to our condition $UJ = JU$, we know that U must have the form

$$U = \begin{pmatrix} U_{11} & -U_{21} \\ U_{21} & U_{22}\end{pmatrix},$$

allowing us to express the relationship for U as

$$\begin{bmatrix} X^t \\ Z^t \end{bmatrix} = \begin{pmatrix} \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{pmatrix}\begin{bmatrix} U_{11}^t \\ U_{21}^t\end{bmatrix}.$$

In order to ensure the unitary normalization $U_{11}^t U_{11} + U_{21}^t U_{21} = I$, we note that we can choose the frame $\mathbf{X}$ to be $\binom{XM}{ZM}$ for any $n \times n$ invertible matrix M. With this choice, we find that U has the form

$$U = \begin{pmatrix} XM & -ZM \\ ZM & XM \end{pmatrix} \mathcal{B},$$

where

$$\mathcal{B} := \begin{bmatrix} \beta_1 & \beta_2 \\ -\beta_2 & \beta_1 \end{bmatrix},$$

and we must have

$$M^t X^t X M + M^t Z^t Z M = I$$
$$M^t X^t Z M - M^t Z^t X M = 0.$$

We will check below that the choices $M = (X^t X + Z^t Z)^{-1/2}$ and $M = X^{-1}(I + M_D^2)^{-1/2}$, where $M_D = ZX^{-1}$ can both be effective. (As discussed in [39] M_D is the Weyl-Titchmarsh function associated with H_s^D.)

For the following calculations we will find it convenient to define two matrices

$$\mathbb{M}(s,\lambda) := I + M_D^2(s,\lambda),$$
$$\mathbb{X}(s,\lambda) := X^t(s,\lambda)X(s,\lambda) + Z^t(s,\lambda)Z(s,\lambda).$$

Lemma 3.3. *The matrix $M_D = ZX^{-1}$ is symmetric whenever X is invertible. Moreover, we have the relations*

$$X\mathbb{X}^{-1}X^t + Z\mathbb{X}^{-1}Z^t = I_n,$$
$$Z\mathbb{X}^{-1}X^t - X\mathbb{X}^{-1}Z^t = 0_n,$$

as well as the commutation

$$\mathbb{M}^{-1/2} M_D = M_D \mathbb{M}^{-1/2}.$$

Proof. For symmetry, we observe that if X is invertible, we can write

$$ZX^{-1}X = (X^t)^{-1} X^t Z X^{-1} X.$$

Recalling the relation $X^t Z - Z^t X = 0$, and interchanging transpose with inverse, we find

$$(X^t)^{-1} X^t Z X^{-1} X = (X^{-1})^t Z^t X X^{-1} X = (X^{-1})^t Z^t X.$$

We see that $ZX^{-1} = (X^{-1})^t Z^t$; i.e., $M_D = M_D^t$.

For the last claim, we first assume $X(s,\lambda)$ and $Z(s,\lambda)$ are invertible. Then,

$$\begin{aligned} X\mathbb{X}^{-1}X^t + Z\mathbb{X}^{-1}Z^t &= X(X^tX + Z^tZ)^{-1}X^t + Z(X^tX + Z^tZ)^{-1}Z^t \\ &= ((X^t)^{-1}(X^tX + Z^tZ)X^{-1})^{-1} + ((Z^t)^{-1}(X^tX + Z^tZ)Z^{-1})^{-1} \\ &= (I + (X^t)^{-1}Z^tZX^{-1})^{-1} + ((Z^t)^{-1}X^tXZ^{-1} + I)^{-1} \\ &= (I + M_D^2)^{-1} + (M_D^{-2} + I)^{-1} \\ &= (I + M_D^2)^{-1} + M_D^2(I + M_D^2)^{-1} = I_n. \end{aligned} \tag{3.9}$$

Since $X\mathbb{X}^{-1}X^t + Z\mathbb{X}^{-1}Z^t$ is continuous with respect to (s,λ), (3.9) holds for any s and λ (i.e., even for pairs with $X(s,\lambda)$ not invertible). Similarly, one can check that

$$Z\mathbb{X}^{-1}X^t - X\mathbb{X}^{-1}Z^t = 0.$$

In order to see the commutation relation, we note that the relation

$$M_D \mathbb{M} = \mathbb{M} M_D$$

is trivial and leads immediately to

$$\mathbb{M}^{-1} M_D = M_D \mathbb{M}^{-1}.$$

The claim now follows from the general observation that if A is positive definite and $AB = BA$ then $A^{1/2}B = BA^{1/2}$ and $BA^{-1/2} = A^{-1/2}B$. $\square$

We will identify two choices of unitary matrix $U_{s,\lambda}$, which will be specified in terms of the matrices

$$\begin{aligned} \mathcal{M}_D &:= \begin{bmatrix} \mathbb{M}^{-1/2} & -\mathbb{M}^{-1/2} M_D \\ \mathbb{M}^{-1/2} M_D & \mathbb{M}^{-1/2} \end{bmatrix} \\ \mathcal{X}_D &:= \begin{bmatrix} X\mathbb{X}^{-1/2} & -Z\mathbb{X}^{-1/2} \\ Z\mathbb{X}^{-1/2} & X\mathbb{X}^{-1/2} \end{bmatrix}. \end{aligned} \tag{3.10}$$

Lemma 3.4. *Suppose* $\mathbf{X}(s,\lambda) = \binom{X(s,\lambda)}{Z(s,\lambda)}$ *is any frame for the Lagrangian subspace* $\Phi_s^\lambda(Y_\lambda)$*. Then*

$$U_{s,\lambda} = \mathcal{M}_D \mathcal{B}$$

is unitary in $\mathbb{R}^{2n}_J$ *and satisfies* $\Phi_s^\lambda(Y_\lambda) = U_{s,\lambda}(\ell_1^\perp)$ *for all* $\lambda \in \mathbb{R} \backslash \sigma(H_s^D)$*, and*

$$Q_{s,\lambda} := \mathcal{X}_D \mathcal{B}$$

is unitary in $\mathbb{R}^{2n}_J$ *and satisifies the same relation for all* $\lambda \in \mathbb{R}$*.*

Proof. First, using (1.3) and (1.4), we see that

$$\mathcal{B}\mathcal{B}^t = \begin{bmatrix} \beta_1\beta_1^t + \beta_2\beta_2^t & -\beta_1\beta_2^t + \beta_2\beta_1^t \\ -\beta_2\beta_1^t + \beta_1\beta_2^t & \beta_2\beta_2^t + \beta_1\beta_1^t \end{bmatrix} = \begin{bmatrix} I_n & 0_n \\ 0_n & I_n \end{bmatrix}.$$

We can now readily check that $U_{s,\lambda}$ is unitary on $\mathbb{R}^{2n}$. We compute

$$\begin{aligned} U_{s,\lambda}U_{s,\lambda}^t &= \mathcal{M}_D \mathcal{B}\mathcal{B}^t \mathcal{M_D}^t = \mathcal{M}_D \mathcal{M_D}^t \\ &= \begin{bmatrix} (I+M_D^2)^{-1} + (I+M_D^2)^{-1}M_D^2 & M_D(I+M_D^2)^{-1} - M_D(I+M_D^2)^{-1} \\ M_D(I+M_D^2)^{-1} - M_D(I+M_D^2)^{-1} & (I+M_D^2)^{-1}M_D^2 + (I+M_D^2)^{-1} \end{bmatrix} = I_{2n}. \end{aligned}$$

Note that we used the fact that M_D is symmetric. Similarly, $U_{s,\lambda}^t U_{s,\lambda} = I_{2n}$, and it is also easy to check that $U_{s,\lambda} J = J U_{s,\lambda}$.

For $Q_{s,\lambda}$, we proceed as with $U_{s,\lambda}$ to find

$$\begin{aligned} Q_{s,\lambda}Q_{s,\lambda}^t &= \tilde{\mathcal{M}}_D \tilde{\mathcal{M}}_D^t \\ &= \begin{bmatrix} X\mathbb{X}^{-1}X^t + Z\mathbb{X}^{-1}Z^t & X\mathbb{X}^{-1}Z^t - Z\mathbb{X}^{-1}X^t \\ Z\mathbb{X}^{-1}X^t - X\mathbb{X}^{-1}Z^t & Z\mathbb{X}^{-1}Z^t + X\mathbb{X}^{-1}X^t \end{bmatrix} = I_{2n}. \end{aligned}$$

Proceeding similarly, we can show that $Q_{s,\lambda}^t Q_{s,\lambda} = I_{2n}$.

In order to check the relation $\Phi_s^\lambda(Y_\lambda) = U_{s,\lambda}(\ell_1^\perp)$, let $u \in \ell_1^\perp$, so that $u = (\beta_1^t x, \beta_2^t x)^\top$ for some $x \in \mathbb{R}^n$. Therefore,

$$U_{s,\lambda}u = ((I+M^2(s,\lambda))^{-1/2}x, M(s,\lambda)(I+M^2(s,\lambda))^{-1/2}x)^\top = (Xy, Zy)^\top \in \Phi_s^\lambda(Y_\lambda),$$

where $y = X^{-1}(I + M^2(s, \lambda))^{-1/2}x$. Similarly,

$$Q_{s,\lambda}u = (X(X^tX + Z^tZ)^{-1/2}x, Z(X^tX + Z^tZ)^{-1/2}x)^\top = (Xy, Zy)^\top \in \Phi_s^\lambda(Y_\lambda).$$

where $y = (X^tX + Z^tZ)^{-1/2}x$ □

Remark 3.5. The matrices $U_{s,\lambda}$ and $Q_{s,\lambda}$ are in $\mathfrak{U}_J$, and as discussed in Section 2, can be associated with $n \times n$ complex-valued unitary matrices. To be precise, notice that we can express the matrix $U_{s,\lambda}$ as

$$U_{s,\lambda} = \begin{bmatrix} \mathbb{M}^{-1/2} & 0 \\ 0 & \mathbb{M}^{-1/2} \end{bmatrix} + J \begin{bmatrix} \mathbb{M}^{-1/2}M_D & 0 \\ 0 & \mathbb{M}^{-1/2}M_D \end{bmatrix},$$

which can be associated with the complex-valued $n \times n$ matrix

$$\tilde{U}_{s,\lambda} = \mathbb{M}^{-1/2} + i\mathbb{M}^{-1/2}M_D = \mathbb{M}^{-1/2}(I + iM_D).$$

Likewise, for $Q_{s,\lambda}$ we can write

$$Q_{s,\lambda} = \begin{bmatrix} X\mathbb{X}^{-1/2} & 0 \\ 0 & X\mathbb{X}^{-1/2} \end{bmatrix} + J \begin{bmatrix} Z\mathbb{X}^{-1/2} & 0 \\ 0 & Z\mathbb{X}^{-1/2} \end{bmatrix},$$

and we associate with this the complex-valued $n \times n$ matrix

$$\tilde{Q}_{s,\lambda} = X\mathbb{X}^{-1/2} + iZ\mathbb{X}^{-1/2} = (X + iZ)\mathbb{X}^{-1/2}.$$

We are now prepared to derive an expression for the matrix $W_{s,\lambda} = U_{s,\lambda}U_{s,\lambda}^T$ described in our definition of the Maslov index. We note at the outset that we can write

$$W_{s,\lambda} = U_{s,\lambda}U_{s,\lambda}^T = U_{s,\lambda}\tau_1U_{s,\lambda}^t\tau_1 = \mathcal{M}_D\mathcal{B}\tau_1\mathcal{B}^t\mathcal{M}_D^t\tau_1.$$

Lemma 3.6. *Under the assumptions of Lemma* 3.4*,*

$$W_{s,\lambda} = \mathcal{M}_D(s, \lambda)^2\mathfrak{B},$$

where

$$\mathfrak{B} = \begin{bmatrix} \beta_1^t\beta_1 - \beta_2^t\beta_2 & 2\beta_2^t\beta_1 \\ -2\beta_2^t\beta_1 & \beta_1^t\beta_1 - \beta_2^t\beta_2 \end{bmatrix}.$$

Proof. First, we would like to find $\tau_1 = 2\Pi_1 - I_{2n}$. It is clear that

$$\Pi_1 = \begin{bmatrix} \beta_2^t\beta_2 & -\beta_2^t\beta_1 \\ -\beta_1^t\beta_2 & \beta_1^t\beta_1 \end{bmatrix}, \tag{3.11}$$

and we can check directly that $\Pi_1^t = \Pi_1$. Moreover, using (1.3) and (1.4), we obtain that

$$\begin{aligned} \Pi_1^2 &= \begin{bmatrix} \beta_2^t\beta_2 & -\beta_2^t\beta_1 \\ -\beta_1^t\beta_2 & \beta_1^t\beta_1 \end{bmatrix} \begin{bmatrix} \beta_2^t\beta_2 & -\beta_2^t\beta_1 \\ -\beta_1^t\beta_2 & \beta_1^t\beta_1 \end{bmatrix} \\ &= \begin{bmatrix} \beta_2^t\beta_2\beta_2^t\beta_2 + \beta_2^t\beta_1\beta_1^t\beta_2 & -\beta_2^t\beta_2\beta_2^t\beta_1 - \beta_2^t\beta_1\beta_1^t\beta_1 \\ -\beta_1^t\beta_2\beta_2^t\beta_2 - \beta_1^t\beta_1\beta_1^t\beta_2 & \beta_1^t\beta_2\beta_2^t\beta_1 + \beta_1^t\beta_1\beta_1^t\beta_1 \end{bmatrix} \\ &= \begin{bmatrix} \beta_2^t\beta_2 & -\beta_2^t\beta_1 \\ -\beta_1^t\beta_2 & \beta_1^t\beta_1 \end{bmatrix} = \Pi_1. \end{aligned}$$

Also,

$$\Pi_1 \begin{bmatrix} -\beta_2^t x \\ \beta_1^t x \end{bmatrix} = \begin{bmatrix} \beta_2^t\beta_2 & -\beta_2^t\beta_1 \\ -\beta_1^t\beta_2 & \beta_1^t\beta_1 \end{bmatrix} \begin{bmatrix} -\beta_2^t x \\ \beta_1^t x \end{bmatrix} = \begin{bmatrix} -\beta_2^t\beta_2\beta_2^t x - \beta_2^t\beta_1\beta_1^t x \\ \beta_1^t\beta_2\beta_2^t x + \beta_1^t\beta_1\beta_1^t x \end{bmatrix} = \begin{bmatrix} -\beta_2^t x \\ \beta_1^t x \end{bmatrix},$$

and

$$\Pi_1 \begin{bmatrix} \beta_1^t x \\ \beta_2^t x \end{bmatrix} = \begin{bmatrix} \beta_2^t\beta_2 & -\beta_2^t\beta_1 \\ -\beta_1^t\beta_2 & \beta_1^t\beta_1 \end{bmatrix} \begin{bmatrix} \beta_1^t x \\ \beta_2^t x \end{bmatrix} = \begin{bmatrix} \beta_2^t\beta_2\beta_1^t x - \beta_2^t\beta_1\beta_2^t x \\ -\beta_1^t\beta_2\beta_1^t x + \beta_1^t\beta_1\beta_2^t x \end{bmatrix} = 0.$$

Therefore,

$$\tau_1 = 2\Pi_1 - I_{2n} = \begin{bmatrix} 2\beta_2^t\beta_2 - I & -2\beta_2^t\beta_1 \\ -2\beta_1^t\beta_2 & 2\beta_1^t\beta_1 - I \end{bmatrix}. \tag{3.12}$$

Computing directly, we find

$$\begin{aligned} \mathcal{B}\tau_1\mathcal{B}^t &= \begin{bmatrix} \beta_1 & \beta_2 \\ -\beta_2 & \beta_1 \end{bmatrix} \begin{bmatrix} 2\beta_2^t\beta_2 - I & -2\beta_2^t\beta_2 \\ -2\beta_1^t\beta_2 & 2\beta_1^t\beta_1 - I \end{bmatrix} \begin{bmatrix} \beta_1^t & -\beta_2^t \\ \beta_2^t & \beta_1^t \end{bmatrix} \\ &= \begin{bmatrix} \beta_1 & \beta_2 \\ -\beta_2 & \beta_1 \end{bmatrix} \begin{bmatrix} -\beta_1^t & -\beta_2^t \\ -\beta_2^t & \beta_1^t \end{bmatrix} = \begin{bmatrix} -I_n & 0 \\ 0 & I_n \end{bmatrix}. \end{aligned}$$

We have, then,

$$\begin{aligned} \mathcal{B}\tau_1\mathcal{B}^t\mathcal{M}_D^t &= \begin{bmatrix} -I_n & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} \mathbb{M}^{-1/2} & \mathbb{M}^{-1/2}M_D \\ -\mathbb{M}^{-1/2}M_D & \mathbb{M}^{-1/2} \end{bmatrix} \\ &= \begin{bmatrix} -\mathbb{M}^{-1/2} & -\mathbb{M}^{-1/2}M_D \\ -\mathbb{M}^{-1/2}M_D & \mathbb{M}^{-1/2} \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{M}^{-1/2} & -\mathbb{M}^{-1/2}M_D \\ \mathbb{M}^{-1/2}M_D & \mathbb{M}^{-1/2} \end{bmatrix} \begin{bmatrix} -I_n & 0 \\ 0 & I_n \end{bmatrix}. \end{aligned}$$

In this way, we see that

$$\begin{aligned} \mathcal{M}_D\mathcal{B}\tau_1\mathcal{B}^t\mathcal{M}_D^t\tau_1 &= \mathcal{M}_D^2 \begin{bmatrix} -I_n & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 2\beta_2^t\beta_2 - I & -2\beta_2^t\beta_2 \\ -2\beta_1^t\beta_2 & 2\beta_1^t\beta_1 - I \end{bmatrix} \\ &= \mathcal{M}_D^2 \begin{bmatrix} \beta_1^t\beta_1 - \beta_2^t\beta_2 & 2\beta_2^t\beta_1 \\ -2\beta_2^t\beta_1 & \beta_1^t\beta_1 - \beta_2^t\beta_2 \end{bmatrix}. \end{aligned}$$

□

We observe that $\mathcal{M}_D$ is a unitary matrix in the form

$$\mathcal{M}_D = \begin{pmatrix} \mathbb{M}^{-1/2} & 0 \\ 0 & \mathbb{M}^{-1/2} \end{pmatrix} + J \begin{pmatrix} \mathbb{M}^{-1/2}M_D & 0 \\ 0 & \mathbb{M}^{-1/2}M_D \end{pmatrix},$$

and can be associated with the $n \times n$ complex unitary matrix

$$\tilde{\mathcal{M}}_D = \mathbb{M}^{-1/2} + i\mathbb{M}^{-1/2}M_D.$$

Likewise,

$$\mathfrak{B} = \begin{bmatrix} \beta_1^t\beta_1 - \beta_2^t\beta_2 & 2\beta_2^t\beta_1 \\ -2\beta_2^t\beta_1 & \beta_1^t\beta_1 - \beta_2^t\beta_2 \end{bmatrix} = \begin{bmatrix} \beta_1^t\beta_1 - \beta_2^t\beta_2 & 0 \\ 0 & \beta_1^t\beta_1 - \beta_2^t\beta_2 \end{bmatrix} + J \begin{bmatrix} -2\beta_2^t\beta_1 & 0 \\ 0 & -2\beta_2^t\beta_1 \end{bmatrix},$$

with corresponding complex matrix

$$\tilde{\mathfrak{B}} = (\beta_1^t \beta_1 - \beta_2^t \beta_2) - i2\beta_2^t \beta_1.$$

In this way, $W_{s,\lambda}$ corresponds with the complex $n \times n$ matrix

$$\tilde{W}_{s,\lambda} = \tilde{\mathcal{M}}_D^2 \tilde{\mathfrak{B}}.$$

Here,

$$\begin{aligned}
\tilde{\mathcal{M}}_D^2 &= (\mathbb{M}^{-1/2} + i\mathbb{M}^{-1/2} M_D)^2 \\
&= (I + M_D)^{-1}(I + iM_D)^2 = ((I + iM_D)(I - iM_D))^{-1}(I + iM_D)^2 \\
&= (I - iM_D)^{-1}(I + iM_D),
\end{aligned}$$

which is the standard Cayley transform of iM_D.

