Asymptotic behavior near planar transition fronts for the Cahn–Hilliard equation

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Abstract

We consider the asymptotic behavior of perturbations of planar wave solutions arising in the Cahn–Hilliard equation in space dimensions $d \geq 2$. Such equations are well known to arise in the study of spinodal decomposition, a phenomenon in which rapid cooling of a homogeneously mixed binary alloy causes separation to occur, resolving the mixture into regions in which one component or the other is dominant, with these regions separated by steep transition layers. A critical feature of the Cahn–Hilliard equation in one space dimension is that the linear operator that arises upon linearization of the equation about a standing wave solution has essential spectrum extending onto the imaginary axis, a feature that is known to complicate the step from spectral to nonlinear stability. The analysis of planar waves in multiple space dimensions is further complicated by the fact that the leading eigenvalue for this linearized operator (leading in the case of stability) moves into the negative-real half plane with cubic scaling $\lambda \sim |\xi|^3$, where $\xi \in \mathbb{R}^{d-1}$ denotes a Fourier variable associated with spatial components transverse to the planar wave. Under the assumption of spectral stability, described in terms of an appropriate Evans function, we develop detailed asymptotics for perturbations from planar wave solutions, establishing asymptotic stability for initial perturbations decaying with appropriate algebraic rate in an L^1 norm of the transverse variables.

1 Introduction

We consider the asymptotic behavior of perturbations of planar wave solutions $\bar{u}(x_1), \bar{u}(\pm\infty) = u_\pm$ arising as equilibrium solutions in the Cahn–Hilliard equation in multiple space dimensions $d \geq 2$

$$\begin{aligned} u_t &= \nabla \cdot \{M(u)\nabla(F'(u) - \nu\triangle u)\}; \quad u, M, F \in \mathbb{R}, t > 0, x \in \mathbb{R}^d \\ u(0,x) &= u_0(x), \end{aligned} \tag{1.1}$$

where ν is a positive constant and for which we will assume

(H0) $F \in C^4(\mathbb{R})$, $M \in C^2(\mathbb{R})$.

(H1) $F''(u_\pm) > 0$, $M(\bar{u}(x_1)) \geq M_0 > 0$, $x_1 \in \mathbb{R}$.

Our restriction to equations of form (1.1), in lieu of the more general form

$$u_t = \sum_{jk} (b^{jk}(u) u_{x_j})_{x_k} - \sum_{jklm} (c^{jklm}(u) u_{x_j x_k x_l})_{x_m}, \tag{1.2}$$

is taken both because of the particular physical interest in (1.1), and because of a particular difficulty that arises for equations of form (1.1) in the step from spectral to nonlinear stability. More precisely, it is well known that for equations of form (1.1), the leading eigenvalue for the linear operator that arises upon linearization about $\bar{u}(x_1)$ (leading in the case of stability) moves into the negative-real half plane with cubic scaling $\lambda \sim |\xi|^3$, where $\xi \in \mathbb{R}^{d-1}$ denotes a Fourier variable associated with spatial components transverse to the planar wave. Such a scaling, which is not necessarily present in the case of (1.2), complicates the step from spectral to nonlinear stability, and serves as the primary obstacle to be overcome in the current analysis.

The Cahn–Hilliard equation—often augmented by a driving or reaction term—arises in the study of several phenomena, including phase separation [10], growth and dispersal of biological populations [13], chemical reaction kinetics [30], image inpainting [3], and as the modulation equation for a viscous incompressible fluid under the action of an external force on an infinite strip [39, 43]. Our analysis is particularly motivated by the distinguished role the Cahn–Hilliard equation plays in the study of spinodal decomposition, a phenomenon in which rapid cooling of a homogeneously mixed binary alloy causes separation to occur, resolving the mixture into regions in which one component or the other is dominant, with these regions separated by steep transition layers. In this context, u typically denotes the concentration of one component of the binary alloy (or a convenient affine transformation of this concentration), and the Cahn–Hilliard equation arises from the conservation law

$$u_t + \nabla \cdot \vec{J} = 0, \tag{1.3}$$

where $\vec{J}$ denotes the flux of u. Letting $E(u)$ denote the total free energy associated with u, a standard phenomenological assumption is

$$\vec{J} = -M(u)\nabla \frac{\delta E}{\delta u}, \tag{1.4}$$

where $M(u)$ denotes the *mobility* associated with component u and is typically assumed positive. That is, the composition of the alloy tends to change from configurations for which a small change in concentration is accompanied by a large change in total free energy into configurations in which a small change in concentration is accompanied by a small change in total free energy. The Cahn–Hilliard equation as we state it arises from these considerations and a form of $E(u)$ proposed in 1958 by Cahn and Hilliard, who were trying to understand the interfacial energy between two components of a binary compound [11]. Taking $F(u)$ to denote the free energy density associated with a homogeneously arranged alloy with composition u, Cahn and Hilliard posed the energy functional

$$E(u) = \int_\Omega F(u) + \frac{\nu}{2}|\nabla u|^2 dx, \tag{1.5}$$

where Ω denotes some bounded open subset of $\mathbb{R}^3$ and the term $\frac{\nu}{2}|\nabla u|^2$ arises as a first order correction accounting for interfacial energy. (In fact, the functional $E(u)$ was originally proposed by van der Waals in [45] as an appropriate energy for a two-phase system.) In the case of relatively high temperatures, we expect $F(u)$ to have a quadratic form, signifying that entropy is minimized for homogenously mixed configurations with $u \equiv u_h =$ constant. (In this case, the second order term corresponds with standard diffusion.) On the other hand, as temperature drops, free energy increases at a rate proportional to entropy, and a pair of wells forms on either side of the original free energy minimum (and entropy maximum), leading to double-well forms of F such as

$$F = \frac{1}{8}u^4 - \frac{1}{4}u^2. \tag{1.6}$$

In this case, the original global minimizer u_h becomes a local maximum of $F(u)$, and consequently small pertubations from u_h do not dissipate, but rather propagate with quite complicated dynamics. (For a more detailed discussion of these dynamics, the reader is referred to Section 5 of [15].) As a step toward understanding the dynamics of such evolution, we would like to understand the stability of the possible stationary solutions, which might correspond with either transient or long time behavior of the system. If we let u_1 and u_2 denote the minimizing values of such an F, then there exist precisely two monotonic planar wave connections between u_1 and u_2, $\bar{u}(x_1)$ and $\bar{u}(-x_1)$. In the case of (1.6) (the case studied in the series of papers [9, 36, 37]) one readily verifies that

$$\bar{u}(x_1) = \tanh(\frac{x_1}{2\sqrt{\nu}}) \tag{1.7}$$

is such a solution. We note that it is clear from (1.7) that as $\nu \to 0$, the standing waves approach non-classical solutions in which the solution has a jump discontinuity.

A critical feature of equations of form (1.1) in one space dimension is that the linear operator that arises upon linearization of the equation about a standing wave solution has essential spectrum extending onto the imaginary axis, a feature that is known to complicate the step from spectral to nonlinear stability. The analysis of standing waves in multiple space dimensions is further complicated by the cubic scaling of the leading eigenvalue for this linearized operator (discussed immediately following (1.2)). The purpose of this paper is first to understand this dispersion relation in terms of an appropriate Evans function, and second to study the step from spectral to nonlinear stability. In particular, under the assumption of spectral stability (described below in terms of an appropriate Evans function and verified in the particular case (1.1) with (1.6), $M(u) \equiv 1$, and $\nu = 1$, and with wave (1.7)), we develop detailed asymptotics for perturbations from standing wave solutions, establishing phase-asymptotic orbital stability for initial perturbations $v_0(x)$ satisfying

$$\int_{\mathbb{R}^{d-1}} |v_0(x)| d\tilde{x} \leq E_0(1 + |x_1|)^{-3/2},$$

for some sufficiently small constant E_0, where $\tilde{x} = (x_2, x_3, ..., x_d)$ is the tranverse coordinate vector.

Our approach to this problem will be to extend to this setting methods developed previously in the context of conservation laws with diffusive and/or dispersive regularity,

$$u_t + f(u)_x = (b(u)u_x)_x + (c(u)u_{xx})_x + ..., \tag{1.8}$$

which also have no spectral gap. More precisely, we proceed by computing pointwise estimates on the Green's function for the linear equation that arises upon linearization of (1.1) about the wave $\bar{u}(x_1)$ (employing a contour-shifting approach introduced in [23, 46] and extended to the case of multiple space dimensions in [21, 22, 27, 28]; see also [38]). Such estimates are dependent upon the spectrum of the linear operator, which we understand here in terms of an appropriate Evans function (see, for example, [14, 16, 34, 46, 47] and the discussion and references below). Finally, we employ the local tracking method developed in [29] (and [27, 28] in the case of multiple space dimensions, though see also the very closely related approach of [21, 22]), an approach through which Green's function estimates on the linearized operator can be used to approximately locate shifts from the planar wave $\bar{u}$. (See also [12] in which tracking is carried out in an entirely different manner, through consideration of the time evolution of the energy (1.5).) Our general approach is similar to the analysis of [9] (in one space dimension), in which case the authors also employ Green's function estimates on the linear operator in order to close an iteration on the perturbation in some appropriately weighted space. A fundamental difference between the two analyses is that in [9], this iteration is carried out by the renormalization group method, a theory that has its origins in particle physics (see [6]) and was introduced in the context of time-asymptotic behavior for nonlinear PDE in [17, 18, 19], and further developed in [7, 8] (it pre-dates the current approach by about eight years). Briefly, the renormalization group method is an approach toward understanding asymptotic behavior of PDE that makes use of certain natural scalings in the PDE. In the context of the equation that arises upon linearization of (1.1) about the wave $\bar{u}(x_1)$, this natural scaling is the same as one would use for the heat equation. The difficulty with such a scaling technique in the context of the Cahn–Hilliard equation arises in the extension to multiple space dimensions, in which case the leading eigenvalue of the linearized operator introduces a new scale into the problem. (See the remarks following (1.2).) A different approach is taken in this setting in [36, 37], quite similar to the method employed here, and the authors conclude stability in dimensions $d \geq 3$ for the planar wave

$$\bar{u}(x_1) = \tanh \frac{x_1}{2},$$

arising in (1.1) with (1.6), $M(u) \equiv 1$ and $\nu = 1$. The primary difference between the approach of [36, 37] and the current analysis is the local tracking function $\delta(t, \tilde{x})$, $\tilde{x} = (x_2, x_3, ..., x_d)$. It is precisely this local tracking that allows us here to obtain stability for the case of dimension $d = 2$, which was left open in [36, 37].

It is well known that for the case of one space dimension solutions $u(t, x)$ of the Cahn–Hilliard equation initialized by $u(0, x)$ near a standing wave solution $\bar{u}(x)$ will not generally approach $\bar{u}(x)$ time-asymptotically, but rather will approach a translate of $\bar{u}(x)$ determined by an integral of the initial perturbation. In [23], a local tracking function $\delta(t)$ was employed to track shifts so that at each time the shapes of $u(t, x)$ and $\bar{u}(x)$ were compared, not the relative positions. In the case $d \geq 2$, $u(t, x)$ does not approach a shifted wave asymptotically, but local shifts along the transition front serve to hinder the analysis (they reduce the rate of decay of the perturbations and consequently nonlinearities become more difficult to control). In the current analysis, we employ a shift function that depends both on t and the transverse variable $\tilde{x}$, defining our perturbation as in [27, 28, 21, 22] by

$$v(t, x) = u(t, x) - \bar{u}(x_1 - \delta(t, \tilde{x})). \tag{1.9}$$

Upon substitution of (1.9) into (1.1), we arrive at the perturbation equation

$$(\partial_t - L)v = (\partial_t - L)(\delta(t,\tilde{x})\bar{u}_{x_1}(x_1)) + \sum_{k=1}^{d} Q_{k\,y_k}, \tag{1.10}$$

with

$$Lv := \nabla \cdot \{M(\bar{u}(x_1))\nabla(F''(\bar{u}(x_1))v - \nu\triangle v)\}, \tag{1.11}$$

where we have used the observation that for $\bar{u}(x_1)$

$$\nabla(F'(\bar{u}(x_1)) - \nu\triangle\bar{u}) = 0,$$

and we define

$$\begin{aligned} Q_1 &= \mathbf{O}(e^{-\eta|x_1|}|\delta\delta_t|) + \mathbf{O}(e^{-\eta|x_1|}v^2) + \mathbf{O}(e^{-\eta|x_1|}|\delta v|) + H_1 \\ Q_k &= H_k, \quad k = 2,3,...,d, \end{aligned}$$

with finally

$$\begin{aligned} H_k(t,\tilde{x}) &= \mathbf{O}(|vv_{x_k}|) + \mathbf{O}(|v\partial_{x_k}\triangle v|) + \mathbf{O}(e^{-\eta|x_1|}|\delta\delta_{x_k}|) + \mathbf{O}(e^{-\eta|x_1|}|v\delta_{x_k}|) + \mathbf{O}(e^{-\eta|x_1|}|\delta v_{x_k}|) \\ &+ \sum_{j=2}^{d}\Big[\mathbf{O}(e^{-\eta|x_1|}|\delta\delta_{x_jx_j}|) + \mathbf{O}(e^{-\eta|x_1|}|\delta_{x_j}|^2) + \mathbf{O}(e^{-\eta|x_1|}|\delta_{x_k}\delta_{x_jx_j}|) + \mathbf{O}(e^{-\eta|x_1|}|\delta\delta_{x_jx_jx_k}|) \\ &+ \mathbf{O}(e^{-\eta|x_1|}|\delta_{x_j}\delta_{x_jx_k}|) + \mathbf{O}(e^{-\eta|x_1|}|v\delta_{x_jx_jx_k}|) + \mathbf{O}(e^{-\eta|x_1|}|\delta\partial_{x_k}\triangle v|) + \mathbf{O}(e^{-\eta|x_1|}|v\delta_{x_jx_j}|)\Big] \end{aligned} \tag{1.12}$$

Letting $G(t,x;y)$ denote a Green's function associated with the operator L,

$$G_t = LG; \quad G(0,x;y) = \delta_y(x) \tag{1.13}$$

we integrate (1.10) to obtain

$$\begin{aligned} v(t,x) &= \int_{\mathbb{R}^d} G(t,x;y)v_0(y)dy \\ &+ \int_0^t \int_{\mathbb{R}^d} G(t-s,x;y)\Big[(\partial_s - L)(\delta(s,\tilde{y})\bar{u}_{y_1}(y_1)) + \sum_{k=1}^{d} Q_{k\,y_k}\Big]dyds. \end{aligned} \tag{1.14}$$

According to Lemma 2.2 of [28], we can simplify this somewhat through the relation

$$\int_0^t \int_{\mathbb{R}^d} G(t-s,x;y)\Big[(\partial_s - L)(\delta(s,\tilde{y})\bar{u}_{y_1}(y_1))\Big]dyds = \delta(t,\tilde{x})\bar{u}_{x_1}(x_1), \tag{1.15}$$

where we have anticipated here a choice of $\delta(t,\tilde{x})$ so that $\delta(0,\tilde{x}) \equiv 0$. We find

$$\begin{aligned} v(t,x) &= \int_{\mathbb{R}^d} G(t,x;y)v_0(y)dy + \delta(t,\tilde{x})\bar{u}_{x_1}(x_1) \\ &+ \int_0^t \int_{\mathbb{R}^d} G(t-s,x;y)\sum_{k=1}^{d} Q_{k\,y_k}dyds. \end{aligned} \tag{1.16}$$

In order to select an appropriate tracking function $\delta(t,\tilde{x})$, we divide $G(t,x;y)$ into two terms

$$G(t,x;y) = \tilde{G}(t,x;y) + \bar{u}_{x_1}(x_1)E(t,\tilde{x},y),$$

where $E(t,\tilde{x},y)$ is characterized by a slower rate of decay in t (than $\tilde{G}(t,x;y)$, see Theorem 1.1); in particular, the best transverse L^1 estimate that we can obtain on E is

$$\|E(t,\tilde{x},y)\|_{L^1_{\tilde{x}}} = \int_{\mathbb{R}^{d-1}} |E(t,\tilde{x},y)|d\tilde{x} \le C.$$

With this notation, our integral equation (1.16) becomes

$$\begin{aligned} v(t,x) &= \int_{\mathbb{R}^d} \tilde{G}(t,x;y)v_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \tilde{G}(t-s,x;y)\sum_{k=1}^d Q_{k_{y_k}} dyds \\ &+ \delta(t,\tilde{x})\bar{u}_{x_1}(x_1) + \bar{u}_{x_1}(x_1)\int_{\mathbb{R}^d} E(t,\tilde{x};y)v_0(y)dy \\ &+ \bar{u}_{x_1}(x_1)\int_0^t \int_{\mathbb{R}^d} E(t-s,\tilde{x};y)\sum_{k=1}^d Q_{k_{y_k}} dyds. \end{aligned} \tag{1.17}$$

We now choose $\delta(t,\tilde{x})$ in such a way that the slowly decaying $E(t,\tilde{x};y)$ are all eliminated from the integral equation for v:

$$\delta(t,\tilde{x}) = -\int_{\mathbb{R}^d} E(t,\tilde{x};y)v_0(y)dy - \int_0^t \int_{\mathbb{R}^d} E(t-s,\tilde{x};y)\sum_{k=1}^d Q_{k_{y_k}} dyds. \tag{1.18}$$

Our final integral equation for v is then

$$v(t,x) = \int_{\mathbb{R}^d} \tilde{G}(t,x;y)v_0(y)dy - \int_0^t \int_{\mathbb{R}^d} \sum_{k=1}^d \tilde{G}_{y_k}(t-s,x;y)Q_k dyds, \tag{1.19}$$

where aside from the cancellation due to our definition of $\delta(t,\tilde{x})$, we have integrated by parts. Integral equations for derivatives of v and δ can be obtained through direct differentiation of (1.18) and (1.19).

Our approach will be to determine estimates on $\tilde{G}(t,x;y)$ and $E(t,\tilde{x};y)$ sufficient for closing a simultaneous iteration on the variables $\partial_x^\alpha v$, for multi-index $|\alpha| \le 1$ and $\partial_{\tilde{x}}^\beta \delta$ for multi-index $|\beta| \le 3$. Observing that the coefficients for our Green's function equation (1.13) depend only on x_1, we take a Fourier transform in $\tilde{x}$ (scaling the transform as $\int_{\mathbb{R}^{d-1}} e^{-i\xi\cdot\tilde{x}} G(t,x;y)d\tilde{x}$) to obtain

$$\begin{aligned} \hat{G}_t = L_\xi \hat{G} &:= \Big(M(\bar{u}(x_1))(F''(\bar{u}(x_1))\hat{G} - \nu \hat{G}_{x_1x_1})_{x_1}\Big)_{x_1} \\ &- |\xi|^2 M(\bar{u}(x_1))F''(\bar{u}(x_1))\hat{G} + \nu|\xi|^2 (M(\bar{u}(x_1))\hat{G}_{x_1})_{x_1} + \nu|\xi|^2 M(\bar{u}(x_1))\hat{G}_{x_1x_1} - \nu|\xi|^4 M(\bar{u}(x_1))\hat{G}, \end{aligned} \tag{1.20}$$

where we will often write L_ξ in the notation of [23],

$$\begin{aligned} L_\xi \phi &= -(c(x_1)\phi_{x_1x_1x_1})_{x_1} + (b(x_1)\phi_{x_1})_{x_1} - (a(x_1)\phi)_{x_1} \\ &+ |\xi|^2 \Big[(c(x_1)\phi_{x_1})_{x_1} + c(x_1)\phi_{x_1x_1}\Big] - \Big[|\xi|^2 b(x_1) + |\xi|^4 c(x_1)\Big]\phi, \end{aligned} \tag{1.21}$$

where

$$\begin{aligned} b(x_1) &= M(\bar{u}(x_1))F''(\bar{u}(x_1)) \\ c(x_1) &= \nu M(\bar{u}(x_1)) \\ a(x_1) &= -M(\bar{u}(x_1))F'''(\bar{u}(x_1))\bar{u}_{x_1}. \end{aligned}$$

According to hypotheses (H0) and (H1), we have that $a \in C^1(\mathbb{R})$, $b, c \in C^2(\mathbb{R})$ and for $k = 0, 1$

$$|\partial_{x_1}^k a(x_1)| = \mathbf{O}(e^{-\alpha|x_1|}); \quad |\partial_{x_1}^k (b(x_1) - b_\pm)| = \mathbf{O}(e^{-\alpha|x_1|}); \quad |\partial_{x_1}^k (c(x_1) - c_\pm)| = \mathbf{O}(e^{-\alpha|x_1|}), \tag{1.22}$$

where $\alpha > 0$ and $\pm$ represent the asymptotic limits as $x_1 \to \pm\infty$. We now analyze $\hat{G}(t,x;y)$ through its Laplace transform $G_{\lambda,\xi}(x,y)$, which satisfies the ODE (t transformed to λ)

$$L_\xi G_{\lambda,\xi} - \lambda G_{\lambda,\xi} = -e^{-i\xi\cdot\tilde{y}}\delta_{y_1}(x_1). \tag{1.23}$$

Letting $\phi_1^-(x_1;\lambda,\xi)$ and $\phi_2^-(x_1;\lambda,\xi)$ denote the two linearly independent asymptotically decaying solutions at $-\infty$ of

$$L_\xi \phi = \lambda \phi \tag{1.24}$$

(for λ away from essential spectrum), and $\phi_1^+(x_1;\lambda,\xi)$ and $\phi_2^+(x_1;\lambda,\xi)$ similarly the two linearly independent asymptotically decaying solutions at $+\infty$, we construct the ODE Green's function as

$$G_{\lambda,\xi}(x_1,y) = \begin{cases} \phi_1^-(x_1;\lambda,\xi)N_1^+(y;\lambda,\xi) + \phi_2^-(x_1;\lambda,\xi)N_2^+(y;\lambda,\xi), & x_1 < y_1 \\ \phi_1^+(x_1;\lambda,\xi)N_1^-(y;\lambda,\xi) + \phi_2^+(x_1;\lambda,\xi)N_2^-(y;\lambda,\xi), & x_1 > y_1. \end{cases}, \tag{1.25}$$

where the $N_k^\pm(y;\lambda,\xi)$ are expansion coefficients determined in the following manner: Insisting, as usual, on the continuity of $G_{\lambda,\xi}(x_1,y)$ and its first two x_1-derivatives in x_1, and on the jump in $\partial_{x_1}^3 G_{\lambda,\xi}(x_1,y)$ at $x_1 = y_1$, we have

$$\begin{aligned} N_1^+(y;\lambda,\xi) &= +(2\pi)^{\frac{1-d}{2}} e^{-i\tilde{y}\cdot\xi} \frac{W(\phi_1^+,\phi_2^+,\phi_2^-)}{c(y_1)W_{\lambda,\xi}(y_1)} \\ N_2^+(y;\lambda,\xi) &= -(2\pi)^{\frac{1-d}{2}} e^{-i\tilde{y}\cdot\xi} \frac{W(\phi_1^+,\phi_2^+,\phi_1^-)}{c(y_1)W_{\lambda,\xi}(y_1)} \\ N_1^-(y;\lambda,\xi) &= -(2\pi)^{\frac{1-d}{2}} e^{-i\tilde{y}\cdot\xi} \frac{W(\phi_1^-,\phi_2^-,\phi_2^+)}{c(y_1)W_{\lambda,\xi}(y_1)} \\ N_2^-(y;\lambda,\xi) &= +(2\pi)^{\frac{1-d}{2}} e^{-i\tilde{y}\cdot\xi} \frac{W(\phi_1^-,\phi_2^-,\phi_1^+)}{c(y_1)W_{\lambda,\xi}(y_1)}, \end{aligned} \tag{1.26}$$

where for example

$$W(\phi_1^+,\phi_2^+,\phi_2^-) = \begin{pmatrix} \phi_1^+ & \phi_2^+ & \phi_2^- \\ \phi_1^{+\prime} & \phi_2^{+\prime} & \phi_2^{-\prime} \\ \phi_1^{+\prime\prime} & \phi_2^{+\prime\prime} & \phi_2^{-\prime\prime}, \end{pmatrix} \tag{1.27}$$

and more generally $W(\phi_1,\phi_2,...,\phi_n)$ denotes a square determinant of column vectors created by augmentation with an appropriate number of x_1-derivatives (i.e., a Wronskian) and $W_{\lambda,\xi}(y_1) := W(\phi_1^-,\phi_2^-,\phi_1^+,\phi_2^+)$. We see immediately from (1.25) and (1.26) that, away from essential spectrum, $G_{\lambda,\xi}(x_1,y)$ is bounded so long as $W_{\lambda,\xi}(y_1)$ is bounded away from 0. Since $W_{\lambda,\xi}(y_1)$ is a Wronskian for (1.24), for each fixed λ and ξ its sign does not change as y_1 varies. In light of this, we define the *Evans function* as

$$D(\lambda,\xi) := W_{\lambda,\xi}(0). \tag{1.28}$$

Introduced by Evans in the context of nerve impulse equations [14] (see also the early analysis of Jones for pulse solutions to the FitzHugh–Nagumo equation [34]), the Evans function serves as a characteristic function for the operator L_ξ. More precisely, away from essential spectrum, zeros of the Evans function correspond in location and multiplicity with eigenvalues of the operator L_ξ, an observation that has been made precise in [2] in the case—pertaining to reaction–diffusion equations—of isolated eigenvalues, and in [16, 46] and [35] in the cases—pertaining respectively to conservation laws and the nonlinear Schrödinger equation—of nonstandard "effective" eigenvalues embedded in essential spectrum. (The latter correspond with resonant poles of L_ξ, as also examined in [41].) Though defined here as a simple Wronskian, the Evans function has been analyzed by Evans, Jones and others more generally as an appropriate wedge product. See, for example, [14, 16, 34, 35], and the remarks in [23], regarding the Evans function in the case of the Cahn–Hilliard equation in one space dimension.

We will observe in Section 3 that the Evans function in this context is not analytic in λ, but can be defined analytically in terms of the variables

$$\begin{aligned} \kappa &:= |\xi|^2 \\ \rho_\pm &:= \frac{\sqrt{\lambda + b_\pm\kappa + c_\pm\kappa^2}}{b_\pm + 2c_\pm\kappa}. \end{aligned} \tag{1.29}$$

Our strategy for the critical case in which $|\lambda|$ and $|\xi|$ are both small will be to identify the triplets κ, ρ_-, and ρ_+ that correspond with $D = 0$ and use the relations (1.29) to express this as a relationship between λ and ξ (see Condition (1) below).

Throughout the analysis, we will refer to the following conditions (1) and (2) as spectral criteria ($\mathcal{D}$).

Condition (1). There is a neighborhood V of the origin in complex ξ-space and a value $r > 0$ so that for all $\xi \in V$ there exists an $L^2(\mathbb{R})$ eigenvalue $\lambda_*(\xi)$ of L_ξ that lies on the curve described by the relations $D(\lambda_*(\xi), \xi) = 0$, $\lambda(0) = 0$ and is contained in the disk $|\lambda| < r$. Moreover, for $\xi \in V$, $\lambda_*(\xi)$ is the only $L^2(\mathbb{R})$ eigenvalue of L_ξ in this disk, and $\lambda_*(\xi)$ satisfies

$$\lambda_*(\xi) = -\lambda_3|\xi|^3 + \mathbf{O}(|\xi|^4), \tag{1.30}$$

for some constant $\lambda_3 > 0$.

Condition (2). Outside the neighborhood described in Condition (1) (i.e., outside the region described by $\xi \in V$ and $|\lambda| < r$), and for $\xi = \xi_R + i\xi_I$, with $|\xi_I|$ sufficiently small, the point spectrum of L_ξ is contained to the left of a wedge described by

$$\text{Re } \lambda = -c_1\Big(|\xi_R|^4 - C_2|\xi_I|^4 + |\text{Im } \lambda|\Big),$$

where c_1 and C_2 are both positive constants.