In the event that X is invertible, we find (using the definition of M_D) that

$$\tilde{\mathcal{M}}_D^2 = (X + iZ)(X - iZ)^{-1},$$

and more generally we can arrive at this form by repeating our calculations using Q in place of U. We conclude with the matrix we'll use for our Maslov index calculations,

$$\tilde{W}_{s,\lambda} = (X + iZ)(X - iZ)^{-1}((\beta_1^t \beta_1 - \beta_2^t \beta_2) - i2\beta_2^t \beta_1).$$

Remark 3.7. We are now in a position to indicate how the Sturm-Liouville oscillation theorem for $n = 1$ follows from Theorem 1.5. In this case (i.e., for $n = 1$) we have

$$\tilde{W}_{s,\lambda} = \frac{y(s;\lambda) + iy'(s;\lambda)}{y(s;\lambda) - iy'(s;\lambda)}(\beta_1^2 - \beta_2^2 - i2\beta_1\beta_2),$$

and for simplicity let's focus on the case in which we have Dirichet boundary conditions at both $x = 0$ and $x = 1$ (so that $\alpha_1, \beta_1 = 1$ and $\alpha_2, \beta_2 = 0$). In this case, we have a crossing at s^* (so that $\tilde{W}_{s^*,\lambda} = -1$) if and only if $y(s^*;\lambda) = 0$. We'll see in Section 3.4 that in this case crossings on S^1 must occur in the clockwise direction, and since $\tilde{W}_{0,\lambda} = -1$ (due to the Dirichlet condition at $x = 0$) we will have $\tilde{W}_{s_0,\lambda} = e^{i(\pi-\epsilon)}$ for s_0 sufficiently small (and some $\epsilon > 0$). The Principal Maslov Index will now be the negative of a count of the number of times $\tilde{W}_{s,0}$ crosses -1 as s goes from s_0 to 1. Moreover, each of these crossings will correspond with a zero of $y(s;0)$ (as noted above), and so we can conclude from Theorem 1.5 that the number of negative eigenvalues of H is precisely the number of zeros of y. (The standard Sturm-Liouville oscillation theorem for $n = 1$ requires $\lambda = 0$ to be an eigenvalue, but we clearly do not need that.) Other cases follow similarly.

Our final preliminary lemma addresses continuity of the path of Lagrangian subspaces $\{\ell(s,\lambda)\}_{(s,\lambda)\in\Gamma}$.

Lemma 3.8. *For system (1.1), let $V \in C([0,1])$ be a symmetric matrix in $\mathbb{R}^{n\times n}$, and let α_1, α_2, β_1, and β_2 be as in (1.2)-(1.3). Then the path of Lagrangian subspaces $\{\ell(s,\lambda)\}_{(s,\lambda)\in\Gamma}$ is continuous.*

Proof. Following [22] (p. 274), we specify our metric on the Lagrangian Grassmannian $\Lambda(n)$ in terms of orthogonal projections onto elements $\ell \in \Lambda(n)$. Precisely, let $\mathcal{P}_i$ denote the orthogonal projection matrix onto $\ell_i \in \Lambda(n)$ for $i = 1, 2$. We take our metric d on $\Lambda(n)$ to be defined by

$$d(\ell_1, \ell_2) := \|\mathcal{P}_1 - \mathcal{P}_2\|,$$

where $\|\cdot\|$ can denote any matrix norm.

For $\ell(s,\lambda)$, we have a frame $\mathbf{X}$, and it follows from elementary matrix theory that the associated orthogonal projection matrix $\mathcal{P}_{s,\lambda}$ satisfies $\mathcal{P}_{s,\lambda} = \mathbf{X}(\mathbf{X}^t\mathbf{X})^{-1}\mathbf{X}^t$. Computing directly, we find

$$\mathcal{P}_{s,\lambda} = \begin{pmatrix} X\mathbb{X}^{-1}X^t & X\mathbb{X}^{-1}X^t \\ Z\mathbb{X}^{-1}X^t & Z\mathbb{X}^{-1}Z^t. \end{pmatrix}$$

We see, then, that continuity of $\ell(s,\lambda)$ follows immediately from the continuity of $\mathcal{P}_{s,\lambda}$, which in turn follows from the continuity of solutions of (1.5) in x and λ. $\square$

3.1. Crossings on Γ_3 ($s = 1, \lambda \in [0, -\lambda_\infty]$).

In this section, we verify our claim in the introduction that along the top shelf Γ_3 the Maslov index is precisely the Morse index of H. The inverted interval $[0, -\lambda_\infty]$ indicates the direction of the path Γ_3.

Lemma 3.9. *Under the assumptions of Lemma* 3.8 *we have*

$$\operatorname{Mor}(H) = \operatorname{Mas}(\ell, \ell_1; \Gamma_3). \tag{3.13}$$

Proof. From Lemma 2.1, we know that $\dim(\Phi_s^\lambda(Y_\lambda) \cap \ell_1) = \dim\ker(\tilde{W}_{s,\lambda} + I)$ for $s = 1, \lambda \in [0, -\lambda_\infty]$. Assume that $\lambda^* \in [0, -\lambda_\infty]$ is a crossing, that is, $\Phi_1^{\lambda^*}(Y_{\lambda^*}) \cap \ell_1 \neq \{0\}$. Then there exists a solution of (1.1) such that the boundary conditions are satisfied. Therefore, λ^* is an eigenvalue of H. Moreover, since $\Phi_1^{\lambda^*}(Y_{\lambda^*})$ are the traces of weak solutions that satisfy the boundary condition at 0, $\dim(\ker(H - \lambda^* I)) = \dim(\Phi_L^{\lambda^*}(Y_{\lambda^*}) \cap \ell_1) = \dim\ker(\tilde{W}_{1,\lambda^*} + I)$.

Next, we would like to compute the Maslov index of the path $\left\{\Phi_1^\lambda(Y_{1,\lambda})\right\}_{\lambda=\lambda^*-\varepsilon}^{\lambda^*+\varepsilon}$, i.e., the net count of the eigenvalues of $\tilde{W}_{1,\lambda}$ crossing the point -1 as λ goes from $\lambda^* - \varepsilon$ to $\lambda^* + \varepsilon$. As a starting point, we differentiate $\tilde{W}_{s,\lambda}$ with respect to λ:

$$\begin{aligned}\frac{\partial}{\partial\lambda}\tilde{W}_{s,\lambda} &= (\dot{X} + i\dot{Z})(X - iZ)^{-1}\tilde{\mathfrak{B}} - (X + iZ)(X - iZ)^{-1}(\dot{X} - i\dot{Z})(X - iZ)^{-1}\tilde{\mathfrak{B}} \\ &= (\dot{X} + i\dot{Z})(X - iZ)^{-1}\tilde{\mathfrak{B}} - \tilde{W}_{s,\lambda}\tilde{\mathfrak{B}}^*(\dot{X} - i\dot{Z})(X - iZ)^{-1}\tilde{\mathfrak{B}},\end{aligned}$$

where $\dot{X}, \dot{Z}$ denote derivatives of X and Z with respect to λ, and we've used the fact that $\tilde{\mathfrak{B}}$ is unitary.

Now, we multiply both sides by $\tilde{W}_{s,\lambda}^*$

$$\begin{aligned}&\tilde{W}_{s,\lambda}^*\dot{\tilde{W}}_{s,\lambda} = \tilde{\mathfrak{B}}^*(X^t + iZ^t)^{-1}(X^t - iZ^t)(\dot{X} + i\dot{Z})(X - iZ)^{-1}\tilde{\mathfrak{B}} - \tilde{\mathfrak{B}}^*(\dot{X} - i\dot{Z})(X - iZ)^{-1}\tilde{\mathfrak{B}} \\ &= \tilde{\mathfrak{B}}^*(X^t + iZ^t)^{-1}[(X^t - iZ^t)(\dot{X} + i\dot{Z}) - (X^t + iZ^t)(\dot{X} - i\dot{Z})](X - iZ)^{-1}\tilde{\mathfrak{B}} \\ &= ((X - iZ)^{-1}\tilde{\mathfrak{B}})^*[2iX^t\dot{Z} - 2iZ^t\dot{X}]((X - iZ)^{-1}\tilde{\mathfrak{B}}).\end{aligned}$$

Multiplying on the left by $\tilde{W}_{s,\lambda}$, and recalling that $\tilde{W}_{s,\lambda}$ is unitary, we find

$$\dot{\tilde{W}}_{s,\lambda} = i\tilde{W}_{s,\lambda}\tilde{\Omega},$$

where

$$\tilde{\Omega} = 2((X-iZ)^{-1}\tilde{\mathfrak{B}})^*[X^t\dot{Z} - Z^t\dot{X}]((X-iZ)^{-1}\tilde{\mathfrak{B}}).$$

Let's take a close look at $X^t\dot{Z} - Z^t\dot{X}$. Taking an s derivative of this quantity, denoted with a prime, and using $(\dot{X})' = \dot{Z}$ and $(\dot{Z})' = (V-\lambda I)\dot{X} - X$, we find

$$\begin{aligned}(X^t\dot{Z} - Z^t\dot{X})' &= Z^t\dot{Z} + X^t((V-\lambda I)\dot{X} - X) - X^t(V-\lambda I)\dot{X} - Z^t\dot{Z} &(3.14)\\ &= -X^tX. &(3.15)\end{aligned}$$

After integration, we arrive at

$$X^t\dot{Z} - Z^t\dot{X} = -\int_0^s X^t(t,\lambda)X(t,\lambda)dt + X^t(0,\lambda)\frac{d}{d\lambda}Z(0,\lambda) - Z^t(0,\lambda)\frac{d}{d\lambda}X(0,\lambda).$$

In the current setting, $X(0,\lambda)$ and $Z(0,\lambda)$ are constant in λ, so that

$$X^t\dot{Z} - Z^t\dot{X} = -\int_0^s X^t(t,\lambda)X(t,\lambda)dt,$$

and

$$\tilde{\Omega} = -2((X-iZ)^{-1}\tilde{\mathfrak{B}})^* \int_0^s X^t(t,\lambda)X(t,\lambda)dt((X-iZ)^{-1}\tilde{\mathfrak{B}}).$$

It's clear that $\tilde{\Omega}$ is self-adjoint, and we also claim that it's negative definite. Indeed, if we temporarily set $A = (X-iZ)^{-1}\tilde{\mathfrak{B}}$ and $B = \int_0^s X^t(s,\lambda)X(s,\lambda)dt$ we see that $\tilde{\Omega} = -2A^*BA$, where B is positive definite (when X is invertible) and A is invertible. It follows immediately that $\tilde{\Omega}$ is positive definite.

Finally, we'll show in Lemma 3.11 below that under these conditions the eigenvalues of $\tilde{W}_{s,\lambda}$ move clockwise on the unit circle as λ increases from $\lambda^*-\varepsilon$ to $\lambda^*+\varepsilon$, or counterclockwise as λ decreases from $\lambda^*+\varepsilon$ to $\lambda^*-\varepsilon$. Therefore, $\mathrm{Mas}(\ell,\ell_1;\Gamma_3) = \dim\ker(\tilde{W}_{1,\lambda^*}+I)$, and so $\mathrm{Mas}(\ell,\ell_1;\Gamma_3) = \mathrm{Mor}(H)$. □

Remark 3.10. As discussed in [22], p. 307, the signature of $\tilde{\Omega}$ corresponds precisely with the signature of the crossing form associated with (ℓ,ℓ_1) at any intersection.

Lemma 3.11. *Let $W(\tau)$ be a smooth family of unitary $n\times n$ matrices on $[0,1]$ and satisfy a differential equation $\frac{d}{d\tau}W(\tau) = iW(\tau)\Omega(\tau)$, where $\Omega(\tau)$ is continuous, self-adjoint and negative-definite. Then the eigenvalues $W(\tau)$ move clockwise on the unit circle as τ increases.*

Proof. As a start, fix some $\tau_0 \in [0,1]$, and denote the eigenvalues of $W(\tau_0)$ by $\{\lambda_k(\tau_0)\}_{k=1}^n$. We claim that for τ near τ_0 we can express $W(\tau)$ as

$$W(\tau) = W(\tau_0)e^{iR(\tau)},$$

for some appropriate matrix $R(\tau)$. Indeed, we know $R(\tau)$ exists, because $W(\tau_0)^{-1}W(\tau)$ is invertible, and so has a logarithm. It's convenient to notice here that $R(\tau_0) = 0$.

Next, we compute $W'(\tau_0)$. For this, we write

$$W(\tau) = W(\tau_0) \sum_{j=1}^{\infty} \frac{i^j}{j!} R(\tau)^j,$$

so that

$$W'(\tau) = W(\tau_0) \sum_{j=1}^{\infty} \frac{i^j}{j!} \frac{d}{d\tau} R(\tau)^j.$$

Generally, we run into a commutation problem when computing derivatives of powers of matrices, but since $R(\tau_0) = 0$ we see that

$$\frac{d}{d\tau} R(\tau)^j \Big|_{\tau=\tau_0} = 0,$$

for $j = 2, 3, \dots$. In this way,

$$W'(\tau_0) = iW(\tau_0) R'(\tau_0),$$

and we recognize that $\Omega(\tau_0) = R'(\tau_0)$.

According to Theorem II.5.4 in [36], if $\Omega(\tau_0)$ is negative definite then the eigenvalues of $R(\tau_0)$, which we denote $\{r_k(\tau_0)\}_{k=1}^n$, are decreasing as τ increases at τ_0. By spectral mapping, the eigenvalues of $e^{iR(\tau_0)}$ are $\{e^{ir_k(\tau_0)}\}_{k=1}^n$.

At this point, we proceed similarly as in [22], p. 306. We fix any θ so that $e^{i\theta} \notin \{\lambda_k^*\}_{k=1}^n$, and set

$$A(\tau) := i(e^{i\theta} I - W(\tau))^{-1} (e^{i\theta} I + W(\tau)),$$

for τ near τ_0. Proceeding as in [22], we claim that

$$A'(\tau_0) = \Big((e^{i\theta} I - W(\tau_0))^{-1} \Big)^* 2R'(\tau_0) \Big(e^{i\theta} I - W(\tau_0)^{-1} \Big)$$

To see this, we compute

$$\begin{aligned}
A'(\tau) &= -i(e^{i\theta} I - W(\tau))^{-1} (-W'(\tau))(e^{i\theta} I - W(\tau))^{-1} (e^{i\theta} I + W(\tau)) \\
&\quad + i(e^{i\theta} I - W(\tau))^{-1} W'(\tau) \\
&= i(e^{i\theta} I - W(\tau))^{-1} W'(\tau) \Big\{ I + (e^{i\theta} I - W(\tau))^{-1} (e^{i\theta} I + W(\tau)) \Big\} \\
&= i(e^{i\theta} I - W(\tau))^{-1} W'(\tau) (e^{i\theta} I - W(\tau))^{-1} \Big\{ (e^{i\theta} I - W(\tau)) + (e^{i\theta} I + W(\tau)) \Big\} \\
&= i(e^{i\theta} I - W(\tau))^{-1} W'(\tau) (e^{i\theta} I - W(\tau))^{-1} 2e^{i\theta}.
\end{aligned}$$

Continuing, we see that

$$\begin{aligned} A'(\tau) &= i(e^{i\theta}I - W(\tau))^{-1}e^{i\theta}2W'(\tau)(e^{i\theta}I - W(\tau))^{-1} \\ &= i\Big(e^{-i\theta}(e^{i\theta}I - W(\tau))\Big)^{-1}2W'(\tau)(e^{i\theta}I - W(\tau))^{-1} \\ &= i(I - e^{-i\theta}W(\tau))^{-1}2W'(\tau)(e^{i\theta}I - W(\tau))^{-1} \\ &= i(I - e^{-i\theta}W(\tau))^{-1}2W(\tau)W(\tau)^{-1}W'(\tau)(e^{i\theta}I - W(\tau))^{-1} \\ &= i\Big(W(\tau)^{-1}(I - e^{-i\theta}W(\tau))\Big)^{-1}2W(\tau)^{-1}W'(\tau)(e^{i\theta}I - W(\tau))^{-1}. \end{aligned}$$

Now, we use the fact that $W(\tau)$ is unitary to see that

$$\begin{aligned} A'(\tau) &= i(W(\tau)^* - e^{-i\theta}I)^{-1}2W(\tau)^*W'(\tau)(e^{i\theta}I - W(\tau))^{-1} \\ &= i\Big((-W(\tau) + e^{i\theta}I)^{-1}\Big)^*(-2W(\tau)^*W'(\tau))\Big(e^{i\theta}I - W(\tau)\Big)^{-1}. \end{aligned}$$

Finally, recalling that $W'(\tau_0) = iW(\tau_0)R'(\tau_0)$, we see that $R'(\tau_0) = -iW(\tau_0)^*W'(\tau_0)$, giving the claim.

We see from (3.1) that $A'(\tau_0)$ is negative definite (since $R'(\tau_0)$ is). We conclude (again, from Theorem II.5.4 in [36]) that the eigenvalues of $A(\tau)$ are decreasing as τ increases at τ_0.

At this point, we would like to relate the motion of the eigenvalues of $A(\tau)$ (which we understand) to the motion of the eigenvalues of $W(\tau)$ (which determine the Maslov index). We denote the eigenvalues of $A(\tau)$ by $\{a_k(\tau)\}_{k=1}^n$ and recall that we are denoting the eigenvalues of $W(\tau)$ by $\{\lambda_k(\tau)\}_{k=1}^n$. By spectral mapping, we have (with an appropriate labeling scheme)

$$a_k(\tau) = i(e^{i\theta} - \lambda_k(\tau))^{-1}(e^{i\theta} + \lambda_k(\tau)),$$

from which we find

$$\lambda_k = -e^{i\theta}\frac{1 + ia_k}{1 - ia_k} = e^{i(\theta+\pi)}\frac{1 + ia_k}{1 - ia_k}.$$

In order to better understand this relationship, let b_k satisfy

$$e^{ib_k} = \frac{1 + ia_k}{1 - ia_k},$$

so that

$$b_k = \tan^{-1}\frac{2a_k}{1 - a_k^2}.$$

As a_k moves from $-\infty$ to -1, b_k corresponds with counterclockwise rotation along S^1 from $(-1,0)$ to $(0,-1)$. Likewise, as a_k moves from -1 to $+1$, b_k corresponds with rotation in the counterclockwise direction from $(0,-1)$ to $(0,1)$. Finally, as a_k moves from 1 to $+\infty$, b_k corresponds with rotation from $(0,1)$ to $(-1,0)$, closing a single full loop around S^1. Summarizing, we see that there is a monotonic relationship between the motion of a_k on $\mathbb{R}$ and the motion of e^{ib_k} on S^1. (This is a standard, well-known property of the Cayley Transform on $\mathbb{R}$.)

We see, then, that at any $\tau^* \in [0,1]$ $a_k(\tau)$ decreases through τ^*, and correspondingly $\lambda_k(\tau)$ rotates in the clockwise direction. Since τ^* is arbitrary, we conclude that the eigenvalues of $W(\tau)$ rotate monotonically clockwise as τ increases from 0 to 1. $\square$

3.2. **No crossings on Γ_4.** Associated with $H(s)$, we introduce the operator family $L(s)$

$$L(s)u = (-\frac{d^2}{dx^2} + s^2 V(sx) - s^2\lambda)u,$$
$$\mathrm{dom}(L(s)) = \Big\{u \in H^2(0,1) : \alpha_1 u(0) + \frac{1}{s}\alpha_2 u'(0) = 0;\ \beta_1 u(1) + \frac{1}{s}\beta_2 u'(1) = 0\Big\}.$$

We would like to show that there are no crossings on Γ_4 provided $\lambda_\infty = \lambda_\infty(s_0)$ is large enough.

Lemma 3.12. *Suppose $V \in C([0,1]; \mathbb{R}^n \times \mathbb{R}^n)$ is symmetric. For each $s_0 \in (0,1]$ there exists a positive $\lambda_\infty = \lambda_\infty(s_0)$ such that the path $\Phi_s^\lambda(Y_\lambda)$ has no crossings for any fixed $\lambda \in (-\infty, -\lambda_\infty]$ as s changes from s_0 to 1. In particular, the path $\Phi_s^\lambda(Y_\lambda)$ has no crossings on Γ_4.*

Proof. It is enough to show that for each $s_0 \in (0,1]$ there exists a positive $\lambda_\infty = \lambda_\infty(s_0)$ such that $0 \notin \mathrm{Spec}(L(s))$ for any $s \in [s_0,1]$ and $\lambda \in (-\infty, -\lambda_\infty]$. In fact, we will show that the operator $L(s)$ is positive-definite for any $s \in [s_0,1]$ and $\lambda \in (-\infty, -\lambda_\infty]$.

Fix $s_0 \in (0,1]$, and let $u \in \mathrm{dom}(L(s))$. We take an inner product (in $L^2(0,1)$) of $L(s)u$ with u and integrate by parts:

$$\langle L(s)u, u\rangle_{L^2(0,1)} = \|u'\|^2_{L^2(0,1)} + s^2\langle (V(sx) - \lambda)u, u\rangle_{L^2(0,1)} - (u(1), u'(1))_{\mathbb{R}^n} + (u(0), u'(0))_{\mathbb{R}^n}. \tag{3.16}$$

For the boundary terms, we follow a calculation from p. 21 of [8], and write

$$\begin{aligned}(u(1), u'(1))_{\mathbb{R}^n} &= ((P_{D_1} + P_{N_1} + P_{R_1})u(1), u'(1))_{\mathbb{R}^n}\\ &= (P_{D_1}u(1), u'(1))_{\mathbb{R}^n} + (P_{N_1}u(1), u'(1))_{\mathbb{R}^n} + (P_{R_1}u(1), u'(1))_{\mathbb{R}^n}\\ &= (u(1), P_{N_1}u'(1))_{\mathbb{R}^n} + (P_{R_1}^2 u(1), u'(1))_{\mathbb{R}^n}\\ &= (P_{R_1}u(1), P_{R_1}u'(1))_{\mathbb{R}^n} = (P_{R_1}u(1), s\Lambda_1 P_{R_1}u(1))_{\mathbb{R}^n}\\ &= s(P_{R_1}\Lambda_1 P_{R_1}u(1), u(1))_{\mathbb{R}^n}.\end{aligned}$$

Proceeding similarly for $(u(0), u'(0))_{\mathbb{R}^n}$ we see that

$$\begin{aligned}-(u(1), u'(1))_{\mathbb{R}^n} + (u(0), u'(0))_{\mathbb{R}^n} &= -s(P_{R_1}\Lambda_1 P_{R_1}u(1), u(1))_{\mathbb{R}^n} + s(P_{R_0}\Lambda_0 P_{R_0}u(0), u(0))_{\mathbb{R}^n}\\ &= -s(\mathcal{P}\gamma_D u, \gamma_D u)_{\mathbb{R}^{2n}},\end{aligned}$$

where

$$\mathcal{P} = \begin{pmatrix} -P_{R_0}\Gamma_0 P_{R_0} & 0 \\ 0 & P_{R_1}\Gamma_1 P_{R_1}\end{pmatrix},$$

and γ_D will denote the Dirichlet trace $\gamma_D u = \binom{u(0)}{u(1)}$.

Let $c_B > 0$ be large enough so that

$$|(\mathcal{P}\gamma_D u, \gamma_D u)_{\mathbb{R}^{2n}}| \le c_B \|\gamma_D u\|^2_{\mathbb{R}^2 n},$$

and also notice that given any $\epsilon > 0$ there is a corresponding $\beta(\epsilon)$ so that

$$\|\gamma_D u\|^2_{\mathbb{R}^{2n}} \leq \epsilon \|u'\|^2_{L^2(0,1)} + \beta(\epsilon)\|u\|^2_{L^2(0,1)}.$$

(See, e.g., [8] Lemma 1.3.8.) In this way, we see that

$$\begin{aligned} -sc_B \|\gamma_D u\|^2_{\mathbb{R}^n} &\geq -sc_B \Big(\beta(\epsilon)\|u\|^2_{L^2(0,1)} + \epsilon\|u'\|^2_{L^2(0,1)}\Big) \\ &\geq -c_B \Big(\beta(\epsilon)\|u\|^2_{L^2(0,1)} + \epsilon\|u'\|^2_{L^2(0,1)}\Big), \end{aligned}$$

where the second inequality uses $s \in (0,1]$.

Choose $\epsilon > 0$ small enough so that $c_B \epsilon < 1$ and set

$$\lambda_\infty := \|V\|_{L^\infty(0,1)} + (1 + c_B \beta(\epsilon)) s_0^{-2}.$$

Then,

$$\begin{aligned} s^2 \langle (V(sx) - \lambda)u, u\rangle_{L^2(0,1)} &= s^2 \Big(\langle V(sx)u, u\rangle_{L^2(0,1)} - \lambda \|u\|^2_{L^2(0,1)} \Big) \\ &\geq s^2 \Big(-\|V\|_{L^\infty(0,1)} + \lambda_\infty \Big) \|u\|^2_{L^2(0,1)}. \end{aligned}$$

Combining these observations, we find

$$\begin{aligned} \langle L(s)u, u\rangle_{L^2(0,1)} &\geq \|u'\|^2_{L^2(0,1)} + (-s^2\|V\|_{L^\infty(0,1)} + s^2\lambda_\infty)\|u\|^2_{L^2(0,1)} \\ &\quad - c_B\epsilon \|u'\|^2_{L^2(0,1)} - c_B\beta(\epsilon)\|u\|^2_{L^2(0,1)} \\ &= (1 - c_B\epsilon)\|u'\|^2_{L^2(0,1)} + \Big(s^2\lambda_\infty - c_B\beta(\epsilon) - s^2\|V\|_{L^\infty(0,1)}\Big)\|u\|^2_{L^2(0,1)}. \end{aligned}$$

We have

$$\begin{aligned} s^2\lambda_\infty - c_B\beta(\epsilon) - s^2\|V\|_{L^\infty(0,1)} &= s^2\|V\|_{L^\infty(0,1)} + \frac{s^2}{s_0^2}(1 + c_B\beta(\epsilon)) - c_B\beta(\epsilon) - s^2\|V\|_{L^\infty(0,1)} \\ &= \frac{s^2}{s_0^2}(1 + c_B\beta(\epsilon)) - c_B\beta(\epsilon) \geq 1, \end{aligned}$$

where in obtaining the final inequality we've observed $s > s_0 > 0$.

We conclude that

$$\langle L(s)u, u\rangle_{L^2(0,1)} \geq (1 - c_B\epsilon)\|u'\|^2_{L^2(0,1)} + \|u\|^2_{L^2(0,1)},$$

from which we see that for $\lambda \leq -\lambda_\infty$, $L(s)$ is positive definite. $\square$

3.3. **Crossings on Γ_1. Asymptotic expansions as $s \to 0$.** Our goal in this section is to show that the Maslov index along Γ_1 can be expressed as

$$\operatorname{Mas}(\ell, \ell_1; \Gamma_1) = -\operatorname{Mor}(H(s_0)) = -\operatorname{Mor}(B) - \operatorname{Mor}(Q(V(0) - (P_{R_0}\Lambda_0 P_{R_0})^2)Q),$$

where B and Q are as in Theorem 1.5. For this discussion, we work with the operator $H(s)$, defined in 2.5, and with the domain

$$\operatorname{dom}(H(s)) = \Big\{u \in H^2(0,1) : \alpha_1 u(0) + \frac{1}{s}\alpha_2 u'(0) = 0;\ \beta_1 u(1) + \frac{1}{s}\beta_2 u'(1) = 0\Big\}.$$

Notice that $H(0) = -\frac{d^2}{dx^2}$, with

$$\mathrm{dom}(H(0)) = \big\{u \in H^2(0,1) : P_{D_i}u(i) = 0,\ P_{N_i}u'(i) = 0,\ P_{R_i}u'(i) = 0,\ i = 0,1\big\}.$$

If—as in the Dirichlet case—$H(0)$ does not have zero as an eigenvalue, then there cannot be any crossings along Γ_1. On the other hand, if zero is an eigenvalue of $H(0)$—as, for example, in the Neumann-based cases—there will be an associated family of eigenvalues of $H(s)$ for small s. Our ultimate goal is an asymptotic formula for the eigenvalues of $H(s)$ that bifurcate from a zero eigenvalue of $H(0)$ as $s \to 0$. As a start, we characterize the eigenspace associated with the zero eigenvalue.

Lemma 3.13. *For $H(0)$ as defined above, zero is an eigenvalue of $H(0)$ if and only if $(\ker P_{D_0}) \cap (\ker P_{D_1}) \neq \{0\}$. Moreover, if zero is an eigenvalue of $H(0)$ then the eigenspace associated with zero is precisely the set of constant vectors characterized by this intersection. I.e.,*

$$\ker H(0) = (\ker P_{D_0}) \cap (\ker P_{D_1}).$$

Proof. It's clear that solutions of $H(0) = 0$ have the form

$$u(x) = ax + b, \quad a, b \in \mathbb{R}^n.$$

According to our boundary conditions, we have $P_{D_0}b = 0$, $P_{N_0}a = 0$, $P_{R_0}a = 0$, $P_{D_1}(a+b) = 0$, $P_{N_1}a = 0$, and $P_{R_1}a = 0$. Since $P_{D_0} + P_{N_0} + P_{R_0} = I$ (and similarly for the right boundary condition), we see that $(I - P_{D_0})a = 0$ and $(I - P_{D_1})a = 0$.

We have, then,

$$(a,b)_{\mathbb{R}^n} = (a, P_{D_0}b + (I - P_{D_0})b)_{\mathbb{R}^n} = (a, (I - P_{D_0})b)_{\mathbb{R}^n} = ((I - P_{D_0})a, b)_{\mathbb{R}^n} = 0,$$

and similarly $(a, a+b)_{\mathbb{R}^n} = 0$. It follows immediately that $|a|^2 = 0$, so that $a = 0$ and $u(x) = b$. Finally, we see that since $a = 0$ we must have both $P_{D_0}b = 0$ and $P_{D_1}b = 0$, and also that if these conditions are satisfied for $b \neq 0$ then zero is certainly an eigenvalue of $H(0)$. □

Remark 3.14. In what follows, we generally won't introduce any notation to distinguish between $(\ker P_{D_0}) \cap (\ker P_{D_1})$ as a subspace of $\mathbb{C}^n$ or a subspace of $L^2(0,1)$. We will denote the dimension of this intersection by d. I.e., $d = \dim[(\ker P_{D_0}) \cap (\ker P_{D_1})]$.

Now, Consider the sesquilinear form $h(s)$ on $L^2(0,1)$, defined for $s \in [0,1]$ by

$$\begin{aligned} h(s)(u,v) &= \langle u', v'\rangle_{L^2(0,1)} + s^2\langle V(sx)u, v\rangle_{L^2(0,1)} - s(\mathcal{P}\gamma_D u, \gamma_D u)_{\mathbb{C}^{2n}}, \\ \mathrm{dom}(h(s)) &= \{(u,v) \in H^1(0,1) \times H^1(0,1) : P_{D_i}u(i) = 0, P_{D_i}v(i) = 0\}, \end{aligned} \tag{3.17}$$

where γ_D is defined in the proof of Lemma 3.12. (See Theorem 1.4.11 in [8] for a discussion of why $h(s)$ with the domain specified here is that natural quadratic form to associate with $H(s)$.)

Remark 3.15. Notice that at this point we begin working with complex inner products in anticipation of employing complex analytic tools, including especially Riesz projections. We keep in mind that even though complex-valued functions and vectors are now allowed, all inner products will ultimately be evaluated at real-valued functions and vectors.

Following the general discussion of holomorphic families of closed, unbounded operators in [36, Section VII.1.2], we introduce our next definition.

Definition 3.16. A family of closed, not necessarily bounded, operators $\{T(s)\}_{s\in\Sigma}$ on a Hilbert space $\mathcal{X}$ is said to be continuous on an interval $\Sigma_0 \subset \Sigma$ if there exists a Hilbert space $\mathcal{X}'$ and continuous families of operators $\{U(s)\}_{s\in\Sigma_0}$ and $\{W(s)\}_{s\in\Sigma_0}$ in $\mathcal{B}(\mathcal{X}', \mathcal{X})$ such that $U(s)$ is a one-to-one map of $\mathcal{X}'$ onto $\mathrm{dom}(T(s))$ and the identity $T(s)U(s) = W(s)$ holds for all $s \in \Sigma_0$.

Before applying this definition, we recall that the continuity of V implies

$$\sup_{x\in[0,1]} \|V(sx) - V(s_0x)\|_{\mathbb{R}^{n\times n}} \to 0 \text{ as } s \to s_0 \text{ for any } s_0 \in [0,1]. \tag{3.18}$$

Lemma 3.17. *Assume $V \in C([0,1])$ is a symmetric matrix in $\mathbb{R}^{n\times n}$. Then the family $\{H(s)\}_{s\in[0,1]}$ is continuous near 0; that is, on some interval Σ_0 that contains 0.*

Proof. We notice that formally we can write

$$H(s)(H(s) + I)^{-1} = I - (H(s) + I)^{-1}. \tag{3.19}$$

In this way it is sufficient to establish that $U(s) := (H(s) + I)^{-1}$ is a continuous family of operators.