The primary difficulty to be overcome in our analysis is the scaling specified in Condition (1), which is generic for standing planar waves in this setting, at least in the sense that we can show that the second order term is always 0. (See Lemma 3.5). Evidence for this scaling seems first to have appeared in the experimental work [20], while early analytic verification appeared in [33] and [44]. In [44] the authors additionally observe that this scaling appropriately corresponds with the experimentally and numerically observed asymptotic-growth law for the average pattern size P in the spinodal decomposition process; namely $P \sim t^{1/3}$ (references are given in [44]). As observed in [33], Condition (1) can be understood in a non-rigorous manner by noting that the eigenvalue problem (1.24) can be written in the form

$$D_\xi H_\xi \phi = -\lambda\phi, \tag{1.31}$$

where

$$H_\xi \phi := -\nu\phi'' + F''(\bar{u})\phi + \nu|\xi|^2\phi \tag{1.32}$$

and D_ξ is the positive, self-adjoint operator

$$D_\xi \phi := -(M(\bar{u})\phi')' + |\xi|^2 M(\bar{u})\phi. \tag{1.33}$$

Since D_ξ is positive and self-adjoint for $|\xi| \neq 0$, it has a well-defined square root that is also self-adjoint, and we are justified in setting $\varphi = D_\xi^{-1/2}\phi$. In this way, φ can be seen to solve the self-adjoint eigenvalue problem

$$\mathcal{L}_\xi \varphi := D_\xi^{1/2} H_\xi D_\xi^{1/2} \varphi = -\lambda\varphi, \tag{1.34}$$

and indeed it is easy to see that for $|\xi| \neq 0$ the eigenvalues of $\mathcal{L}_\xi$ correspond precisely with those of L_ξ. Letting now $\langle \cdot, \cdot \rangle$ denote the $L^2(\mathbb{R})$ inner product, we have

$$\langle \varphi, \mathcal{L}_\xi \varphi \rangle = \langle \varphi, D_\xi^{1/2} H_\xi D_\xi^{1/2} \varphi \rangle = \langle D_\xi^{1/2}\varphi, H_\xi D_\xi^{1/2}\varphi \rangle = \langle D_\xi^{1/2}\varphi, H_0 D_\xi^{1/2}\varphi \rangle + \kappa|\xi|^2 \langle D_\xi^{1/2}\varphi, D_\xi^{1/2}\varphi \rangle,$$

where $H_0 := -\nu\partial^2_{x_1 x_1} + F''(\bar{u})$ is known from [26] to be a positive operator. Since $\mathcal{L}_\xi$ is a self-adjoint operator, bounded from below, the min–max principle (see e.g. [42], Theorem XIII.1) gives that the leading eigenvalue $-\lambda_*(\xi)$ satisfies

$$-\lambda_*(\xi) = \inf_{\varphi \in H^2 \setminus \{0\}} \frac{\langle \varphi, \mathcal{L}_\xi \varphi \rangle}{\langle \varphi, \varphi \rangle} = \inf_{\varphi \in H^2 \setminus \{0\}} \Big[\frac{\langle D_\xi^{1/2}\varphi, H_0 D_\xi^{1/2}\varphi \rangle}{\langle \varphi, \varphi \rangle} + \kappa|\xi|^2 \frac{\langle D_\xi \varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} \Big]. \tag{1.35}$$

Since the eigenfunction associated with $\lambda_*(0)$ is $\bar{u}_{x_1}$ (or can be scaled as such), we might expect that the leading eigenvalue of $\mathcal{L}_\xi$ is roughly $D_\xi^{-1/2}\bar{u}_{x_1}$, giving

$$-\lambda_*(\xi) \sim \frac{\langle D_\xi^{-1/2}\bar{u}_{x_1}, \mathcal{L}_\xi D_\xi^{-1/2}\bar{u}_{x_1} \rangle}{\langle D_\xi^{-1/2}\bar{u}_{x_1}, D_\xi^{-1/2}\bar{u}_{x_1} \rangle}. \tag{1.36}$$

Noting that $H_0\bar{u}_{x_1} = 0$, we heuristically have

$$-\lambda_*(\xi) \sim \kappa|\xi|^2 \frac{\langle \bar{u}_{x_1}, \bar{u}_{x_1}\rangle}{\langle D_\xi^{-1}\bar{u}_{x_1}, \bar{u}_{x_1}\rangle}. \tag{1.37}$$

Finally, the inner product $\langle D_\xi^{-1}\bar{u}_{x_1}, \bar{u}_{x_1}\rangle$ is straightforward to compute in terms of the Green's function $g(x_1, y_1; \xi)$ associated with D_ξ, which asymptotically satisfies

$$g(x_1, y_1; \xi) \sim \frac{1}{|\xi|} e^{-|\xi||x_1 - y_1|},$$

and it is easily concluded that

$$\langle \bar{u}_{x_1}, \bar{u}_{x_1}\rangle \langle D_\xi^{-1}\bar{u}_{x_1}, \bar{u}_{x_1}\rangle \sim \begin{cases} \frac{1}{|\xi|} & |\xi| \le \epsilon \\ \frac{1}{|\xi|^2} & |\xi| \ge \epsilon, \end{cases}$$

for some sufficiently small $\epsilon > 0$. The scaling in Condition (1) is immediate from this last expression, and we also see that there is a change in form as $|\xi|$ grows. (The difficulty in making this argument rigorous lies with the essential spectrum of L_ξ, which for $\xi = 0$ extends all the way to the leading eigenvalue $\lambda_*(0) = 0$. Consequently, standard perturbation methods do not apply, and we cannot necessarily regard this as the lowest order of a valid expansion.)

In the case of (1.1) with (1.6) and $M(u) \equiv 1$, spectral conditions (1) and (2) have been (rigorously) shown to hold in [36] (Lemma 1.3; see also [37]). These conditions have also been established in [44], aside from one small gap in the analysis (see the final paragraph in the first column of p. 806). Arguments based on the perturbation ideas outlined above appear in [5, 33]). More generally, such conditions can be verified numerically [4, 32, 31, 40].

We are now in a position to state the first theorem of the paper.

Theorem 1.1. *Suppose $\bar{u}(x_1)$ is a planar wave solution to (1.1) and suppose structural hypotheses (H0)–(H1) hold, as well as spectral criterion ($\mathcal{D}$). Then for some fixed C, and for positive contants M and K sufficiently large, and for $\eta > 0$, depending only on the spectrum and coefficients of L_ξ, the Green's function $G(t, x; y)$ described in (1.13) satisfies the following estimates for $y_1 \le 0$ (with symmetric estimates in the case $y_1 \ge 0$).*

(I) For either $|x - y| \ge Kt$ or $t \le 1$, and for α a multi-index in the variables x and y,

$$\|\partial^\alpha G(t, x; y)\|_{L^p_{\tilde{x}}} \le C t^{-\frac{d-1}{4}(1-\frac{1}{p}) - \frac{1+|\alpha|}{4}} e^{-\frac{|x_1-y_1|^{4/3}}{Mt^{1/3}}} + Ce^{-\eta(|x_1-y_1|+t)}, \quad |\alpha| \le 3.$$

(II) For $|x - y| \le Kt$ and $t \ge 1$, there exists a splitting

$$G(t, x; y) = \tilde{G}(t, x; y) + \bar{u}_{x_1}(x_1)E(t, \tilde{x}; y),$$

such that the following estimates hold: For $\tilde{G}(t, x; y)$, let β denote a multi-index in the transverse variables $\tilde{x}$ and $\tilde{y}$, $|\beta| \le 1$. Then for $\sigma = 0$ in the case $d = 2$ and any $\sigma > 0$ sufficiently small in the cases $d \ge 3$,

(i) $y_1, x_1 \le 0$

$$\|\partial^\beta \tilde{G}(t, x; y)\|_{L^p_{\tilde{x}}} \le C t^{-\frac{d-1}{2}(1-\frac{1}{p}) - \frac{|\beta|+1}{2}} \Big[e^{-\frac{(x_1-y_1)^2}{4b_- t}} - e^{-\frac{(x_1+y_1)^2}{4b_- t}} \Big] + Ct^{-\frac{d-1}{2}(1-\frac{1}{p}) - \frac{|\beta|+2}{2}} e^{-\frac{(x_1-y_1)^2}{Mt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p}) - \frac{|\beta|+2}{3} + \sigma} I_{\{|x_1-y_1| \le t^{1/2}\}}.$$

(ii) $y_1 \le 0 \le x_1$

$$\|\partial^\beta \tilde{G}(t, x; y)\|_{L^p_{\tilde{x}}} \le Ct^{-\frac{d-1}{2}(1-\frac{1}{p}) - \frac{|\beta|+2}{2}} e^{-\frac{(x_1-y_1)^2}{Mt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p}) - \frac{|\beta|+2}{3} + \sigma} I_{\{|x_1-y_1| \le t^{1/2}\}}.$$

For the estimates on $E(t,\tilde{x},y)$ the range of β can be extended to $|\beta|\le 3$, and we have the estimates,

$$\|\partial^\beta E(t,\tilde{x};y)\|_{L^p_{\tilde{x}}} \le Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|}{2}}e^{-\frac{y_1^2}{Mt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|}{3}+\sigma}I_{\{|y_1|\le t^{1/2}\}},$$

with also

$$\|\partial^\beta E_{y_1}(t,\tilde{x};y)\|_{L^p_{\tilde{x}}} \le Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|}{2}}e^{-\frac{y_1^2}{Mt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|}{3}+\sigma}I_{\{|y_1|\le t^{1/2}\}}.$$

Moreover,

$$\|\partial_t E(t,\tilde{x};y)\|_{L^p_{\tilde{x}}} \le Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-1}e^{-\frac{y_1^2}{Mt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{2}{3}+\sigma}I_{\{|y_1|\le t^{1/2}\}}.$$

Employing the estimates of Theorem 1.1 (augmented by the similar derivative estimates of Theorem 5.1) with the integral representations (1.19) and (1.18), we can establish the following theorem regarding the perturbation $v(t,x)$ and the local tracking function $\delta(t,\tilde{x})$.

Theorem 1.2. *Suppose $\bar{u}(x_1)$ is a planar wave solution to (1.1) and suppose structural hypotheses (H0)–(H1) hold, as well as spectral criterion $(\mathcal{D})$. Then for Hölder continuous initial perturbations $(u(0,x)-\bar{u}(x))\in C^{0+\gamma}(\mathbb{R}^d)$, $\gamma>0$, with*

$$\|u(0,x)-\bar{u}(x)\|_{L^1_{\tilde{x}}} \le E_0(1+|x_1|)^{-3/2}, \tag{1.38}$$

for some E_0 sufficiently small, and for $\delta(t,\tilde{x})$ as implicitly defined in (1.18) and σ as in Theorem 1.1, there holds

$$\|u(t,x)-\bar{u}(x_1-\delta(t,\tilde{x}))\|_{L^p_{\tilde{x}}} \le CE_0\Big[(1+t)^{-\frac{d-1}{2}(1-\frac{1}{p})} + (1+t)^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{6}+\sigma}h_d(t)\Big]\Theta(t,x_1),$$

with

$$\|\partial^\beta_{\tilde{x}}\delta(t,\tilde{x})\|_{L^p_{\tilde{x}}} \le CE_0(1+t)^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|}{3}+\sigma},$$

where $|\beta|\le 1$,

$$\Theta(t,x_1) = (1+t)^{-1/2}e^{-\frac{x_1^2}{Lt}} + (1+|x_1|+\sqrt{t})^{-\frac{3}{2}},$$

and

$$h_d(t) = \begin{cases} \ln t & d=2 \\ 1 & d\ge 3 \end{cases}$$

Moreover, we have the derivative estimates

$$\|u_{x_1}(t,x)-\bar{u}'(x_1-\delta(t,\tilde{x}))\|_{L^p_{\tilde{x}}} \le CE_0t^{-1/4}\Big[(1+t)^{-\frac{d-1}{3}(1-\frac{1}{p})+\frac{1}{12}+\sigma}\Theta(t,x_1) + (1+t)^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{5}{12}}h_d(t)e^{-\eta|x_1|}\Big]$$

and for $k=2,3,...,d$,

$$\|\partial_{x_k}\Big(u(t,x)-\bar{u}(x_1-\delta(t,\tilde{x}))\Big)\|_{L^p_{\tilde{x}}} \le CE_0t^{-1/4}(1+t)^{-\frac{d-1}{3}(1-\frac{1}{p})+\frac{1}{12}}\Theta(t,x_1).$$

Outline of the paper. In Section 2, we state a lemma regarding the existence of standing waves arising as solutions of equations of form (1.1), while in Section 3, we analyze the Evans function associated with such waves. The proofs of Theorem 1.1 and Theorem 1.2 are given respectively in the final two sections.

2 Existence and structure of the planar waves

In this section, we re-state for the current setting a result of Aifantis and Serrin on the existence and structure of planar wave solutions $\bar{u}(x_1)$ to (1.1) (see [1]). We first state precisely what we will mean by a double well function.

Definition 2.1. *We will say that a function F has a double well form if there exist real numbers $\alpha_1<\alpha_2<\alpha_3<\alpha_4<\alpha_5$ so that F is strictly decreasing on $(-\infty,\alpha_1)$ and on (α_3,α_5) and strictly increasing on (α_1,α_3) and $(\alpha_5,+\infty)$, and additionally F is concave up on $(-\infty,\alpha_2)\cap(\alpha_4,+\infty)$ and concave down on (α_2,α_4).*

The main result of this section is Lemma 2.1.

Lemma 2.1. *Suppose (H0) and (H1) hold for a pair of functions $F(u)$ and $M(u)$, and that $F(u)$ has a double well form as defined in Definition 2.1. Then there exist unique values u_1 and u_2, with $u_1 < u_2$ and*

$$F'(u_1) = F'(u_2) = \frac{[F]}{[u]}, \tag{2.1}$$

where

$$[F] := F(u_2) - F(u_1); \quad [u] := (u_2 - u_1),$$

so that a monotonic increasing standing wave solution $\bar{u}(x_1)$ to (1.1) exists, connecting u_1 on the left to u_2 on the right. Moreover, the only other stationary solutions to (1.1) for which

$$\lim_{x_1 \to \pm\infty} u(x_1) = u_\pm; \quad u_+ \neq u_-$$

are the monotonic decreasing wave $\bar{u}(-x_1)$ and constant shifts of $\bar{u}(x_1)$ and $\bar{u}(-x_1)$.

Condition (2.1) is easily understood graphically: given a double well form $F(u)$, the points u_1 and u_2 satisfying condition (2.1) correspond precisely with the unique pair of points at which a line raised from below the graph of F would touch F simultaneously with the same tangency.

3 The Evans function

In this section, we analyze the Evans function as defined in (1.28). We begin by writing our eigenvalue problem (1.24) as a first order system

$$W' = \mathbb{A}(x_1; \lambda, \xi) W, \tag{3.1}$$

where

$$\mathbb{A}(x_1;\lambda,\xi) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\lambda + a'(x_1) + |\xi|^2 b(x_1) + |\xi|^4 c(x_1)}{c(x_1)} & +\frac{b'(x_1) - a(x_1) + |\xi|^2 c'(x_1)}{c(x_1)} & \frac{b(x_1) + 2|\xi|^2 c(x_1)}{c(x_1)} & -\frac{c'(x_1)}{c(x_1)} \end{pmatrix}.$$

Under assumptions (H0) and (H1), $\mathbb{A}(x_1; \lambda, \xi)$ has the asymptotic behavior

$$\mathbb{A}(x_1;\lambda,\xi) = \begin{cases} \mathbb{A}_-(\lambda,\xi) + \mathbb{E}(x_1;\lambda,\xi), & x_1 < 0 \\ \mathbb{A}_+(\lambda,\xi) + \mathbb{E}(x_1;\lambda,\xi), & x_1 > 0, \end{cases}$$

where

$$\mathbb{A}_\pm(\lambda,\xi) := \lim_{x_1 \to \pm\infty} \mathbb{A}(x_1;\lambda,\xi) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\lambda + b_\pm |\xi|^2 + c_\pm |\xi|^4}{c_\pm} & 0 & \frac{b_\pm + 2|\xi|^2 c_\pm}{c_\pm} & 0 \end{pmatrix}, \tag{3.2}$$

and for $|\lambda|$ and $|\xi|$ both bounded $\mathbb{E}(x_1;\lambda,\xi) = \mathbf{O}(e^{-\alpha|x_1|})$. The eigenvalues of the matrices $\mathbb{A}_\pm(\lambda,\xi)$, denoted here by $\mu_\pm$ satisfy

$$c_\pm \mu_\pm^4 - (b_\pm + 2|\xi|^2 c_\pm)\mu_\pm^2 + (\lambda + b_\pm|\xi|^2 + c_\pm|\xi|^4) = 0, \tag{3.3}$$

or equivalently one of

$$\mu_\pm^2 = \frac{(b_\pm + 2|\xi|^2 c_\pm) - \sqrt{b_\pm^2 - 4c_\pm\lambda}}{2c_\pm},$$

$$\mu_\pm^2 = \frac{(b_\pm + 2|\xi|^2 c_\pm) + \sqrt{b_\pm^2 - 4c_\pm\lambda}}{2c_\pm}.$$

In terms of the variables (1.29), we can write these eigenvalues as

$$
\begin{aligned}
\mu_1^\pm &= -\sqrt{(\frac{b_\pm}{2c_\pm}+\kappa)}\sqrt{1+\sqrt{1-4c_\pm\rho_\pm^2}}\\
\mu_2^\pm &= -\sqrt{(\frac{b_\pm}{2c_\pm}+\kappa)}\frac{2\sqrt{c_\pm}\rho_\pm}{\sqrt{1+\sqrt{1-4c_\pm\rho_\pm^2}}}\\
\mu_3^\pm &= +\sqrt{(\frac{b_\pm}{2c_\pm}+\kappa)}\frac{2\sqrt{c_\pm}\rho_\pm}{\sqrt{1+\sqrt{1-4c_\pm\rho_\pm^2}}}\\
\mu_4^\pm &= +\sqrt{(\frac{b_\pm}{2c_\pm}+\kappa)}\sqrt{1+\sqrt{1-4c_\pm\rho_\pm^2}},
\end{aligned}
\tag{3.4}
$$

where the slow eigenvalues $\mu_2^\pm$ and $\mu_3^\pm$ have been written in a form from which analyticity in κ and $\rho_\pm$ is apparent. (See the discussion of [23] just above Lemma 2.1. This development follows closely the notation of [23]; the reader is also referred to the almost identical development of [36], p. 11 and [37], p. 20, in which κ is replaced by k^2 and $\rho_\pm$ is replaced by $i\tau$.)

We are now in a position to state our main lemma on the asymptotic (in x_1) behavior of the growth and decay solutions of (1.24).

Lemma 3.1. *For the eigenvalue problem (1.24), with L_ξ as defined in (1.21) assume $a \in C^1(\mathbb{R})$, $b, c \in C^2(\mathbb{R})$, with $b_\pm > 0$ and $c_\pm > 0$, and additionally that (1.22) holds. Then for some $\bar{\alpha} > 0$ and $k = 0, 1, 2, 3$, we have the following estimates on a choice of linearly independent solutions of (1.24). For $|\lambda| + |\xi|^2 \le r$, some $r > 0$ sufficiently small, there holds:*

(i) For $x_1 \le 0$

$$
\begin{aligned}
\partial_{x_1}^k \phi_1^-(x_1;\lambda,\xi) &= e^{\mu_3^-(\lambda,\xi)x_1}(\mu_3^-(\lambda,\xi)^k + \mathbf{O}(e^{-\bar{\alpha}|x_1|}))\\
\partial_{x_1}^k \phi_2^-(x_1;\lambda,\xi) &= e^{\mu_4^-(\lambda,\xi)x_1}(\mu_4^-(\lambda,\xi)^k + \mathbf{O}(e^{-\bar{\alpha}|x_1|}))\\
\partial_{x_1}^k \psi_1^-(x_1;\lambda,\xi) &= e^{\mu_1^-(\lambda,\xi)x_1}(\mu_1^-(\lambda,\xi)^k + \mathbf{O}(e^{-\bar{\alpha}|x_1|}))\\
\partial_{x_1}^k \psi_2^-(x_1;\lambda,\xi) &= \frac{1}{\mu_2^-(\lambda,\xi)}\Big(\mu_2^-(\lambda,\xi)^k e^{\mu_2^-(\lambda,\xi)x_1} - \mu_3^-(\lambda,\xi)^k e^{\mu_3^-(\lambda,\xi)x_1}\Big) + \mathbf{O}(e^{-\bar{\alpha}|x_1|}).
\end{aligned}
$$

(ii) For $x_1 \ge 0$

$$
\begin{aligned}
\partial_{x_1}^k \phi_1^+(x_1;\lambda,\xi) &= e^{\mu_1^+(\lambda,\xi)x_1}(\mu_1^+(\lambda,\xi)^k + \mathbf{O}(e^{-\bar{\alpha}|x_1|}))\\
\partial_{x_1}^k \phi_2^+(x_1;\lambda,\xi) &= e^{\mu_2^+(\lambda,\xi)x_1}(\mu_2^+(\lambda,\xi)^k + \mathbf{O}(e^{-\bar{\alpha}|x_1|}))\\
\partial_{x_1}^k \psi_1^+(x_1;\lambda,\xi) &= \frac{1}{\mu_3^+(\lambda,\xi)}\Big(\mu_3^+(\lambda,\xi)^k e^{\mu_3^+(\lambda,\xi)x_1} - \mu_2^+(\lambda,\xi)^k e^{\mu_2^+(\lambda,\xi)x_1}\Big) + \mathbf{O}(e^{-\bar{\alpha}|x_1|})\\
\partial_{x_1}^k \psi_2^+(x_1;\lambda,\xi) &= e^{\mu_4^+(\lambda,\xi)x_1}(\mu_4^+(\lambda,\xi)^k + \mathbf{O}(e^{-\bar{\alpha}|x_1|})).
\end{aligned}
$$

The proof of Lemma 3.1 is almost identical to that of Lemma 2.1 in [23] and we omit it here.

In the analysis that follows, we will also require estimates on the Wronskian quotients that appear in the definitions of $N_k^\pm$, and we gather these in Lemma 3.2.

Lemma 3.2. *Under the assumptions of Lemma 3.1, and for $\phi_k^\pm$, $\psi_k^\pm$ as in Lemma 3.1, with $k = 0, 1$ and for $D(\lambda, \xi)$ as in (1.28) we have the following estimates. For some $\tilde{\alpha} > 0$,* *(i) For $x_1 \le 0$*

$$\partial_{x_1}^k \frac{W(\phi_1^-,\phi_2^-,\psi_1^-)}{c(x_1)W_{\lambda,\xi}(x_1)} = \frac{1}{c(0)D(\lambda,\xi)} e^{-\mu_2^- x_1}\Big((-\mu_2^-)^k(\mu_1^- - \mu_4^-)(\mu_1^- - \mu_3^-)(\mu_4^- - \mu_3^-) + \mathbf{O}(e^{-\tilde\alpha|x_1|})\Big)$$
$$\partial_{x_1}^k \frac{W(\phi_1^-,\phi_2^-,\psi_2^-)}{c(x_1)W_{\lambda,\xi}(x_1)} = \frac{1}{c(0)D(\lambda,\xi)} e^{-\mu_1^- x_1}\Big((-\mu_1^-)^k(\mu_2^- - \mu_4^-)(\frac{\mu_2^- - \mu_3^-}{\mu_2^-})(\mu_4^- - \mu_3^-) + \mathbf{O}(e^{-\tilde\alpha|x_1|})\Big)$$
$$\partial_{x_1}^k \frac{W(\phi_1^-,\psi_1^-,\psi_2^-)}{c(x_1)W_{\lambda,\xi}(x_1)} = \frac{1}{c(0)D(\lambda,\xi)} e^{-\mu_4^- x_1}\Big((-\mu_4^-)^k(\mu_2^- - \mu_1^-)(\frac{\mu_2^- - \mu_3^-}{\mu_2^-})(\mu_1^- - \mu_3^-) + \mathbf{O}(e^{-\tilde\alpha|x_1|})\Big)$$
$$\partial_{x_1}^k \frac{W(\phi_2^-,\psi_1^-,\psi_2^-)}{c(x_1)W_{\lambda}(x_1)} = \frac{1}{c(0)D(\lambda,\xi)}\Big(\frac{(\mu_2^- - \mu_1^-)(\mu_2^- - \mu_4^-)(\mu_1^- - \mu_4^-)}{\mu_2^-}\Big)\Big(e^{-\mu_3^- x_1}(-\mu_3^-)^k - e^{-\mu_2^- x_1}(-\mu_2^-)^k + \mu_2^-\mathbf{O}(e^{-\tilde\alpha|x_1|})\Big).$$

(ii) For $x_1 \ge 0$

$$\partial_{x_1}^k \frac{W(\phi_1^+,\phi_2^+,\psi_1^+)}{c(x_1)W_{\lambda,\xi}(x_1)} = \frac{1}{c(0)D(\lambda,\xi)} e^{-\mu_4^+ x_1}\Big((-\mu_4^+)^k(\frac{\mu_3^+ - \mu_2^+}{\mu_3^+})(\mu_3^+ - \mu_1^+)(\mu_2^+ - \mu_1^+) + \mathbf{O}(e^{-\tilde\alpha|x_1|})\Big)$$
$$\partial_{x_1}^k \frac{W(\phi_1^+,\phi_2^+,\psi_2^+)}{c(x_1)W_{\lambda}(x_1)} = \frac{1}{c(0)D(\lambda,\xi)} e^{-\mu_3^+ x_1}\Big((-\mu_3^+)^k(\mu_4^+ - \mu_2^+)(\mu_4^+ - \mu_1^+)(\mu_2^+ - \mu_1^+) + \mathbf{O}(e^{-\tilde\alpha|x_1|})\Big)$$
$$\partial_{x_1}^k \frac{W(\phi_1^+,\psi_1^+,\psi_2^+)}{c(x_1)W_{\lambda,\xi}(x_1)} = \frac{1}{c(0)D(\lambda,\xi)}\Big(\frac{(\mu_4^+ - \mu_3^+)(\mu_4^+ - \mu_1^+)(\mu_3^+ - \mu_1^+)}{\mu_3^+}\Big)\Big(e^{-\mu_2^+ x_1}(-\mu_2^+)^k - e^{-\mu_3^+ x_1}(-\mu_3^+)^k + \mu_3^+\mathbf{O}(e^{-\tilde\alpha|x_1|})\Big)$$
$$\partial_{x_1}^k \frac{W(\phi_2^+,\psi_1^+,\psi_2^+)}{c(x_1)W_{\lambda,\xi}(x_1)} = \frac{1}{c(0)D(\lambda,\xi)} e^{-\mu_1^+ x_1}\Big((-\mu_1^+)^k(\mu_4^+ - \mu_3^+)(\mu_4^+ - \mu_2^+)(\frac{\mu_3^+ - \mu_2^+}{\mu_3^+}) + \mathbf{O}(e^{-\tilde\alpha|x_1|})).$$

Here, the dependence of the $\mu_k^\pm$ on λ and ξ (or alternatively on $\rho_\pm$ and κ) has been suppressed for notational brevity.

The proof of Lemma 3.2 is almost identical to the proof of Lemma 2.2 of [23] and we omit it here. We mention, however, that while the estimates of Lemma 3.1 continue to hold in the more general setting of (1.2), the estimates of Lemma 3.2 take advantage of the fact that L_ξ in the restricted case of (1.1) has no terms that are $\mathbf{O}(|\xi|)$.

In addition to the estimates of Lemma 3.2, we require slightly refined estimates on the particular combinations $N_k^\pm$. The critical issue here is that while the estimates of Lemma 3.2 all include exponentially decaying error terms that do not vanish as $(|\lambda| + |\xi|^2) \to 0$, cancellation occurs in the x_1-derivatives of the $N_k^\pm$ whereby appropriate combinations of the estimates of Lemma 3.2 do vanish as $(|\lambda| + |\xi|^2) \to 0$.

Lemma 3.3. *Under the assumptions of Theorem 1.1, we have the following estimates on the $N_k^\pm$ of (1.26).*

For $x_1 \le 0$, and for $|\lambda| + |\xi|^2 \le r$, some r sufficiently small,

$$\partial_{x_1} N_1^-(x_1;\lambda,\xi) = -(2\pi)^{\frac{1-d}{2}} e^{-i\tilde y\cdot\xi} \frac{\mathbf{O}((|\lambda| + |\xi|^2)^{\frac12})}{c(0)D(\lambda,\xi)} e^{-\mu_2^-(\lambda,\xi)x_1}$$
$$\partial_{x_1} N_2^-(x_1;\lambda,\xi) = (2\pi)^{\frac{1-d}{2}} e^{-i\tilde y\cdot\xi} \frac{\mathbf{O}(|\lambda| + |\xi|^3)}{c(0)D(\lambda,\xi)} e^{-\mu_2^-(\lambda,\xi)x_1}$$

The proof of Lemma 3.3 is almost identical to that of Lemma 2.3 in [23], and we omit it here.

In practice, our analysis of the Evans function $D(\lambda,\xi)$ can be divided into two pieces, a calculation near the complex origin $(0,0)$, through which the eigenvalue $\lambda_*(\xi)$ is understood for $|\xi|$ sufficiently small, and a calculation away from the complex origin, based on stardard mimimax arguments such as have been employed in [36, 37, 44]. (For the latter case, we will state a result from [36, 37]). For our analysis near the origin, it will be convenient to consider the Evans function as a function of the variables κ, ρ_-, and ρ_+, defined in (1.29). We observe that our eigenvalue problem (1.24) can be written in the form

$$\begin{aligned} &-(c(x_1)\phi''')' + (b(x_1)\phi')' - (a(x_1)\phi)' + \kappa[(c(x_1)\phi')' + c(x_1)\phi''] \\ &\quad -[\kappa(b(x_1)-b_\pm) + \kappa^2(c(x_1)-c_\pm)]\phi = (b_\pm + 2c_\pm\kappa)^2\rho_\pm^2\phi, \end{aligned} \tag{3.5}$$

from which it is apparent that for each x_1, the ϕ_k^- and the ψ_k^- can be expressed analytically in terms of κ and ρ_-, while the ϕ_k^+ and the ψ_k^+ can be expressed analytically in terms of κ and ρ_+. (That is, we can construct each of these solutions analytically for sufficiently large $|x_1|$, and conclude analyticity for all x_1 by analytic continuation; see [46] Proposition 3.1.) In this way, the Evans function can be expressed as a function $D(\kappa,\rho_-,\rho_+)$, analytic in each of its arguments. We have the following lemma.