First, we note that it's clear from our construction of h that we have the identity

$$h(s)(u, v) = \langle H(s)u, v\rangle_{L^2(0,1)},$$

for all $s \in [0,1]$. In particular, we have

$$\langle (H(0) + I)u, u\rangle_{L^2(0,1)} = (h(0) + 1)(u, u) = \|u'\|^2_{L^2(0,1)} + \|u\|^2_{L^2(0,1)},$$

where we have used the convenient operator notation $1(u, u) = \|u\|^2_{L^2(0,1)}$. It follows that the operator $(H(0) + I)$ is self-adjoint, invertible and positive definite, with a well-defined square root, which we denote

$$G := (H(0) + I)^{1/2}; \quad G : \mathrm{dom}(h(s)) \to L^2(0,1).$$

We notice that for any $u \in \mathrm{dom}(H(s))$ we have

$$\begin{aligned} \|Gu\|^2_{L^2(0,1)} &= \langle Gu, Gu\rangle_{L^2(0,1)} = \langle G^2u, u\rangle_{L^2(0,1)} \\ &= \langle H(0)u + u, u\rangle_{L^2(0,1)} = \|u'\|^2_{L^2(0,1)} + \|u\|^2_{L^2(0,1)} = \|u\|^2_{H^1(0,1)}, \end{aligned} \tag{3.20}$$

from which we conclude that G is an (invertible) isometry.

Now, take any $u, v \in L^2(0,1)$ such that $\|u\|_{L^2(0,1)}, \|v\|_{L^2(0,1)} \le 1$, and compute

$$\big|(\mathcal{P}\gamma_D G^{-1}u, \gamma_D G^{-1}v)_{\mathbb{C}^{2n}}\big| \le C_1 \|\gamma_D G^{-1}u\|_{\mathbb{C}^{2n}} \|\gamma_D G^{-1}v\|_{\mathbb{C}^{2n}} \tag{3.21}$$

$$\le C_2 \|G^{-1}u\|_{H^1(0,1)} \|G^{-1}v\|_{H^1(0,1)} = C_2 \|u\|_{L^2(0,1)} \|v\|_{L^2(0,1)} \le C_2, \tag{3.22}$$

where we've used the observation from the proof of Lemma 3.12 that γ_D is bounded as a map from $H^1(0,1)$ to $\mathbb{C}^{2n}$.

We introduce a new sesquilinear form

$$\widetilde{h}(s)(u,v) := h(s)\left(G^{-1}u, G^{-1}v\right), \tag{3.23}$$

$$\operatorname{dom}(\widetilde{h}(s)) = L^2(0,1) \times L^2(0,1). \tag{3.24}$$

From (3.20) and (3.21) it is easy to see that $\widetilde{h}(s)$ is bounded on $L^2(0,1)\times L^2(0,1)$. Let $\widetilde{H}(s) \in \mathcal{B}(L^2(0,1))$ be the self-adjoint operator associated with $\widetilde{h}(s)$ by the First Representation Theorem [36, Theorem VI.2.1]. Then

$$\begin{aligned}\langle \widetilde{H}(s)u, v\rangle_{L^2(0,1)} &= \widetilde{h}(s)(u,v) \\ &= h(s)\left(G^{-1}u, G^{-1}v\right) \text{ for all } u, v \in L^2(0,1).\end{aligned} \tag{3.25}$$

Taking into account (3.21) and (3.18), we conclude that

$$\langle \widetilde{H}(s)u, v\rangle_{L^2(0,1)} \to \langle \widetilde{H}(s_0)u, v\rangle_{L^2(0,1)} \text{ as } s \to s_0$$

uniformly with respect to u and v satisfying $\|u\|_{L^2(0,1)}, \|v\|_{L^2(0,1)} \le 1$. Hence

$$\|\widetilde{H}(s) - \widetilde{H}(s_0)\|_{\mathcal{B}(L^2(0,1))} \to 0 \text{ as } s \to s_0, \tag{3.26}$$

which implies $\widetilde{H}(s) \in \mathcal{B}(L^2(0,1))$ is a continuous family on $[0,1]$.

Replacing u in (3.25) by Gu (and similarly for v), we conclude that

$$h(s)(u,v) = \left\langle \widetilde{H}(s)Gu, Gv\right\rangle_{L^2(0,1)}$$

for any $u, v \in \operatorname{dom}(h(s))$. Therefore, cf. [36, VII-(4.4), (4.5)], for all $u \in \operatorname{dom}(H(s))$

$$H(s)u = G\widetilde{H}(s)Gu, \tag{3.27}$$

when G is viewed as an unbounded, self-adjoint operator on $L^2(0,1)$. Adding I to both sides, we find

$$H(s) + I = G\widetilde{H}(s)G + I = G\Big(\widetilde{H}(s) + G^{-2}\Big)G.$$

Now, $H(0) + I = G^2$, so $G^2 = G(\tilde{H}(0) + G^{-2})G$, giving $\widetilde{H}(0) + G^{-2} = I$. We've seen that $\tilde{H}(s) \in \mathcal{B}(L^2(0,1))$ is a continuous family, and since $\tilde{H}(0) + G^{-2} = I$ it follows that $\widetilde{H}(s) + G^{-2}$ is boundedly invertible for s near 0. We conclude that near $s = 0$

$$(H(s) + I)^{-1} = G^{-1}\Big(\widetilde{H}(s) + G^{-2}\Big)^{-1}G^{-1}.$$

Thus $U(s) = (H(s) + I)^{-1}$ and $W(s) = I - U(s)$ are both continuous families near $s = 0$, and it is now clear that

$$H(s)U(s) = W(s) \text{ for } s \text{ near } 0. \tag{3.28}$$

Hence, (3.19) is justified, and according to Definition 3.16 the family $\{H(s)\}$ is continuous near 0. ☐

For $\zeta \in \mathbb{C} \setminus \sigma(H(s))$, we denote the resolvent

$$R(\zeta, s) = \big(H(s) - \zeta I\big)^{-1} \in \mathcal{B}(L^2(0,1)).$$

Lemma 3.18. *Let $\zeta \in \mathbb{C} \setminus \sigma(H(0))$. Then $\zeta \in \mathbb{C} \setminus \sigma(H(s))$ for s near 0. Moreover, the function $s \mapsto R(\zeta, s)$ is continuous for s near 0, uniformly for ζ in compact subsets of $\mathbb{C} \setminus \sigma(H(s))$.*

Proof. Let $\zeta \in \mathbb{C} \setminus \sigma(H(0))$. Since $H(s)U(s) = W(s)$, we have (for s near 0)

$$(H(s) - \zeta I)U(s) = W(s) - \zeta U(s). \tag{3.29}$$

The operator

$$W(0) - \zeta U(0) = (H(0) - \zeta I)U(0) = (H(0) - \zeta I)(H(0) + I)^{-1}$$

is a bijection of $L^2(0,1)$ onto $L^2(0,1)$ (because $H(0) + I$ and $H(0) - \zeta I$ are both boundedly invertible). By continuity, the operator $W(s) - \zeta U(s)$ is boundedly invertible for s near 0. This implies that $(H(s) - \zeta I)U(s)$ is boundedly invertible with inverse $U(s)^{-1}(H(s) - \zeta I)^{-1}$. In this way, we see that

$$(H(s) - \zeta)^{-1} = U(s)(W(s) - \zeta U(s))^{-1}, \tag{3.30}$$

the product of two bounded operators. Hence, $\zeta \in \mathbb{C} \setminus \sigma(H(s))$, and the function $s \mapsto R(\zeta, s)$ is continuous for s near 0 in the operator norm, uniformly in ζ. ☐

Our next lemma gives an asymptotic result for the difference of the resolvents of the operators $H(s)$ and $H(0)$ as $s \to 0$, which involves the value $V(0)$ of the potential at zero. We observe at the outset that since $R(\zeta, 0)$ is a bounded linear operator, it has a bounded linear adjoint (both on $L^2(0,1)$). Consider the composite map $\gamma_D R(\zeta, 0)^* : L^2(0,1) \to \mathbb{C}^{2n}$, which for any $u \in L^2(0,1)$, $z \in \mathbb{C}^{2n}$ satisfies

$$(z, \gamma_D R(\zeta, 0)^* u)_{\mathbb{C}^{2n}} = \langle (\gamma_D R(\zeta, 0)^*)^* z, u \rangle_{L^2(0,1)}. \tag{3.31}$$

Lemma 3.19. *If $\zeta \in \mathbb{C} \setminus \sigma(H(0))$ and $\|u\|_{L^2(0,1)} \leq 1$, then*

$$\begin{aligned} R(\zeta, s)u - R(\zeta, 0)u = s[\gamma_{_D} R(\zeta, 0)^*]^* \mathcal{P} \gamma_{_D} R(\zeta, 0)u - s^2 R(\zeta, 0)V(0)R(\zeta, 0)u \\ + s^2 [\gamma_{_D} R(\zeta, 0)^*]^* \mathcal{P} \gamma_{_D} [\gamma_{_D} R(\zeta, 0)^*]^* \mathcal{P} \gamma_{_D} R(\zeta, 0)u + r(s), \end{aligned} \tag{3.32}$$

where $\|r(s)\|_{L^2(0,1)} = \mathrm{o}(s^2)$ as $s \to 0$, uniformly for ζ in compact subsets of $\mathbb{C} \setminus \sigma(H(0))$ and $\|u\|_{L^2(0,1)} \leq 1$.

Proof. We recall that $\zeta \in \mathbb{C} \setminus \sigma(H(s))$ for s near 0 by Lemma 3.18, since $\zeta \in \mathbb{C} \setminus \sigma(H(0))$. For $\|u\|_{L^2(0,1)} \leq 1$, we set $w := R(\zeta, s)u - R(\zeta, 0)u$. Since $R(\zeta, 0) : L^2(0,1) \to \mathcal{D}(H(0))$, we see that for any $u \in L^2(0,1)$, $v \in H^1(0,1)$ we have

$$h(0)(R(\zeta, 0)u, v) = \langle H(0)R(\zeta, 0)u, v \rangle_{L^2(0,1)},$$

and likewise (since $H(0)$ is self-adjoint)

$$h(0)(v, R(\zeta, 0)^* u) = \langle v, H(0)R(\zeta, 0)^* u \rangle_{L^2(0,1)}.$$

In this way, we have

$$\begin{aligned} (h(0) - \zeta)(R(\zeta, 0)u, v) &= \langle (H(0) - \zeta I)R(\zeta, 0)u, v \rangle_{L^2(0,1)} \\ &= \langle u, v \rangle_{L^2(0,1)}, \end{aligned}$$

and likewise

$$\begin{aligned}(h(0)-\zeta)(v, R(\zeta,0)^*u) &= \langle v, (H(0)-\bar{\zeta}I)R(\zeta,0)^*u\rangle_{L^2(0,1)} \\ &= \langle v, u\rangle_{L^2(0,1)}.\end{aligned}$$

We compute

$$\begin{aligned}\langle w, v\rangle_{L^2(0,1)} &= (h(0)-\zeta)(w, R(\zeta,0)^*v) = (h(0)-\zeta)(R(\zeta,s)u - R(\zeta,0)u, R(\zeta,0)^*v) \\ &= (h(0)-\zeta)(R(\zeta,s)u, R(\zeta,0)^*v) - (h(0)-\zeta)(R(\zeta,0)u, R(\zeta,0)^*v).\end{aligned}$$

At this stage, we notice that

$$h(s)(u,v) = h(0)(u,v) + s^2\langle V(sx)u, v\rangle_{L^2(0,1)} - s(\mathcal{P}\gamma_D u, \gamma_D v)_{\mathbb{C}^{2n}}.$$

Using this, we can write

$$\begin{aligned}\langle w, v\rangle_{L^2(0,1)} &= (h(s)-\zeta)(R(\zeta,s)u, R(\zeta,0)^*v) - s^2\langle V(sx)R(\zeta,s)u, R(\zeta,0)^*v\rangle_{L^2(0,1)} \\ &\quad + s(\mathcal{P}\gamma_D R(\zeta,s)u, \gamma_D R(\zeta,0)^*v)_{\mathbb{C}^{2n}} - (h(0)-\zeta)(R(\zeta,0)u, R(\zeta,0)^*v).\end{aligned}$$

In this way, we obtain

$$\begin{aligned}\langle w, v\rangle_{L^2(0,1)} &= \langle (H(s)-\zeta I)R(\zeta,s)u, R(\zeta,0)^*v\rangle_{L^2(0,1)} - s^2\langle V(sx)R(\zeta,s)u, R(\zeta,0)^*v\rangle_{L^2(0,1)} \\ &\quad + s(\mathcal{P}\gamma_D R(\zeta,s)u, \gamma_D R(\zeta,0)^*v)_{\mathbb{C}^{2n}} - \langle (H(0)-\zeta I)R(\zeta,0)u, R(\zeta,0)^*v\rangle_{L^2(0,1)} \\ &= \langle u, R(\zeta,0)^*v\rangle_{L^2(0,1)} - s^2\langle V(sx)R(\zeta,s)u, R(\zeta,0)^*v\rangle_{L^2(0,1)} \\ &\quad + s(\mathcal{P}\gamma_D R(\zeta,s)u, \gamma_D R(\zeta,0)^*v)_{\mathbb{C}^{2n}} - \langle u, R(\zeta,0)^*v\rangle_{L^2(0,1)} \\ &= -s^2\langle V(sx)R(\zeta,s)u, R(\zeta,0)^*v\rangle_{L^2(0,1)} + s(\mathcal{P}\gamma_D R(\zeta,s)u, \gamma_D R(\zeta,0)^*v)_{\mathbb{C}^{2n}}.\end{aligned}$$

Using (3.31), we find

$$\langle w, v\rangle_{L^2(0,1)} = -s^2\langle R(\zeta,0)V(sx)R(\zeta,s)u, v\rangle_{L^2(0,1)} + s((\gamma_D R(\zeta,0)^*)^*\mathcal{P}\gamma_D R(\zeta,s)u, v)_{L^2(0,1)}.$$

Since this is true for all $v \in L^2(0,1)$, we have

$$w = -s^2 R(\zeta,0)V(sx)R(\zeta,s)u + s(\gamma_D R(\zeta,0)^*)^*\mathcal{P}\gamma_D R(\zeta,s)u,$$

and recalling the definition of w, we arrive at

$$R(\zeta,s)u = R(\zeta,0)u - s^2 R(\zeta,0)V(sx)R(\zeta,s)u + s(\gamma_D R(\zeta,0)^*)^*\mathcal{P}\gamma_D R(\zeta,s)u. \tag{3.33}$$

Replacing $R(\zeta,s)u$ in the right-hand side of (3.33) again by (3.33) yields

$$\begin{aligned}R(\zeta,s)u &= R(\zeta,0)u - s^2 R(\zeta,0)V(sx)\Big(R(\zeta,0)u - s^2 R(\zeta,0)V(sx)R(\zeta,s)u \\ &\qquad + s(\gamma_D R(\zeta,0)^*)^*\mathcal{P}\gamma_D R(\zeta,s)u\Big) \\ &\quad + s(\gamma_D R(\zeta,0)^*)^*\mathcal{P}\gamma_D\Big(R(\zeta,0)u - s^2 R(\zeta,0)V(sx)R(\zeta,s)u \\ &\qquad + s(\gamma_D R(\zeta,0)^*)^*\mathcal{P}\gamma_D R(\zeta,s)u\Big) \\ &= R(\zeta,0)u + s(\gamma_D R(\zeta,0)^*)^*\mathcal{P}\gamma_D R(\zeta,0)u - s^2 R(\zeta,0)V(0)R(\zeta,0)u \\ &\qquad + s^2(\gamma_D R(\zeta,0)^*)^*\mathcal{P}\gamma_D(\gamma_D R(\zeta,0)^*)^*\mathcal{P}\gamma_D R(\zeta,0)u + r(s),\end{aligned} \tag{3.34}$$

where

$$
\begin{aligned}
r(s) = &-s^2 R(\zeta,0)\big(V(sx)-V(0)\big)R(\zeta,0)u \\
&+ s^2(\gamma_{_D} R(\zeta,0)^*)^*\mathcal{P}\gamma_{_D}(\gamma_{_D} R(\zeta,0)^*)^*\mathcal{P}\gamma_{_D}\big(R(\zeta,s)u - R(\zeta,0)u\big) \\
&- s^3 R(\zeta,0)V(sx)(\gamma_{_D} R(\zeta,0)^*)^*\mathcal{P}\gamma_{_D} R(\zeta,s)u \\
&- s^3(\gamma_{_D} R(\zeta,0)^*)^*\mathcal{P}\gamma_{_D} R(\zeta,0)V(sx)R(\zeta,s)u \\
&+ s^4 R(\zeta,0)V(sx)R(\zeta,0)V(sx)R(\zeta,s)u.
\end{aligned}
\tag{3.35}
$$

Finally, we remark that $\|w\|_{L^2(0,1)} \to 0$ and $\|R(\zeta,s)\|_{\mathcal{B}(L^2(0,1))}$ is bounded as $s \to 0$ by Lemma 3.18 and thus, using (3.18) for $s_0 = 0$, we conclude that $\|r(s)\|_{L^2(0,1)} = \mathrm{o}(s^2)$ as $s \to 0$, uniformly for ζ in compact subsets of $\mathbb{C}\setminus\sigma(H(s))$ and $\|u\|_{L^2(0,1)} \le 1$. $\square$

We've already noted that $H(0)$ may have $\lambda = 0$ as an eigenvalue (for example, in the Neumann-based case), and our next goal is to understand the corresponding family of eigenvalues $\{\lambda_j(s)\}$, with $\lambda_j(0) = 0$. To begin, we will separate the spectrum of $H(s)$. First, we note that 0 is the only possible nonpositive eigenvalue in $\sigma(H(0))$. We would like to appeal to the continuity of eigenvalues with respect to s, but since $H(s)$ is unbounded we must take care with our argument. We will proceed by shifting the spectrum so that it lies entirely to the left of 0, and then inverting our operator to work with a resolvent (which will be bounded).