Lemma 3.4. *Suppose $\bar{u}(x_1)$ is a planar wave solution to (1.1) and suppose (H0)–(H1) hold. Then there exists a neighborhood V of $(\kappa,\rho_-,\rho_+) = (0,0,0)$ so that the Evans function*

$$D(\kappa,\rho_-,\rho_+) = W(\phi_1^-(x_1,\kappa,\rho_-),\phi_2^-(x_1,\kappa,\rho_-),\phi_1^+(x_1,\kappa,\rho_+),\phi_2^+(x_1,\kappa,\rho_+))\Big|_{x_1=0},$$

is analytic in V. Moreover, if (without loss of generality) we specify the choice

$$\phi_1^+(x;0,0) = \bar{u}_{x_1}(x_1) = \phi_2^-(x;0,0), \tag{3.6}$$

there holds

$$D(\kappa,\rho_-,\rho_+) = D(0,0,0) + \sum_{k=1}^{\infty}\frac{1}{k!}(\kappa\partial_\kappa + \rho_-\partial_{\rho_-} + \rho_+\partial_{\rho_+})^k D(0,0,0), \tag{3.7}$$

with

$$\begin{aligned} D(0,0,0) &= \frac{\partial D}{\partial\rho_\pm}(0,0,0) = \frac{\partial D}{\partial\rho_-\partial\rho_+}(0,0,0) = 0; \\ \frac{\partial D}{\partial\kappa}(0,0,0) &= -\frac{1}{c(0)}(-[bu] + [b]\bar{u}(0))W(\phi_1^-,\bar{u}_{x_1},\phi_2^+)(0,0,0,0) \\ \frac{\partial^2 D}{\partial\rho_\pm\partial\rho_\pm}(0,0,0) &= \pm\frac{1}{c(0)}2b_\pm^2(\bar{u}(0)-u_\pm)W(\phi_1^-,\bar{u}_{x_1},\phi_2^+)(0,0,0,0) \\ \frac{\partial^2 D}{\partial\kappa\partial\rho_-}(0,0,0) &= \frac{1}{c(0)}W(b_-^{3/2}\partial_\kappa(\phi_2^- - \phi_1^+) + ([bu]-[b]\bar{u}(0))\partial_{\rho_-}\phi_1^-),\bar{u}_{x_1},\phi_2^+) \\ \frac{\partial^2 D}{\partial\kappa\partial\rho_+}(0,0,0) &= \frac{1}{c(0)}W(-b_+^{3/2}\partial_\kappa(\phi_2^- - \phi_1^+) + ([bu]-[b]\bar{u}(0))\partial_{\rho_+}\phi_2^+),\phi_1^-,\bar{u}_{x_1}), \end{aligned} \tag{3.8}$$

and additionally

$$\begin{aligned} \frac{\partial^3 D}{\partial\rho_-\partial\rho_-\partial\rho_-}(0,0,0) &= \frac{3}{c(0)}W(b_-^{3/2}\partial_{\rho_-\rho_-}\phi_2^- - 2b_-^2(\bar{u}(0)-u_-)\partial_{\rho_-}\phi_1^-,\bar{u}_{x_1},\phi_2^+) \\ \frac{\partial^3 D}{\partial\rho_+\partial\rho_+\partial\rho_+}(0,0,0) &= \frac{3}{c(0)}W(\phi_1^-,\bar{u}_{x_1},b_+^{3/2}\partial_{\rho_+\rho_+}\phi_1^+ - 2b_+^2(u_+-\bar{u}(0))\partial_{\rho_+}\phi_2^+) \\ \frac{\partial^3 D}{\partial\rho_-\partial\rho_+\partial\rho_+}(0,0,0) &= -\frac{1}{c(0)}W(b_-^{3/2}\partial_{\rho_+\rho_+}\phi_1^+ + 2b_+^2(u_+-\bar{u}(0))\partial_{\rho_-}\phi_1^-,\bar{u}_{x_1},\phi_2^+) \\ \frac{\partial^3 D}{\partial\rho_-\partial\rho_-\partial\rho_+}(0,0,0) &= \frac{1}{c(0)}W(\phi_1^-,\bar{u}_{x_1},-2b_-^2(\bar{u}(0)-u_-)\partial_{\rho_+}\phi_2^+ - b_+^{3/2}\partial_{\rho_-\rho_-}\phi_2^-). \end{aligned} \tag{3.9}$$

Here, $[u] = (u_+ - u_-)$ and $[bu] = (b_+u_+ - b_-u_-)$ and for notational brevity evaluation at $(\kappa,\rho_-,\rho_+,x_1) = (0,0,0,0)$ has been suppressed in the final six terms.

Proof. We first observe that the relation $D(0,0,0) = 0$ is an immediate consequence of (3.6). We next compute

$$\partial_{\rho_-}D(\kappa,\rho_-,\rho_+) = W(\partial_{\rho_-}\phi_1^-,\phi_2^-,\phi_1^+,\phi_2^+)\Big|_{x_1=0} + W(\phi_1^-,\partial_{\rho_-}\phi_2^-,\phi_1^+,\phi_2^+)\Big|_{x_1=0}, \tag{3.10}$$

the first of which is 0 at $(\kappa, \rho_-, \rho_+) = (0,0,0)$ due to (3.6), while the second vanishes at this point because ϕ_2^- is analytic in ρ_-^2. Proceeding similarly for the first derivative with respect to ρ_+, and for mixed partials, we obtain the first line of (3.8).

We proceed now with the first κ derivative, for which we compute

$$\begin{aligned}\partial_\kappa D(\kappa, \rho_-, \rho_+) &= W(\partial_\kappa \phi_1^-, \phi_2^-, \phi_1^+, \phi_2^+)\Big|_{x_1=0} + W(\phi_1^-, \partial_\kappa \phi_2^-, \phi_1^+, \phi_2^+)\Big|_{x_1=0} \\ &+ W(\phi_1^-, \phi_2^-, \partial_\kappa \phi_1^+, \phi_2^+)\Big|_{x_1=0} + W(\phi_1^-, \phi_2^-, \phi_1^+, \partial_\kappa \phi_2^+)\Big|_{x_1=0}.\end{aligned} \tag{3.11}$$

Upon rearranging terms and employing (3.6), we find

$$\begin{aligned}\partial_\kappa D(0,0,0) &= W(\phi_1^-, \partial_\kappa(\phi_2^- - \phi_1^+), \bar{u}_{x_1}, \phi_2^+)\Big|_{x_1=0} \\ &= \det \begin{pmatrix} \phi_1^- & \partial_\kappa(\phi_2^- - \phi_1^+) & \bar{u}' & \phi_2^+ \\ \phi_1^{-\prime} & \partial_\kappa(\phi_2^- - \phi_1^+)' & \bar{u}'' & \phi_2^{+\prime} \\ \phi_1^{-\prime\prime} & \partial_\kappa(\phi_2^- - \phi_1^+)'' & \bar{u}''' & \phi_2^{+\prime\prime} \\ \phi_1^{-\prime\prime\prime} & \partial_\kappa(\phi_2^- - \phi_1^+)''' & \bar{u}'''' & \phi_2^{+\prime\prime\prime} \end{pmatrix}\end{aligned} \tag{3.12}$$

where explicit variable dependence has been omitted for notational brevity (the entire right-hand side is evaluated at $(x_1, \kappa, \rho_-, \rho_+) = (0,0,0,0)$). In order to evaluate (3.12), we will obtain representations for third order differentiation directly from (3.5). Working first with ϕ_1^-, we integrate (3.5) for $y_1 \in (-\infty, x_1]$, $x_1 \leq 0$, obtaining

$$\begin{aligned}&-c(x_1)\phi_1^{-\prime\prime\prime} + b(x_1)\phi_1^{-\prime} - a(x_1)\phi_1^- \\ &= -2\kappa c_- \phi_1^{-\prime} + \int_{-\infty}^{x_1} (b_- + 2c_-\kappa)^2 \rho_-^2 \phi_1^-(y_1, \kappa, \rho_-) dy_1 + \int_{-\infty}^{x_1} \mathbf{O}(\kappa e^{-\eta|y_1|}) dy_1.\end{aligned} \tag{3.13}$$

Following the notation of [23], we denote the right hand side of this last equation $\mathcal{W}_1^-$. Upon direct integration, and using the estimate on ϕ_1^- from Lemma 3.1, we obtain the order relation

$$\mathcal{W}_1^-(x_1, \kappa, \rho_-) = \mathbf{O}(|\rho_-|) e^{\mu_3^- x_1} + \mathbf{O}(|\kappa| e^{-\eta|x_1|}). \tag{3.14}$$

Similarly, we can show that $x_1 \geq 0$

$$-c(x_1)\phi_2^{+\prime\prime\prime} + b(x_1)\phi_2^{+\prime} - a(x_1)\phi_2^+ = \mathcal{W}_2^+(x_1, \kappa, \rho_+) = \mathbf{O}(|\rho_+|) e^{\mu_2^+ x_1} + \mathbf{O}(|\kappa| e^{-\eta|x_1|}). \tag{3.15}$$

In order to analyze the term $\partial_\kappa(\phi_2^- - \phi_1^+)'''$, we take a κ derivative of (3.5) to obtain

$$\begin{aligned}&-(c(x_1)(\partial_\kappa \phi)''')' + (b(x_1)(\partial_\kappa \phi)')' - (a(x_1)\partial_\kappa \phi)' \\ &\quad + [(c(x_1)\phi')' + c(x_1)\phi''] + \kappa[(c(x_1)(\partial_\kappa \phi)')' + c(x_1)(\partial_\kappa \phi)''] \\ &\quad - [\kappa(b(x_1) - b_\pm) + \kappa^2(c(x_1) - c_\pm)]\partial_\kappa \phi - [(b(x_1) - b_\pm) + 2\kappa(c(x_1) - c_\pm)]\phi \\ &= 2(b_\pm + 2c_\pm\kappa) 2c_\pm \rho_\pm^2 \phi + (b_\pm + 2c_\pm\kappa)^2 \rho_\pm^2 \partial_\kappa \phi.\end{aligned} \tag{3.16}$$

Focusing first on ϕ_2^-, we set $(\kappa, \rho_-) = (0,0)$ and integrate over $y_1 \in (-\infty, x_1]$ to obtain

$$-c(x_1)\partial_\kappa \phi_2^{-\prime\prime\prime} + b(x_1)\partial_\kappa \phi_2^{-\prime} - a(x_1)\partial_\kappa \phi_2^- = -c(x_1)\bar{u}_{x_1x_1} - b_-(\bar{u}(x_1) - u_-), \tag{3.17}$$

where we have used here the observation

$$b(y_1)\bar{u}_{y_1} - c(y_1)\bar{u}_{y_1y_1y_1} = 0 \tag{3.18}$$

(see the proof of Lemma 2.1). Proceeding similarly with ϕ_1^+, we find

$$-c(x_1)\partial_\kappa \phi_1^{+\prime\prime\prime} + b(x_1)\partial_\kappa \phi_1^{+\prime} - a(x_1)\partial_\kappa \phi_1^+ = -c(x_1)\bar{u}_{x_1x_1} + b_+(u_+ - \bar{u}(x_1)). \tag{3.19}$$

Upon subtraction of (3.19) from (3.17), we obtain

$$\begin{aligned}&-c(x_1)\partial_\kappa(\phi_2^- - \phi_1^+)''' + b(x_1)\partial_\kappa(\phi_2^- - \phi_1^+)' - a(x_1)\partial_\kappa(\phi_2^- - \phi_1^+)\\ &= -[bu] + [b]\bar{u}(x_1),\end{aligned} \tag{3.20}$$

where $[bu] = (b_+u_+ - b_-u_-)$ and $[b] = (b_+ - b_-)$.

Returning now to (3.12), a short calculation (see, for example, the proof of Lemma 2.4 in [23]) reveals

$$\begin{aligned}\partial_\kappa D(0,0,0) &= -\frac{1}{c(0)}\det\begin{pmatrix}\phi_1^- & \partial_\kappa(\phi_2^- - \phi_1^+) & \bar{u}_x & \phi_2^+\\ \phi_1^{-\prime} & \partial_\kappa(\phi_2^- - \phi_1^+)' & \bar{u}_{xx} & \phi_2^{+\prime}\\ \phi_1^{-\prime\prime} & \partial_\kappa(\phi_2^- - \phi_1^+)'' & \bar{u}_{xxx} & \phi_2^{+\prime\prime}\\ 0 & -[bu]+[b]\bar{u}(0) & 0 & 0\end{pmatrix}.\\ &= -\frac{1}{c(0)}(-[bu]+[b]\bar{u}(0))W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(0,0,0,0).\end{aligned} \tag{3.21}$$

For the second order ρ_- derivative, we have

$$\begin{aligned}\partial_{\rho_-\rho_-}D(0,0,0) &= W(\partial_{\rho_-\rho_-}\phi_1^-, \phi_2^-, \phi_1^+, \phi_2^+)\Big|_{x_1=0} + 2W(\partial_{\rho_-}\phi_1^-, \partial_{\rho_-}\phi_2^-, \phi_1^+, \phi_2^+)\Big|_{x_1=0}\\ &+ W(\phi_1^-, \partial_{\rho_-\rho_-}\phi_2^-, \phi_1^+, \phi_2^+)\Big|_{x_1=0} = W(\phi_1^-, \partial_{\rho_-\rho_-}\phi_2^-, \bar{u}_{x_1}, \phi_2^+)\Big|_{x_1=0}.\end{aligned} \tag{3.22}$$

Proceeding similarly as with the first κ derivative, we take two ρ_- derivatives of (3.5), set $(\kappa, \rho_-) = (0,0)$ and integrate on $y_1 \in (-\infty, x_1]$ to obtain the relation

$$-c(x_1)(\partial_{\rho_-\rho_-}\phi_2^-)''' + b(x_1)(\partial_{\rho_-\rho_-}\phi_2^-)' - a(x_1)(\partial_{\rho_-\rho_-}\phi_2^-) = 2b_-^2(\bar{u}(x_1) - u_-). \tag{3.23}$$

Combining this observation with (3.22), (3.13), and (3.15), and proceeding as in (3.21), we obtain

$$\partial_{\rho_-\rho_-}D(0,0,0) = -\frac{1}{c(0)}2b_-^2(\bar{u}(0) - u_-)W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(0,0,0,0).$$

An almost identical calculation reveals

$$\partial_{\rho_+\rho_+}D(0,0,0) = -\frac{1}{c(0)}2b_+^2(u_+ - \bar{u}(0))W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(0,0,0,0).$$

For the mixed partial $\partial_{\kappa\rho_-}$, we proceed as in previous cases to obtain

$$\partial_{\kappa\rho_-}D(0,0,0) = W(\partial_{\rho_-}\phi_1^-, \partial_\kappa(\phi_2^- - \phi_1^+), \bar{u}_{x_1}, \phi_2^+)\Big|_{x_1=0}. \tag{3.24}$$

In order to understand the behavior of $\partial_{\rho_-}\phi_1^-$, we differentiate (3.5) with respect to ρ_-, set $\kappa = 0$ (though not yet $\rho_- = 0$), and integrate on $y_1 \in (-\infty, x_1]$ to obtain

$$\begin{aligned}&-c(x_1)(\partial_{\rho_-}\phi_1^-)''' + b(x_1)(\partial_{\rho_-}\phi_1^-)' - a(x_1)(\partial_{\rho_-}\phi_1^-)\\ &= 2\rho_- b_-^2\int_{-\infty}^{x_1}\phi_1^-(y_1;0,\rho_-)dy_1 + b_-^2\rho_-^2\int_{-\infty}^{x_1}\partial_{\rho_-}\phi_1^-(y_1;0,\rho_-)dy_1.\end{aligned}$$

We evaluate the right hand side of this last equation by direct integration over our estimate from Lemma 3.1

$$\phi_1^- = e^{\mu_3^- x_1}(1 + \mathbf{O}(e^{-\bar{\alpha}|x_1|})),$$

and using analyticity of ϕ_1^- in ρ_-. We find

$$\lim_{\rho_-\to 0} 2\rho_- b_-^2\int_{-\infty}^{0}\phi_1^-(y_1;0,\rho_-)dy_1 + b_-^2\rho_-^2\int_{-\infty}^{0}\partial_{\rho_-}\phi_1^-(y_1;0,\rho_-)dy_1 = b_-^{3/2},$$

from which we conclude

$$-c(0)(\partial_{\rho_-}\phi_1^-)''' + b(0)(\partial_{\rho_-}\phi_1^-)' - a(0)(\partial_{\rho_-}\phi_1^-) = b_-^{3/2}.$$

Combining this with (3.24), we find

$$\partial_{\kappa\rho_-}D(0,0,0) = \frac{1}{c(0)}W(b_-^{3/2}\partial_\kappa(\phi_2^- - \phi_1^+) + ([bu] - [b]\bar{u}(0))\partial_{\rho_-}\phi_1^-), \bar{u}_{x_1}, \phi_2^+).$$

The expression for $\partial_{\kappa\rho_+}D(0,0,0)$ can be derived by a calculation almost identical to that of $\partial_{\kappa\rho_-}D(0,0,0)$. Similarly, the expressions in (3.9) can now be derived in a straightforward fashion using the methods employed in the case of (3.8). □

Our next lemma asserts that in the setting of (1.1), we do not generally have a quadratic scaling for $\lambda_*(\xi)$.

Lemma 3.5. *Suppose $\bar{u}(x_1)$ denotes a planar wave solution to (1.1), that (H0)–(H1) hold, and additionally that*

$$W(\phi_1^-(x_1;0,0), \bar{u}_{x_1}(x_1), \phi_2^+(x_1;0,0)) \neq 0. \tag{3.25}$$

Then there exists a neighborhood V of $(\lambda, |\xi|) = (0,0)$ so that the curve $\lambda_*(|\xi|)$ defined by $D(\lambda_*(|\xi|), |\xi|) = 0$, $\lambda_*(0) = 0$, (where D is as in (1.28)) satisfies

$$\lambda_*(\xi) = -\lambda_3|\xi|^3 + \mathbf{O}(|\xi|^4),$$

where

$$\lambda_3 = 2b_-^{3/2}\frac{\frac{1}{\sqrt{b_-}}D_{110} + \frac{1}{\sqrt{b_+}}D_{101} + \frac{1}{6b_-^{3/2}}D_{030} + \frac{1}{6b_+^{3/2}}D_{003} + \frac{1}{2\sqrt{b_-}b_+}D_{012} + \frac{1}{2\sqrt{b_+}b_-}D_{021}}{\frac{1}{\sqrt{b_-}}D_{020} + \frac{b_-^{3/2}}{b_+^2}D_{002}}.$$

Here

$$D_\alpha := \frac{\partial^{|\alpha|}D}{\partial\kappa^{\alpha_1}\partial\rho_-^{\alpha_2}\partial\rho_+^{\alpha_3}}(0,0,0).$$

Remark 3.1 (Remark on 3.25)**.** *Condition (3.25) can be proven in all cases for which F has the double well form of Definition 2.1. Briefly, we can observe that $\phi_1^-(x_1;0,0)$, $\bar{u}_{x_1}(x_1)$, and $\phi_2^+(x_1;0,0)$ are all solutions of the third order equation*

$$(F''(\bar{u})\phi - \nu\phi'')' = 0,$$

and additionally that the three functions $\bar{u}_{x_1}$, $\phi_A(x_1) := \bar{u}_{x_1}\int_0^{x_1}\frac{dy}{\bar{u}_y^2}$, and $\phi_B(x_1) := \bar{u}_{x_1}\int_0^{x_1}\frac{\bar{u}(y)}{\bar{u}_y^2}dy$ form a basis of solutions for this equation. Combining these observations, we find that no three functions with the asymptotic properties of $\phi_1^-(x_1;0,0)$, $\bar{u}_{x_1}(x_1)$, and $\phi_2^+(x_1;0,0)$ can be linearly dependent. (The full argument is detailed in [26].)

Before proving Lemma 3.5, we note that it is not as strong as Condition 1 of $(\mathcal{D})$. Rather, it asserts only that in this case the scaling is certainly not quadratic (that is, at this level of generality, λ_3 could be 0). In the case of (1.2) quadratic scaling is possible, and it is primarily Lemma 3.5 that has prompted our restriction to the study of (1.1). We note that these same considerations are studied in a different manner in [44].

Proof of Lemma 3.5. We proceed by expanding $D(\kappa, \rho_-, \rho_+)$ with (3.7)

$$\begin{aligned} D(\kappa,\rho_-,\rho_+) &= D_{100}\kappa + \frac{1}{2}D_{020}\rho_-^2 + \frac{1}{2}D_{002}\rho_+^2 + D_{110}\kappa\rho_- + D_{101}\kappa\rho_+ \\ &+ \frac{1}{6}D_{030}\rho_-^3 + \frac{1}{6}D_{003}\rho_+^3 + \frac{1}{2}D_{021}\rho_-^2\rho_+ + \frac{1}{2}D_{012}\rho_-\rho_+^2 + ..., \end{aligned} \tag{3.26}$$

where we have dropped off higher order terms that will not be relevant to the calculation. We now expand each of the $\rho_\pm$ as a power series in $|\xi| = \sqrt{\kappa}$,

$$\rho_-(|\xi|) = a_1|\xi| + a_2|\xi|^2 + \mathbf{O}(|\xi|^3)$$
$$\rho_+(|\xi|) = b_1|\xi| + b_2|\xi|^2 + \mathbf{O}(|\xi|^3).$$

In order to determine the values of a_1, a_2, b_1, and b_2 that coincide with $D(|\xi|^2, \rho_-(|\xi|), \rho_+(|\xi|)) = 0$, we substitute our expansions for $\rho_\pm$ into the right hand side of (3.26) and set the result to 0. In the resulting equation, we equate coefficients of matching powers of $|\xi|$. For $|\xi|^2$ (the lowest power that appears), we obtain

$$D_{100} + \frac{1}{2} D_{020} a_1^2 + \frac{1}{2} D_{002} b_1^2 = 0,$$

while for $|\xi|^3$, we obtain

$$\begin{aligned} &D_{020} a_1 a_2 + D_{002} b_1 b_2 + D_{110} a_1 + D_{101} b_1 + \frac{1}{6} D_{030} a_1^3 + \frac{1}{6} D_{003} b_1^3 \\ &\quad + \frac{1}{2} D_{021} a_1^2 b_1 + \frac{1}{2} D_{012} a_1 b_1^2 = 0. \end{aligned}$$

Finally, we augment these last two expressions with the assertion that λ must be the same whether defined in terms of ρ_- or ρ_+. Upon solving (1.29) for λ in terms of ρ_- and similarly for ρ_+, and equating the results, we find

$$\begin{aligned} &(b_- + 2c_-|\xi|^2)^2 (a_1|\xi| + a_2|\xi|^2 + \mathbf{O}(|\xi|^3)) - b_-|\xi|^2 - c_-|\xi|^4 \\ &= (b_+ + 2c_+|\xi|^2)^2 (b_1|\xi| + b_2|\xi|^2 + \mathbf{O}(|\xi|^3)) - b_+|\xi|^2 - c_+|\xi|^4, \end{aligned}$$

from which we have the additional relations

$$\begin{aligned} b_-^2 a_1^2 - b_- &= b_+^2 b_1^2 - b_+ \\ b_-^2 2a_1 a_2 &= b_+^2 2b_1 b_2. \end{aligned}$$

We now have a system of four equations and four unknowns, which can be solved for a_1, a_2, b_1, and b_2. We find $a_1 = 1/\sqrt{b_-}$ and $a_2 = \lambda_3/(2b_-^{3/2})$, where λ_3 is given in the statement of the lemma. The lemma follows immediately upon substitution of these values into the relation

$$\lambda = (b_- + 2c_-|\xi|^2)^2 \rho_-^2 - b_-|\xi|^2 - c_-|\xi|^4.$$

□

In order to provide an indication of how Lemma 3.5 can be used in specific cases, we apply it to the case of (1.1) with $M \equiv 1$, $\nu = 1$ and (1.6), recovering, then, the result of [36, 37].

Lemma 3.6. *In the case of (1.1) with $M \equiv 1$, $\nu = 1$ and (1.6), and for the solution*

$$\bar{u}(x_1) = \tanh(\frac{x_1}{2}),$$

we have

$$\lambda_*(|\xi|) = -\frac{1}{3}|\xi|^3 + \mathbf{O}(|\xi|^4).$$

Proof. In this case, we have the relations $b_\pm = 1$, $c_\pm = 1$, so that $\rho_- = \rho_+$, and also $[u] = 2$. In this way, the number of terms to evaluate is considerably reduced by symmetry. Also, in this case, for $(\lambda, \xi) = (0, 0)$, we can solve (1.24) exactly, and we find

$$\begin{aligned} \phi_1^-(x_1; 0, 0) &= -(\phi_2^0(x_1) + \phi_3^0(x_1)) \\ \phi_2^-(x_1; 0, 0) &= \phi_1^+(x_1; 0, 0) = \bar{u}_{x_1}(x_1) \\ \phi_1^+(x_1; 0, 0) &= \phi_2^0(x_1) + \phi_3^0(x_1), \end{aligned} \tag{3.27}$$

where ϕ_2^0 and ϕ_3^0 are taken from [23],

$$\begin{aligned} \phi_2^0(x_1) &= \bar{u}_{x_1}(x_1) \Big[2 \sinh \frac{x_1}{2} \cosh^3 \frac{x_1}{2} + 3 \sinh \frac{x_1}{2} \cosh \frac{x_1}{2} + \frac{3}{2} x_1 \Big] \\ \phi_3^0(x_1) &= \cosh^2 \frac{x_1}{2}. \end{aligned}$$

With this choice, the determinant $W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+)$ can be computed directly, and we obtain

$$W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+)\Big|_{(x_1;\lambda,\xi)=(0,0,0)} = -2.$$

We next consider the Wronskian

$$W(b_-^{3/2}\partial_\kappa(\phi_2^- - \phi_1^+) + ([bu] - [b]\bar{u}(0))\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+),$$

which in this case becomes more simply

$$W(\partial_\kappa(\phi_2^- - \phi_1^+) + 2\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+). \tag{3.28}$$

In the analysis that follows, we will take advantage of the observation that this last Wronskian is a Wronskian of three solutions to the third order equation

$$-c(x_1)\phi''' + b(x_1)\phi' - a(x_1)\phi = 0,$$

and as such is independent of x_1. Closely following an argument of Korvola ([36], Section 3.4), we write (3.28) as

$$\begin{aligned} -W(\partial_\kappa\phi_1^+, \bar{u}_{x_1}, \phi_2^+) &+ W(\partial_\kappa\phi_2^- + 2\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+) \\ &=: \Phi_1(x_1) + \Phi_2(x_1), \end{aligned}$$

where the defined terms are defined respectively. Since we can evaluate this last expression at any choice of x_1, we will proceed by taking a limit as $x_1 \to \infty$. Noting, however, that $\Phi_2'(x_1)$ is more readily understood than $\Phi_2(x_1)$, we write

$$\Phi_1(x_1) + \Phi_2(x_1) = \Phi_1(x_1) + \Phi_2(\bar{x}_1) + \int_{\bar{x}_1}^{x_1} \Phi_2'(y_1)dy_1,$$

which is valid for any $\bar{x}_1 \le x_1$. Taking a limit now $x_1 \to \infty$, and observing that in such a limit Φ_1 vanishes, we have

$$W(\partial_\kappa(\phi_2^- - \phi_1^+) + 2\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+)\Big|_{(\lambda,|\xi|)=(0,0)} = \Phi_2(\bar{x}_1) + \int_{\bar{x}_1}^{+\infty} \Phi_2'(y_1)dy_1,$$

where $\bar{x}_1$ is now arbitrary. We now complete the argument by taking the limit of this last expression as $\bar{x}_1$ approaches $-\infty$. In order to evaluate $\Phi_2(\bar{x}_1)$ in this limit, we note directly from (3.27) the asymptotic relation for $\bar{x}_1 \le 0$,

$$\phi_2^+(\bar{x}_1; 0, 0) = -\frac{1}{2}e^{-\bar{x}_1}(1 + \mathbf{O}(e^{-\alpha|\bar{x}_1|})).$$

For $\bar{x}_1 \le 0$, the expressions $\partial_\kappa\phi_2^-$ and $\partial_{\rho_-}\phi_1^-$ can be evaluated directly from the estimates of Lemma 3.1 (and analyticity) and we have

$$\partial_\kappa{\phi_2^-}^{(j)}(\bar{x}_1; 0, 0) = \mathbf{O}(e^{-\alpha|\bar{x}_1|}); \quad j = 1, 2, 3$$

and also

$$\begin{aligned} \partial_{\rho_-}\phi_1^-(\bar{x}_1; 0, 0) &= \frac{\partial\mu_3^-}{\partial\rho_-}(0,0)\bar{x}_1 + \mathbf{O}(e^{-\alpha|\bar{x}_1|}) \\ (\partial_{\rho_-}\phi_1^-(\bar{x}_1; 0, 0))' &= \frac{\partial\mu_3^-}{\partial\rho_-}(0,0) + \mathbf{O}(e^{-\alpha|\bar{x}_1|}) \\ (\partial_{\rho_-}\phi_1^-(\bar{x}_1; 0, 0))'' &= \mathbf{O}(e^{-\alpha|\bar{x}_1|}). \end{aligned}$$

Combining these observations, we have

$$\begin{aligned} \Phi_2(\bar{x}_1) &= \begin{pmatrix} 2\bar{x}_1 & 2e^{\bar{x}_1} & -\frac{1}{2}e^{-\bar{x}_1} \\ 2 & 2e^{\bar{x}_1} & \frac{1}{2}e^{-\bar{x}_1} \\ 0 & 2e^{\bar{x}_1} & -\frac{1}{2}e^{-\bar{x}_1} \end{pmatrix} + \mathbf{O}(e^{-\alpha|\bar{x}_1|}) \\ &= -4\bar{x}_1 + \mathbf{O}(e^{-\alpha|\bar{x}_1|}). \end{aligned}$$

We have, then,

$$W(\partial_\kappa(\phi_2^- - \phi_1^+) + 2\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+)\Big|_{(\lambda,|\xi|)=(0,0)} = \int_{-\infty}^0 (\Phi_2'(y_1) + 4)dy_1 + \int_0^{+\infty} \Phi_2'(y_1)dy_1. \quad (3.29)$$

Finally, we analyze $\Phi_2'(\bar{x}_1)$, for which we have

$$\begin{aligned}
\Phi_2'(\bar{x}_1) &= \begin{pmatrix} \partial_\kappa\phi_2^- + 2\partial_{\rho_-}\phi_1^- & \bar{u}_{\bar{x}_1} & \phi_2^+ \\ (\partial_\kappa\phi_2^- + 2\partial_{\rho_-}\phi_1^-)' & \bar{u}_{\bar{x}_1\bar{x}_1} & \phi_2^{+\prime} \\ (\partial_\kappa\phi_2^- + 2\partial_{\rho_-}\phi_1^-)''' & \bar{u}_{\bar{x}_1\bar{x}_1\bar{x}_1\bar{x}_1} & \phi_2^{+\prime\prime\prime} \end{pmatrix}\Big|_{(\lambda,\xi)=(0,0)} \\
&= \begin{pmatrix} \partial_\kappa\phi_2^- + 2\partial_{\rho_-}\phi_1^- & \bar{u}_{\bar{x}_1} & \phi_2^+ \\ (\partial_\kappa\phi_2^- + 2\partial_{\rho_-}\phi_1^-)' & \bar{u}_{\bar{x}_1\bar{x}_1} & \phi_2^{+\prime} \\ \bar{u}_{\bar{x}_1\bar{x}_1} + b_+(u_+ - \bar{u}(\bar{x}_1)) - 2 & 0 & 0 \end{pmatrix}\Big|_{(\lambda,\xi)=(0,0)} \\
&= \Big[\bar{u}_{\bar{x}_1\bar{x}_1} + 1(1 - \bar{u}(\bar{x}_1)) - 2\Big] W(\bar{u}_{\bar{x}_1}, \phi_2^+(\bar{x}_1; 0, 0)).
\end{aligned}$$

Combining this last expression with (3.29), now proceed by direct integration to obtain

$$W(\partial_\kappa(\phi_2^- - \phi_1^+) + 2\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+)\Big|_{(\lambda,|\xi|)=(0,0)} = \frac{14}{3}.$$

We conclude that in this case

$$\frac{\partial^2 D}{\partial\kappa\partial\rho_-}(0,0,0) = \frac{14}{3}.$$

Proceeding similarly as in this last calculation, we can additionally establish

$$\begin{aligned}
&\frac{\partial^2 D}{\partial\kappa\partial\rho_+}(0,0,0) = \frac{14}{3} \\
&\frac{1}{3}\frac{\partial^3 D}{\partial\rho_-\partial\rho_-\partial\rho_-}(0,0,0) + \frac{\partial^3 D}{\partial\rho_-\partial\rho_+\partial\rho_+}(0,0,0) = -8 \\
&\frac{1}{3}\frac{\partial^3 D}{\partial\rho_+\partial\rho_+\partial\rho_+}(0,0,0) + \frac{\partial^3 D}{\partial\rho_-\partial\rho_-\partial\rho_+}(0,0,0) = -8.
\end{aligned}$$

In this way, we can directly compute λ_3 from Lemma 3.5 to be -1/3. □

4 Estimates on $G_\lambda(x, y)$

In this section, we combine the observations of Section 3 in order to develop estimates on the ODE Green's function $G_{\lambda,\xi}(x, y)$. We begin by noting that in the event that $x_1 \le 0$, we will expand ϕ_k^+ as a linear combination of the ϕ_k^- and ψ_k^-,

$$\begin{aligned}
\phi_k^+(x_1; \lambda, \xi) = A_k^+(\lambda,\xi)\phi_1^-(x_1;\lambda,\xi) + B_k^+(\lambda,\xi)\phi_2^-(x_1;\lambda,\xi) \\
+ C_k^+(\lambda,\xi)\psi_1^-(x_1;\lambda,\xi) + D_k^+(\lambda,\xi)\psi_2^-(x_1;\lambda,\xi).
\end{aligned} \quad (4.1)$$

For such expansions, we have the following lemma regarding expansion coefficients.

Lemma 4.1. *Under the assumptions of Lemma 3.5, and for $\phi_k^\pm$, $\psi_k^\pm$ as in Lemma 3.1, there holds*

$$A_1^+(\lambda,\xi) = \mathbf{O}(|\lambda| + |\xi|^2); \quad B_1^+(\lambda,\xi) = \mathbf{O}(1); \quad C_1^+(\lambda,\xi) = \mathbf{O}(|\lambda| + |\xi|^2); \quad D_1^+(\lambda,\xi) = \mathbf{O}(|\lambda| + |\xi|^2).$$

$$A_2^+(\lambda,\xi) = \mathbf{O}(1); \quad B_2^+(\lambda,\xi) = \mathbf{O}(1); \quad C_2^+(\lambda,\xi) = \mathbf{O}(1); \quad D_2^+(\lambda,\xi) = \mathbf{O}(1),$$

with additionally

$$\begin{aligned}
C_1^+(\lambda,\xi)D_2^+(\lambda,\xi) - D_1^+(\lambda,\xi)C_2^+(\lambda,\xi) &= D(\lambda,\xi) + \mathbf{O}(|\lambda| + |\xi|^2)^{3/2}, \\
C_1^+(\lambda,\xi)A_2^+(\lambda,\xi) - A_1^+(\lambda,\xi)C_2^+(\lambda,\xi) &= \mathbf{O}(|\lambda| + |\xi|^2), \\
D_1^+(\lambda,\xi)A_2^+(\lambda,\xi) - D_2^+(\lambda,\xi)A_1^+(\lambda,\xi) &= \mathbf{O}(|\lambda| + |\xi|^2).
\end{aligned}$$

All order relations are associated with behavior as $|\lambda| + |\xi|^2 \to 0$.

Proof. The proof of each estimate in Lemma 4.1 is similar, and so we consider only the first cross-term estimates

$$C_1^+(\lambda,\xi)D_2^+(\lambda,\xi) - D_1^+(\lambda,\xi)C_2^+(\lambda,\xi) = \mathbf{O}(|\lambda| + |\xi|^3).$$

According to (4.1), and employing the notation defined in the paragraph immediately following (1.26), we have

$$\begin{aligned} &C_1^+(\lambda,\xi)D_2^+(\lambda,\xi) - D_1^+(\lambda,\xi)C_2^+(\lambda,\xi) \\ &= \frac{W(\phi_1^-,\phi_2^-,\phi_1^+,\psi_2^-)W(\phi_1^-,\phi_2^-,\psi_1^-,\phi_2^+) - W(\phi_1^-,\phi_2^-,\psi_1^-,\phi_1^+)W(\phi_1^-,\phi_2^-,\phi_2^+,\psi_2^-)}{W(\phi_1^-,\phi_2^-,\psi_1^-,\psi_2^-)}. \end{aligned} \tag{4.2}$$

The denominator $W(\phi_1^-,\phi_2^-,\psi_1^-,\psi_2^-)$ is non-vanishing by construction as $|\lambda|+|\xi|^2 \to 0$, and so we need focus only on the numerator, which we will analyze with respect to the variables κ, ρ_-, and ρ_+. For sufficiently small values of κ, ρ_-, and ρ_+ by Taylor expansion around the origin, we have

$$\mathcal{N}(\kappa,\rho_-,\rho_+) = \mathcal{N}(0,0,0) + \sum_{k=1}^{\infty} \frac{1}{k!}(\kappa\partial_\kappa + \rho_-\partial_{\rho_-} + \rho_+\partial_{\rho_+})^k \mathcal{N}(0,0,0).$$

We first observe that by virtue of (3.6), we immediately have

$$\mathcal{N}(0,0,0) = 0.$$

We next compute

$$\begin{aligned} \frac{\partial \mathcal{N}}{\partial \rho_-}(0,0,0) &= \partial_{\rho_-} W(\phi_1^-,\phi_2^-,\phi_1^+,\psi_2^-)W(\phi_1^-,\phi_2^-,\psi_1^-,\phi_2^+) \\ &+ W(\phi_1^-,\phi_2^-,\phi_1^+,\psi_2^-)\partial_{\rho_-} W(\phi_1^-,\phi_2^-,\psi_1^-,\phi_2^+) \\ &- \partial_{\rho_-} W(\phi_1^-,\phi_2^-,\psi_1^-,\phi_1^+)W(\phi_1^-,\phi_2^-,\phi_2^+,\psi_2^-) \\ &- W(\phi_1^-,\phi_2^-,\psi_1^-,\phi_1^+)\partial_{\rho_-} W(\phi_1^-,\phi_2^-,\phi_2^+,\psi_2^-). \end{aligned} \tag{4.3}$$

The second and fourth or these are clearly 0 by (3.6), while the first and third are 0 due to (3.6) and the fact that ϕ_2^- and ψ_1^- are both analytic in ρ_-^2. By this, and a similar argument for differentiation with respect to ρ_+, we conclude

$$\frac{\partial \mathcal{N}}{\partial \rho_-}(0,0,0) = \frac{\partial \mathcal{N}}{\partial \rho_+}(0,0,0) = 0.$$

For the first κ derivative, we proceed similarly as in (4.3) to obtain

$$\begin{aligned} \frac{\partial \mathcal{N}}{\partial \kappa}(0,0,0) &= W(\phi_1^-,\partial_\kappa(\phi_2^- - \phi_1^+),\bar{u}_{x_1},\psi_2^-)W(\phi_1^-,\phi_2^-,\psi_1^-,\phi_2^+) \\ &- W(\phi_1^-,\partial_\kappa(\phi_2^- - \phi_1^+),\psi_1^-,\bar{u}_{x_1})W(\phi_1^-,\phi_2^-,\phi_2^+,\psi_2^-). \end{aligned}$$

Proceeding as in (3.12), we find

$$\begin{aligned} &W(\phi_1^-,\partial_\kappa(\phi_2^- - \phi_1^+),\bar{u}_{x_1},\psi_2^-) = \\ &-\frac{1}{c(0)}\Big[(-[bu] + [b]\bar{u}(0))W(\phi_1^-,\bar{u}_{x_1},\psi_2^-) - K_2 W(\phi_1^-,\partial_\kappa(\phi_2^- - \phi_1^+),\bar{u}_{x_1})\Big], \end{aligned}$$

where K_2 is the constant value defined by the relationship

$$-c(x_1){\psi_2^-}''' + b(x_1){\psi_2^-}' - a(x_1)\psi_2^- = K_2.$$

Similarly,

$$\begin{aligned} &W(\phi_1^-,\partial_\kappa(\phi_2^- - \phi_1^+),\psi_1^-,\bar{u}_{x_1}) = \\ &-\frac{1}{c(0)}\Big[(-[bu] + [b]\bar{u}(0))W(\phi_1^-,\bar{u}_{x_1},\psi_2^-) + K_1 W(\phi_1^-,\partial_\kappa(\phi_2^- - \phi_1^+),\bar{u}_{x_1})\Big], \end{aligned}$$

where K_1 is the constant value defined by the relationship

$$-c(x_1){\psi_1^-}''' + b(x_1){\psi_1^-}' - a(x_1)\psi_1^- = K_1.$$

Combining these expressions with the undifferentiated expressions

$$W(\phi_1^-, \phi_2^-, \psi_1^-, \phi_2^+) = \frac{K_1}{c(0)} W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+)$$
$$W(\phi_1^-, \phi_2^-, \phi_2^+, \psi_2^-) = -\frac{K_2}{c(0)} W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+),$$

we find

$$\frac{\partial \mathcal{N}}{\partial \kappa}(0,0,0) = \frac{[bu] - [b]\bar{u}(0)}{c(0)} W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+) W(\phi_1^-, \bar{u}_{x_1}, K_1\psi_2^- - K_2\psi_2^-).$$

Comparing this with Lemma 3.4, we see immediately the relation

$$\frac{\partial \mathcal{N}}{\partial \kappa}(0,0,0) = \frac{1}{c(0)} W(\phi_1^-, \bar{u}_{x_1}, K_1\psi_2^- - K_2\psi_2^-) \frac{\partial D}{\partial \kappa}(0,0,0).$$

Observing in addition that a similar calculation on the denominator of (4.2) establishes

$$W(\phi_1^-, \phi_2^-, \psi_1^-, \psi_2^-)\Big|_{(\kappa,\rho_-,\rho_+)=(0,0,0)} = -\frac{1}{c(0)} W(\phi_1^-, \bar{u}_{x_1}, K_2\psi_1^- - K_1\psi_2^-), \tag{4.4}$$

we conclude

$$\frac{\partial \mathcal{N}}{\partial \kappa}(0,0,0) = \frac{\partial D}{\partial \kappa}(0,0,0). \tag{4.5}$$

For the second ρ_- derivative, we proceed similarly as in (4.3) to obtain

$$\begin{aligned}
&\frac{\partial \mathcal{N}}{\partial \rho_- \partial \rho_-}(0,0,0) \\
&= W(\phi_1^-, \partial_{\rho_-\rho_-}\phi_2^-, \phi_1^+, \psi_2^-) W(\phi_1^-, \phi_2^-, \psi_1^-, \phi_2^+) - W(\phi_1^-, \partial_{\rho_-\rho_-}\phi_2^-, \psi_1^-, \phi_1^+) W(\phi_1^-, \phi_2^-, \phi_2^+, \psi_2^-),
\end{aligned}$$

which can be analyzed simiarly as was the first κ derivative. Using (3.23), we find

$$\begin{aligned}
&\frac{\partial \mathcal{N}}{\partial \rho_- \partial \rho_-}(0,0,0) = \\
&- \frac{1}{c(0)} \Big[2b_-^2(\bar{u}(0) - u_-) W(\phi_1^-, \bar{u}_{x_1}, \psi_2^-) + K_2 W(\phi_1^-, \partial_{\rho_-\rho_-}\phi_2^-, \bar{u}_{x_1}) \Big] \frac{K_1}{c(0)} W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+) \\
&+ \frac{1}{c(0)} \Big[2b_-^2(\bar{u}(0) - u_-) W(\phi_1^-, \psi_1^-, \bar{u}_{x_1}) - K_1 W(\phi_1^-, \partial_{\rho_-\rho_-}\phi_2^-, \bar{u}_{x_1}) \Big] \frac{K_2}{c(0)} W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+) \\
&= -\frac{2b_-^2(\bar{u}(0) - u_-)}{c(0)^2} W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+) W(\phi_1^-, \bar{u}_{x_1}, K_1\psi_2^- - K_2\psi_1^-).
\end{aligned}$$

Similarly,

$$\frac{\partial \mathcal{N}}{\partial \rho_+ \partial \rho_+}(0,0,0) = \frac{2b_+^2(u_+ - \bar{u}(0))}{c(0)^2} W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+) W(\phi_1^-, \bar{u}_{x_1}, K_1\psi_2^- - K_2\psi_1^-),$$

and

$$\frac{\partial \mathcal{N}}{\partial \rho_- \partial \rho_+}(0,0,0) = 0.$$

Comparing this with Lemma 3.4, and using (4.4), we conclude

$$\frac{\partial \mathcal{N}}{\partial \rho_\pm \partial \rho_\pm}(0,0,0) = \frac{\partial D}{\partial \rho_\pm \partial \rho_\pm}(0,0,0). \tag{4.6}$$

The estimate on the cross term $C_1^+(\lambda,\xi)D_2^+(\lambda,\xi) - C_2^+(\lambda,\xi)D_1^+(\lambda,\xi)$ is an immediate consequence of (4.5) and (4.6). The remaining relations of Lemma 4.1 can be proven similarly. □

We now combine the estimates of Section 3 with those of Lemma 4.1 to establish estimates on the ODE Green's function $G_{\lambda,\xi}(x_1, y)$.

Lemma 4.2. *Suppose $\bar{u}(x_1)$ denotes a planar wave solution to (1.1) and that structural hypotheses (H0)–(H1) hold. Then for $G_{\lambda,\xi}(x_1, y)$ as defined in (1.23), there exists a splitting*

$$(2\pi)^{\frac{d-1}{2}} e^{i\tilde{y}\cdot\xi} G_{\lambda,\xi}(x_1, y) = \tilde{G}_{\lambda,\xi}(x_1, y_1) + E_{\lambda,\xi}(x_1, y_1),$$

such that for $(|\lambda| + |\xi|^2) < r$, where $r > 0$ is some suitably small constant, the following estimates hold:

(i) For $y_1, x_1 \leq 0$

$$\begin{aligned}\tilde{G}_{\lambda,\xi}(x_1, y_1) = &\Big[\frac{C_S}{\mu_2^-} + \frac{\mathbf{O}(|\lambda| + |\xi|^2)}{\mu_2^-} + \frac{\mathbf{O}(|\lambda| + |\xi|^2)^{3/2}}{D(\lambda,\xi)\mu_2^-}\Big]\Big(e^{\mu_2^-|x_1-y_1|} - e^{\mu_2^-|x_1+y_1|}\Big)\\ &+ \frac{\mathbf{O}(|\lambda| + |\xi|^2)}{D(\lambda,\xi)} e^{\mu_2^-|x_1+y_1|} + \mathbf{O}(e^{-\eta|x_1|})e^{-\mu_2^- y_1} + \mathbf{O}(e^{-\eta|y_1|})e^{-\mu_2^- x_1}\\ &+ \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(|\lambda| + |\xi|^2)^{3/2}}{D(\lambda,\xi)}\mathbf{O}(e^{-\eta|x_1-y_1|})\end{aligned}$$

(ii) For $y_1 \leq 0 \leq x_1$

$$\tilde{G}_{\lambda,\xi}(x_1, y_1) = \frac{\mathbf{O}(|\lambda| + |\xi|^2)}{D(\lambda,\xi)} e^{\mu_2^+ x_1 - \mu_2^- y_1}.$$

Here

$$C_S = \frac{2}{c(0)}\Big(\frac{b_-}{c_-}\Big)^{3/2}.$$

In both cases,

$$\begin{aligned}E_{\lambda,\xi}(x_1, y_1) &= \frac{\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_2^- y_1}\Big(c_E + \mathbf{O}((|\lambda| + |\xi|^2)^{1/2}) + \mathbf{O}(e^{-\eta|y_1|})\Big)\\ \partial_{y_1} E_{\lambda,\xi}(x_1, y_1) &= \frac{\mathbf{O}((|\lambda| + |\xi|^2)^{1/2})\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_2^- y_1},\end{aligned}$$

where C_E is an expansion coefficient that is not specified in the analysis.

Proof. In each case, the stated estimates are obtained in straightforward fashion from (1.25) and (4.1). In the case $y_1 \leq x_1 \leq 0$, we obtain (omitting λ and ξ dependence for notational brevity)

$$\begin{aligned}&(2\pi)^{\frac{d-1}{2}} e^{i\tilde{y}\cdot\xi} G_{\lambda,\xi}(x_1, y) = N_1^-(y_1)\phi_1^+(x_1) + N_2^-(y_1)\phi_2^+(x_1)\\ &= \Big[(A_2^+C_1^+ - A_1^+C_2^+)\phi_1^-(x_1) + (B_2^+C_1^+ - B_1^+C_2^+)\phi_2^-(x_1) + (D_2^+C_1^+ - D_1^+C_2^+)\psi_2^-(x_1)\Big]\\ &\quad\times \frac{W(\phi_1^-, \phi_2^-, \psi_1^-)(y_1)}{c(0)D(\lambda,\xi)}\\ &+ \Big[(A_2^+D_1^+ - A_1^+D_2^+)\phi_1^-(x_1) + (B_2^+D_1^+ - B_1^+D_2^+)\phi_2^-(x_1) + (D_1^+C_2^+ - D_2^+C_1^+)\psi_1^-(x_1)\Big]\\ &\quad\times \frac{W(\phi_1^-, \phi_2^-, \psi_2^-)(y_1)}{c(0)D(\lambda,\xi)}.\end{aligned} \tag{4.7}$$

We first analyze the expression

$$(D_2^+C_1^+ - D_1^+C_2^+)\psi_2^-(x_1)\frac{W(\phi_1^-, \phi_2^-, \psi_1^-)(y_1)}{c(0)D(\lambda,\xi)},$$

which according to Lemma 4.1 can be written as

$$\Big[\frac{1}{c(0)} + \frac{\mathbf{O}((|\lambda| + |\xi|^2)^{3/2})}{D(\lambda,\xi)}\Big]\psi_2^-(x_1)W(\phi_1^-, \phi_2^-, \psi_1^-)(y_1). \tag{4.8}$$

Employing now the estimates of Lemma 3.1 and Lemma 3.2, and additionally the relation

$$(\mu_1^- - \mu_4^-)(\mu_1^- - \mu_3^-)(\mu_4^- - \mu_3^-) = 2\mu_1^-({\mu_3^-}^2 - {\mu_1^-}^2) = 2(\frac{b_-}{c_-})^{3/2} + \mathbf{O}(|\lambda| + |\xi|^2),$$

we see that (4.8) is equivalent to

$$\begin{aligned}
&\Big[\frac{1}{c(0)} + \frac{\mathbf{O}((|\lambda|+|\xi|^2)^{3/2})}{D(\lambda,\xi)}\Big]\Big[\frac{1}{\mu_2^-}(e^{\mu_2^- x_1} - e^{-\mu_2^- x_1}) + \mathbf{O}(e^{-\bar{\alpha}|x_1|})\Big] \\
&\quad \times e^{-\mu_2^- y_1}\Big(2(\frac{b_-}{c_-})^{3/2} + \mathbf{O}(|\lambda|+|\xi|^2) + \mathbf{O}(e^{-\tilde{\alpha}|y_1|})\Big) \\
&= \frac{2}{c(0)\mu_2^-}(\frac{b_-}{c_-})^{3/2}\Big(e^{\mu_2^-(x_1-y_1)} - e^{-\mu_2^-(x_1+y_1)}\Big) + \frac{\mathbf{O}(|\lambda|+|\xi|^2)}{\mu_2^-}\Big(e^{\mu_2^-(x_1-y_1)} - e^{-\mu_2^-(x_1+y_1)}\Big) \\
&+ \frac{\mathbf{O}(e^{-\eta|y_1|})}{\mu_2^-}\Big(e^{\mu_2^-(x_1-y_1)} - e^{-\mu_2^-(x_1+y_1)}\Big) + \frac{\mathbf{O}(|\lambda|+|\xi|^2)^{3/2}}{D(\lambda,\xi)\mu_2^-}\Big(e^{\mu_2^-(x_1-y_1)} - e^{-\mu_2^-(x_1+y_1)}\Big) \\
&+ \mathbf{O}(e^{-\eta|x_1|})e^{-\mu_2^- y_1} + \frac{\mathbf{O}((|\lambda|+|\xi|^2)^{3/2})}{D(\lambda,\xi)}\mathbf{O}(e^{-\eta|x_1|})e^{-\mu_2^- y_1}.
\end{aligned}$$

We next consider the combination

$$-B_1^+\phi_2^-(x_1)\Big[C_2^+\frac{W(\phi_1^-,\phi_2^-,\psi_1^-)(y_1)}{c(y_1)W_{\lambda,\xi}(y_1)} + D_2^+\frac{W(\phi_1^-,\phi_2^-,\psi_2^-)(y_1)}{c(y_1)W_{\lambda,\xi}(y_1)}\Big].$$

According to (3.6) and the analyticity of $\phi_2^-(x_1)$ in λ and $|\xi|^2$, we have

$$\phi_2^-(x_1;\lambda,\xi) = \bar{u}_{x_1}(x_1) + \mathbf{O}((|\lambda|+|\xi|^2)e^{-\eta|x_1|}). \tag{4.9}$$

In light of this, we define

$$\begin{aligned}
E_{\lambda,\xi}(x_1,y_1) &:= -B_1^+\bar{u}_{x_1}(x_1)\Big[C_2^+\frac{W(\phi_1^-,\phi_2^-,\psi_1^-)(y_1)}{c(y_1)W_{\lambda,\xi}(y_1)} + D_2^+\frac{W(\phi_1^-,\phi_2^-,\psi_2^-)(y_1)}{c(y_1)W_{\lambda,\xi}(y_1)}\Big] \\
&= \frac{\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)}e^{-\mu_2^- y_1}\Big(c_E + \mathbf{O}((|\lambda|+|\xi|^2)^{1/2}) + \mathbf{O}(e^{-\eta|y_1|})\Big),
\end{aligned}$$

where the order relation is a direct consequence of the estimates of Lemma 3.1 and Lemma 3.2 and where c_E is defined by

$$-2(\frac{b_-}{c_-})^{3/2}B_1^+(\lambda,\xi)C_1^+(\lambda,\xi) = c_E + \mathbf{O}((|\lambda|+|\xi|^2)^{1/2}).$$

For the remainder from (4.9), we can proceed in a much less refined manner to establish an estimate of the form

$$\frac{\mathbf{O}(|\lambda|+|\xi|^2)e^{-\eta|x_1|}}{D(\lambda,\xi)}e^{-\mu_2^- y_1}.$$

In the case $x_1 \le y_1 \le 0$, we have

$$\begin{aligned}
&(2\pi)^{\frac{d-1}{2}}e^{i\tilde{y}\cdot\xi}G_{\lambda,\xi}(x_1,y) = N_1^+(y_1)\phi_1^-(x_1) + N_2^+(y_1)\phi_2^-(x_1) \\
&= \frac{\phi_1^-(x_1)}{c(0)D(\lambda,\xi)}\Big[(A_1^+C_2^+ - A_2^+C_1^+)W(\phi_1^-,\psi_1^-,\phi_2^-) + (A_1^+D_2^+ - A_2^+D_1^+)W(\phi_1^-,\psi_2^-,\phi_2^-) \\
&\quad + (C_1^+D_2^+ - C_2^+D_1^+)W(\psi_1^-,\psi_2^-,\phi_2^-)\Big] \\
&- \frac{\phi_2^-(x_1)}{c(0)D(\lambda,\xi)}\Big[(B_1^+C_2^+ - B_2^+C_1^+)W(\phi_2^-,\psi_1^-,\phi_1^-) + (B_1^+D_2^+ - B_2^+D_1^+)W(\phi_2^-,\psi_2^-,\phi_1^-) \\
&\quad + (C_1^+D_2^+ - C_2^+D_1^+)W(\psi_1^-,\psi_2^-,\phi_1^-)\Big],
\end{aligned} \tag{4.10}$$

from which we can proceed almost precisely as in the case $y_1 \le x_1 \le 0$, noting in particular that the excited estimate is the same in each case.

The case $y_1 \le 0 \le x_1$ can be analyzed in a similar manner. □

In addition to Lemma 4.2, we have the following lemma regarding derivatives of $\tilde{G}_{\lambda,\xi}$.