We clarify that in contrast with the setting of Lemma 3.12, we are concerned here with eigenvalues of $H(s)$ so that $H(s)u = \lambda u$ (i.e., the s^2 scaling from Lemma 3.12 does not appear on λ). Nonetheless, a calculation similar to the proof of Lemma 3.12 shows that any eigenvalue of $H(s)$ must satisfy

$$\lambda \ge -(\|V\|_{L^\infty(0,1)} + c_B\beta(\epsilon)),$$

for constants c_B and $\beta(\epsilon)$ that arise precisely as in the proof of Lemma 3.12. It's clear, then, that there exists a value $\Lambda > 0$ sufficiently large so that $-\Lambda/2 < \lambda$ for all $\lambda \in \sigma(H(s))$ and $s \in [0,1]$. By the spectral mapping theorem, we infer

$$\sigma\big((-\Lambda - H(s))^{-1}\big)\setminus\{0\} = \big\{(-\Lambda-\lambda)^{-1} : \lambda \in \sigma(H(s))\big\},\ s \in [0,1]. \tag{3.36}$$

In particular, if $0 \in \sigma(H(0))$, then $-1/\Lambda \in \sigma\big((-\Lambda - H(0))^{-1}\big)$.

Now fix a sufficiently small $\varepsilon \in (0, 1/(2\Lambda))$ such that the disc of radius 2ε centered at the point $-1/\Lambda$ does not contain any other eigenvalues in $\sigma\big((-\Lambda - H(0))^{-1}\big)$ except $-1/\Lambda$. Using Lemma 3.18 we know that $(-\Lambda - H(s))^{-1} \to (-\Lambda - H(0))^{-1}$ in $\mathcal{B}(L^2(0,1))$ as $s \to 0$. By the upper semicontinuity of the spectra of bounded operators, see, e.g., [36, Theorem IV.3.1], there exists a $\delta = \delta(\varepsilon)$ such that if $s \in [0,\delta]$, then

$$\sigma\big((-\Lambda - H(s))^{-1}\big) \subset \big\{\mu : \operatorname{dist}\big(\mu, \sigma\big((-\Lambda - H(0))^{-1}\big)\big) < \varepsilon\big\}. \tag{3.37}$$

In the remaining part of this section we take $s \le \delta$. Let $\{\nu_\ell(s)\}_{\ell=1}^{\tilde n} \subset \sigma\big((-\Lambda - H(s))^{-1}\big)$ denote the eigenvalues of $(-\Lambda - H(s))^{-1}$ which are located inside of the disc of radius ε centered at the point $-1/\Lambda$, and let $\lambda_\ell(s) = -\Lambda - 1/\nu_\ell(s)$ be the respective eigenvalues of $H(s)$. Let γ be a small circle centered at zero which encloses the eigenvalues $\lambda_\ell(s)$ for all $\ell = 1, \dots, \tilde n$ and $s \in [0,\delta]$ and separates them from the rest of the spectrum of $H(s)$.

By choosing ε sufficiently small, we can ensure that $\{\lambda_\ell(s)\}_{l=1}^{\tilde{n}}$ are precisely the eigenvalues bifurcating from $\lambda(0) = 0$, and also that γ separates $0 \in \sigma(H(0))$ from the rest of the spectrum of $H(0)$.

We denote by P_0 the orthogonal Riesz projection for $H(0)$ corresponding to the eigenvalue $0 \in \sigma(H(0))$, with $\operatorname{ran}(P_0) = \ker(H(0)) = (\ker P_{D_0}) \cap (\ker P_{D_1})$. (If $0 \notin \sigma(H(0))$ then $P_0 \equiv 0$.) Also, we let $\{P(s)\}_{s\in[0,\delta]}$ denote the family of Riesz spectral protections for $H(s)$ corresponding to the eigenvalues $\{\lambda_j(s)\}_{j=1}^d \subset \sigma(H(s))$, where d denotes the dimension of the subspace $(\ker P_{D_0}) \cap (\ker P_{D_1})$. That is,

$$P_0 = \frac{1}{2\pi i}\int_\gamma (\zeta - H(0))^{-1}\, d\zeta,\ P(s) = \frac{1}{2\pi i}\int_\gamma (\zeta - H(s))^{-1}\, d\zeta, \tag{3.38}$$

where γ encloses the set $\{\lambda_j(s)\}_{j=1}^d$.

Our objective is to establish an asymptotic formula for the eigenvalues $\lambda_j(s)$ as $s \to 0$ similar to [36, Theorem II.5.11], which is valid for families of bounded operators on finite-dimensional spaces. We stress that one cannot directly use a related result [36, Theorem VIII.2.9] for families of unbounded operators, as the s-dependence of $H(s)$ in our case is more complicated than allowed in the latter theorem. We are thus forced to mimic the main strategy of [36] in order to extend the relevant results to the family $\{H(s)\}_{s\in[0,\delta]}$.

Keeping in mind that our main goal for Γ_1 is to count the number of negative eigenvalues of the operator $H(s)$ for s near zero, we next establish the following claim.

Claim 3.20. For $s \in [0, \delta]$, the number of negative eigenvalues of $H(s)$ is equivalent to the number of negative eigenvalues of $H(s)P(s)$; that is, the restriction of $H(s)$ to the finite-dimensional subspace $\operatorname{ran}(P(s))$.

Proof. By the spectral mapping theorem (3.36), $\lambda < 0$ is in $\sigma(H(s))$ if and only if $(-\Lambda - \lambda)^{-1} < -1/\Lambda$. Thus for s near zero the negative eigenvalues of $H(s)$ are in one-to-one correspondence with the eigenvalues $\nu_j(s) \in \sigma\big((-\Lambda - H(s))^{-1}\big)$ that satisfy the inequality $\nu_j(s) < -1/\Lambda$, and therefore with the negative eigenvalues among $\lambda_j(s) \in \sigma\big(H(s)P(s)\big)$ as claimed. □

Next, we would like to work with a Neumann-type expansion for $R(\zeta, 0)$. From [36, Section III.6.5], we can write

$$R(\zeta, 0) = (-\zeta)^{-1}P_0 + \sum_{n=0}^{\infty} \zeta^n S^{n+1}, \tag{3.39}$$

where

$$S = \frac{1}{2\pi i}\int_\gamma \zeta^{-1} R(\zeta, 0)\, d\zeta \tag{3.40}$$

is the *reduced resolvent* for the operator $H(0)$ in $L^2(0, 1)$ (this uses equations (III.6.32) and (III.6.33) in [36]). Moreover, we have from [36] the useful relation $P_0S = SP_0 = 0$. (We'll say much more about the nature of the reduced resolvent at the end of this section.)

We introduce the notation

$$D(s) = P(s) - P_0 = -\frac{1}{2\pi i}\int_\gamma R(\zeta, s) - R(\zeta, 0) d\zeta, \tag{3.41}$$

and it's clear from Lemma 3.19 that this is $\mathbf{O}(s)$. This implies that $I - D(s)^2$ is strictly positive for s near 0, and following [36, Section I.4.6], we may introduce mutually inverse operators $U(s)$ and $U(s)^{-1}$ in $\mathcal{B}(L^2(0,1))$ as follows:

$$\begin{aligned} U(s) &= (I - D^2(s))^{-1/2}\big((I - P(s))(I - P_0) + P(s)P_0\big), \\ U(s)^{-1} &= (I - D^2(s))^{-1/2}\big((I - P_0)(I - P(s)) + P_0P(s)\big), \end{aligned} \tag{3.42}$$

for which

$$U(s)P_0 = P(s)U(s) \tag{3.43}$$

(equation (I.4.42) in [36]). We see that $U(s)$ is an isomorphism of the d-dimensional subspace $\mathrm{ran}(P_0)$ onto the subspace $\mathrm{ran}(P(s))$.

We isolate the main technical steps of our perturbation analysis in the following lemma, for which the statement and proof have been adapted with only minor changes from [15].

Lemma 3.21. *Let P_0 be the Riesz projection for $H(0)$ onto the subspace $\mathrm{ran}(P_0) = \ker(H(0))$ and $P(s)$ the respective Riesz projection for $H(s)$ from (3.38). Let S be the reduced resolvent for $H(0)$ defined in (3.40), and let the transformation operators $U(s)$ and $U(s)^{-1}$ be defined in (3.42). Then*

$$\begin{aligned} P_0U(s)^{-1}H(s)P(s)U(s)P_0 = &-s(\gamma_{_D} P_0)^*\mathcal{P}\gamma_{_D} P_0 + s^2P_0V(0)P_0 \\ &- s^2(\gamma_{_D} P_0)^*\mathcal{P}\gamma_{_D}(\gamma_{_D} S)^*\mathcal{P}\gamma_{_D} P_0 + \mathrm{o}(s^2) \ \text{as } s \to 0. \end{aligned} \tag{3.44}$$

Proof. We will split the proof into several steps.

Step 1. We first claim the following four asymptotic relations for $\zeta \in \gamma$:

$$R(\zeta, s)P_0 = (-\zeta)^{-1}P_0 + s(-\zeta)^{-1}(\gamma_{_D} R(\zeta,0)^*)^*\mathcal{P}\gamma_{_D} P_0 + \mathrm{o}(s)_u, \tag{3.45}$$

$$P_0R(\zeta, s) = (-\zeta)^{-1}P_0 + s(-\zeta)^{-1}(\gamma_{_D} P_0)^*\mathcal{P}\gamma_{_D} R(\zeta,0) + \mathrm{o}(s)_u, \tag{3.46}$$

$$\begin{aligned} P_0R(\zeta, s)P_0 = (-\zeta)^{-1}P_0 &+ s(-\zeta)^{-2}(\gamma_{_D} P_0)^*\mathcal{P}\gamma_{_D} P_0 - s^2(-\zeta)^{-2}P_0V(0)P_0 \\ &+ s^2(-\zeta)^{-2}(\gamma_{_D} P_0)^*\mathcal{P}\gamma_{_D}(\gamma_{_D} R(\zeta,0)^*)^*\mathcal{P}\gamma_{_D} P_0 + \mathrm{o}(s^2)_u, \end{aligned} \tag{3.47}$$

$$(I - P_0)R(\zeta, s)P_0 = s(-\zeta)^{-1}(I - P_0)(\gamma_{_D} R(\zeta,0)^*)^*\mathcal{P}\gamma_{_D} P_0 + \mathrm{o}(s)_u. \tag{3.48}$$

Here and below we write $\mathrm{o}(s^\alpha)_u$ to indicate a term which is $\mathrm{o}(s^\alpha)$ as $s \to 0$ uniformly for $\zeta \in \gamma$.

To prove (3.45) we note that $R(\zeta,0)P_0 = (-\zeta)^{-1}P_0$, by (3.39) and the relation $SP_0 = 0$. Using Lemma 3.19 with $u = P_0v$, we see that

$$R(\zeta, s)P_0 - (-\zeta)^{-1}P_0 = s(-\zeta)^{-1}(\gamma_{_D} R(\zeta,0)^*)^*\mathcal{P}\gamma_{_D} P_0 + \mathrm{O}(s^2)_u,$$

which gives (3.45) with a slightly better error. (Our errors are stated generally as $\mathrm{o}(\cdot)_u$ for consistency.)

For (3.46) we observe

$$(\gamma_D(P_0R(\zeta,0))^*)^* = P_0R(\zeta,0)\gamma_D^* = P_0(\gamma_D R(\zeta,0)^*)^*,$$

and likewise

$$(\gamma_D(P_0R(\zeta,0))^*)^* = (\gamma_D((-\bar{\zeta})^{-1}P_0))^* = (-\zeta)^{-1}(\gamma_D P_0)^*,$$

so that

$$P_0(\gamma_D R(\zeta,0)^*)^* = (-\zeta)^{-1}(\gamma_D P_0)^*.$$

If we apply P_0 on the left to the identity in Lemma 3.19, and use this last relation, we arrive at (3.46).

For (3.47) we again take $u = P_0 v$ in Lemma 3.19, and we apply P_0 on the left of the resulting expression. Finally, (3.48) is a straightforward consequence of (3.45) and (3.47).

Step 2. We claim the following asymptotic relations for the Riesz projections:

$$P(s)P_0 = P_0 + s(\gamma_{_D} S)^*\mathcal{P}\gamma_{_D} P_0 + \mathrm{o}(s)_u, \tag{3.49}$$

$$P_0 P(s) = P_0 + s(\gamma_{_D} P)^*\mathcal{P}\gamma_{_D} S + \mathrm{o}(s)_u, \tag{3.50}$$

$$P_0 P(s) P_0 = P_0 - s^2(\gamma_{_D} P)^*\mathcal{P}\gamma_{_D}(\gamma_{_D}(S^2))^*\mathcal{P}\gamma_{_D} P_0 + \mathrm{o}(s^2)_u. \tag{3.51}$$

To see (3.49), we integrate (3.45) with $-\frac{1}{2\pi i}\int_\gamma(\cdot)\,d\zeta$. We find

$$\begin{aligned} P(s)P_0 &= -\frac{1}{2\pi i}\int_\gamma (-\zeta)^{-1} d\zeta P_0 - \frac{s}{2\pi i}\int_\gamma (-\zeta)^{-1}(\gamma_D R(\zeta,0)^*)^* d\zeta \mathcal{P}\gamma_D P_0 + \mathrm{o}(s)_u \\ &= P_0 - s\Big(\gamma_D \frac{1}{2\pi i}\int_\gamma (-\zeta)^{-1} R(\zeta,0)^* d\zeta\Big)^* \mathcal{P}\gamma_D P_0 + \mathrm{o}(s)_u \\ &= P_0 + s(\gamma_D S^*)^*\mathcal{P}\gamma_D P_0 + \mathrm{o}(s)_u, \end{aligned}$$

from which (3.49) follows because S is self-adjoint. Likewise, (3.50) and (3.51) follow respectively by applying $-\frac{1}{2\pi i}\int_\gamma(\cdot)\,d\zeta$ to (3.46) and (3.47).

Step 3. We next claim the following asymptotic relations for the transformation operators defined in (3.42):

$$U(s) = I + s\Big((\gamma_{_D} S)^*\mathcal{P}\gamma_{_D} P_0 - (\gamma_{_D} P_0)^*\mathcal{P}\gamma_{_D} S\Big) + \mathrm{o}(s)_u, \tag{3.52}$$

$$U(s)^{-1} = I + s\Big((\gamma_{_D} P_0)^*\mathcal{P}\gamma_{_D} S - (\gamma_{_D} S)^*\mathcal{P}\gamma_{_D} P_0)\Big) + \mathrm{o}(s)_u, \tag{3.53}$$

$$P_0 U(s) P_0 = P_0 - \frac{1}{2}s^2(\gamma_{_D} P_0)^*\mathcal{P}\gamma_{_D}(\gamma_{_D}(S^2))^*\mathcal{P}\gamma_{_D} P_0 + \mathrm{o}(s^2)_u, \tag{3.54}$$

$$P_0 U(s)^{-1} P_0 = P_0 - \frac{1}{2}s^2(\gamma_{_D} P_0)^*\mathcal{P}\gamma_{_D}(\gamma_{_D}(S^2))^*\mathcal{P}\gamma_{_D} P_0 + \mathrm{o}(s^2)_u, \tag{3.55}$$

$$P_0 U(s)^{-1}(I - P_0) = s(\gamma_{_D} P_0)^*\mathcal{P}\gamma_{_D} S + \mathrm{o}(s)_u. \tag{3.56}$$

Indeed, recalling that $D(s) = P(s) - P_0$ and using (3.49) and (3.50) yields

$$\begin{aligned} D^2(s) &= (P(s) - P_0)(P(s) - P_0) = P(s) + P_0 - P(s)P_0 - P_0 P(s) \\ &= (P(s) - P_0) + (P_0 - P(s)P_0) + (P_0 - P_0 P(s)) = D(s) - sP^{(1)} + \mathrm{o}(s)_u, \end{aligned} \tag{3.57}$$

where from Step 2

$$(P_0 - P(s)P_0) + (P_0 - P_0 P(s)) = -(\gamma_D S)^*\mathcal{P}\gamma_D P_0 - s(\gamma_D P_0)^*\mathcal{P}\gamma_D S + \mathrm{o}(s)_u,$$

and we define

$$P^{(1)} = (\gamma_{_D} S)^*\mathcal{P}\gamma_{_D} P_0 + (\gamma_{_D} P_0)^*\mathcal{P}\gamma_{_D} S.$$