Lemma 4.3. *Under the assumptions of Lemma 4.2, we have the following additional estimates on derivatives of $\tilde{G}_{\lambda,\xi}(x_1,y_1)$.*

(i) For $y_1 \le x_1 \le 0$

$$\begin{aligned}\partial_{y_1}\tilde{G}_{\lambda,\xi}(x_1,y_1) &= \Big[-C_S + \mathbf{O}(|\lambda|+|\xi|^2) + \frac{\mathbf{O}(|\lambda|+|\xi|^2)^{3/2}}{D(\lambda,\xi)}\Big]\Big(e^{\mu_2^-|x_1-y_1|} - e^{\mu_2^-|x_1+y_1|}\Big)\\ &+ \frac{\mathbf{O}(|\lambda|+|\xi|^3)}{D(\lambda,\xi)}e^{\mu_2^-|x_1+y_1|} + \mathbf{O}(e^{-\eta|x_1|})\mu_2^- e^{-\mu_2^- y_1}\\ &+ \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(|\lambda|+|\xi|^2)^{3/2}}{D(\lambda,\xi)}\mathbf{O}(e^{-\eta|x_1-y_1|})\end{aligned}$$

$$\begin{aligned}\partial_{x_1}\tilde{G}_{\lambda,\xi}(x_1,y_1) &= C_S\Big(e^{\mu_2^-|x_1-y_1|} + e^{\mu_2^-|x_1+y_1|}\Big) + \mathbf{O}(|\lambda|+|\xi|^2)e^{\mu_2^-|x_1-y_1|}\\ &+ \mathbf{O}(e^{-\eta|x_1|})e^{-\mu_2^- y_1} + \frac{\mathbf{O}(|\lambda|+|\xi|^3)}{D(\lambda,\xi)}e^{\mu_2^-|x_1-y_1|} + \frac{\mathbf{O}(|\lambda|+|\xi|^2)}{D(\lambda,\xi)}\mathbf{O}(e^{-\eta|x_1|})e^{-\mu_2^- y_1}\\ &+ \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(|\lambda|+|\xi|^2)^{3/2}}{D(\lambda,\xi)}\mathbf{O}(e^{-\eta|x_1-y_1|})\end{aligned}$$

$$\begin{aligned}\partial_{x_1y_1}\tilde{G}_{\lambda,\xi}(x_1,y_1) &= -2C_S\mu_2^-\Big(e^{\mu_2^-|x_1-y_1|} - e^{\mu_2^-|x_1+y_1|}\Big) + \mathbf{O}(|\lambda|+|\xi|^2)\mu_2^- e^{\mu_2^-|x_1-y_1|}\\ &+ \mathbf{O}(e^{-\eta|x_1|})\mu_2^- e^{-\mu_2^- y_1} + \frac{\mathbf{O}(|\lambda|+|\xi|^3)\mu_2^-}{D(\lambda,\xi)}e^{\mu_2^-|x_1-y_1|} + \frac{\mathbf{O}(|\lambda|+|\xi|^3)}{D(\lambda,\xi)}\mathbf{O}(e^{-\eta|x_1|})e^{-\mu_2^- y_1}\\ &+ \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(|\lambda|+|\xi|^2)^{3/2}}{D(\lambda,\xi)}\mathbf{O}(e^{-\eta|x_1-y_1|}).\end{aligned}$$

(ii) For $x_1 \le y_1 \le 0$,

$$\begin{aligned}\partial_{y_1}\tilde{G}_{\lambda,\xi}(x_1,y_1) &= C_S\Big(e^{\mu_2^-|x_1-y_1|} + e^{\mu_2^-|x_1+y_1|}\Big) + \mathbf{O}(|\lambda|+|\xi|^2)e^{\mu_2^-|x_1-y_1|}\\ &+ \mathbf{O}(e^{-\eta|y_1|})e^{-\mu_2^- x_1} + \frac{\mathbf{O}(|\lambda|+|\xi|^3)}{D(\lambda,\xi)}e^{\mu_2^-|x_1-y_1|}\\ &+ \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(|\lambda|+|\xi|^2)^{3/2}}{D(\lambda,\xi)}\mathbf{O}(e^{-\eta|x_1-y_1|})\end{aligned}$$

$$\begin{aligned}\partial_{x_1}\tilde{G}_{\lambda,\xi}(x_1,y_1) &= \Big[-C_S + \mathbf{O}(|\lambda|+|\xi|^2) + \frac{\mathbf{O}(|\lambda|+|\xi|^2)^{3/2}}{D(\lambda,\xi)}\Big]\Big(e^{\mu_2^-|x_1-y_1|} - e^{\mu_2^-|x_1+y_1|}\Big)\\ &+ \frac{\mathbf{O}(|\lambda|+|\xi|^3)}{D(\lambda,\xi)}e^{\mu_2^-|x_1+y_1|} + \mathbf{O}(e^{-\eta|y_1|})\mu_2^- e^{-\mu_2^- x_1} + \frac{\mathbf{O}(|\lambda|+|\xi|^2)}{D(\lambda,\xi)}\mathbf{O}(e^{-\eta|x_1|})e^{-\mu_2^- y_1}\\ &+ \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(|\lambda|+|\xi|^2)^{3/2}}{D(\lambda,\xi)}\mathbf{O}(e^{-\eta|x_1-y_1|})\end{aligned}$$

$$\begin{aligned}\partial_{x_1y_1}\tilde{G}_{\lambda,\xi}(x_1,y_1) &= -2\mu_2^- C_S\Big(e^{\mu_2^-|x_1-y_1|} - e^{\mu_2^-|x_1+y_1|}\Big) + \mathbf{O}(|\lambda|+|\xi|^2)\mu_2^- e^{\mu_2^-|x_1-y_1|}\\ &+ \mathbf{O}(e^{-\eta|y_1|})\mu_2^- e^{-\mu_2^- x_1} + \frac{\mathbf{O}(|\lambda|+|\xi|^3)\mu_2^-}{D(\lambda,\xi)}e^{\mu_2^-|x_1-y_1|} + \frac{\mathbf{O}(|\lambda|+|\xi|^3)}{D(\lambda,\xi)}\mathbf{O}(e^{-\eta|x_1|})e^{-\mu_2^- y_1}\\ &+ \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(|\lambda|+|\xi|^2)^{3/2}}{D(\lambda,\xi)}\mathbf{O}(e^{-\eta|x_1-y_1|}).\end{aligned}$$

(iii) For $y_1 \le 0 \le x_1$,

$$\partial_{y_1}\tilde{G}_{\lambda,\xi}(x_1,y_1) = \frac{\mathbf{O}(|\lambda|+|\xi|^3)}{D(\lambda,\xi)} e^{\mu_2^+ x_1 - \mu_2^- y_1} + \frac{\mathbf{O}((|\lambda|+|\xi|^2)^{3/2})}{D(\lambda,\xi)} \mathbf{O}(e^{-\eta|x_1|}) e^{-\mu_2^- y_1}$$

$$\partial_{x_1}\tilde{G}_{\lambda,\xi}(x_1,y_1) = \frac{\mathbf{O}(|\lambda|+|\xi|^2)\mu_2^+}{D(\lambda,\xi)} e^{\mu_2^+ x_1 - \mu_2^- y_1} + \frac{\mathbf{O}(|\lambda|+|\xi|^2)}{D(\lambda,\xi)} \mathbf{O}(e^{-\eta|x_1|}) e^{-\mu_2^- y_1}$$

$$\partial_{x_1 y_1}\tilde{G}_{\lambda,\xi}(x_1,y_1) = \frac{\mathbf{O}(|\lambda|+|\xi|^3)\mu_2^+}{D(\lambda,\xi)} e^{\mu_2^+ x_1 - \mu_2^- y_1} + \frac{\mathbf{O}(|\lambda|+|\xi|^3)}{D(\lambda,\xi)} \mathbf{O}(e^{-\eta|x_1|}) e^{-\mu_2^- y_1}.$$

Here C_S is as in the statement of Lemma 4.2.

Regarding the proof of Lemma 4.3, we note only that the method is almost identical of that employed in the proof of Lemma 4.2 and that additional details can be found in [23].

We now state a pair of lemmas corresponding with estimates on $G_{\lambda,\xi}(x_1,y)$ for (respectively) large values of $|\lambda|+|\xi|^2$ and for medium values of this quantity.

Lemma 4.4. *Under the assumptions of Theorem 1.1, and for $G_{\lambda,\xi}(x_1,y)$ as defined in (1.23), we have the following estimates. For $(|\lambda|+|\xi|^2) > R$, where R is an appropriately large constant, and for λ bounded to the right of the contour*

$$Re\ \lambda = -\frac{c_1}{K_1}\Big(|Re\ \xi|^4 - C_2|Im\ \xi|^4 + |Im\ \lambda|\Big),$$

where c_1 and C_2 are as in Condition 2 of spectral criteria ($\mathcal{D}$), and K_1 is some suitably large constant, there holds

$$|e^{i\xi\cdot\tilde{y}}\partial^\alpha G_{\lambda,\xi}(x_1,y)| \le C(|\lambda|+|\xi|^4)^{\frac{|\alpha|-3}{4}} e^{-\beta(|\lambda|+|\xi|^4)^{1/4}|x_1-y_1|},$$

where α is a multi-index in the variables (x_1,y_1), with $|\alpha|\le 3$.

Lemma 4.5. *Under the assumptions of Theorem 1.1, and for $G_{\lambda,\xi}(x_1,y)$ as defined in (1.23), we have the following estimates. For $r < (|\lambda|+|\xi|^2) < R$, where r is as in Lemma 4.2 and R is as in Lemma 4.4, and for λ bounded away from the essential spectrum of L_ξ, there holds*

$$|e^{i\xi\cdot\tilde{y}}\partial^\alpha G_{\lambda,\xi}(x_1,y)| \le C,$$

for some appropriately large constant C.

Regarding the proofs of Lemmas 4.4 and 4.5, we note that large $|\lambda|+|\xi|^2$ behavior corresponds with small t behavior, for which the fourth order effects dominate. Consequently the proofs of these lemmas are almost precisely the same as those of the corresponding Lemmas 3.2 and 3.3 of [27], carried out in the context of equations

$$u_t + \sum_{j=1}^d f^j(u)_{x_j} = -\sum_{jklm}(c^{jklm}(u)u_{x_j x_k x_l})_{x_m}.$$

5 Proof of Theorem 1.1

In this section, we use the estimates of Lemmas 4.2–4.5 to prove Theorem 1.1. We proceed through consideration of the Fourier–Laplace inversion formula,

$$G(t,x;y) = \frac{1}{(2\pi)^d i}\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot\tilde{x}}\int_\Gamma e^{\lambda t}G_{\lambda,\xi}(x_1,y)d\lambda d\xi, \tag{5.1}$$

where for each $\xi\in\mathbb{R}^{d-1}$ the contour Γ must encircle the poles of $G_{\lambda,\xi}(x_1,y)$ (which correspond with point spectrum of the operator L_ξ). The validity of (5.1) can be established in a straightforward manner from the estimates of Lemmas 4.2–4.5. For details (in the setting of second order regularization only), see Corollary 7.4 of [46].

Before beginning the detailed proof of Theorem 1.1, we give a brief overview of the approach taken and set some notation. In each case of Lemma 4.2, the estimate on $G_{\lambda,\xi}(x_1,y)$ is divided into a number of terms

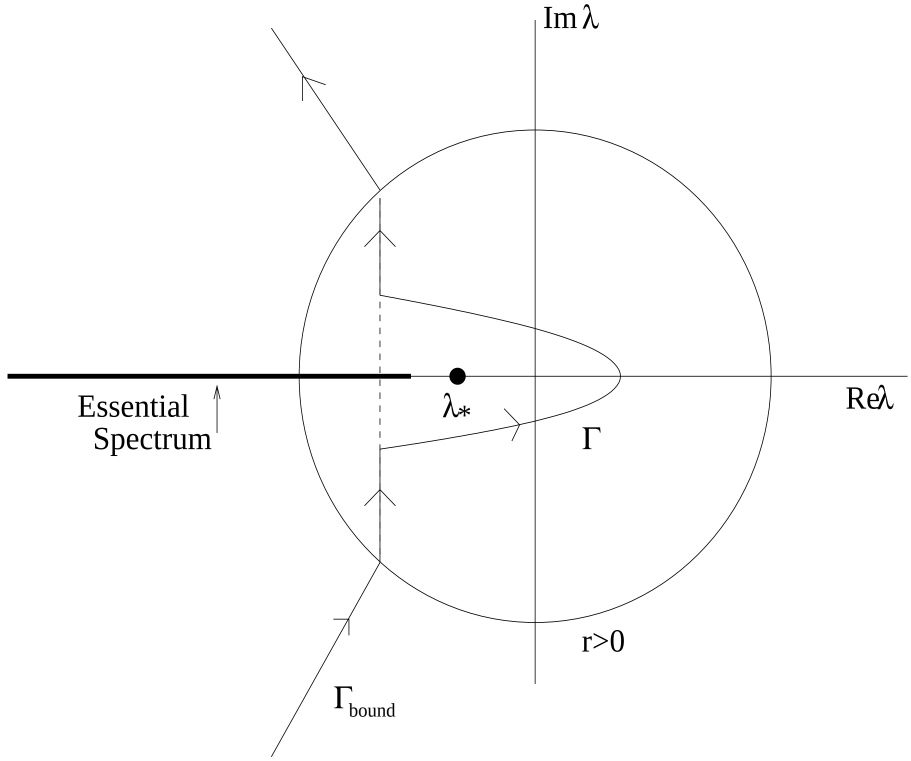

Figure 1: Contours Γ and Γ_{bound}

that can each be integrated separately against $e^{\lambda t + i\xi\cdot\tilde{x}}$. For each of these terms, the contour of integration Γ will depend on t, x, y, and ξ, and we will rely on the Cauchy theorem for integration of analytic functions for invariance of the result. Moreover, in certain cases, we will complexify the ξ integration as well, taking $\xi = \xi_R + i\xi_I$, where the complex part will depend on t, x, and y. This complexification of the ξ integration—which follows the analysis of Hoff and Zumbrun [21, 22]—allows us to obtain a more refined description of $G(t,x;y)$ than that obtained in [36, 37]. Throughout this analysis, we must bear in mind that we will not generally be able to follow our optimally chosen contour out to the point at ∞. Except in our neighborhood of the origin $|\lambda| + |\xi|^2 < r$, where the point spectrum of L_ξ is understood in terms of Condition (1) of $\mathcal{D}$, we must remain to the right of our boundary contour

$$\text{Re } \lambda = -c_1\Big(|\text{Re } \xi|^4 - C_2|\text{Im } \xi|^4 + |\text{Im } \lambda|\Big), \tag{5.2}$$

defined in Condition (2) of ($\mathcal{D}$). In this way, our approach will be to follow an optimal contour until it interects (5.2) and then to follow (5.2) out to the point at ∞ (see Figure 1).

Our analysis is divided in principle into two regimes (corresponding with cases (I) and (II) of Theorem 1.1): (I) $|x-y| \geq Kt$ or $t \leq 1$ (some K sufficiently large) and (II) $|x-y| \leq Kt$ and $t \geq 1$. For Case (I), our estimate on $G_{\lambda,\xi}(x_1,y)$ is qualitatively the same as that of [27], and we can proceed as there (see Section 4.1) to recover

$$\|\partial^\alpha G(t,x;y)\|_{L^p_{\tilde{x}}} \leq Ct^{-\frac{d-1}{4}(1-\frac{1}{p})-\frac{1+|\alpha|}{4}} e^{-\frac{|x_1-y_1|^{4/3}}{Mt^{1/3}}}, \quad |\alpha| \leq 3.$$

The remainder of this section will be devoted to Case (II), $|x-y| \leq Kt$.

Outline of the section. The details of our proof are quite involved, and in order to clarify the discussion the section is divided into seven subsections, each of which contains a different aspect of the analysis. In Subsection 5.1, we give the main idea of the contour-shifting argument, considering for simplicity only the leading order terms in $G_{\lambda,\xi}(x_1;y)$. In this subsection, there is no difficulty with the zero of the Evans function at $\lambda_*(\xi)$, and the analysis is relatively straightforward. Nonetheless, the main idea is established.

In Subsection 5.2 we briefly discuss the treatment of higher order terms that were omitted, for the sake of clarity, from the discussion of Subsection 5.1. The argument of Subsection 5.1 is based entirely on the small $|\lambda|+|\xi|^2$ estimates of Lemma 4.2, though our contours of integration necessarily exceed this region. In Subsection 5.3 we correct for the error that arises from this analysis (the *continuation correction*). Subsection 5.4, the most technical part of the paper, contains the general argument of the proof. Though much of this subsection is necessarily technical in nature, it also contains one important insight: this is where we deal with the cubic scaling of $\lambda_*(\xi)$. In Subsection 5.5 we deal particularly with the excited terms; that is, the terms in $G(t,x;y)$ collected as $\bar{u}_{x_1}(x_1)E(t,x;y)$, that decay at minimal rate. The principle idea in this subsection is that the case $d=2$ must be dealt with in a slightly more refined manner than we require for the cases $d\geq 3$, and we take advantage of the fact that for $d=2$, ξ is not a vector, and many calculations simplify. In Subsection 5.6, we briefly mention how the analyses of the previous subsections extend to the case $y_1\leq 0\leq y_1$, and finally in Subsection 5.7 we state a general theorem on higher order derivatives of $G(t,x;y)$ that can be proved by the methods of this section.

5.1 The First Order Approximation

As a straightforward baseline case, we begin with the case $x_1, y_1 \leq 0$, for which the first order estimate is

$$\frac{C_S}{\mu_2^-}\Big(e^{\mu_2^-|x_1-y_1|}-e^{\mu_2^-|x_1+y_1|}\Big), \tag{5.3}$$

which can of course be regarded as two terms that can each be analyzed in the same manner. Combining the first part of this estimate with integral representation (5.1), we obtain

$$\frac{1}{(2\pi)^d i}C_S\int_{\mathbb{R}^{d-1}}e^{i\xi\cdot\tilde{x}}\int_\Gamma\frac{1}{\mu_2^-}e^{\lambda t+\mu_2^-|x_1-y_1|}d\lambda d\xi. \tag{5.4}$$

Our approach will be to choose the contour Γ so as to optimally fix the real part of μ_2^-:

$$-\mu_2^-(\lambda,\xi)=\sqrt{R}+ik;\quad k\in\mathbb{R}. \tag{5.5}$$

Upon squaring this expression and solving for λ, we obtain the contour

$$\begin{aligned}\lambda(k)&=b_-R-b_-(k^2+|\xi|^2)+2ib_-\sqrt{R}k\\&\quad -c_-R^2+4c_-Rk^2-c_-(k^2+|\xi|^2)^2+2c_-R(k^2+|\xi|^2)+4ic_-k\sqrt{R}(k^2+|\xi|^2-R),\end{aligned} \tag{5.6}$$

where the second line consists of higher order terms that will have little affect on the analysis. (In most cases, we will simply omit these terms, but in this baseline case, we will carry them through so as to establish that they can indeed be neglected.) In this way, our choice of contour will correspond with a choice of R. In the case of (5.4), we take

$$\sqrt{R}=\frac{|x_1-y_1|}{2b_-t},$$

for which (5.6) becomes

$$\begin{aligned}\lambda(k)&=\frac{|x_1-y_1|^2}{4b_-t^2}-b_-(k^2+|\xi|^2)+2ib_-\frac{|x_1-y_1|}{2b_-t}k\\&\quad -c_-(\frac{|x_1-y_1|}{2b_-t})^4+4c_-k^2(\frac{|x_1-y_1|}{2b_-t})^2-c_-(k^2+|\xi|^2)^2+2c_-(\frac{|x_1-y_1|}{2b_-t})^2(k^2+|\xi|^2)\\&\quad +4ic_-k\frac{|x_1-y_1|}{2b_-t}(k^2+|\xi|^2-(\frac{|x_1-y_1|}{2b_-t})^2).\end{aligned} \tag{5.7}$$

In this last expression, the first three terms on the right-hand side play the most significant role, and for $|k|$, $|\xi|$, and $\frac{|x_1-y_1|}{t}$ all small, the remaining terms can be considered higher order corrections (see Subsection 5.2). Observing additionally the relation

$$\begin{aligned}\frac{d\lambda}{dk}&=-2b_-k+2ib_-\sqrt{R}\\&\quad +8c_-kR-4kc_-(k^2+|\xi|^2)+4c_-kR+4ic_-\sqrt{R}(k^2+|\xi|^2)-R)+8ic_-k^2\sqrt{R},\end{aligned}$$

we find that (5.4) becomes (to lowest order, and omitting for the moment higher order effects and the constant outside the integration)

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y})} \int_\Gamma \frac{(-2b_-k + 2ib_-\frac{|x_1-y_1|}{2b_-t})}{-\frac{|x_1-y_1|}{2b_-t} - ik} e^{\frac{|x_1-y_1|^2}{4b_-t} - b_-(k^2+|\xi|^2)t + 2ib_-\frac{|x_1-y_1|}{2b_-}k - (\frac{|x_1-y_1|}{2b_-t}+ik)|x_1-y_1|} dk d\xi \\
&\quad = \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y})} \int_\Gamma -2ib_- e^{-\frac{|x_1-y_1|^2}{4b_-{}^2t} - b_-(k^2+|\xi|^2)t} dk d\xi \qquad (5.8) \\
&\quad = -2ib_- e^{-\frac{|x_1-y_1|^2}{4b_-t}} (\frac{\pi}{b_-t})^{1/2} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y}) - b_-|\xi|^2 t} d\xi.
\end{aligned}$$

The remaining $d-1$ integrals can be evaluated iteratively, and we obtain

$$-2ib_-(\frac{\pi}{b_-t})^{d/2} e^{-\frac{|x-y|^2}{4b_-t}}.$$

Returning our constants from (5.4), we conclude that to first order the integral (5.4) is

$$-2b_-(4\pi b_-t)^{-d/2} e^{-\frac{|x-y|^2}{4b_-t}} (\frac{\pi}{b_-t})^{d/2}.$$

Replacing $|x-y|$ in the above calculation with $|x+y|$, we obtain a similar expression for the subtracted expression in 5.3. Finally, we obtain the first estimate of Case (i) of Theorem 1.1 by direct L^p integration of the transverse variable $\tilde{x}$.

5.2 Higher order corrections

We next consider the terms that have been omitted in (5.8). First, we have used

$$\begin{aligned}
e^{\lambda t + \mu_2^-|x_1-y_1|} &= e^{b_-Rt - b_-(k^2+|\xi|^2)t + 2ib_-\sqrt{R}kt - \sqrt{R}|x_1-y_1| - ik|x_1-y_1|} \\
&\quad \times e^{-c_-R^2t + 4c_-Rk^2t - c_-(k^2+|\xi|^2)^2t + 2c_-R(k^2+|\xi|^2)t + 4ic_-k\sqrt{R}(k^2+|\xi|^2-R)t},
\end{aligned}$$

where the second line has been regarded as a higher order correction. Upon expansion of this exponentiation, we have several order terms, beginning with $\mathbf{O}(R^2t)$. Carrying this correction through the calculation (5.8), we obtain an estimate by

$$C\frac{|x_1-y_1|^4}{16b_-{}^4t^3} t^{-d/2} e^{-\frac{|x-y|^2}{4b_-t}} \le C_1 t^{-d/2-1} e^{-\frac{|x-y|^2}{4b_-t}},$$

whose transverse $L^p_{\tilde{x}}$ norm can be absorbed into the second summand in the first estimate of Case (i) of Theorem 1.1. The remaining order terms from this exponentiation can be analyzed similarly.

In addition to the corrections discussed in the previous paragraph, we have corrections arising from $d\lambda$. We mention here only that since these are not multiplied by t (as are the corrections in the previous paragraph), they are more easily accomodated and give a smaller correction.

5.3 The Continuation Correction

More precisely, the estimates of Lemma 4.2 are only valid for $|\lambda| + |\xi|^2 < r$, for some suitable constant r. In this way, our integration for $|\lambda| + |\xi|^2 \ge r$ must be carried out in terms of the estimates of Lemmas 4.5 and 4.4. However, in the case $|\lambda| + |\xi|^2 \ge r$, we have exponential decay in t along both our optimal contour and along (5.2), and in the current setting of $|x-y| \le Kt$ exponential decay in t is sufficient to give an estimate that can easily be subsumed into those of Theorem 1.1.

5.4 The General Argument

We next consider the more general setting in which $G_{\lambda,\xi}(x_1, y)$ involves division by the Evans function. In this case, we must insure that our contour Γ both continues to contain $\lambda_*(\xi)$ and remains bounded away from this point. As each case is similar, we focus on the estimate

$$\frac{\mathbf{O}(|\lambda| + |\xi|^2)}{D(\lambda, \xi)} e^{\mu_2^- |x_1+y_1|},$$

from which the main points of our general contour-shifting argument will be apparent. In this case, prior to striking Γ_{bound}, our integral (5.1) takes the form

$$\frac{1}{(2\pi)^d i} \int_{R(r_2)} e^{i\xi\cdot(\tilde{x}-\tilde{y})} \int_{|\lambda|\le r_1} e^{\lambda t} \frac{\mathbf{O}(|\lambda| + |\xi|^2)}{D(\lambda, \xi)} e^{\mu_2^- |x_1+y_1|} d\lambda d\xi, \tag{5.9}$$

where R here denotes the region

$$R(r_2) := \{\xi : |\xi_k| \le r_2, k = 2, 3, ..., d\}, \tag{5.10}$$

with r_1 and r_2 chosen sufficiently small so that $|\lambda| + |\xi|^2 < r$, and the truncated part can be analyzed as a *continuation correction* (see section 5.3).

The analysis will be divided into four subcases, as follows:

$$\begin{aligned}
&1. \quad t^{1/2} \le |x_1 + y_1| \quad \text{and} \quad |x_1 + y_1| \ge |\tilde{x} - \tilde{y}| \\
&2. \quad t^{1/2} \le |x_1 + y_1| \le |\tilde{x} - \tilde{y}| \\
&3. \quad |x_1 + y_1| \le t^{1/2} \le |\tilde{x} - \tilde{y}| \\
&4. \quad |x_1 + y_1| \le t^{1/2}; \quad |\tilde{x} - \tilde{y}| \le t^{1/2}.
\end{aligned} \tag{5.11}$$

Case (5.11-1.) For the first case in (5.11), and for $\xi \in R(c_1 \frac{|x_1+y_1|}{t})$, for some suitably small constant c_1, we take contour (5.5), choosing in this case

$$\sqrt{R} = \frac{|x_1 + y_1|}{L_1 t}, \tag{5.12}$$

where L_1 will denote a large constant that will be chosen during the analysis. Upon substitution of this value of R into (5.6), and omitting higher order corrections, we arrive at integrals of the form

$$\int_{R(c_1 \frac{|x_1+y_1|}{t})} e^{i\xi\cdot(\tilde{x}-\tilde{y})} \int_{|\lambda|\le r_1} e^{b_- \frac{|x_1+y_1|^2}{L_1^2 t} - b_-(k^2+|\xi|^2)t + 2ib_- k \frac{|x_1+y_1|}{L_1} - \frac{|x_1+y_1|}{L_1 t}|x_1+y_1| - ik|x_1+y_1|} \frac{\mathbf{O}(|\lambda| + |\xi|^2)}{D(\lambda, \xi)} d\lambda d\xi. \tag{5.13}$$

Here,

$$\mathbf{O}(|\lambda|) = \mathbf{O}\Big(\frac{|x_1 + y_1|^2}{t^2} + k^2 + |\xi|^2\Big),$$

and

$$|d\lambda| = \mathbf{O}\Big(|k| + \frac{|x_1 + y_1|}{t}\Big)|dk|.$$

Moreover, for c_1 sufficiently small, we can insure that λ is bounded away from the leading eigenvalue $\lambda_*(\xi)$, passing to the right of it as Γ crosses the real axis. In the neighborhood of the origin under consideration,

$$D(\lambda, \xi) \sim (\lambda - \lambda_*(\xi)),$$

(see Condition 1 of $(\mathcal{D})$ and Lemma 3.5) so that

$$|D(\lambda, \xi)| \ge m_1 |\lambda| \ge m_2 \frac{|x_1 + y_1|^2}{t^2} \ge m_2 t,$$

and consequently

$$|D(\lambda,\xi)|^{-1} \le Ct$$

for some appropriately large constant C. In this way, (5.13) is bounded in absolute value by

$$\begin{aligned} &C_1 t e^{-\frac{|x_1+y_1|^2}{M_1 t}} \int_{R(c_1\frac{|x_1+y_1|}{t})} \int_{|k|\le r_3} e^{-b_-(k^2+|\xi|^2)t}\Big(\frac{|x_1+y_1|^3}{t^3} + |k^3| + |\xi|^3\Big)|dk||d\xi| \\ &\le C_2 t^{-\frac{d}{2}-\frac{1}{2}} e^{-\frac{|x_1+y_1|^2}{Mt}}, \end{aligned} \tag{5.14}$$

for some suitably large constants C_1, C_2, M_1, and M. In this case, decay in $|x_1+y_1|$ yields decay in $|\tilde{x}-\tilde{y}|$, and we immediately have an estimate by

$$C_2 t^{-\frac{d}{2}-\frac{1}{2}} e^{-\frac{|x_1+y_1|^2}{M_1 t}} e^{-\frac{|\tilde{x}-\tilde{y}|^2}{Mt}}.$$

Taking the transverse norm of this last expression yields an estimate that can be absorbed into the second summand in the first estimate of Case (i) of Theorem 1.1.

For $\xi \in R(r_2)\backslash R(c_1\frac{|x_1+y_1|}{t}))$, we choose

$$\sqrt{R} = c_2 \frac{|x_1+y_1|}{L_1 t},$$

where both c_2 and L_1 will be chosen during the argument. In this case,

$$\lambda(k) = b_- c_2^2 \frac{|x_1+y_1|^2}{L_1^2 t^2} - b_-(k^2+|\xi|^2) + 2ib_- c_2 \frac{|x_1+y_1|}{L_1} k + \text{H. O. T.}, \tag{5.15}$$

and we see that for $\xi \in R(r_2)\backslash R(c_1\frac{|x_1+y_1|}{t}))$ we can choose c_2 sufficiently small so that Γ passes entirely to the left of $\lambda_*(\xi)$. (See Figure 2, which we note is quite similar to Figure 2.2 of [37]. Indeed, though this approach of "leaping" the leading eigenvalue was employed in [24, 25], we have certainly been inspired in the current implementation by [37].)