Hence,

$$(I - D(s))(D(s) - sP^{(1)}) = sD(s)P^{(1)} + \mathrm{o}(s)_u = \mathrm{o}(s)_u, \tag{3.58}$$

and therefore $D(s) = sP^{(1)} + (I - D(s))^{-1}\mathrm{o}(s)_u$, yielding

$$D(s) = sP^{(1)} + \mathrm{o}(s)_u. \tag{3.59}$$

Turning now to (3.52), we have

$$\begin{aligned}
U(s) &= (I - D(s)^2)^{-1/2}\Big((I - P(s))(I - P_0) + P(s)P_0\Big)\\
&= I - P_0 - P(s) + 2P(s)P_0 + \mathrm{O}(s^2)_u\\
&= I - P_0 - P(s) + 2P_0 + 2s(\gamma_D S)^*\mathcal{P}\gamma_D P_0 + \mathrm{o}(s)_u\\
&= I - D(s) + 2s(\gamma_D S)^*\mathcal{P}\gamma_D P_0 + \mathrm{o}(s)_u\\
&= I - sP^{(1)} + 2s(\gamma_D S)^*\mathcal{P}\gamma_D P_0 + \mathrm{o}(s)_u\\
&= I - s(\gamma_D S)^*\mathcal{P}\gamma_D P_0 - s(\gamma_D P_0)^*\mathcal{P}\gamma_D S + 2s(\gamma_D S)^*\mathcal{P}\gamma_D P_0 + \mathrm{o}(s)_u\\
&= I + s(\gamma_D S)^*\mathcal{P}\gamma_D P_0 - s(\gamma_D P_0)^*\mathcal{P}\gamma_D S + \mathrm{o}(s)_u,
\end{aligned}$$

which is (3.52). Likewise, (3.53) is established by a similar calculation, beginning with

$$U(s)^{-1} = (I - D(s))^{-1/2}\Big((I - P_0)(I - P(s)) + P_0P(s)\Big).$$

Formula (3.54) follows from the calculation

$$\begin{aligned}
P_0U(s)P_0 &= P_0(I - D^2(s))^{-1/2}P(s)P_0 = P_0P(s)P_0 + \frac{1}{2}P_0D(s)^2P_0 + O(s^3)_u\\
&= P_0 - s^2(\gamma_{{}_D} P_0)^*\mathcal{P}\gamma_{{}_D}(\gamma_{{}_D}(S^2))^*\mathcal{P}\gamma_{{}_D} P_0 + \frac{1}{2}s^2P_0(P^{(1)})^2P_0 + \mathrm{o}(s^2)_u\\
&= P_0 - s^2(\gamma_{{}_D} P_0)^*\mathcal{P}\gamma_{{}_D}(\gamma_{{}_D}(S^2))^*\mathcal{P}\gamma_{{}_D} P_0\\
&\quad + \frac{1}{2}s^2P_0\Big(((\gamma_{{}_D} S)^*\mathcal{P}\gamma_{{}_D} P_0 + (\gamma_D P_0)^*\mathcal{P}\gamma_{{}_D} S\Big)^2P_0 + \mathrm{o}(s^2)_u,
\end{aligned}$$

from which we see that

$$\begin{aligned}
P_0U(s)P_0 = P_0 &- s^2(\gamma_{{}_D} P_0)^*\mathcal{P}\gamma_{{}_D}(\gamma_{{}_D}(S^2))^*\mathcal{P}\gamma_{{}_D} P_0\\
&+ \frac{1}{2}s^2P_0\Big\{(\gamma_{{}_D} S)^*\mathcal{P}\gamma_{{}_D} P_0(\gamma_{{}_D} S)^*\mathcal{P}\gamma_{{}_D} P_0 + (\gamma_{{}_D} S)^*\mathcal{P}\gamma_{{}_D} P_0(\gamma_{{}_D} P_0)^*\mathcal{P}\gamma_{{}_D} S\\
&+ (\gamma_{{}_D} P_0)^*\mathcal{P}\gamma_{{}_D} S(\gamma_{{}_D} S)^*\mathcal{P}\gamma_{{}_D} P_0 + (\gamma_{{}_D} P_0)^*\mathcal{P}\gamma_{{}_D} S(\gamma_{{}_D} P_0)^*\mathcal{P}\gamma_{{}_D} S\Big\}P_0 + \mathrm{o}(s^2)_u.
\end{aligned}$$

Three terms are eliminated by the relation $SP_0 = 0$ (first, second, fourth), and we also have the identity $S(\gamma_D S)^* = (\gamma_D S^2)^*$. Combining these observations, we obtain (3.54).

A similar argument yields (3.55), and (3.56) follows using (3.53).

Step 4. We now claim the following asymptotic relation for the resolvent:

$$\begin{aligned} P_0U(s)^{-1}R(\zeta,s)U(s)P_0 &= (-\zeta)^{-1}P_0 + (-\zeta)^{-2}s(\gamma_{_D}P_0)^*\mathcal{P}\gamma_{_D}P_0 \\ &- (-\zeta)^{-2}s^2P_0V(0)P_0 - (-\zeta)^{-1}s^2(\gamma_{_D}P_0)^*\mathcal{P}\gamma_{_D}(\gamma_{_D}(S^2))^*\mathcal{P}\gamma_{_D}P_0 \\ &+ (-\zeta)^{-2}s^2(\gamma_{_D}P_0)^*\mathcal{P}\gamma_{_D}\Big\{\gamma_{_D}\Big[R(\zeta,0)(I+2(-\zeta)S+(-\zeta)^2S^2)\Big]^*\Big\}^*\mathcal{P}\gamma_DP_0 + \mathrm{o}(s^2)_u. \end{aligned} \tag{3.60}$$

To see this, we begin by writing

$$\begin{aligned} P_0U(s)^{-1}R(\zeta,s)U(s)P_0 &= P_0U(s)^{-1}\Big(P_0R(\zeta,s)P_0 + (I-P_0)R(\zeta,s)P_0 \\ &+ P_0R(\zeta,s)(I-P_0) + (I-P_0)R(\zeta,s)(I-P_0)\Big)U(s)P_0 \\ &= A_1 + A_2 + A_3 + A_4, \end{aligned}$$

where we denote

$$\begin{aligned} A_1 &= P_0U(s)^{-1}P_0R(\zeta,s)P_0U(s)P_0, \\ A_2 &= P_0U(s)^{-1}(I-P_0)R(\zeta,s)P_0U(s)P_0, \\ A_3 &= P_0U(s)^{-1}P_0R(\zeta,s)(I-P_0)U(s)P_0, \\ A_4 &= P_0U(s)^{-1}(I-P_0)R(\zeta,s)(I-P_0)U(s)P_0. \end{aligned}$$

For A_1, we use the fact that P_0 is a projection, along with (3.55), (3.47) and (3.54), to obtain

$$\begin{aligned} A_1 &= (P_0U(s)^{-1}P_0)(P_0R(\zeta,s)P_0)(P_0U(s)P_0) \\ &= \Big(P_0 - \frac{1}{2}s^2(\gamma_{_D}P_0)^*\mathcal{P}\gamma_{_D}(\gamma_{_D}(S^2))^*\mathcal{P}\gamma_{_D}P_0 + \mathrm{o}(s^2)_u\Big) \\ &\quad\times\Big((-\zeta)^{-1}P_0 + (-\zeta)^{-2}s(\gamma_{_D}P_0)^*\mathcal{P}\gamma_{_D}P_0 - s^2(-\zeta)^{-2}P_0V(0)P_0 \\ &+ s^2(-\zeta)^{-2}(\gamma_{_D}P_0)^*\mathcal{P}\gamma_{_D}(\gamma_{_D}R(\zeta,0)^*)^*\mathcal{P}\gamma_{_D}P_0 + \mathrm{o}(s^2)_u\Big) \\ &\quad\times\Big(P_0 - \frac{1}{2}s^2(\gamma_{_D}P_0)^*\mathcal{P}\gamma_{_D}(\gamma_{_D}(S^2))^*\mathcal{P}\gamma_{_D}P_0 + \mathrm{o}(s^2)_u\Big) \\ &= (-\zeta)^{-1}P_0 + s(-\zeta)^{-2}(\gamma_{_D}P_0)^*\mathcal{P}\gamma_{_D}P_0 - s^2(-\zeta)^{-2}P_0V(0)P_0 \\ &\quad+ s^2(-\zeta)^{-2}(\gamma_{_D}P_0)^*\mathcal{P}\gamma_{_D}(\gamma_{_D}R(\zeta,0)^*)^*\mathcal{P}\gamma_{_D}P_0 \\ &\quad- s^2(-\zeta)^{-1}(\gamma_{_D}P_0)^*\mathcal{P}\gamma_{_D}(\gamma_{_D}(S^2))^*\mathcal{P}\gamma_{_D}P_0 + \mathrm{o}(s^2)_u. \end{aligned}$$

Likewise, it follows from (3.56) and (3.48) that

$$\begin{aligned} A_2 &= (P_0U(s)^{-1}(I-P_0))((I-P_0)R(\zeta,s)P_0)(P_0U(s)P_0) \\ &= \Big(s(\gamma_{_D}P_0)^*\mathcal{P}\gamma_{_D}S + \mathrm{o}(s)_u\Big)\Big(s(-\zeta)^{-1}(I-P_0)(\gamma_{_D}R(\zeta,0)^*)^*\mathcal{P}\gamma_{_D}P_0 + \mathrm{o}(s)_u\Big) \\ &\quad\times\Big(P_0 - s^2\frac{1}{2}(\gamma_{_D}P_0)^*\mathcal{P}\gamma_{_D}(\gamma_{_D}(S^2))^*\mathcal{P}\gamma_{_D}P_0 + \mathrm{o}(s^2)_u\Big) \\ &= s^2(-\zeta)^{-1}(\gamma_{_D}P_0)^*\mathcal{P}\gamma_{_D}S(\gamma_{_D}R(\zeta,0)^*)^*\mathcal{P}\gamma_{_D}P_0 + \mathrm{o}(s^2)_u. \end{aligned}$$

For A_3, we have

$$\begin{aligned}
A_3 &= (P_0U(s)^{-1}P_0)(P_0R(\zeta,s))((I-P_0)U(s)P_0) \\
&= \Big(P_0 - \frac{1}{2}s^2(\gamma_{{}_D}P_0)^*\mathcal{P}\gamma_{{}_D}(\gamma_{{}_D}(S^2))^*\mathcal{P}\gamma_{{}_D}P_0 + \mathrm{o}(s^2)_u\Big) \\
&\quad\times\Big((-\zeta)^{-1}P_0 + s(-\zeta)^{-1}(\gamma_{{}_D}P_0)^*\mathcal{P}\gamma_{{}_D}R(\zeta,0) + \mathrm{o}(s)_s\Big)\Big(s(\gamma_{{}_D}S)^*\mathcal{P}\gamma_{{}_D}P_0 + \mathrm{o}(s)_s\Big) \\
&= s^2(-\zeta)^{-1}(\gamma_{{}_D}P_0)^*\mathcal{P}\gamma_{{}_D}R(\zeta,0)(\gamma_{{}_D}S)^*\mathcal{P}\gamma_{{}_D}P_0 + \mathrm{o}(s^2)_u,
\end{aligned}$$

while for A_4 we have

$$\begin{aligned}
A_4 &= (P_0U(s)^{-1}(I-P_0))R(\zeta,s)((I-P_0)U(s)P_0 \\
&= s^2(\gamma_{{}_D}P_0)^*\mathcal{P}\gamma_{{}_D}SR(\zeta,0)(\gamma_{{}_D}S)^*\mathcal{P}\gamma_{{}_D}P_0 + \mathrm{o}(s^2)_u.
\end{aligned}$$

Collecting all these terms, we obtain (3.60):

$$\begin{aligned}
P_0U(s)^{-1}R(\zeta,s)U(s)P_0 &= (-\zeta)^{-1}P_0 + s(-\zeta)^{-2}(\gamma_{{}_D}P_0)^*\mathcal{P}\gamma_{{}_D}P_0 - s^2(-\zeta)^{-2}P_0V(0)P_0 \\
&\quad + s^2(-\zeta)^{-2}(\gamma_{{}_D}P_0)^*\mathcal{P}\gamma_{{}_D}(\gamma_{{}_D}R(\zeta,0)^*)^*\mathcal{P}\gamma_{{}_D}P_0 \\
&\quad - s^2(-\zeta)^{-1}(\gamma_{{}_D}P_0)^*\mathcal{P}\gamma_{{}_D}(\gamma_{{}_D}(S^2))^*\mathcal{P}\gamma_{{}_D}P_0 + \mathrm{o}(s^2) \\
&\quad + s^2(-\zeta)^{-1}(\gamma_{{}_D}P_0)^*\mathcal{P}\gamma_{{}_D}S(\gamma_{{}_D}R(\zeta,0)^*)^*\mathcal{P}\gamma_{{}_D}P_0 + \mathrm{o}(s^2)_u \\
&\quad + s^2(-\zeta)^{-1}(\gamma_{{}_D}P_0)^*\mathcal{P}\gamma_{{}_D}R(\zeta,0)(\gamma_{{}_D}S)^*\mathcal{P}\gamma_{{}_D}P_0 + \mathrm{o}(s^2)_u \\
&\quad + s^2(\gamma_{{}_D}P_0)^*\mathcal{P}\gamma_{{}_D}SR(\zeta,0)(\gamma_{{}_D}S)^*\mathcal{P}\gamma_{{}_D}P_0 + \mathrm{o}(s^2)_u,
\end{aligned}$$

from which the claim is immediate.

Step 5. We are ready to finish the proof of the lemma. Using the standard relation from [36, Equation (III.6.24)] we have

$$H(s)P(s) = -\frac{1}{2\pi i}\int_\gamma \zeta R(\zeta,s)\,d\zeta,$$

and applying integration $-\frac{1}{2\pi i}\int_\gamma \zeta(\cdot)\,d\zeta$ in (3.60), we find

$$\begin{aligned}
&-\frac{1}{2\pi i}\int_\gamma \zeta P_0U(s)^{-1}R(\zeta,s)U(s)P_0\,d\zeta \\
&\qquad = -s(\gamma_DP_0)^*\mathcal{P}\gamma_DP_0 + s^2P_0V(0)P_0 \\
&\qquad\quad - s^2(\gamma_DP_0)^*\mathcal{P}\gamma_D\Big(\gamma_D\frac{1}{2\pi i}\int_\gamma \zeta^{-1}R(\zeta,s)\,d\zeta\Big)^*\mathcal{P}\gamma_DP_0 + \mathrm{o}(s^2) \\
&\qquad = -s(\gamma_{{}_D}P_0)^*\mathcal{P}\gamma_{{}_D}P_0 + s^2P_0V(0)P_0 - s^2(\gamma_{{}_D}P_0)^*\mathcal{P}\gamma_{{}_D}(\gamma_{{}_D}S)^*\mathcal{P}\gamma_{{}_D}P_0 + \mathrm{o}(s^2).
\end{aligned}$$

□

We now complete our perturbation analysis with the following lemma.

Lemma 3.22. *Under the assumptions of Theorem* 1.5*, we have*

$$\operatorname{Mor}H(s) = \operatorname{Mor}(B) + \operatorname{Mor}(Q(V(0) - (P_{R_0}\Lambda_0P_{R_0})^2)Q),$$

for $s > 0$ sufficiently small.

Proof. By Claim 3.20, it suffices to count the negative eigenvalues of the finite-dimensional operator $H(s)P(s)$. By Lemma 3.21, it is enough to obtain an asymptotic formula for the eigenvalues of the operator $T(s) := P_0 U(s)^{-1} H(s) P(s) U(s) P_0$, where

$$T(s) = T + sT^{(1)} + s^2 T^{(2)} + \mathrm{o}(s^2) \text{ as } s \to 0$$

and we denote

$$\begin{aligned} T = 0, \quad T^{(1)} &= -(\gamma_{{}_D} P_0)^* \mathcal{P} \gamma_{{}_D} P_0, \quad T^{(2)} = T_1^{(2)} + T_2^{(2)}, \\ T_1^{(2)} = P_0 V(0) P_0, \quad T_2^{(2)} &= -(\gamma_{{}_D} P_0)^* \mathcal{P} \gamma_{{}_D} (\gamma_{{}_D} S)^* \mathcal{P} \gamma_{{}_D} P_0. \end{aligned} \tag{3.61}$$

(For this calculation, we're following [36], along with some notation from that reference.) These operators act on the d-dimensional space $\operatorname{ran}(P_0) = \ker(H(0)) = (\ker P_{D_0}) \cap (\ker P_{D_1})$. We will apply a well known finite-dimensional perturbation result [36, Theorem II.5.1] to the family $\{T(s)\}$ for s near zero. For this we will need some more notations and preliminaries.

Let $\{\lambda_j^{(1)}\}_{j=1}^{\mathfrak{m}(1)}$ denote the $\mathfrak{m}(1)$ distinct eigenvalues of the operator $T^{(1)}$, let $m_j^{(1)}$ denote their multiplicities, and let $P_j^{(1)}$ denote the respective orthogonal Riesz spectral projections. We define the bilinear form

$$\mathfrak{b}(p, q) = (Bp, q)_{\mathbb{C}^n}, \quad \forall p, q \in (\ker P_{D_0}) \cap (\ker P_{D_1}),$$

where we recall that we denote by B the operator obtained by restricting $(P_{R_0} \Lambda_0 P_{R_0} - P_{R_1} \Lambda_1 P_{R_1})$ to the space $(\ker P_{D_0}) \cap (\ker P_{D_1})$.

The quadratic form on $\operatorname{ran}(P)$ associated with $T^{(1)}$ is given by

$$\begin{aligned} \mathfrak{t}^{(1)}(p, q) &= \langle T^{(1)} p, q \rangle_{L^2(0,1)} = -\langle (\gamma_{{}_D} P_0)^* \mathcal{P} \gamma_{{}_D} P_0 p, q \rangle_{L^2(0,1)} \\ &= -\Big(\mathcal{P} \gamma_{{}_D} P_0 p, \gamma_{{}_D} P_0 q \Big)_{\mathbb{C}^{2n}} = -\left(\begin{pmatrix} -P_{R_0} \Lambda_0 P_{R_0} & 0 \\ 0 & P_{R_1} \Lambda_1 P_{R_1} \end{pmatrix} \begin{pmatrix} p \\ p \end{pmatrix}, \begin{pmatrix} q \\ q \end{pmatrix} \right)_{\mathbb{C}^{2n}} \\ &= \Big((P_{R_0} \Lambda_0 P_{R_0} - P_{R_1} \Lambda_1 P_{R_1}) p, q \Big)_{\mathbb{C}^n} = \Big(Bp, q \Big)_{\mathbb{C}^n} =: \mathfrak{b}(p, q). \end{aligned}$$

In particular, we see that the number of negative values in $\{\lambda_j^{(1)}\}_{j=1}^{\mathfrak{m}(1)}$, including multiplicities, is $n_-(\mathfrak{b})$ (the number of negative values of B, including multiplicities), and likewise for the number of positive and zero values in $\{\lambda_j^{(1)}\}_{j=1}^{\mathfrak{m}(1)}$ with the respective values $n_+(\mathfrak{b})$ and $n_0(\mathfrak{b})$.