We now have two contours to consider, one that lies entirely to the left of $\lambda_*(\xi)$ (described in (5.15)) and one that encircles $\lambda_*(\xi)$ in such a way that a union of the contours encircles the entire spectrum of L_ξ (see Figure 2). For the contour lying entirely to the left of λ_*, we can proceed almost precisely as in our analysis for the case $\xi \in R(c_1|x_1+y_1|/t)$. For the contour encircling $\lambda_*(\xi)$, we obtain the residue integral

$$\int_{\{\xi\in R(r_2)\backslash R(c_1\frac{|x_1+y_1|}{t}))\}} e^{i(\tilde{x}-\tilde{y})\cdot\xi+\lambda_*(\xi)t+\mu_2^-(\lambda_*(\xi),\xi))|x_1+y_1|}\mathbf{O}(|\xi|^2)d\xi. \tag{5.16}$$

Here,

$$\begin{aligned} \lambda_*(\xi) &= -\lambda_3|\xi|^3 + \mathbf{O}(|\xi|^4) \\ \text{Re }\mu_2^-(\lambda_*(\xi),\xi) &= -\sqrt{|\xi|^2+\mathbf{O}(|\xi|^3)} + \mathbf{O}(|\xi|^2) \le -\theta|\xi|, \end{aligned} \tag{5.17}$$

for some $\theta > 0$. In this way, (5.16) can be estimated by

$$\begin{aligned} &C\int_{\{\xi\in R(r_2)\backslash R(c_1\frac{|x_1+y_1|}{t}))\}} e^{-\lambda_3^0|\xi|^3 t-\theta|\xi||x_1+y_1|}|\xi|^2|d\xi| \\ &\le C_1\Big[|x_1+y_1|^{-(d+1)} + t^{-\frac{3}{2}}|x_1+y_1|^{-(d-2)}\Big]e^{-\frac{(x_1+y_1)^2}{Mt}}, \end{aligned} \tag{5.18}$$

where the algebraic decay in $|x_1+y_1|$ arises upon integration of the exponent involving θ, and the exponential decay arises from the same exponent and the observation that on this region of integration $|\xi| \ge c_1|x_1+y_1|/t$. (Here, we have simply integrated by parts and applied Young's inequality to collect terms.) Finally, in this case $|x_1+y_1| \ge t^{1/2}$, and we have an estimate by

$$C_2 t^{-\frac{d+1}{2}} e^{-\frac{(x_1+y_1)^2}{Mt}},$$

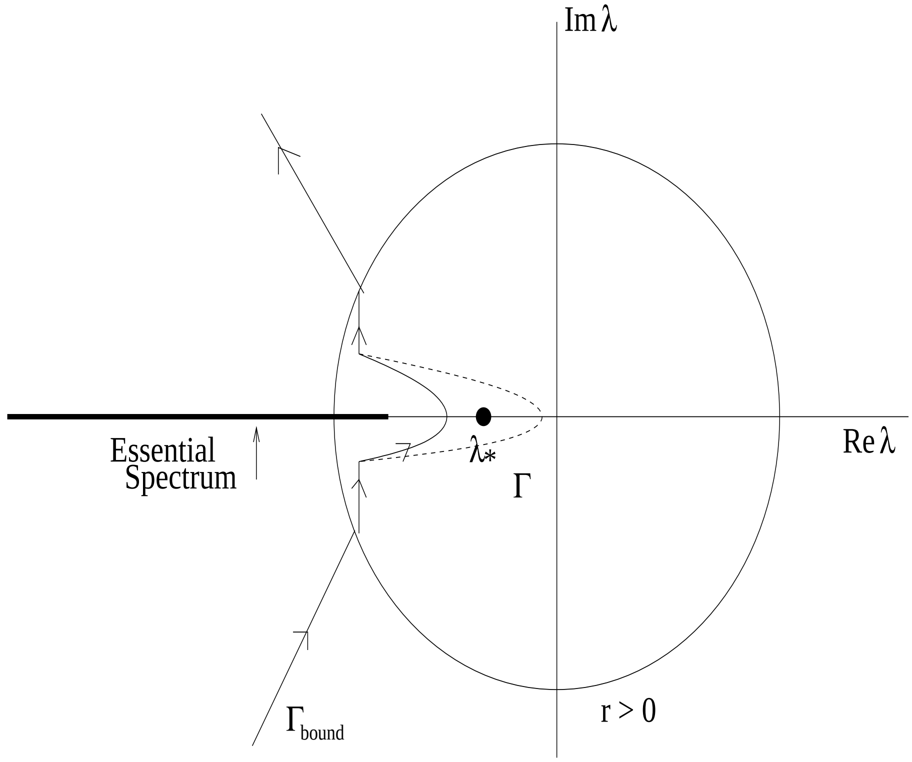

Figure 2: Contours Γ and Γ_{bound} for $\xi \in R(r_2)\backslash R(c_1\frac{|x_1+y_1|}{t}))$

precisely as in (5.14).

Case (5.11)–2. In cases (2)–(4) of (5.11) exponential decay in $|\tilde{x}-\tilde{y}|$ will not follow from exponential decay in $|x_1+y_1|$, and we will proceed similarly as in [21, 27] by complexifying ξ as

$$\xi = \xi_R + i\xi_I. \tag{5.19}$$

Suppose without loss of generality that we are in the case

$$|x_2 - y_2| = \max_{k=2,3,\dots,d} |x_k - y_k|. \tag{5.20}$$

For $\xi_R \in R(c_1(|x_2-y_2|)/(L_3 t))$, with $c_1 > 0$ to be chosen sufficiently small during the analysis, we select our contour by the choices

$$\begin{aligned} \sqrt{R} &= \frac{|x_2-y_2|}{L_3 t} \\ \xi_I &= (\frac{(x_2-y_2)}{L_3 t}, 0, ..., 0). \end{aligned} \tag{5.21}$$

After complexification, (5.6) becomes

$$\lambda(k,\xi) = b_- R - b_-(k^2 + |\xi_R|^2) + b_-|\xi_I|^2 + 2ib_-\Big[\sqrt{R}k - \xi_R\cdot\xi_I\Big] + \text{H. O. T.} \tag{5.22}$$

With this choice, and for c_1 sufficiently small, we can insure that Γ passes entirely to the right of $\lambda_*(\xi)$, and as in the analysis of Case (5.11)–1 we have $|D(\lambda,\xi)|^{-1} \le Ct$. Also, according to (5.22), we have

$$\begin{aligned}
&\text{Re }\Big(\lambda t + i(\tilde{x}-\tilde{y})\cdot\xi + \mu_2^-(\lambda,\xi)|x_1+y_1|\Big)\\
&= b_- \frac{|x_2-y_2|^2}{L_3^2 t} - b_-(k^2+|\xi_R|^2) + b_- \frac{|x_2-y_2|^2}{L_3^2 t} - \frac{(x_2-y_2)^2}{L_3 t} - \frac{|x_2-y_2|}{L_3 t}|x_1+y_1| + \text{H.O.T.}\\
&\le -\frac{(x_2-y_2)^2}{L_2 t} - \theta(k^2+|\xi_R|^2)t,
\end{aligned}$$

for some $\theta > 0$, where this last inequality is true for sufficiently large values of the constants L_2 and L_3. In this way, (5.9) can be estimated by

$$\begin{aligned}
&Ct \int_{\xi_R \in R(c_1(|x_2-y_2|)/(L_3 t))} \int_{|k|\le r_3} e^{-\frac{(x_2-y_2)^2}{L_2 t} - \theta(k^2+|\xi_R|^2)t} \Big(\frac{|x_2+y_2|^3}{t^3} + |k^3| + |\xi|^3\Big)|dk||d\xi|\\
&\le C_1 t^{-\frac{d+1}{2}} e^{-\frac{(x_2-y_2)^2}{Mt}},
\end{aligned}$$

for which we observe that the dominance of $|x_2-y_2|$ allows us to conclude exponential decay in all components, and hence the transverse norm is bounded by the second expression in the first estimate of Theorem 1.1 Case (i). In the event that $\xi \in R(r_2)\backslash R((c_1(|x_2-y_2|)/(L_3 t)))$, we alternatively choose

$$\begin{aligned}
\sqrt{R} &= c_2 \frac{|x_2-y_2|}{L_3 t}\\
\xi_I &= (\frac{(x_2-y_2)}{L_3 t}, 0, ..., 0),
\end{aligned} \tag{5.23}$$

where by choosing c_2 sufficiently small we can insure that our contour passes entirely to the left of $\lambda_*(\xi)$. As in Case (5.11)–1, this yields two contours to consider, one passing entirely to the left of $\lambda_*(\xi)$ and one encircling $\lambda_*(\xi)$. For the contour passing to the left of $\lambda_*(\xi)$, we can proceed almost precisely as in the case $\xi \in R((c_1(|x_2-y_2|)/(L_3 t)))$, while for the contour encircling $\lambda_*(\xi)$ we obtain the residue integral (5.16) with

$$\begin{aligned}
\text{Re } \lambda_*(\xi) &\le -\bar{c}|\xi_R|^3 + \bar{C}\frac{|x_2-y_2|^3}{L_3^3 t^3}\\
\text{Re } \mu_2^-(\lambda_*(\xi),\xi) &\le -\bar{\theta}|\xi_R|,
\end{aligned} \tag{5.24}$$

for some constants $\bar{c}$, $\bar{C}$, and $\bar{\theta}$. In this way, (5.16) is bounded by

$$\int_{\{\xi_R \in R(r_2)\backslash R(c_1 \frac{|x_2-y_2|}{L_3 t})\}} e^{-\frac{(x_2-y_2)^2}{L_3 t} - \bar{c}|\xi_R|^3 t + \bar{C}\frac{|x_2-y_2|^3}{L_3^3 t^3} - \bar{\theta}|\xi_R||x_1+y_1|} \Big(|\xi_R|^2 + \frac{|x_2-y_2|^2}{t^2}\Big)|d\xi_R|.$$

We are currently working in the case $|x-y| \le Kt$, for which $|x_2-y_2| \le Kt$, and so

$$\bar{C}\frac{|x_2-y_2|^3}{L_3^3 t^3} \le \bar{C}K^2 \frac{|x_2-y_2|}{L_3^3 t}.$$

Observing that K, $\bar{C}$ and L_3 are independent constants, we can choose L_3 sufficiently large so that

$$-\frac{(x_2-y_2)^2}{L_3 t} + \bar{C}K^2 \frac{|x_2-y_2|}{L_3^3 t} \le -\frac{(x_2-y_2)^2}{M_1 t},$$

for some sufficiently large constant M_1. Similarly as in (5.18) we obtain an estimate on (5.4) by

$$C\Big(|x_1+y_1|^{-(d+1)} + t^{-\frac{3}{2}}|x_1+y_1|^{-(d-2)}\Big)e^{-\frac{(x_2-y_2)^2}{Mt}},$$

for some M sufficiently large. Keeping in mind that in this case $|x_1+y_1| \ge t^{1/2}$, and that $|x_2-y_2|$ is the dominant spatial term, we obtain an estimate by the second expression in Case (i) of Theorem 1.1.

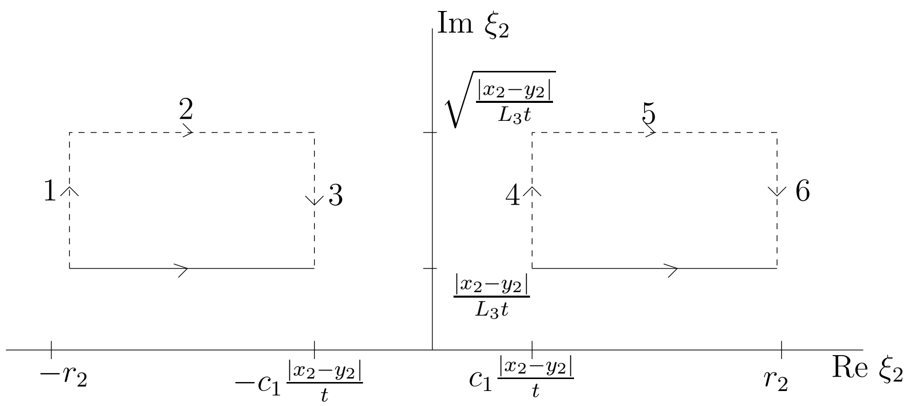

Figure 3: Re-complexification for $x_2 - y_2 > 0$.

Case (5.11)–3. In the case (5.11)–3, we again take (5.20). For $\xi_R \in R(c_1|x_2 - y_2|/t)$, we proceed almost precisely as in (5.11)–2 to obtain the same estimate as obtained there. For $\xi_R \in R(c_2)\backslash R(c_1|x_2 - y_2|/t)$, we can choose c_1 sufficiently small so that we have two contours, one passing entirely to the left of $\lambda_*(\xi)$ and one encircling $\lambda_*(\xi)$. For the contour passing to the left of $\lambda_*(\xi)$, we can proceed similarly as with the contour passing to the right of $\lambda_*(\xi)$ in (5.11)–2 to get the same estimate as obtained there, while for the contour encircling $\lambda_*(\xi)$, we obtain the residue integral (5.16) with ξ replaced by ξ_R. In this case, we have $|x_1 + y_1| \le \sqrt{t}$, and consequently we will no longer be able to convert our spatial decay into $t^{-1/2}$ decay. In light of this, we require a different exponential scaling, appropriate to the $t^{-1/3}$ decay we expect to get from our exponentiation of $\lambda_*(\xi)t$. We obtain this new scaling by re-complexifying ξ, lifting it appropriately to

$$\xi_{I2} = \text{sgn}(x_2 - y_2)\sqrt{\frac{|x_2 - y_2|}{L_3 t}} \tag{5.25}$$

(keeping in mind that $|x_2 - y_2|$ is assumed dominant). This complexification leads to six additional intervals of integration, which can be written as

$$\begin{aligned}
&(1) \text{ and } (6): \Big\{\xi : \xi_{R2} = \pm r, \frac{|x_2 - y_2|}{L_3 t} \le |\xi_{I2}| \le \sqrt{\frac{|x_2 - y_2|}{L_3 t}}\Big\} \\
&(2) \text{ and } (5): \Big\{\xi : c_1\frac{|x_2 - y_2|}{t} \le |\xi_{R2}| \le r_2, \xi_{I2} = \sqrt{\frac{|x_2 - y_2|}{L_3 t}}\Big\} \\
&(3) \text{ and } (4): \Big\{\xi : \xi_{R2} = \pm c_1\frac{|x_2 - y_2|}{t}, \frac{|x_2 - y_2|}{L_3 t} \le |\xi_{I2}| \le \sqrt{\frac{|x_2 - y_2|}{L_3 t}}\Big\}
\end{aligned} \tag{5.26}$$

(see Figure 3).

For contours (1) and (6), we have exponential decay in t and immediately obtain an estimate that can be subsumed into those of Theorem 1.1. The analysis for contour (4) is almost precisely the same as that of contour (2), and we consider only the latter, along which, similarly as in (5.24)

$$\begin{aligned}
\text{Re } \lambda_*(\xi) &\le -\bar{c}|\xi_R|^3 + \bar{C}\frac{|x_2 - y_2|^{3/2}}{L_3^{3/2} t^{3/2}} \\
\text{Re } \mu_2^-(\lambda_*(\xi), \xi) &\le 0.
\end{aligned} \tag{5.27}$$

In this way, (5.16) (for contours (2) and (5)) becomes

$$\int_{\{\xi_R \in R(r_2)\backslash R(c_1\frac{|x_2-y_2|}{L_3 t})\}} e^{-\frac{|x_2-y_2|^{3/2}}{L_3\sqrt{t}} - \bar{c}|\xi_R|^3 t + \bar{C}\frac{|x_2-y_2|^{3/2}}{L_3^{3/2} t^{3/2}}} \Big(|\xi_R|^2 + |\frac{x_2 - y_2}{t}|\Big)|d\xi_R|.$$

Upon choosing L_3 sufficiently large, we obtain an estimate by

$$Ct^{-\frac{d+1}{3}} e^{-\frac{|x_2-y_2|^{3/2}}{M\sqrt{t}}},$$

for some sufficiently large constant M. Noting again the dominance of $|x_2 - y_2|$ in this case, we obtain an estimate of the form

$$Ct^{-\frac{d+1}{3}} e^{-\frac{|x-y|^{3/2}}{M\sqrt{t}}},$$

which can be integrated in the transverse norm to give a result that can be subsumed into the third summand in Case (i) of Theorem 1.1.

We now turn to the critical case of contours (3) and (4), for which we first consider the case $d = 2$. Proceeding similarly as with (2) and (5), and focusing on the case $(x_2 - y_2) \geq 0$, we find that (5.16) becomes

$$\int_{\frac{x_2-y_2}{L_3 t}}^{\sqrt{\frac{x_2-y_2}{L_3 t}}} e^{-(x_2-y_2)\xi_I - \bar{c}\frac{(x_2-y_2)^3}{t^2} + \bar{C}(x_2-y_2)\xi_I{}^2} \Big(\frac{(x_2-y_2)^2}{t^2} + |\xi_I|^2\Big) d\xi_I.$$

We obtain an estimate by

$$Ce^{-\frac{(x_2-y_2)^2}{Mt}} \Big(t^{-3/2} + (x_2-y_2)^{-3}\Big).$$

Observing that we remain in the case $|x_2 - y_2| \geq \sqrt{t}$, we can conclude from this an estimate with the required rate of decay. For notational convenient, we now focus on the case $d = 3$, from which the general argument will be apparent. We first observe that for $|x_3 - y_3| \leq t^{1/3}$, we can regard boundedness by a constant as boundedness by

$$Ce^{-\frac{|x_3-y_3|^{-3/2}}{M\sqrt{t}}}. \tag{5.28}$$

and consequently the algebraic decay rate $t^{-1/3}$ is sufficient for an $L^1_{x_3}$ estimate. Accordingly, for (5.26)–3 (and similarly for (5.26)–4 (5.16) can be estimated by

$$\int_{\{c_1\frac{|x_2-y_2|}{t} \leq |\xi_3| \leq r_2\}} \int_{\sqrt{\frac{|x_2-y_2|}{L_3 t}}}^{\frac{|x_2-y_2|}{L_3 t}} e^{-(x_2-y_2)\xi_{I2} - \bar{c}(\frac{|x_2-y_2|^3}{(L_3 t)^3} + |\xi_3|^3)t + \bar{C}|\xi_{I2}|^3 t} \Big(\frac{|x_2-y_2|^2}{t^2} + |\xi_{I2}|^2 + |\xi_3|^2\Big) d\xi_{I2} d\xi_3.$$

Here, $|\xi_{I2}| \leq \sqrt{\frac{|x_2-y_2|}{L_3 t}}$, and consequently upon taking L_3 sufficiently large, we obtain

$$e^{-(x_2-y_2)\xi_{I2} + \bar{C}|\xi_{I2}|^3 t} \leq e^{-|x_2-y_2||\xi_{I2}| + \bar{C}|\xi_{I2}|\frac{|x_2-y_2|}{L_3}} \leq e^{-\theta|x_2-y_2||\xi_{I2}|},$$

for some $\theta > 0$. We have, then, an estimate by

$$\begin{aligned}
&\int_{\{c_1\frac{|x_2-y_2|}{t} \leq |\xi_3| \leq r_2\}} \int_{\sqrt{\frac{|x_2-y_2|}{L_3 t}}}^{\frac{|x_2-y_2|}{L_3 t}} e^{-\theta|x_2-y_2||\xi_{I2}| - \bar{c}|\xi_3|^3 t} \Big(\frac{|x_2-y_2|^2}{t^2} + |\xi_{I2}|^2 + |\xi_3|^2\Big) d\xi_{I2} d\xi_3 \\
&\leq Ct^{-1/3} e^{-\frac{|x_2-y_2|^2}{Mt}} e^{-\frac{|x_3-y_3|^{3/2}}{M\sqrt{t}}} \Big(t^{-3/2} + |x_2-y_2|^{-3} + |x_2-y_2|^{-1} t^{-2/3}\Big).
\end{aligned}$$

We recall that we are in the case $|x_2 - y_2| \geq |x_1 + y_1|$, and so we have additionally $\exp(-|x_1 + y_1|^2/(Mt))$ decay. Keeping in mind that in this case $|x_2 - y_2| \geq t^{1/2}$, we can take a transverse $L^p_{\tilde{x}}$ norm of this last expression, to obtain an estimate by

$$t^{-1-\frac{1}{3}(1-\frac{1}{p})}$$

which is bounded by the third estimate in Case (i) of Theorem 1.1 (for $d = 3$). For $|x_3 - y_3| \geq t^{1/3}$, we complexify ξ_3 in a manner almost identical to the complexification of ξ_2 above, with

$$\begin{aligned}
&(1) \text{ and } (6): \Big\{\xi_3 : \xi_{R3} = \pm r_2, 0 \leq |\xi_{I3}| \leq \sqrt{\frac{|x_3-y_3|}{L_3 t}}\Big\} \\
&(2) \text{ and } (5): \Big\{\xi_3 : c_1\frac{|x_2-y_2|}{t} \leq |\xi_{R3}| \leq r_2, \xi_{I3}\sqrt{\frac{|x_3-y_3|}{L_3 t}}\Big\} \\
&(3) \text{ and } (4): \Big\{\xi_3 : \xi_{R3} = \pm c_1\frac{|x_3-y_3|}{t}, 0 \leq |\xi_{I3}| \leq \sqrt{\frac{|x_3-y_3|}{L_3 t}}\Big\}.
\end{aligned} \tag{5.29}$$

In this way, we have in principle twelve cases to consider, each of the contours (5.26)–3, 4 combined with the six cases of (5.29). First, we observe that for (5.29)–1,6, we have exponential decay in t, and consequently we obtain an estimate easily subsumed into those of Theorem 1.1. The four combinations (5.26)–3,4 and (5.29)–2, 5 can each be analyzed in a similar manner, and we consider only the case (5.26)–3 with (5.29)–2. Here, (5.16) becomes

$$\int_{-r}^{-c_1\frac{|x_2-y_2|}{t}}\int_{\sqrt{\frac{(x_2-y_2)}{L_3t}}}^{\frac{(x_2-y_2)}{L_3t}} e^{i(\tilde{x}-\tilde{y})+\lambda_*(\xi)t+\mu_2^-(\lambda_*(\xi),\xi))|x_1+y_1|}\mathbf{O}(|\xi|^2)d\xi. \tag{5.30}$$

In this case, we have

$$\begin{aligned}\text{Re }(i(\tilde{x}-\tilde{y})\cdot\xi) &= -(x_2-y_2)\xi_{I2}-\frac{|x_3-y_3|^{3/2}}{\sqrt{L_3t}}\\ \text{Re }(\lambda_*(\xi)) &\le -\bar{c}(\frac{|x_2-y_2|^3}{t^3}+|\xi_{R3}|^3)t+\bar{C}(|\xi_{I2}|^3+(\frac{|x_3-y_3|}{L_3t})^{3/2})t.\end{aligned}$$

First, we observe that by choosing L_3 sufficiently large, we can insure

$$e^{-\frac{|x_3-y_3|^{3/2}}{\sqrt{L_3t}}+\bar{C}(\frac{|x_3-y_3|}{L_3t})^{3/2}t}\le Ce^{-\frac{|x_3-y_3|^{3/2}}{\sqrt{Mt}}},$$

for some constant M sufficiently large. In addition, for this range of ξ_{I2}, we have

$$e^{-(x_2-y_2)\xi_{I2}+\bar{C}|\xi_{I2}|^3t}\le e^{-(x_2-y_2)\xi_{I2}+\bar{C}|\xi_{I2}|\frac{(x_2-y_2)}{L_3t}t}\le e^{-\theta(x_2-y_2)\xi_{I2}},$$

again for L_3 sufficiently large. In this way, (5.30) can be estimated by

$$\begin{aligned}&Ce^{-\frac{|x_3-y_3|^{3/2}}{\sqrt{Mt}}}\int_{-r}^{-c_1\frac{|x_2-y_2|}{t}}\int_{\sqrt{\frac{(x_2-y_2)}{L_3t}}}^{\frac{(x_2-y_2)}{L_3t}} e^{-\theta(x_2-y_2)\xi_{I2}-\bar{c}|\xi_{R3}|^3t}\\ &\times\Big((\frac{|x_2-y_2|}{t})^2+|\xi_{I2}|^2+|\xi_{R3}|^2+\frac{|x_3-y_3|}{t}\Big)d\xi_{I2}d\xi_{R3}\\ &\le Ct^{-1/3}e^{-\frac{|x_3-y_3|^{3/2}}{\sqrt{Mt}}}e^{-\frac{(x_2-y_2)^2}{Mt}}\Big(t^{-3/2}+|x_2-y_2|^{-3}+|x_2-y_2|^{-1}t^{-2/3}+|x_2-y_2|t^{-2/3}\Big)\\ &\le Ct^{-4/3}e^{-\frac{|x_3-y_3|^{3/2}}{\sqrt{Mt}}}e^{-\frac{(x_2-y_2)^2}{Mt}}.\end{aligned}$$

Recalling that we are in the case $|x_2-y_2|\ge|x_1+y_1|$, with additionally $|x_2-y_2|\ge\sqrt{t}$, we observe that a transverse norm of this last expression is bounded by

$$Ct^{-\frac{2}{3}-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{3}(1-\frac{1}{p})}e^{-\frac{|x_1+y_1|^2}{Mt}},$$

which is bounded by the third estimate in Case (i) of Theorem 1.1.

We are left now with the combination (5.26)–3, 4 with (5.29)–3, 4, for which we consider only the pairing (5.26(4)) and (5.29(4)). For this, we have

$$\int_0^{\sqrt{\frac{(x_3-y_3)}{L_3t}}}\int_{\frac{(x_2-y_2)}{L_3t}}^{\sqrt{\frac{(x_2-y_2)}{L_3t}}} e^{i(\tilde{x}-\tilde{y})+\lambda_*(\xi)t+\mu_2^-(\lambda_*(\xi),\xi))|x_1+y_1|}\mathbf{O}(|\xi|^2)d\xi_{I2}d\xi_{I3}. \tag{5.31}$$

Along these contours, we have the relations

$$\begin{aligned}\text{Re }(i(\tilde{x}-\tilde{y})\cdot\xi) &= -(x_2-y_2)\xi_{I2}-(x_3-y_3)\xi_{I3}\\ \text{Re }(\lambda_*(\xi)) &\le -\bar{c}(\frac{|x_2-y_2|^3}{t^3}+\frac{|x_3-y_3|^3}{t^3})t+\bar{C}(|\xi_{I2}|^3+|\xi_{I2}|^3)t,\end{aligned}$$

with which (5.31) can be estimated by

$$C\int_0^{\sqrt{\frac{(x_3-y_3)}{L_3 t}}}\int_{\frac{(x_2-y_2)}{L_3 t}}^{\sqrt{\frac{(x_2-y_2)}{L_3 t}}} e^{-\theta(x_2-y_2)\xi_{I2}-\theta(x_3-y_3)\xi_{I3}}\Big(\frac{|x_2-y_2|^2}{t^2}+|\xi_{I2}|^2+|\xi_{I3}|^2\Big)d\xi_{I2}d\xi_{I3}$$
$$\le Ce^{-\frac{|x_2-y_2|^2}{Mt}}\Big(|x_3-y_3|^{-1}t^{-3/2}+|x_2-y_2|^{-3}|x_3-y_3|^{-1}+|x_2-y_2|^{-1}|x_3-y_3|^{-3}\Big). \tag{5.32}$$

We recall that we are in the case $|x_2-y_2|\ge\sqrt{t}$ and $|x_3-y_3|\ge t^{1/3}$, from which we have an estimate by

$$Ce^{-\frac{|x_2-y_2|^2}{Mt}}\Big(t^{-3/2}|x_3-y_3|^{-1}+t^{-1/2}(|x_3-y_3|+t^{1/3})^{-3}\Big). \tag{5.33}$$

Bearing in mind that $|x_2-y_2|\ge|x_3-y_3|$, we can take a transverse L^1 norm of this last expression to obtain an estimate by $Ct^{-2/3}$. On the other hand, we also have the L^∞ transverse norm estimate $Ct^{-4/3}$. Interpolating between these last two estimates, we obtain an estimate smaller than the third estimate in Case (i) of Theorem 1.1.

Case (5.11)–4. In the case (5.11)–4, and for $\xi\in R(c_1t^{-1/2})$, for some constant c_1, we take

$$\sqrt{R}=t^{-1/2},$$

with no complexification of ξ. By choosing $c_1>0$ sufficiently small, we can insure as in previous cases

$$|D(\lambda,\xi)|^{-1}\le Ct$$
$$\text{Re}\Big(i(\tilde{x}-\tilde{y})\cdot\xi+\lambda t-\mu_2^-(\lambda,\xi)(x_1+y_1)\Big)\le b_- -\theta(k^2+|\xi|^2)t,$$

for some $\theta>0$. In this way, we obtain an estimate on (5.9) by

$$Ct^{-\frac{d+1}{2}}.$$

In this case, we can multiply any estimate by

$$e^{-\frac{|\tilde{x}-\tilde{y}|^2}{Mt}}e^{-\frac{|x_1+y_1|^2}{Mt}},$$

and consequently we can immediately obtain a transverse L^p estimate bounded by the second estimate in Case (i) of Theorem 1.1.

For $\xi\in R(r_2)\backslash R(c_1t^{-1/2})$, we take

$$\sqrt{R}=c_2t^{-1/2},$$

where by choosing c_2 small we can insure that our contour lies entirely to the left of $\lambda_*(\xi)$, and consequently must be augmented by a contour that encircles $\lambda_*(\xi)$. For the contour passing to the left of $\lambda_*(\xi)$, we can proceed as we did in the previous paragraph concerning $\xi\in R(c_1t^{-1/2})$ to obtain the same estimate as we found there. For the contour encircling $\lambda_*(\xi)$, we obtain the residue integral

$$\int_{\{\xi\in R(r_2)\backslash R(\frac{c_1}{\sqrt{t}})\}} e^{i(\tilde{x}-\tilde{y})\cdot\xi+\lambda_*(\xi)t-\mu_2^-(\lambda_*(\xi),\xi)(x_1+y_1)}\mathbf{O}(|\xi|^2)d\xi.$$

According to $(\mathcal{D})$ Condition (1), we immediately have an L^∞ tranverse norm estimate on this integral of

$$Ct^{-\frac{d+1}{3}}.$$

In the event that $|x_2-y_2|\le t^{1/3}$, this is equivalent to an estimate by

$$C_1t^{-\frac{d+1}{3}}e^{-\frac{|\tilde{x}-\tilde{y}|^{3/2}}{M\sqrt{t}}},$$

for some constants C_1 and M, and this is sufficient to give the estimate claimed in Theorem 1.1. In order to obtain an appropriate L^1 transverse norm in the case $|x_2 - y_2| \geq t^{1/3}$, we must complexify ξ. Taking $|x_2 - y_2|$ as in (5.20), we proceed iteratively, complexifying ξ_2 first as

$$\begin{aligned}
&(1) \text{ and } (6): \Big\{\xi_2 : \xi_{R2} = \pm r_2, 0 \leq |\xi_{I2}| \leq \sqrt{\frac{|x_2 - y_2|}{L_3 t}}\Big\} \\
&(2) \text{ and } (5): \Big\{\xi_2 : c_1 t^{-1/2} \leq |\xi_{R2}| \leq r, \xi_{I2} = \sqrt{\frac{|x_2 - y_2|}{L_3 t}}\Big\} \\
&(3) \text{ and } (4): \Big\{\xi_2 : \xi_{R2} = \pm c_1 t^{-1/2}, 0 \leq |\xi_{I2}| \leq \sqrt{\frac{|x_2 - y_2|}{L_3 t}}\Big\}.
\end{aligned} \tag{5.34}$$

For (5.34)–1,6, we obtain exponential decay in t, and this gives an estimate that can be subsumed into those of Theorem 1.1. For (5.34)–2,5, we can proceed as in (5.26)–2,5, observing as there that decay in $|x_3 - y_3|$ is given by decay in $|x_2 - y_2|$. The critical case is (5.34)–3,4, in which case we cannot obtain $|\tilde{x} - \tilde{y}|$ decay from $|x_2 - y_2|$ decay and must complexify $\xi_3, \xi_4, ..., \xi_d$ as well. In order to simplify notation, we carry out details in this case only for $d = 3$. We take

$$\begin{aligned}
&(1) \text{ and } (6): \Big\{\xi_3 : \xi_{R3} = \pm r_2, 0 \leq |\xi_{I3}| \leq \sqrt{\frac{|x_3 - y_3|}{L_3 t}}\Big\} \\
&(2) \text{ and } (5): \Big\{\xi_3 : c_1 t^{-1/2} \leq |\xi_{R3}| \leq r_2, \xi_{I3} = \sqrt{\frac{|x_3 - y_3|}{L_3 t}}\Big\} \\
&(3) \text{ and } (4): \Big\{\xi_3 : \xi_{R3} = \pm c_1 t^{-1/2}, 0 \leq |\xi_{I3}| \leq \sqrt{\frac{|x_3 - y_3|}{L_3 t}}\Big\}.
\end{aligned} \tag{5.35}$$

In the case (5.35)–1,6, we have exponential decay in t and consequently obtain an estimate that can easily be absorbed into those of Theorem 1.1. The cases (5.35)–2,5 are similar, and we consider only (5.35)–5, for which we have

$$\int_{c_1 t^{-1/2}}^{r_1} \int_0^{\sqrt{\frac{|x_2 - y_2|}{L_3 t}}} e^{i(\tilde{x} - \tilde{y}) \cdot \xi + \lambda_*(\xi) t - \mu_2^-(\lambda_*(\xi), \xi)(x_1 + y_1)} \mathbf{O}(|\xi|^2) d\xi_{I2} d\xi_{R3}.$$

Here,

$$\begin{aligned}
\text{Re } (i(\tilde{x} - \tilde{y}) \cdot \xi) &= -(x_2 - y_2)\xi_{I2} - \frac{|x_3 - y_3|^{3/2}}{\sqrt{L_3 t}} \\
\text{Re } (\lambda_*(\xi) t) &\leq -\bar{c}(t^{-3/2} + |\xi_{R3}|^3)t + \bar{C}(|\xi_{I2}|^3 + \frac{|x_3 - y_3|^{3/2}}{(L_3 t)^{3/2}})t,
\end{aligned}$$

for some constants $\bar{c}$ and $\bar{C}$ from which we observe that by taking L_3 sufficiently large we can insure that the decay from Re $(i(\tilde{x} - \tilde{y}) \cdot \xi)$ dominates the growth from Re $(\lambda_*(\xi) t)$. In this way, we obtain an estimate by

$$\begin{aligned}
&C \int_{c_1 t^{-1/2}}^{r_1} \int_0^{\sqrt{\frac{|x_2 - y_2|}{L_3 t}}} e^{-\theta |x_2 - y_2| \xi_{I2} - \frac{|x_3 - y_3|^{3/2}}{\sqrt{Mt}} - \bar{c}|\xi_{R3}|^3 t} \Big(t^{-1} + |\xi_{I2}|^2 + |\xi_{R3}|^2 + \frac{|x_3 - y_3|}{t}\Big) d\xi_{I2} d\xi_{R3} \\
&\leq C t^{-1/3} e^{-\frac{|x_3 - y_3|^{3/2}}{\sqrt{Mt}}} \Big(|x_2 - y_2|^{-1} t^{-1} + |x_2 - y_2|^{-3} + |x_2 - y_2|^{-1} t^{-2/3}\Big).
\end{aligned}$$

Here, $t^{1/3} \leq |x_2 - y_2| \leq t^{1/2}$, and so we have an estimate by

$$C t^{-1} |x_2 - y_2|^{-1} e^{-\frac{|x_2 - y_2|^2}{Mt}} e^{-\frac{|x_3 - y_3|^{3/2}}{\sqrt{Mt}}}.$$

Taking an L^1 transverse norm of this last estimate, we obtain

$$C_1 t^{-2/3} \ln(e + t),$$

which is smaller than the third estimate in Case (i) of Theorem 1.1.

We close our analysis of (5.9) by considering the combination (5.34)–3,4 and (5.35)–3,4. Since each pairing can be analyzed in the same way, we focus on the case (5.34)–4 with (5.35)–4, for which we have integrals

$$\int_0^{\sqrt{\frac{|x_3-y_3|}{L_3 t}}} \int_0^{\sqrt{\frac{|x_2-y_2|}{L_3 t}}} e^{i(\tilde{x}-\tilde{y})\cdot\xi+\lambda_*(\xi)t-\mu_2^-(\lambda_*(\xi),\xi)(x_1+y_1)}\mathbf{O}(|\xi|^2)d\xi_{I2}d\xi_{I3}.$$

Here,

$$\begin{aligned}\text{Re }(i(\tilde{x}-\tilde{y})\cdot\xi) &= -(x_2-y_2)\xi_{I2}-(x_3-y_3)\xi_{I3}-\\ \text{Re }(\lambda_*(\xi)t) &\le -\bar{c}t^{-1/2}+\bar{C}(|\xi_{I2}|^3+|\xi_{I3}|^3)t,\end{aligned}$$

from which we observe that over this range of integration, L_3 can be chosen sufficiently large so that the growth arising from Re $(\lambda_*(\xi)t)$ is dominated by the decay in Re $(i(\tilde{x}-\tilde{y})\cdot\xi)$. In this way, we have an estimate by

$$\begin{aligned}&\int_0^{\sqrt{\frac{|x_3-y_3|}{L_3 t}}} \int_0^{\sqrt{\frac{|x_2-y_2|}{L_3 t}}} e^{-\theta(x_2-y_2)\xi_{I2}-\theta(x_3-y_3)\xi_{I3}}\Big(t^{-1}+|\xi_{I2}|^2+|\xi_{I3}|^2\Big)d\xi_{I2}d\xi_{I3}\\ &\le C\Big(t^{-1}|x_2-y_2|^{-1}+|x_2-y_2|^{-1}|x_3-y_3|^{-3/2}t^{-1/2}+|x_2-y_2|^{-1}|x_3-y_3|^{-3}\Big)\\ &+C\Big(t^{-1}|x_3-y_3|^{-1}+|x_3-y_3|^{-1}|x_2-y_2|^{-3/2}t^{-1/2}+|x_2-y_2|^{-3}|x_3-y_3|^{-1}\Big)\Big).\end{aligned}$$

We recall that in this setting $t^{1/3}\le|x_2-y_2|\le t^{1/2}$ and $t^{1/3}\le|x_3-y_3|\le t^{1/2}$, so that our estimate becomes

$$Ct^{-2/3}|x_2-y_2|^{-1}|x_3-y_3|^{-1}[\ln t]^2.$$

Finally, the L^1 transverse norm of this expression gives an estimate less than the third summand in Case (i) of Theorem 1.1.

While the argument above can be extended to all dimensions $d\ge 2$, we note that for $d=2$ a slight (and necessary) refinement is possible: the $\ln t$ growth can be eliminated. As this only arises in the case (5.34)–4, we need only consider that in the case $d=2$, we have

$$\begin{aligned}&\int_0^{\sqrt{\frac{|x_2-y_2|}{L_3 t}}} e^{-\theta(x_2-y_2)\xi_{I2}}\Big(t^{-1}+|\xi_{I2}|^2\Big)d\xi_{I2}\\ &\le C\Big(t^{-1}|x_2-y_2|^{-1}+(t^{-1}+|x_2-y_2|^{-3/2}t^{-1/2})e^{-\frac{(x_2-y_2)^2}{Mt}}+|x_2-y_2|^{-3}\Big).\end{aligned}$$

Here, we are in the case $t^{1/3}\le|x_2-y_2|\le t^{1/2}$, and consequently we have an estimate by

$$\begin{aligned}&C_1\Big(t^{-1}(|x_2-y_2|+t^{1/3})^{-1}+(t^{-1}+(|x_2-y_2|+t^{1/3})^{-3/2}t^{-1/2})e^{-\frac{(x_2-y_2)^2}{2Mt}}e^{-\frac{t^{1/6}}{2M}}\\ &+(|x_2-y_2|+t^{1/3})^{-3}\Big)I_{\{t^{1/3}\le|x_2-y_2|\le t^{1/2}\}}\end{aligned}$$

For the $L^1_{x_2}$ transverse norm, we have an estimate by $Ct^{-2/3}$, without logarithmic growth.

This concludes our general argument, which is applicable with only slight modifications to all expressions in $G_{\lambda,\xi}(x_1,y)$ except for the exact case examined in Subsection 5.1 and the excited estimates, which we analyze below in Subsection 5.5.

5.5 The Excited Terms

We next consider the expression from Lemma 4.2

$$E_(\lambda,\xi)(x_1,y_1)=\frac{\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)}e^{-\mu_2^- y_1}\Big(c_E+\mathbf{O}((|\lambda|+|\xi|^2)^{1/2})+\mathbf{O}(e^{-\eta|y_1|})\Big). \tag{5.36}$$

For the first of these, we have integrals

$$\frac{c_E \bar{u}_{x_1}(x_1)}{(2\pi)^d i} \int_{R(r_2)} e^{i\xi\cdot(\tilde{x}-\tilde{y})} \int_{|\lambda|\le r_1} \frac{e^{\lambda t - \mu_2^- y_1}}{D(\lambda,\xi)} d\lambda d\xi, \tag{5.37}$$

where $R(r_2)$ is as in (5.10), with r_1 and r_2 chosen sufficiently small so that $|\lambda| + |\xi|^2 < r$, and the truncated part can be analyzed as a *continuation correction*. In almost all cases, the integrals (5.37) can be analyzed as in Subsection 5.4. There is a critical difficulty, however, in the case $d = 2$, with

$$|y_1| \le t^{-1/2}$$
$$|x_2 - y_2| \le t^{-1/2}$$

(case (5.11)–4 in this setting). In this case, and for $|\xi_2| \ge c_1 t^{-1/2}$, with c_1 chosen sufficiently small, we obtain similarly as in (5.11–4) the residue integral

$$\int_{\{|\xi_2|\ge c_1 t^{-1/2}\}} e^{i(x_2-y_2)\xi_2 + \lambda_*(\xi_2)t - \mu_2^-(\lambda_*(\xi),\xi)y_1} d\xi_2.$$

In order to obtain the proper scaling in $|x_2 - y_2|$, we complexify ξ_2 precisely as in (5.34), which gives six parts to consider. For (5.34)–1,6 and for (5.34)–2,5, we can proceed as in Subsection 5.4. The difficulty lies with (5.34)–3,4, in which case the analysis of Subsection 5.4 leads to an estimate in transverse norm of the form $Ct^{-1/3}\log(e+t)$, which is not sufficient for closing our nonlinear iteration. (For $d = 2$, the iteration is sharp, and the logarithm cannot be accomodated, as it can be in higher dimensions.) Here, we proceed with a refined analysis, focusing first on (5.34)–4, for which we have

$$\int_0^{\sqrt{\frac{|x_2-y_2|}{L_3 t}}} e^{i(x_2-y_2)(c_1 t^{-\frac{1}{2}} + i\xi_{I2}) + \lambda_*(c_1 t^{-\frac{1}{2}} + i\xi_{I2})t} e^{-\mu_2^-(\lambda_*(c_1 t^{-\frac{1}{2}} + i\xi_{I2}), c_1 t^{-\frac{1}{2}} + i\xi_{I2})} i d\xi_{I2}, \tag{5.38}$$

where in this case

$$\begin{aligned} &\lambda_*(c_1 t^{-\frac{1}{2}} + i\xi_{I2}) = -\lambda_3(c_1 t^{-\frac{1}{2}} + i\xi_{I2})^3 + \mathbf{O}(t^{-2} + |\xi_{I2}|^2) \\ &\mu_2^-(\lambda_*(c_1 t^{-\frac{1}{2}} + i\xi_{I2}), c_1 t^{-\frac{1}{2}} + i\xi_{I2}) = -\sqrt{(c_1 t^{-1/2} + i\xi_{I2})^2} + \mathbf{O}(t^{-1} + |\xi_{I2}|^2). \end{aligned} \tag{5.39}$$

First, in the event that $\xi_{I2} \ge c_3 t^{-1/3}$ for some constant $c_3 > 0$, we immediately obtain exponential decay of the form

$$e^{-c_3(x_2-y_2)t^{-1/3}}.$$

In this case, the boundedness of the integrand allows us to crudely estimate (5.38) (for integration over this restricted region) by

$$C\sqrt{\frac{|x_2-y_2|}{L_3 t}} e^{-c_3(x_2-y_2)t^{-1/3}} \le C_1 t^{-1/3} e^{-\frac{c_3}{2}(x_2-y_2)t^{-1/3}},$$

whose transverse L^1 norm does not have logarithmic growth in t.

For $\xi_{I2} \le c_3 t^{-1/3}$, we first observe that since $|x_2 - y_2| \le \sqrt{t}$ and we have

$$e^{i(x_2-y_2)c_1 t^{-\frac{1}{2}}} = 1 + \mathbf{O}(\frac{|x_2-y_2|}{\sqrt{t}}).$$

For the order term, we have an estimate by

$$C\frac{|x_2-y_2|}{\sqrt{t}} \int_0^{c_3 t^{-1/3}} e^{-(x_2-y_2)\xi_{I2}} d\xi_{I2} \le C_1 t^{-1/2},$$

for which in the current case of $|x_2 - y_2| \le \sqrt{t}$, we have a transverse L^1 norm that does not grow logarithmically with t. Similarly, we have

$$e^{\lambda_*(c_1 t^{-\frac{1}{2}} + i\xi_{I2})t} = 1 + \mathbf{O}(t^{-1/2} + |\xi_{I2}|^3 t).$$

For the order term, we have an estimate by

$$
\begin{aligned}
&C\int_0^{c_3 t^{-1/3}} e^{-(x_2-y_2)\xi_{I2}}\Big(t^{-1/2}+|\xi_{I2}|^3 t\Big)d\xi_{I2}\\
&\quad\le C_1\Big(t^{-1/2}|x_2-y_2|^{-1}+(|x_2-y_2|^{-1}+t^{1/3}|x_2-y_2|^{-2}+t^{2/3}|x_2-y_2|^{-3})e^{-c_3(x_2-y_2)t^{-1/3}}+\frac{t}{|x_2-y_2|^4}\Big).
\end{aligned}
$$

In the current setting, for which $t^{1/3}\le|x_2-y_2|\le t^{1/2}$, the L^1 transverse norm of this last expression is bounded by a constant.

Proceedingly similarly for the case (5.34)–3, we find that the remaining integrals are

$$
\begin{aligned}
&\int_{\sqrt{\frac{|x_2-y_2|}{L_3 t}}}^0 e^{-(x_2-y_2)\xi_{I2}-\mu_2^-(\lambda_*(-c_1t^{-\frac12}+i\xi_{I2}),-c_1t^{-\frac12}+i\xi_{I2})y_1}id\xi_{I2}\\
&+\int_0^{\sqrt{\frac{|x_2-y_2|}{L_3 t}}} e^{-(x_2-y_2)\xi_{I2}-\mu_2^-(\lambda_*(c_1t^{-\frac12}+i\xi_{I2}),c_1t^{-\frac12}+i\xi_{I2})y_1}id\xi_{I2}\\
&=\int_0^{\sqrt{\frac{|x_2-y_2|}{L_3 t}}} e^{-(x_2-y_2)\xi_{I2}}H(t,y_1,\xi_{I2})id\xi_{I2},
\end{aligned}
\tag{5.40}
$$

where

$$
H(t,y_1,\xi_{I2})=e^{-\mu_2^-(\lambda_*(c_1t^{-\frac12}+i\xi_{I2}),c_1t^{-\frac12}+i\xi_{I2})y_1}-e^{-\mu_2^-(\lambda_*(-c_1t^{-\frac12}+i\xi_{I2}),-c_1t^{-\frac12}+i\xi_{I2})y_1}.
$$

According to (5.39), we can write

$$
\begin{aligned}
H(t,y_1,\xi_{I2})&=e^{(c_1t^{-1/2}+i\xi_{I2})y_1}(1+\mathbf{O}(\frac{|y_1|}{t}+|\xi_{I2}|^2y_1))-e^{(c_1t^{-1/2}-i\xi_{I2})y_1}(1+\mathbf{O}(\frac{|y_1|}{t}+|\xi_{I2}|^2y_1))\\
&=e^{c_1\frac{y_1}{\sqrt t}}\Big(e^{i\xi_{I2}y_1}-e^{-i\xi_{I2}y_1}\Big)+\mathbf{O}(\frac{|y_1|}{t}+|\xi_{I2}|^2y_1).
\end{aligned}
$$

The critical term is

$$
\begin{aligned}
&\Big|\int_0^{c_3t^{-1/3}} e^{-(x_2-y_2)\xi_{I2}}e^{c_1\frac{y_1}{\sqrt t}}\Big(e^{i\xi_{I2}y_1}-e^{-i\xi_{I2}y_1}\Big)id\xi_{I2}\Big|\\
&=\Big|\int_0^{c_3t^{-1/3}} e^{-(x_2-y_2)\xi_{I2}}e^{c_1\frac{y_1}{\sqrt t}}2i\sin(\xi_{I2}y_1)id\xi_{I2}\Big|\\
&\le C\Big[|x_2-y_2|^{-1}e^{-c_1(x_2-y_2)t^{-1/3}}+\frac{|y_1|}{(x_2-y_2)^2+y_1^2}\Big],
\end{aligned}
$$

where in establishing this final inequality, we have carried out the integration exactly and then combined terms. We conclude by observing that the transverse L^1 estimate of this last expression is bounded by a constant.

5.6 The Case $y_1\le 0\le x_1$

We remark finally that the case $y_1\le 0\le x_1$ can be analyzed almost precisely as was the case $x_1,y_1\le 0$. As an example term, consider

$$
\frac{\mathbf{O}(|\lambda|+|\xi|^2)}{D(\lambda,\xi)}e^{\mu_2^+x_1-\mu_2^-y_1}.
$$

The only new aspect of the analysis in this case regards the appearance of both μ_2^- and μ_2^+, which makes the choice of a single optimal contour problematic. This difficulty is easily overcome, however, by observing that in the event that $|y_1|\le|x_1|$, we choose our optimal contour based soley on μ_2^+, while for $|y_1|\ge|x_1$—, we choose our optimal contour based soley on μ_2^-.

This completes the proof of Theorem 1.1. □

5.7 Higher Order Estimates

Proceeding as in the proof of Theorem 1.1, we can additionally establish the following theorem on derivative estimates. The proof of Theorem 5.1 is quite similar to the proof of Theorem 1.1 and we omit it here.

Theorem 5.1. *Under the assumptions of Theorem 1.1, the Green's function $G(t,x;y)$ described in (1.13) satisfies the following estimates for $y \le 0$ (with symmetric estimates in the case $y \ge 0$).*

For $|x-y| \le Kt$ and $t \ge 1$, K as in Theorem 1.1,

$$G(t,x;y) = \tilde{G}(t,x;y) + \bar{u}_{x_1}(x_1)E(t,\tilde{x};y),$$

where for β a multi-index in the transverse variables $\tilde{x}$ and $\tilde{y}$, $|\beta| \le 1$, and for $\sigma = 0$ in the case $d = 2$ and any $\sigma > 0$ in the cases $d \ge 3$,

(i) $y_1 \le x_1 \le 0$

$$\begin{aligned}\|\partial^\beta \tilde{G}_{y_1}(t,x;y)\|_{L^p_{\tilde{x}}} &\le Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+1}{2}} \partial_{y_1}\Big[e^{-\frac{(x_1-y_1)^2}{4b\_t}} - e^{-\frac{(x_1+y_1)^2}{4b\_t}}\Big] \\ &+ Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+3}{2}} e^{-\frac{(x_1-y_1)^2}{Lt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+3}{3}+\sigma} I_{\{|x_1-y_1|\le t^{1/2}\}} \\ &+ Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+2}{2}} e^{-\eta|x_1|} e^{-\frac{y_1^2}{Lt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+3}{3}+\sigma} e^{-\eta|x_1|} I_{\{|y_1|\le t^{1/2}\}}.\end{aligned}$$

$$\begin{aligned}\|\partial^\beta \tilde{G}_{x_1}(t,x;y)\|_{L^p_{x_2}} &\le Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+2}{2}} e^{-\frac{(x_1-y_1)^2}{Mt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+3}{3}+\sigma} I_{\{|x_1-y_1|\le t^{1/2}\}} \\ &+ Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+2}{3}+\sigma} e^{-\eta|x_1|} I_{\{|y_1|\le t^{1/2}\}}.\end{aligned}$$

$$\begin{aligned}\|\partial^\beta \tilde{G}_{x_1 y_1}(t,x;y)\|_{L^p_{x_2}} &\le Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+3}{2}} e^{-\frac{(x_1-y_1)^2}{Mt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+4}{3}+\sigma} I_{\{|x_1-y_1|\le t^{1/2}\}} \\ &+ Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+3}{3}+\sigma} e^{-\eta|x_1-y_1|} I_{\{|y_1|\le t^{1/2}\}} + Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+2}{2}} e^{-\eta|x_1|} e^{-\frac{y_1^2}{Mt}} \\ &+ Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+3}{3}+\sigma} e^{-\eta|x_1|} I_{\{|y_1|\le t^{1/2}\}}.\end{aligned}$$

(ii) $x_1 \le y_1 \le 0$

$$\begin{aligned}\|\partial^\beta \tilde{G}_{y_1}(t,x;y)\|_{L^p_{\tilde{x}}} &\le Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+2}{2}} e^{-\frac{(x_1-y_1)^2}{Mt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+3}{3}+\sigma} I_{\{|x_1-y_1|\le t^{1/2}\}} \\ &+ Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+2}{3}+\sigma} e^{-\eta|x_1|}.\end{aligned}$$

$$\begin{aligned}\|\partial^\beta \tilde{G}_{x_1}(t,x;y)\|_{L^p_{\tilde{x}}} &\le Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+1}{2}} \partial_{x_1}\Big[e^{-\frac{(x_1-y_1)^2}{4b\_t}} - e^{-\frac{(x_1+y_1)^2}{4b\_t}}\Big] \\ &+ Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+3}{2}} e^{-\frac{(x_1-y_1)^2}{Mt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+3}{3}+\sigma} I_{\{|x_1-y_1|\le t^{1/2}\}} \\ &+ Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+2}{2}} e^{-\eta|x_1-y_1|} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+2}{3}+\sigma} e^{-\eta|x_1|}.\end{aligned}$$

$$\begin{aligned}\|\partial^\beta \tilde{G}_{x_1 y_1}(t,x;y)\|_{L^p_{\tilde{x}}} &\le Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+3}{2}} e^{-\frac{(x_1-y_1)^2}{Mt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+4}{3}+\sigma} I_{\{|x_1-y_1|\le t^{1/2}\}} \\ &+ Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+3}{3}+\sigma} e^{-\eta|x_1-y_1|} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+3}{3}+\sigma} e^{-\eta|y_1|} I_{\{|x_1+y_1|\le t^{1/2}\}} \\ &+ Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+2}{3}+\sigma} e^{-\eta(|x_1|+|y_1|}.\end{aligned}$$

(iii) $y_1 \le 0 \le x_1$

$$\|\partial^\beta \tilde{G}_{y_1}(t,x;y)\|_{L^p_{\tilde{x}}} \le Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+2}{2}} e^{-\frac{(x_1-y_1)^2}{Mt}} Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+3}{3}+\sigma} I_{\{|x_1-y_1|\le t^{1/2}\}}.$$

$$\|\partial^\beta \tilde{G}_{x_1}(t,x;y)\|_{L^p_{\tilde{x}}} \le Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+3}{2}} e^{-\frac{(x_1-y_1)^2}{Mt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+3}{3}+\sigma} I_{\{|x_1-y_1|\le t^{1/2}\}}$$
$$+ Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+2}{2}} e^{-\eta|x_1|} e^{-\frac{(x_1-y_1)^2}{Mt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+2}{3}+\sigma} e^{-\eta|x_1|} I_{\{|x_1-y_1|\le t^{1/2}\}}.$$

$$\|\partial^\beta \tilde{G}_{x_1 y_1}(t,x;y)\|_{L^p_{\tilde{x}}} \le Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+3}{2}} e^{-\frac{(x_1-y_1)^2}{Mt}} + Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+4}{3}+\sigma} I_{\{|x_1-y_1|\le t^{1/2}\}}$$
$$+ Ct^{-\frac{d-1}{2}(1-\frac{1}{p})-\frac{|\beta|+2}{2}} e^{-\eta|x_1|} e^{-\frac{(x_1-y_1)^2}{Mt}} Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|+3}{3}+\sigma} e^{-\eta|x_1|}.$$

6 Proof of Theorem 1.2

In this section, we combine the estimates of Theorem 1.1 with the integral representations (1.19) and (1.18) to prove Theorem 1.2. We begin by stating a lemma regarding the linear estimates: integration of the Green's function estimates of Theorem 1.1 against the initial perturbation $v_0(y)$.