Turning now to $T^{(2)}$, and following [36, Section II.5], we let $\lambda_{jk}^{(2)}$, $j = 1, \dots, \mathfrak{m}(1)$, $k = 1, \dots, m_j^{(1)}$, denote the eigenvalues of the family of operators $P_j^{(1)} T^{(2)} P_j^{(1)}$ in $\operatorname{ran}(P_j^{(1)})$ (recall that in our case the unperturbed operator is just $T = 0$ and thus its reduced resolvent is zero and $P_j^{(1)} \widetilde{T}^{(2)} P_j^{(1)} = P_j^{(1)} T^{(2)} P_j^{(1)}$ using the notations from [36, Section II.5]). By [36, Theorem II.5.11] the eigenvalues $\lambda_{jk}(s)$ of the operator $T(s)$ are given by the formula

$$\lambda_{jk}(s) = s\lambda_j^{(1)} + s^2 \lambda_{jk}^{(2)} + \mathrm{o}(s^2) \text{ as } s \to 0, \ j = 1, \dots, \mathfrak{m}(1), k = 1, \dots, m_j^{(1)}. \tag{3.62}$$

It's clear from (3.62) that if $\lambda_j^{(1)} \neq 0$ the value of $\lambda_{jk}^{(2)}$ will be inconsequential for s sufficiently small. In particular, if $\lambda_j^{(1)} < 0$ then $T(s)$ (and hence $H(s)$) will have a negative eigenvalue, while if $\lambda_j^{(1)} > 0$ then $T(s)$ (and hence $H(s)$) will have a positive eigenvalue. Since our convention takes the Morse index to be a count of negative eigenvalues, we conclude that $\mathrm{Mor}(B)$ is precisely a count of the negative eigenvalues of $H(s)$ corresponding with $\lambda_j^{(1)} < 0$.

In the event that $\lambda_j^{(1)} = 0$ we need a sign for $\lambda_{jk}^{(2)}$ (which will be non-zero by our non-degeneracy assumption). For notational convenience, we index the eigenvalues so that $\lambda_1^{(1)} = 0$, with corresponding Riesz projection $P_1^{(1)}$ onto the $m_1^{(1)}$-dimensional eigenspace $\ker B$. The corresponding values $\{\lambda_{1k}^{(2)}\}_{k=1}^{m_1^{(1)}}$ will be eigenvalues of $T^{(2)}$, and in particular will be precisely the $m_1^{(1)}$ eigenvalues of $P_1^{(1)} T^{(2)} P_1^{(1)}$. For $p, q \in (\ker P_{D_0}) \cap (\ker P_{D_1})$, we define

$$\mathfrak{t}^{(2)}(p,q) = \langle P_1^{(1)} T^{(2)} P_1^{(1)} p, q \rangle_{L^2(0,1)} = \langle P_1^{(1)} T_1^{(2)} P_1^{(1)} p, q \rangle_{L^2(0,1)} + \langle P_1^{(1)} T_2^{(2)} P_1^{(1)} p, q \rangle_{L^2(0,1)}. \tag{3.63}$$

For the first summand on the right-hand side of (3.63), we have

$$\langle P_1^{(1)} P_0 V(0) P_0 P_1^{(1)} p, q \rangle_{L^2(0,1)} = (P_1^{(1)} P_0 V(0) P_0 P_1^{(1)} p, q)_{\mathbb{C}^n} = (P_1^{(1)} V(0) P_1^{(1)} p, q)_{\mathbb{C}^n}, \tag{3.64}$$

where in the first equality we've observed that the $L^2(0,1)$ inner product is equivalent to the $\mathbb{C}^n$ inner product for constant vectors, and in the second we've observed that since $P_1^{(1)}$ projects onto a subspace of $\mathrm{ran} P_0$ we have $P_0 P_1^{(1)} = P_1^{(1)}$ and $P_1^{(1)} P_0 = P_1^{(1)}$.

For the second summand on the right-hand side, we have

$$\begin{aligned} \langle P_1^{(1)} T_2^{(2)} P_1^{(1)} p, q \rangle_{L^2(0,1)} &= -\langle P_1^{(1)} (\gamma_{_D} P_0)^* \mathcal{P} \gamma_{_D} (\gamma_{_D} S)^* \mathcal{P} \gamma_{_D} P_0 P_1^{(1)} p, q \rangle_{L^2(0,1)} \\ &= -\Big(\gamma_{_D} (\gamma_{_D} S)^* \mathcal{P} \gamma_{_D} P_0 P_1^{(1)} p, \mathcal{P} \gamma_{_D} P_0 P_1^{(1)} q \Big)_{\mathbb{C}^{2n}}. \end{aligned} \tag{3.65}$$

We notice that if we denote $P_1^{(1)} p = p_1^{(1)} \in \ker B$ then

$$\mathcal{P} \gamma_{_D} P_0 P_1^{(1)} p = \begin{pmatrix} -P_{R_0} \Lambda_0 P_{R_0} & 0 \\ 0 & P_{R_1} \Lambda_1 P_{R_1} p \end{pmatrix} \begin{pmatrix} p_1^{(1)} \\ p_1^{(1)} \end{pmatrix} = \begin{pmatrix} -P_{R_0} \Lambda_0 P_{R_0} p_1^{(1)} \\ P_{R_1} \Lambda_1 P_{R_1} p_1^{(1)} \end{pmatrix}.$$

Since $p_1^{(1)} \in \ker B$, we have $P_{R_0} \Lambda_0 P_{R_0} p_1^{(1)} = P_{R_1} \Lambda_1 P_{R_1} p_1^{(1)}$, so that

$$\mathcal{P} \gamma_{_D} P_0 P_1^{(1)} p = \begin{pmatrix} -P_{R_0} \Lambda_0 P_{R_0} p_1^{(1)} \\ P_{R_0} \Lambda_0 P_{R_0} p_1^{(1)} \end{pmatrix}. \tag{3.66}$$

Of course the same calculation hold for q as well. Setting

$$\psi = (\gamma_{_D} S)^* \begin{pmatrix} -P_{R_0} \Lambda_0 P_{R_0} p_1^{(1)} \\ P_{R_0} \Lambda_0 P_{R_0} p_1^{(1)} \end{pmatrix},$$

we see that

$$\begin{aligned} \langle P_1^{(1)} T_2^{(2)} P_1^{(1)} p, q \rangle_{L^2(0,1)} &= -\Big(\gamma_{_D} \psi, \begin{pmatrix} -P_{R_0} \Lambda_0 P_{R_0} q_1^{(1)} \\ P_{R_0} \Lambda_0 P_{R_0} q_1^{(1)} \end{pmatrix} \Big)_{\mathbb{C}^{2n}} \\ &= (\psi(0) - \psi(1), P_{R_0} \Lambda_0 P_{R_0} P_1^{(1)} q)_{\mathbb{C}^n}. \end{aligned} \tag{3.67}$$

At this point, we need to understand the action of $\gamma_{_D} (\gamma_{_D} S)^*$ on vectors in the form on the right-hand side of (3.66). This problem has been studied in detail in [26] for the case of multiple space dimensions, and the current setting is much easier (though a bit different). We will organize the main points of our discussion into a pair of propositions.

Proposition 3.23. *Suppose $v = \binom{v_1}{v_2} \in \mathbb{C}^{2n}$, with $v_1 \in \operatorname{ran} P_{R_0}$ and $v_2 \in \operatorname{ran} P_{R_1}$. Then*

$$(\gamma_{_D} S)^* v = \frac{1}{2\pi i} \int_\Gamma \zeta^{-1} w(x;\zeta) d\zeta,$$

where Γ is a small enough loop around $\zeta = 0$ so that it encloses no other eigenvalues of $H(0)$, and for each $\zeta \in \Gamma$, w is the unique solution to $-w'' - \zeta w = 0$, with boundary conditions

$$\begin{aligned} P_{D_0} w(0) &= 0; & P_{D_1} w(1) &= 0; \\ P_{N_0} w'(0) &= 0; & P_{N_1} w'(1) &= 0; \\ P_{R_0} w'(0) &= -v_1; & P_{R_1} w'(1) &= v_2. \end{aligned} \tag{3.68}$$

Proof. We note at the outset that by the definition of S as the reduced resolvent for $H(0)$, we have

$$\begin{aligned} (\gamma_{_D} S)^* = S\gamma_{_D}^* &= \frac{1}{2\pi i} \int_\Gamma \zeta^{-1} R(\zeta, 0) \gamma_{_D}^* d\zeta \\ &= \frac{1}{2\pi i} \int_\Gamma \zeta^{-1} (\gamma_{_D} R(\zeta, 0)^*)^* d\zeta. \end{aligned} \tag{3.69}$$

Let $f \in L^2(0,1)$ and consider the equation $-u'' - \bar{\zeta} u = f$, with boundary conditions

$$\begin{aligned} P_{D_0} u(0) &= 0; & P_{D_1} u(1) &= 0; \\ P_{N_0} u'(0) &= 0; & P_{N_1} u'(1) &= 0; \\ P_{R_0} u'(0) &= 0; & P_{R_1} u'(1) &= 0, \end{aligned}$$

which is solved by $u(x) = R(\zeta, 0)^* f$. Notice that for any $v \in \mathbb{C}^{2n}$ we can compute

$$(\gamma_{_D} R(\zeta, 0)^* f, v)_{\mathbb{C}^{2n}} = (\gamma_{_D} u, v)_{\mathbb{C}^{2n}} = (u(0), v_1)_{\mathbb{C}^n} + (u(1), v_2)_{\mathbb{C}^n}. \tag{3.70}$$

On the other hand,

$$(\gamma_{_D} R(\zeta, 0)^* f, v)_{\mathbb{C}^{2n}} = \langle f, (\gamma_{_D} R(\zeta, 0)^*)^* v \rangle_{L^2(0,1)}.$$

Motivated by the analysis of [26], we set

$$w := (\gamma_{_D} R(\zeta, 0)^*)^* v,$$

so that

$$\begin{aligned} \langle f, (\gamma_{_D} R(\zeta, 0)^*)^* v \rangle_{L^2(0,1)} &= \langle -u'' - \bar{\zeta} u, w \rangle_{L^2(0,1)} \\ &= -(u', w)_{\mathbb{C}^n} \Big|_0^1 + (u, w')_{\mathbb{C}^n} \Big|_0^1 - \langle u, w'' \rangle_{L^2(0,1)} - \bar{\zeta} \langle u, w \rangle_{L^2(0,1)}. \end{aligned} \tag{3.71}$$

In order to eliminate the $L^2(0,1)$ inner products, we take w to solve $-w'' - \zeta w = 0$, and in order to make (3.70) correspond with (3.71) we choose the boundary conditions (3.68).

With this choice of w, we have

$$\begin{aligned}(u'(1), w(1))_{\mathbb{C}^n} &= (u'(1), P_{D_1}w(1) + P_{N_1}w(1) + P_{R_1}w(1))_{\mathbb{C}^n} \\ &= (u'(1), P_{D_1}w(1))_{\mathbb{C}^n} + (P_{N_1}u'(1), w(1))_{\mathbb{C}^n} + (P_{R_1}u'(1), w(1))_{\mathbb{C}^n} = 0,\end{aligned}$$

and likewise $(u'(0), w(0))_{\mathbb{C}^n} = 0$. Proceeding by an almost identical calculation we find $(u(1), w'(1))_{\mathbb{C}^n} = (u(1), v_2)_{\mathbb{C}^n}$ and $(u(0), w'(0))_{\mathbb{C}^n} = -(u(0), v_1)_{\mathbb{C}^n}$.

Combining with (3.69), we see that the proposition follows. □

Recalling (3.66) we see that we need to solve for w with $v_1 = -P_{R_0}\Lambda_0 P_{R_0} p_1^{(1)}$ and $v_2 = P_{R_0}\Lambda_0 P_{R_0} p_1^{(1)}$. We do this with the following proposition.

Proposition 3.24. *If $v_1 = -v_2$ in (3.68), with $v_1, v_2 \in (\mathrm{ran} P_{R_0}) \cap (\mathrm{ran} P_{R_1})$, then*

$$w(x;0) = v_2 x - \frac{1}{2} v_2.$$

Proof. First, notice that if we set $\breve{w}(x;\zeta) = -w(1-x;\zeta)$, we find that w and $\breve{w}$ solve the same equation, so that by uniqueness (for $|\zeta| > 0$ sufficiently small) we have

$$w(x;\zeta) = -w(1-x;\zeta). \tag{3.72}$$

Next, we set $\tilde{w} = w - v_2 x$, so that

$$-\tilde{w}'' - \zeta\tilde{w} = \zeta v_2 x,$$

with homogeneous boundary conditions

$$\begin{aligned} P_{D_0}\tilde{w}(0) &= 0; & P_{D_1}\tilde{w}(1) &= 0; \\ P_{N_0}\tilde{w}'(0) &= 0; & P_{N_1}\tilde{w}'(1) &= 0; \\ P_{R_0}\tilde{w}'(0) &= 0; & P_{R_1}\tilde{w}'(1) &= 0. \end{aligned}$$

We see from Lemma 3.13 that $\tilde{w}(x;0)$ is a constant function $\tilde{w}_c$, with $\tilde{w}_c \in \ker H(0) = (\ker P_{D_0}) \cap (\ker P_{D_1})$. In this way, we see that

$$w(x;0) = \tilde{w}_c + v_2 x,$$

and taking $\zeta \to 0$ in (3.72) we see that

$$\tilde{w}_c + v_2 x = -(\tilde{w}_c + v_2(1-x)),$$

from which we find

$$\tilde{w}_c = -\frac{1}{2} v_2,$$

giving precisely the claim. □

Combining Proposition 3.23 with Proposition 3.24 see that

$$\begin{aligned}\psi(0) - \psi(1) &= \frac{1}{2\pi i}\int_\Gamma \zeta^{-1}\Big(w(0;\zeta) - w(1;\zeta)\Big)d\zeta \\ &= w(0;0) - w(1;0) == -P_{R_0}\Lambda_0 P_{R_0} P_1^{(1)} p.\end{aligned}$$

Using (3.67), we compute

$$\begin{aligned}\langle P_1^{(1)} T_2^{(2)} P_1^{(1)} p, q\rangle_{L^2(0,1)} &= -(P_{R_0}\Lambda_0 P_{R_0} P_1^{(1)} p, P_{R_0}\Lambda_0 P_{R_0} P_1^{(1)} q)_{\mathbb{C}^n} \\ &= -(P_1^{(1)} (P_{R_0}\Lambda_0 P_{R_0})^2 P_1^{(1)} p, q)_{\mathbb{C}^n}.\end{aligned}$$

Combining with (3.64), we conclude that

$$\langle P_1^{(1)} T^{(2)} P_1^{(1)} p, q\rangle_{L^2(0,1)} = (P_1^{(1)} (V(0) - (P_{R_0}\Lambda_0 P_{R_0})^2) P_1^{(1)} p, q)_{\mathbb{C}^n}.$$

Remark 3.25. We emphasize that in this section, we have been working with eigenvalues $\lambda(s)$ of $H(s)$, and as discussed in Remark 2.3 these are related to the eigenvalues λ_s of H_s by $\lambda_s = \lambda(s)/s^2$.

In view of expansion (3.62), we see that for any $\lambda_j^{(1)} < 0$ we will have $\lambda(s) \sim \lambda_j^{(1)} s$, and so we will have a crossing along Γ_1 at $\lambda_{s_0} \sim \lambda_j^{(1)}/s_0$. I.e., *each negative eigenvalue of B corresponds with a crossing of* Γ_1. In addition, for $\lambda_1^{(1)} = 0$, if $\lambda_{1k}^{(2)} < 0$ then $\lambda(s) \sim \lambda_{1k}^{(2)} s^2$, and so we will have a crossing along Γ_1 at $\lambda_{s_0} \sim \lambda_{1k}^{(2)}$. I.e., *each negative eigenvalue of $P_1^{(1)}(V(0) - (P_{R_0}\Lambda_0 P_{R_0})^2) P_1^{(1)}$ corresponds with a crossing of* Γ_1. We conclude that

$$\operatorname{Mas}(\ell, \ell_1; \Gamma_1) = -\operatorname{Mor}(H(s)) = -\operatorname{Mor}(B) - \operatorname{Mor}\Big(Q(V(0) - (P_{R_0}\Lambda_0 P_{R_0})^2) Q\Big),$$

where for notational convenience we've taken $Q = P_1^{(1)}$ in the statement of Theorem 1.5, and we use that notation here for clarity. $\square$

3.4. **Monotoncity in s.** In our proof of Lemma 3.9, we established that the rotation of the eigenvalues of $\tilde{W}_{s,\lambda}$ is monotonic along S^1 as λ increases or decreases. This is not generally the case as s increases or decreases, but we'll see that it does hold under certain conditions. In order to see when this is possible, we differeniate $\tilde{W}_{s,\lambda}$ with respect to s.

Lemma 3.26. *Under the assumptions of Lemma* 3.8*, we have*

$$\frac{\partial}{\partial s} \tilde{W}_{s,\lambda} = i \tilde{W}_{s,\lambda} \tilde{\Omega}(s,\lambda), \tag{3.73}$$

where

$$\tilde{\Omega}(s,\lambda) = 2\Big((X(s,\lambda) - iZ(s,\lambda))^{-1}\tilde{\mathfrak{B}}\Big)^* [X^t (V - \lambda I) X - Z^t Z] \Big((X(s,\lambda) - iZ(s,\lambda))^{-1}\tilde{\mathfrak{B}}\Big),$$

is a self-adjoint matrix.

Proof. First, we recall the notation

$$\tilde{\mathfrak{B}} = (\beta_1^t \beta_1 - \beta_2^t \beta_2) - i2\beta_2^t \beta_1.$$

We begin by computing

$$\begin{aligned}\frac{\partial}{\partial s} \tilde{W}_{s,\lambda} &= (X' + iZ')(X - iZ)^{-1}\tilde{\mathfrak{B}} - (X + iZ)(X - iZ)^{-1}(X' - iZ')(X - iZ)^{-1}\tilde{\mathfrak{B}} \\ &= (X' + iZ')(X - iZ)^{-1}\tilde{\mathfrak{B}} - \tilde{W}_{s,\lambda}\tilde{\mathfrak{B}}^{-1}(X' - iZ')(X - iZ)^{-1}\tilde{\mathfrak{B}},\end{aligned}$$

where $'$ denotes differentiation with respect to s. Upon multiplication of both sides by $\tilde{W}_{s,\lambda}^*$, we find

$$\begin{aligned}
\tilde{W}_{s,\lambda}^* \frac{\partial}{\partial s} \tilde{W}_{s,\lambda} &= \tilde{\mathfrak{B}}^* (X^t + iZ^t)^{-1} (X^t - iZ^t)(X' + iZ')(X - iZ)^{-1} \tilde{\mathfrak{B}} \\
&\quad - \tilde{\mathfrak{B}}^{-1} (X' - iZ')(X - iZ)^{-1} \tilde{\mathfrak{B}} \\
&= \tilde{\mathfrak{B}}^* (X^t + iZ^t)^{-1} \Big[(X^t - iZ^t)(X' + iZ') - (X^t + iZ^t)(X' - iZ') \Big] (X - iZ)^{-1} \tilde{\mathfrak{B}} \\
&= \Big((X - iZ)^{-1} \tilde{\mathfrak{B}} \Big)^* [2iX^t Z' - 2iZ^t X'] \Big((X - iZ)^{-1} \tilde{\mathfrak{B}} \Big) \\
&= i \Big((X - iZ)^{-1} \tilde{\mathfrak{B}} \Big)^* [2X^t (V - \lambda I) X - 2Z^t Z] \Big((X - iZ)^{-1} \tilde{\mathfrak{B}} \Big) = i \tilde{\Omega}.
\end{aligned}$$

We now multiply both sides by $\tilde{W}_{s,\lambda}$ and use the fact that $\tilde{W}_{s,\lambda}$ is unitary to see the claim. □

The Dirichlet case at $x = 1$. In the event that the boundary conditions at $x = 1$ are Dirichlet, the frame for our target space is $\binom{0}{I}$. Even in this special case, we won't generally have monotonicity in s, but we'll check that we have monotoncity at crossings.

Fix $\lambda \in [-\lambda_\infty, 0]$ and suppose there is a crossing at $s^* \in (s_0, 1)$, so that $\tilde{W}_{s^*,\lambda}$ has -1 as an eigenvalue (possibly with multiplicity greater than 1). Let V^* denote the eigenspace associated with -1, so that

$$\tilde{W}_{s^*,\lambda} v = -v \quad \forall v \in V^*,$$

and correspondingly (by the definition of $\tilde{W}_{s^*,\lambda}$) we have

$$(X(s^*, \lambda) - iZ(s^*, \lambda))^{-1} (X(s^*, \lambda) + iZ(s^*, \lambda)) v = -v, \tag{3.74}$$

so that

$$(X(s^*, \lambda) + iZ(s^*, \lambda)) v = -(X(s^*, \lambda) - iZ(s^*, \lambda)) v.$$

Rearranging terms, we see that $X(s^*, \lambda) v = 0$, so that V^* corresponds with the null space of $X(s^*, \lambda)$. Moreover, if we set $w = (X - iZ)^{-1} v$ and substitute $v = (X - iZ) w$ into (3.74), we see that

$$(X(s^*, \lambda) + iZ(s^*, \lambda)) w = -(X(s^*, \lambda) - iZ(s^*, \lambda)) w,$$

where we've recalled that $(X - iZ)^{-1}$ and $(X + iZ)$ commute. We see that w is also in V^*, so $(X(s^*, \lambda) - iZ(s^*, \lambda))^{-1}$ maps V^* to V^*.

Recall from our proof of Lemma 3.11 that the rotation of the eigenvalues of $\tilde{W}$ can be determined by the motion of the eigenvalues of

$$A_{s,\lambda} := i(e^{i\theta} I - \tilde{W}_{s,\lambda})^{-1} (e^{i\theta} I + \tilde{W}_{s,\lambda}),$$

for which we've seen

$$\frac{\partial}{\partial s} A_{s,\lambda} \Big|_{s=s^*} = 2 \Big((e^{i\theta} I - \tilde{W}_{s^*,\lambda})^{-1} \Big)^* \tilde{\Omega}_{s^*,\lambda} (e^{i\theta} I - \tilde{W}_{s^*,\lambda})^{-1}.$$

According to the Spectral Mapping Theorem, the eigenvalue -1 of $\tilde{W}_{s^*,\lambda}$ corresponds with the eigenvalue

$$a = i(e^{i\theta} + 1)^{-1} (e^{i\theta} I - 1),$$

and both eigenvalues correspond with the eigenspace V^*. Let P denote projection onto this space. According, then, to Theorem II.5.4 in [36] the motion of a as s varies near s^* is determined by the eigenvalues of $PA'_{s^*,\lambda}P$, where prime denotes differentiation with respect to s. In order to get a sign for these eigenvalues, we take any vector $v \in \mathbb{C}^n$ and compute

$$\begin{aligned}(PA'_{s^*,\lambda}Pv, v)_{\mathbb{C}^n} &= (A'_{s^*,\lambda}Pv, Pv)_{\mathbb{C}^n} \\ &= 2\Big(\Big((e^{i\theta}I - \tilde{W}_{s^*,\lambda})^{-1}\Big)^* \tilde{\Omega}_{s^*,\lambda}(e^{i\theta}I - \tilde{W}_{s^*,\lambda})^{-1}Pv, Pv\Big)_{\mathbb{C}^n} \\ &= 2\Big(\tilde{\Omega}_{s^*,\lambda}(e^{i\theta}I - \tilde{W}_{s^*,\lambda})^{-1}Pv, (e^{i\theta}I - \tilde{W}_{s^*,\lambda})^{-1}Pv\Big)_{\mathbb{C}^n}.\end{aligned}$$

Using $\tilde{W}_{s^*,\lambda}Pv = -Pv$, we arrive at

$$(PA'_{s^*,\lambda}Pv, v)_{\mathbb{C}^n} = \frac{2}{|e^{i\theta}+1|^2}(\tilde{\Omega}_{s^*,\lambda}Pv, Pv)_{\mathbb{C}^n}.$$

We see that we need to determine a sign for the matrix $\tilde{\Omega}_{s^*,\lambda}$, restricted to the space V^*. To this end, we compute (with all evaluations at (s^*, λ))

$$\begin{aligned}(\tilde{\Omega}Pv, Pv)_{\mathbb{C}^n} &= \Big(2\Big((X-iZ)^{-1}\Big)^*[X^t(V-\lambda I)X - Z^tZ]\Big((X-iZ)^{-1}\Big)Pv, Pv\Big)_{\mathbb{C}^n} \\ &= 2\Big([X^t(V-\lambda I)X - Z^tZ](X-iZ)^{-1}Pv, (X-iZ)^{-1}Pv\Big)_{\mathbb{C}^n}.\end{aligned}$$

where we've observed that for the Dirichlet case $\tilde{\mathfrak{B}} = I$. Recalling that $(X-iZ)^{-1}$ maps V^* to V^*, and that V^* is the kernel of X, we see that

$$\Big([X^t(V-\lambda I)X](X-iZ)^{-1}Pv, (X-iZ)^{-1}Pv\Big)_{\mathbb{C}^n} = 0,$$

and so

$$(\tilde{\Omega}Pv, Pv)_{\mathbb{C}^n} = -2\Big(Z^tZ(X-iZ)^{-1}Pv, (X-iZ)^{-1}Pv\Big)_{\mathbb{C}^n} \leq 0.$$

We conclude that crossings for the Dirichlet case must proceed in the clockwise direction as s increases. (We emphasize that we only require Dirichlet conditions at $x = 1$.) In particular, the Maslov index will always be non-increasing as s increases in this case. (See Figure 2.) Combining this observation with our definition of the Maslov index, we see that in the Dirichlet case we can write

$$\operatorname{Mor}(H) = \sum_{s\in[s_0,1)} \dim\ker(-\frac{d^2}{dx^2} + s^2V(sx)).$$

Remark 3.27. The preceding discussion illuminates the manner in which the current analysis is a generalization of the Sturm-Liouville oscillation theorem for $n = 1$. We see that in the case of Dirichlet conditions at $x = 1$, the relation of negative eigenvalues to zeros of the eigenfunction associated with $\lambda = 0$ is replaced by a relation of negative eigenvalues to the kernel of $X(s, 0)$. Precisely, we have

$$\operatorname{Mor}(H) = \sum_{s\in[s_0,1)} \dim\ker X(s, 0).$$

4. Applications

In this section we apply our framework to four illustrative examples. All calculations were carried out in MATLAB, and the figures were created in MATLAB.

We note at the outset that these calculations have been carried out to highlight certain observations in our analysis, and that in practice Theorem 1.5 only requires a calculation of the Principal Maslov Index (along with some matrix eigenvalues). Such a calculation is quite straightforward, and for convenient reference, we summarize it here.

Calculation of the Principal Maslov Index. We construct a frame $\mathbf{X} = \binom{X}{Z}$ by solving the ODE system (1.5) with initial values $\binom{X}{Z} = \binom{\alpha_2^t}{-\alpha_1^t}$. We then compute the spectral flow of $\tilde{W}_{s,\lambda}$ through the point $(-1,0)$; that is, we count the number of eigenvalues, including multiplicities, crossing $(-1,0)$ in the counterclockwise direction, and subtract the number crossing $(-1,0)$ in the clockwise direction.

Example 1 (Dirichlet Case). We consider (1.1) with

$$V(x) = \begin{pmatrix} -22 & 10\sin x \\ x & -20 \end{pmatrix},$$

and Dirichlet boundary conditions specified by $\alpha_1, \beta_1 = I$, $\alpha_2, \beta_2 = 0$. In this case, there can be no crossings along the bottom shelf, and indeed the only allowable behavior is for the eigenvalue curves to enter the box through Γ_2 and move upward until exiting through Γ_3. See Figure 2. The Principal Maslov Index in this case is -2, and according to Theorem 1.5 this means the Morse index is 2, consistent with our figure.

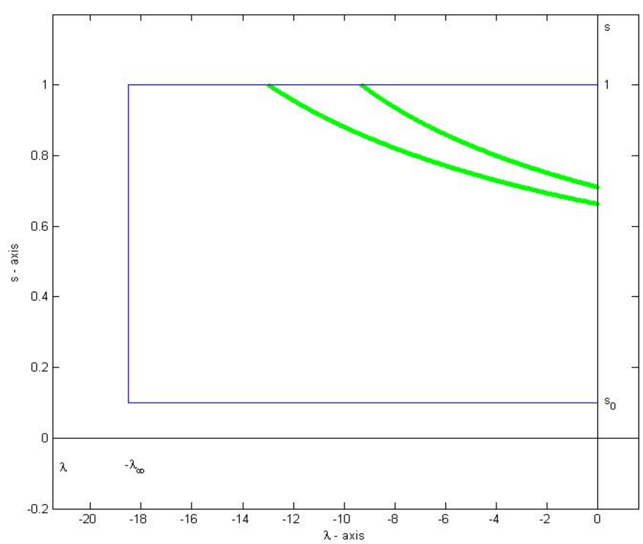

Figure 2. Eigenvalue curves for Example 1: Dirichlet case.

Example 2 (Neumann Case). We consider (1.1) with

$$V(x) = \begin{pmatrix} -.13 - \frac{.7\cos(6\pi x)}{2+\cos(6\pi x)} & 0 \\ -\frac{\cos(\pi x)}{2+\cos(4\pi x)} & 1 \end{pmatrix},$$

and Neumann boundary conditions specified by $\alpha_1, \beta_1 = 0$, $\alpha_2, \beta_2 = I$. In this case, we see the emergence of an eigenvalue from the bottom shelf (corresponding with the second order term in our perturbation series), and we notice a very distinct loss of the monotonicity in s associated with the Dirichlet case. See Figure 3. The Principal Maslov Index in this case is 0, and according to Theorem 1.5 the Morse index of H is the Morse index of $V(0)$ (because $B = 0$ and $Q = I$). The eigenvalues of $V(0)$ are $-.3633$ and 1, so that $\mathrm{Mor}(V(0)) = 1$, and indeed we see that the eigenvalue emerges from $s = 0$ at $-.3633$.

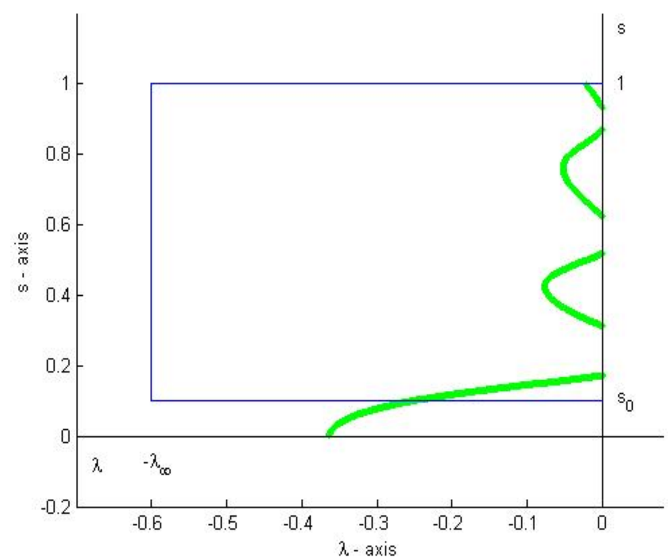

FIGURE 3. Eigenvalue curves for Example 2: Neumann case.

Example 3 (Neumann-based Case, I: First Order Perturbation Terms). We consider (1.1) with

$$V(x) = \begin{pmatrix} -13 + 12x^2 & -7\cos x \\ -x & -9 \end{pmatrix},$$

and Neumann-based boundary conditions specified by $\alpha_1 = \frac{1}{\sqrt{2}}I$, $\beta_1 = 0$, $\alpha_2 = \frac{1}{\sqrt{2}}I$, and $\beta_2 = I$. In this case, we see an eigenvalue curve entering through Γ_2, and also two curves entering through Γ_1 (corresponding with the first order term in our perturbation series). The Principal Maslov Index in this case is -1, and according to Theorem 1.5 the contribution from the bottom shelf to the Morse index of H will be the Morse index of $B = -\alpha_2^{-1}\alpha_2 = -I$, which is clearly 2. We conclude that $\mathrm{Mor}(H) = 3$, as indicated by Figure 4.

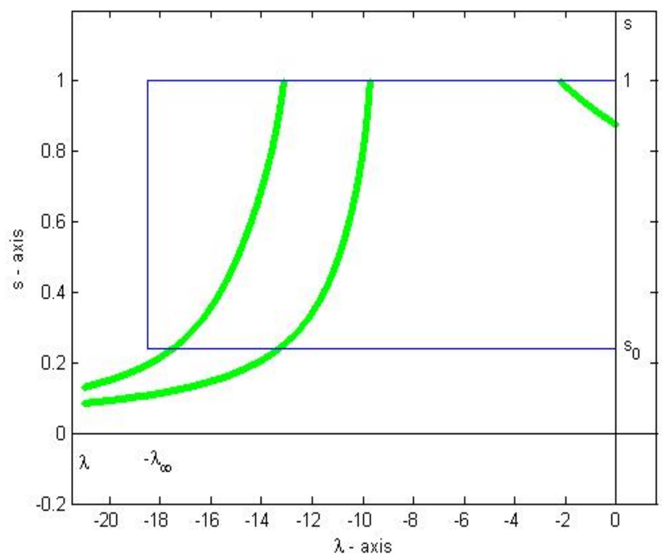

FIGURE 4. Eigenvalue curves for Example 3: First Order Perturbation Terms.

Example 4 (Neumann-based Case, II: Second Order Perturbation Terms). We consider (1.1) with

$$V(x) = \begin{pmatrix} -10 - 5x^2 & -3x \\ -9\sin x & -5 - 7x^2 \end{pmatrix},$$

and Neumann-based boundary conditions specified by $\alpha_1 = \frac{1}{\sqrt{2}} I$, $\beta_1 = \frac{1}{\sqrt{2}} I$, $\alpha_2 = \frac{1}{\sqrt{2}} I$, and $\beta_2 = \frac{1}{\sqrt{2}} I$. In this case, we see an eigenvalue curve entering through Γ_2, and two eigenvalue curves entering through Γ_1 (corresponding with the second order term in our perturbation series). The Principal Maslov Index in this case is -1, and according to Theorem 1.5 the contribution from the bottom shelf to the Morse index of H will be the Morse index of $V(0) - (\alpha_2^{-1}\alpha_1)^2$ (because $B = 0$ and $Q = I$). The eigenvalues of $V(0) - (\alpha_2^{-1}\alpha_1)^2$ are -11 and -6. We see that the Morse index of this matrix is 2, and indeed that the eigenvalues that come in through the bottom shelf originate when $s = 0$ at $\lambda = -11$ and $\lambda = -6$. We conclude that the Morse index of H is 3 in this case, as indicated in Figure 5.

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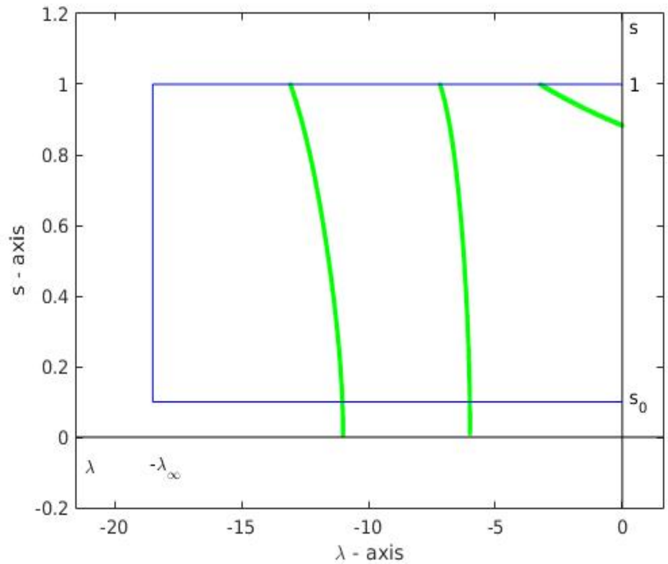

FIGURE 5. Eigenvalue curves for Example 4: Second Order Perturbation Terms.

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