Lemma 6.1. *For $G(t,x;y)$ as described in Theorem 1.1, and for $v_0(y)$ as described in Theorem 1.2, there holds*

$$\Big\| \int_{\mathbb{R}^d} \tilde{G}(t,x;y) v_0(y) dy \Big\|_{L^p_{\tilde{x}}} \le CE_0 t^{-\frac{d-1}{2}(1-\frac{1}{p})} \Theta(t,x_1) + CE_0 t^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{6}+\sigma} \Theta(t,x_1)$$
$$\Big\| \int_{\mathbb{R}^d} \tilde{G}_{x_1}(t,x;y) v_0(y) dy \Big\|_{L^p_{\tilde{x}}} \le CE_0 t^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{2}+\sigma} \Theta(t,x_1) + CE_0 t^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{6}+\sigma} e^{-\eta|x_1|}$$

while for $k=2,3,...,d$

$$\Big\| \int_{\mathbb{R}^d} \tilde{G}_{x_k}(t,x;y) v_0(y) dy \Big\|_{L^p_{\tilde{x}}} \le CE_0 t^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{2}+\sigma} \Theta(t,x_1)$$

Here, $\Theta(t,x_1)$ is as in Theorem 1.2 and σ is as in Theorem 1.1. For the excited estimates, we have

$$\Big\| \int_{\mathbb{R}^d} \partial^\beta_{\tilde{x}} E(t,\tilde{x};y) v_0(y) dy \Big\|_{L^p_{\tilde{x}}} \le CE_0 t^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\beta|}{3}+\sigma},$$

where β is a multi-index in the transverse $\tilde{x}$ variables with $|\beta| \le 3$.

We also have the following lemma regarding the interaction between our nonlinearities and the Green's kernels.

Lemma 6.2. *For $G(t,x;y)$ as described in Theorem 1.1, and for any $\Psi(s,y)$ satisfying*

$$\Big\| \Psi(s,y) \Big\|_{L^p_{\tilde{y}}} \le s^{-3/4}(1+s)^{-\frac{d-1}{3}(2-\frac{1}{p})+\frac{5}{12}+2\sigma} e^{-\eta|y_1|}$$
$$+ s^{-3/4}(1+s)^{-\frac{d-1}{3}(2-\frac{1}{p})+\frac{5}{12}+2\sigma} h_d(s) \Theta(s,y_1)^2,$$

where $\Theta(s,y_1)$ and $h_d(s)$ are as in Theorem 1.2, the following estimates hold. For each $k=1,2,...,d$,

$$\Big\| \int_0^t \int_{\mathbb{R}^d} \tilde{G}_{y_k}(t-s,x;y) \Psi(s,y) dy ds \Big\|_{L^p_{\tilde{x}}} \le Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{6}+\sigma} h_d(t) \Theta(t,x_1)$$
$$\Big\| \int_0^t \int_{\mathbb{R}^d} \tilde{G}_{x_1 y_k}(t-s,x;y) \Psi(s,y) dy ds \Big\|_{L^p_{\tilde{x}}}$$
$$\le CE_0 t^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{6}+\sigma} \Theta(t,x_1) + CE_0 t^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{2}{3}} e^{-\eta|x_1|},$$

while for $j=2,3,...,d$ we have

$$\Big\| \int_0^t \int_{\mathbb{R}^d} \tilde{G}_{x_j y_k}(t-s,x;y) \Psi(s,y) dy ds \Big\|_{L^p_{\tilde{x}}} \le CE_0 t^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{6}+\sigma} \Theta(t,x_1).$$

For the excited estimates, we have

$$\Big\| \int_0^t \int_{\mathbb{R}^d} \partial_{\tilde{x}}^{\beta} E_{y_k}(t-s,x;y)\Psi(s,y)dyds \Big\|_{L^p_{\tilde{x}}} \le Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{3}},$$

where β is a multi-index associated with the transverse variable $\tilde{x}$ and $|\beta| \le 3$.

Briefly, we remark on the selection of $\Psi(s,y)$. Generally speaking, the process is simply to take the linear estimates on δ and v and to use these to develop first approximations for the nonlinear combinations. These combinations are then filtered through the nonlinear integrals, and any new terms that arise are taken to alter the approximations on v and δ. The estimate on $\Psi(s,y)$ is taken from values of v and δ that did not change when filtered through the nonlinear integrations. More precisely, the first summand in the estimate on $\Psi(s,y)$ arises from the transverse L^p estimate of the nonlinearities of the form $e^{-\eta|x_1|}\delta\partial_{x_k}\delta$, $k=2,3,...,d$, which have significantly slower decay than do any of the other nonlinearities. In order to complete the form of $\Psi(s,y_1)$, we estimate the nonlinear integrals associated with both v and v_{x_1} with the nonlinearities of the form $e^{-\eta|x_1|}\delta\partial_{x_k}\delta$. The results of these estimates are taken as a first estimate on v and v_{x_1}, and assuming these, we take a transverse L^p estimate of the nonlinearity vv_{x_1}. Though several more terms arise in the nonlinearity expressions H_k (see (1.12)), they can all be bounded by the expressions arising from these two critical cases. Finally, for later reference, we can regard the estimate on $\Psi(s,y)$ as a sum of three nonlinearities

$$\Big\|\Psi(s,y)\Big\|_{L^p_{\tilde{y}}} \le C\Big[\Psi_1(s,y_1)+\Psi_1(s,y_1)+\Psi_1(s,y_1)\Big], \tag{6.1}$$

where

$$\begin{aligned}
\Psi_1(s,y_1) &:= s^{-3/4}(1+s)^{-\frac{d-1}{3}(2-\frac{1}{p})+\frac{5}{12}+2\sigma}e^{-\eta|y_1|}\\
\Psi_2(s,y_2) &:= s^{-3/4}(1+s)^{-\frac{d-1}{3}(2-\frac{1}{p})-\frac{7}{12}+2\sigma}h_d(s)e^{-2\frac{y_1^2}{Ls}}\\
\Psi_2(s,y_3) &:= s^{-3/4}(1+s)^{-\frac{d-1}{3}(2-\frac{1}{p})+\frac{5}{12}+2\sigma}h_d(s)(1+|y_1|+\sqrt{s})^{-3}.
\end{aligned}$$

In order to set some notation, we define the iteration variable

$$\begin{aligned}
\zeta(t) := \sup_{1\le p\le\infty}\sup_{y_1\in\mathbb{R},0\le s\le t} \Big[&\frac{\|v(s,y_1,\cdot)\|_{L^p_{\tilde{y}}}}{B(s,y_1)} + \sum_{k=1}^d \frac{\|v_{y_k}(s,y_1,\cdot)\|_{L^p_{\tilde{y}}}}{B_k(s,y_1)}\\
&+ \sum_{|\beta|\le 3}\frac{\|\partial_{\tilde{y}}^{\beta}\delta(s,\cdot)\|_{L^p_{\tilde{y}}}}{C_\beta(s,y_1)} + \frac{\|\partial_s\delta(s,\cdot)\|_{L^p_{\tilde{y}}}}{C(s,y_1)}\Big],
\end{aligned} \tag{6.2}$$

where for $k=2,3,...,d$,

$$\begin{aligned}
B(s,y_1) &:= (1+s)^{-\frac{d-1}{2}(1-\frac{1}{p})}\Theta(s,y_1) + (1+s)^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{6}+\sigma}h_p(s)\Theta(s,y_1)\\
B_1(s,y_1) &:= (1+s)^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{6}+\sigma}\Theta(s,y_1) + (1+s)^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{2}{3}}h_p(s)e^{-\eta|y_1|}\\
B_k(s,y_1) &:= (1+s)^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{6}+\sigma}\Theta(s,y_1),
\end{aligned}$$

with additionally

$$\begin{aligned}
C_\beta(s,y_1) &:= \begin{cases} (1+s)^{-\frac{d-1}{3}(1-\frac{1}{p})+\sigma} & |\alpha|=0\\ (1+s)^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{3}+\sigma} & |\alpha|>0 \end{cases}\\
C(s,y_1) &:= (1+s)^{(1+s)^{-\frac{d-1}{3}(1-\frac{1}{p})-1+\sigma}}.
\end{aligned}$$

In addition to Lemma 6.1 and Lemma 6.2, we require the following lemma regarding small time behavior of solutions to our perturbation equation (1.10). In the case of small t, the fourth order effects control the qualitative behavior of our solutions, and consequently this lemma is almost identical to the analogous Lemma 3.4 of [28]. We omit the proof, but specify the salient points: 1. For small t, the behavior of $v(t,x)$ is controlled by fourth order effects, and 2. The behavior of derivatives $\partial_x^{\alpha} v$ can be linked to that of the undifferentiated v.

Lemma 6.3. *Under the assumptions of Theorem 1.1, and under the additional restriction of Hölder continuity on the initial perturbation* v_0 *(that is,* $v_0 \in C^{0+\gamma}(\mathbb{R}^d)$*, for some* $\gamma > 0$*), the integral equations (1.19) and (1.18) have a unique local solution for some interval of time* $t \in [0, T]$ *so that*

$$v \in C^{0+\frac{\gamma}{4}}([0,T]) \cap C^{0+\gamma}(\mathbb{R}^d)$$
$$\delta \in C^{0+\frac{\gamma}{4}}([0,T]) \cap C^{3+\gamma}(\mathbb{R}^{d-1}),$$

extending so long as v *remains bounded in Hölder norm. Moreover, for* $t \in [0, T]$*,*

$$\sup_{x_1 \in \mathbb{R}} \frac{\|v\|_{L^p_{\tilde{x}}}}{B(t, x_1)}$$

remains continuous so long as it and $|\delta_t(t+1, \tilde{x})|$ *are uniformly bounded, and for* $\tau > 0$ *sufficiently small, with* $\tau \le t \in [0, T]$*, there holds*

$$\sup_{x_1 \in \mathbb{R}} \frac{\|\partial_x^\alpha v(t,x)\|_{L^p_{\tilde{x}}}}{B(t,x_1)} \le C\tau^{-\frac{|\alpha|}{4}} \sup_{x_1 \in \mathbb{R}} \frac{\|v(t-\tau,x)\|_{L^p_{\tilde{x}}}}{B(t-\tau,x_1)}.$$

Likewise, for $|\alpha| = 2$*,*

$$\sup_{x_1 \in \mathbb{R}} \frac{\|\partial_x^\alpha \partial_{x_1} v(t,x)\|_{L^p_{\tilde{x}}}}{B_1(t,x_1)} \le C\tau^{-\frac{|\alpha|}{4}} \sum_{k=1}^d \sup_{x_1 \in \mathbb{R}} \frac{\|\partial_{x_1} v(t-\tau,x)\|_{L^p_{\tilde{x}}}}{B_1(t-\tau,x_1)},$$

while for $|\alpha| = 3$*, with* $\alpha_1 = 0$*,*

$$\sup_{x_1 \in \mathbb{R}} \frac{\|\partial_x^\alpha v(t,x)\|_{L^p_{\tilde{x}}}}{B_2(t,x_1)} \le C\tau^{-\frac{|\alpha|-1}{4}} \sum_{k=1}^d \sup_{x_1 \in \mathbb{R}} \frac{\|\partial_{x_1} v(t-\tau,x)\|_{L^p_{\tilde{x}}}}{B_2(t-\tau,x_1)}.$$

The following claim can be proved in a straightforward manner similar to the proof of Claim 4.1 in [23].

Claim 1. *Suppose there exists some constant* C *so that*

$$\zeta(t) \le C(E_0 + \zeta(t)^2),$$

where E_0 *is as in the statement of Theorem 1.2. Then for* E_0 *sufficiently small, there holds*

$$\zeta(t) < 2CE_0.$$

We proceed now by verifying the assumption of Claim 1 for $\zeta(t)$ as defined in (6.2). In order to carry this out, we will use the integral equations (1.19) and (1.18), and additionally integral equations for $\partial_x^\alpha v$, with $|\alpha| = 1$, δ_t, and $\partial_{\tilde{x}}^\beta \delta$, with $|\beta| \le 3$. These additional integral equations are easily obtained by direct differentiation of (1.19) and (1.18) with respect to the appropriate variables. For third order derivatives on v, which appear in our nonlinearity as expressions of the form $\partial_{x_k} \triangle v$, for $k = 2, 3, ..., d$, (see (1.12)), we employ Lemma 6.3 to write

$$\|\partial^\alpha v(s,y)\|_{L^p_{\tilde{y}}} \le Cs^{-1/2} B_1(s, y_1),$$

for $|\alpha| = 3$. For the remaining expressions in the nonlinearity, we observe that by virtue of our definition of $\zeta(t)$, we have

$$\begin{aligned} \|v(s,y_1,\cdot)\|_{L^p_{\tilde{y}}} &\le \zeta(t) B(s,y_1) \\ \|v_{y_k}(s,y_1,\cdot)\|_{L^p_{\tilde{y}}} &\le \zeta(t) B_k(s,y_1) \\ \|\partial_{\tilde{y}}^\beta \delta(s,\cdot)\|_{L^p_{\tilde{y}}} &\le \zeta(t) C_\beta(s,y_1) \\ \|\partial_s \delta(s,\cdot)\|_{L^p_{\tilde{x}}} &\le \zeta(t) C(s,y_1). \end{aligned} \tag{6.3}$$

In this way,

$$\|Q_k\|_{L^p_{\tilde{y}}} \le C\zeta(t)^2 \|\Psi(s,y_1)\|_{L^p_{\tilde{y}}},$$

where Q_k is as in (1.10) and $\Psi(s, y_1)$ is as in Lemma 6.2. Begining with the case of $v(t,x)$ (that is, with integral equation (1.19)), we estimate

$$\begin{aligned}\|v(t,x)\|_{L^p_{\tilde{x}}} &\le \Big\| \int_{\mathbb{R}^d} |\tilde{G}(t,x;y) v_0(y)| dy \Big\| + \Big\| \int_0^t \int_{\mathbb{R}^d} \sum_{k=1}^d |G_{y_k}(t-s,x;y)| |Q_k| dy ds \Big\| \\ &\le C_1 E_0 \Big[(1+t)^{-\frac{d-1}{2}(1-\frac{1}{p})} + (1+t)^{-\frac{d-1}{3}(1-\frac{1}{p}) - \frac{1}{6} + \sigma} h_d(t) \Big] \Theta(t,x_1) \\ &+ C_1 \zeta(t)^2 (1+t)^{-\frac{d-1}{3}(1-\frac{1}{p}) - \frac{1}{6} + \sigma} h_d(t) \Theta(t,x_1) \\ &\le C_1 B(t,x_1)(E_0 + \zeta(t)^2).\end{aligned}$$

We have, then,

$$\frac{\|v(t,x)\|_{L^p_{\tilde{x}}}}{B(t,x_1)} \le C_1(E_0 + \zeta(t)^2),$$

and since $\zeta(t)$ is non-decreasing in t,

$$\sup_{y_1 \in \mathbb{R}, 0 \le s \le t} \frac{\|v(t,x)\|_{L^p_{\tilde{x}}}}{B(t,x_1)} \le C_1(E_0 + \zeta(t)^2).$$

Proceeding in an almost identical fashion, we can similarly establish such estimates for the remaining quotients in the definition is $\zeta(t)$. Combining these, we conclude that the assumption of Claim 1 holds. Theorem 1.2 follows directly. □

7 Proofs of Lemmas 6.1 and 6.2

In this section, we establish the estimates of Lemmas 6.1 and 6.2. We note at the outset that for the case $|x-y| \ge Kt$ the Green's function estimates of Theorem 1.1 give exponential decay in t. For more details, the reader is referred to [23].

7.1 Proof of Lemma 6.1.

Since the analysis is almost identical in each case, we proceed only with the undifferentiated $\tilde{G}(t,x;y)$ estimate. We compute

$$\Big\| \int_{\mathbb{R}^d} \tilde{G}(t,x;y) v_0(y) dy \Big\|_{L^p_{\tilde{x}}} \le \int_{-\infty}^{+\infty} \sup_{\tilde{y} \in \mathbb{R}^{d-1}} \Big\| \tilde{G}(t,x;y) \Big\|_{L^p_{\tilde{x}}} \Big\| v_0(y) \Big\|_{L^1_{\tilde{y}}} dy_1.$$

For $x_1, y_1 \le 0$, we can take from Theorem 1.1 the estimate

$$\Big\| \tilde{G}(t,x;y) \Big\|_{L^p_{\tilde{x}}} \le C \Big[t^{-\frac{d-1}{2}(1-\frac{1}{p}) - \frac{1}{2}} e^{-\frac{|x_1-y_1|^2}{Mt}} + t^{-\frac{d-1}{3}(1-\frac{1}{p}) - \frac{2}{3} + \sigma} I_{\{|x_1-y_1| \le t^{1/2}\}} \Big], \tag{7.1}$$

where $\sigma = 0$ for $d = 2$ and can be taken arbitrarily small for all $d \ge 3$. (This estimate clearly gives up cancellation in the first estimate of Case (i) of Theorem 1.1, but since it is the second estimate here that is dominant, nothing is lost.) Combining this with our initial perburbation estimate

$$\Big\| v_0(y) \Big\|_{L^1_{\tilde{y}}} \le E_0 (1+|y_1|)^{-3/2},$$

we have, for the first estimate in (7.1), integrals of the form

$$\begin{aligned}\int_{-\infty}^0 & t^{-\frac{d-1}{2}(1-\frac{1}{p}) - \frac{1}{2}} e^{-\frac{|x_1-y_1|^2}{Mt}} E_0 (1+|y_1|)^{-3/2} dy_1 \\ &\le C_1 E_0 t^{-\frac{d-1}{2}(1-\frac{1}{p})} \Big[t^{-1/2} e^{-\frac{x_1^2}{Lt}} + (1+|x_1|+\sqrt{t})^{-3/2} \Big],\end{aligned}$$

where the estimate can be established by straightforward considerations similar to those of the proof of Lemma 4.1 in [23]. In addition to this estimate, we have for the second term in (7.1)

$$\int_{x_1-t^{1/2}}^{\min\{0,x_1+t^{1/2}\}} t^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{2}{3}+\sigma} E_0(1+|y_1|)^{-3/2}dy_1 \leq Ct^{-\frac{d-1}{3}(1-\frac{1}{p})}\Big[t^{-2/3+\sigma}e^{-\frac{x_1^2}{Lt}}+t^{-\frac{1}{6}+\sigma}(1+|x_1|+\sqrt{t})^{-3/2}\Big].$$

Lemma 6.1 is established by proceeding similarly in the remaining cases. □

7.2 Proof of Lemma 6.2

We begin by observing two possible transverse L^p estimates on our nonlinear interation integrals. First, for each $k=1,2,...,d$, we have (inside the integration over s)

$$\Big\|\int_{\mathbb{R}^d}\tilde{G}_{y_k}(t-s,x;y)\Psi(s,y)dy\Big\|_{L^p_{\tilde{x}}} \leq \int_{-\infty}^{+\infty}\sup_{\tilde{y}\in\mathbb{R}^{d-1}}\Big\|\tilde{G}_{y_k}(t-s,x;y)\Big\|_{L^p_{\tilde{x}}}\Big\|\Psi(s,y)\Big\|_{L^1_{\tilde{y}}}dy_1. \tag{7.2}$$

On the other hand, Hölder's inequality can be used to establish

$$\begin{aligned}&\Big\|\int_{\mathbb{R}^d}\tilde{G}_{y_k}(t-s,x;y)\Psi(s,y)dy\Big\|_{L^p_{\tilde{x}}}\\ &\leq \int_{-\infty}^{+\infty}\sup_{\tilde{y}\in\mathbb{R}^{d-1}}\Big\|\tilde{G}_{y_k}(t-s,x;y)\Big\|_{L^1_{\tilde{x}}}^{\frac{1}{p}}\sup_{\tilde{x}\in\mathbb{R}^{d-1}}\Big\|\tilde{G}_{y_k}(t-s,x;y)\Big\|_{L^1_{\tilde{y}}}^{\frac{1}{q}}\Big\|\Psi(s,y)\Big\|_{L^p_{\tilde{y}}}dy_1.\end{aligned} \tag{7.3}$$

Precisely the same estimates clearly hold with $\tilde{G}_{y_k}$ replaced everywhere by E_{y_k}. Our general approach will be to employ (7.2) for $s\in[0,t/2]$, in which case $t-s\geq t/2$, and to employ (7.3) in the case $s\in[t/2,t]$.

The tracking estimates. We begin by considering the tracking iteration (1.18), and the nonlinearity with minimal decay in s, for which we have, according to (7.2) and (7.3) (omitting some spatial dependences for notational brevity),

$$\begin{aligned}&\int_0^{t/2}\int_{-\infty}^{+\infty}\sup_{\tilde{y}\in\mathbb{R}^{d-1}}\Big\|E_{y_k}(t-s)\Big\|_{L^p_{\tilde{x}}}e^{-\eta|y_1|}s^{-3/4}(1+s)^{-\frac{d-1}{3}+\frac{5}{12}+2\sigma}dy_1ds\\ &+\int_{t/2}^{t}\int_{-\infty}^{+\infty}\sup_{\tilde{y}\in\mathbb{R}^{d-1}}\Big\|E_{y_k}(t-s)\Big\|_{L^1_{\tilde{x}}}^{\frac{1}{p}}\sup_{\tilde{x}\in\mathbb{R}^{d-1}}\Big\|E_{y_k}(t-s)\Big\|_{L^1_{\tilde{y}}}^{\frac{1}{q}}e^{-\eta|y_1|}s^{-3/4}(1+s)^{-\frac{d-1}{3}(2-\frac{1}{p})+\frac{5}{12}+2\sigma}dy_1ds.\end{aligned}$$

According to the estimates of Theorem 1.1, these integrals can be estimated by

$$\begin{aligned}&C\int_0^{t/2}\int_{-\infty}^{+\infty}(t-s)^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{3}+\sigma}e^{-\eta|y_1|}s^{-\frac{d-1}{3}+\frac{5}{12}+2\sigma}dy_1ds\\ &+C\int_{t/2}^{t}\int_{-\infty}^{+\infty}(t-s)^{-\frac{1}{3}+\sigma}e^{-\eta|y_1|}s^{-3/4}(1+s)^{-\frac{d-1}{3}(2-\frac{1}{p})+\frac{5}{12}+2\sigma}dy_1ds\\ &\leq C_1t^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{3}+\sigma}\int_0^{t/2}s^{-3/4}(1+s)^{-\frac{d-1}{3}+\frac{5}{12}+2\sigma}dy_1ds\\ &+C_1t^{-\frac{d-1}{3}(2-\frac{1}{p})+\frac{5}{12}+2\sigma}\int_{t/2}^{t}(t-s)^{-\frac{1}{3}+\sigma}ds.\end{aligned}$$

(Here, $t\geq 1$, and so while $s^{-3/4}$, which is a result of Lemma 6.3, corresponds with (at least the possibility of) genuine blow-up, division by t in these estimates does not.) Integrating directly, we obtain respective estimates

$$C_2t^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{3}+\sigma}\max\{1,t^{-\frac{d-1}{3}+\frac{2}{3}+2\sigma}\}+C_2t^{-\frac{d-1}{3}(2-\frac{1}{p})-\frac{1}{3}+2\sigma}t^{\frac{2}{3}+\sigma}. \tag{7.4}$$

For the case $d = 2$, we have $\sigma = 0$, and consequently we obtain an estimate rate $t^{-\frac{1}{3}(1-\frac{1}{p})}$, which is precisely enough. For the case $d = 3$, we have an estimate rate $t^{-\frac{2}{3}(1-\frac{1}{p})-\frac{1}{3}+3\sigma}$, which, for σ taken sufficiently small, is better than required. Indeed, it is clear from this calculation, that the nonlinear estimates are only improved as dimension increases, and consequently that the calculation for $d = 2$ is the critical case. Finally, we note that integration of E_{y_k} against the remaining nonlinearities Ψ_2 and Ψ_3 gives rates of decay significantly better than required.

We conclude the analysis of the tracking estimates by observing that the derivative estimates follow in an almost identical fashion.

The perturbation estimates. We next consider the main perturbation iteration (1.19), and the nonlinearity with minimal decay in s, Ψ_1, for which we have (omitting some spatial dependences for notational brevity),

$$\int_0^{t/2}\int_{-\infty}^{+\infty}\sup_{\tilde{y}\in\mathbb{R}^{d-1}}\Big\|\tilde{G}_{y_k}(t-s)\Big\|_{L^p_{\tilde{x}}}e^{-\eta|y_1|}s^{-3/4}(1+s)^{-\frac{d-1}{3}+\frac{5}{12}+2\sigma}dy_1ds$$
$$+\int_{t/2}^{t}\int_{-\infty}^{+\infty}\sup_{\tilde{y}\in\mathbb{R}^{d-1}}\Big\|\tilde{G}_{y_k}(t-s)\Big\|^{\frac{1}{p}}_{L^1_{\tilde{x}}}\sup_{\tilde{x}\in\mathbb{R}^{d-1}}\Big\|\tilde{G}_{y_k}(t-s)\Big\|^{\frac{1}{q}}_{L^1_{\tilde{y}}}e^{-\eta|y_1|}s^{-3/4}(1+s)^{-\frac{d-1}{3}(2-\frac{1}{p})+\frac{5}{12}+2\sigma}dy_1ds.$$

According to the estimates of Theorems 1.1 and 5.1, these integrals can be estimated by

$$\begin{aligned}&C\int_0^{t/2}\int_{-\infty}^{+\infty}(t-s)^{-\frac{d-1}{3}(1-\frac{1}{p})-1+\sigma}e^{-\frac{(x_1-y_1)^2}{M(t-s)}}e^{-\eta|y_1|}s^{-3/4}(1+s)^{-\frac{d-1}{3}+\frac{5}{12}+2\sigma}dy_1ds\\&+C\int_{t/2}^{t}\int_{-\infty}^{+\infty}(t-s)^{-1+\sigma}e^{-\frac{(x_1-y_1)^2}{M(t-s)}}e^{-\eta|y_1|}s^{-3/4}(1+s)^{-\frac{d-1}{3}(2-\frac{1}{p})+\frac{5}{12}+2\sigma}dy_1ds.\end{aligned}\tag{7.5}$$

In this case, we must keep track of spatial decay in x_1. To this end, we note the straightforward inequality

$$e^{-\frac{(x_1-y_1)^2}{M(t-s)}}e^{-\eta|y_1|}\le e^{-\frac{\eta}{2}|y_1|}\Big[e^{-\frac{x_1^2}{4M(t-s)}}+e^{-\frac{\eta}{4}|x_1|}\Big].\tag{7.6}$$

Upon substitution of (7.6) into (7.5), we obtain an estimate on (7.5) by

$$C_1\Big[t^{-\frac{d-1}{3}(1-\frac{1}{p})-1+\sigma}\max\{1,t^{-\frac{d-1}{3}+\frac{2}{3}+2\sigma}\}+t^{-\frac{d-1}{3}(2-\frac{1}{p})-\frac{1}{3}+3\sigma}h_d(t)\Big]\Big[e^{-\frac{x_1^2}{4Mt}}+e^{-\frac{\eta}{4}|x_1|}\Big],$$

where as in Theorem 1.2 $h_d(t)$ is $\ln t$ for $d = 2$ and 1 for $d \ge 3$. (This results from integration of the expression $(t-s)^{-1+\sigma}$.) For the case $d = 2$, we have $\sigma = 0$, and consequently we obtain an estimate by

$$C_1t^{-\frac{1}{3}(1-\frac{1}{p})-\frac{2}{3}}\ln t\Big[e^{-\frac{x_1^2}{4Mt}}+e^{-\frac{\eta}{4}|x_1|}\Big].$$

Taking $L \ge 4M$, we can insure that the expression involving heat kernel type scaling is sufficient, while for the term involving exponential decay, we observe that for $|x_1| \ge \sqrt{t}$, we have exponential decay in both $|x_1|$ and $\sqrt{t}$, while for $|x_1| \le \sqrt{t}$ we can take heat kernel type scaling without loss of generality. As in the discussion following (7.4), the estimate is only improved for the cases $d \ge 3$. Integration of $\tilde{G}_{y_k}$ against the remaining nonlinearities Ψ_2 and Ψ_3 can be carried out similarly.

We next consider the case of x_1-differentiation, and the nonlinearity with minimal decay in s, Ψ_1, for which we have (omitting some spatial dependence for brevity of nottation)

$$\int_0^{t/2}\int_{-\infty}^{+\infty}\sup_{\tilde{y}\in\mathbb{R}^{d-1}}\Big\|\tilde{G}_{x_1y_k}(t-s)\Big\|_{L^p_{\tilde{x}}}e^{-\eta|y_1|}s^{-3/4}(1+s)^{-\frac{d-1}{3}+\frac{5}{12}+2\sigma}dy_1ds$$
$$+\int_{t/2}^{t}\int_{-\infty}^{+\infty}\sup_{\tilde{y}\in\mathbb{R}^{d-1}}\Big\|\tilde{G}_{x_1y_k}(t-s)\Big\|^{\frac{1}{p}}_{L^1_{\tilde{x}}}\sup_{\tilde{x}\in\mathbb{R}^{d-1}}\Big\|\tilde{G}_{x_1y_k}(t-s)\Big\|^{\frac{1}{q}}_{L^1_{\tilde{y}}}e^{-\eta|y_1|}s^{-3/4}(1+s)^{-\frac{d-1}{3}(2-\frac{1}{p})+\frac{5}{12}+2\sigma}dy_1ds.$$

According to Theorem 5.1, we have

$$\begin{aligned}\Big\|\tilde{G}_{x_1y_k}(t,x;y)\Big\|_{L^p_{\tilde{x}}}&\le Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{4}{3}+\sigma}e^{-\eta|x_1|}e^{-\frac{y_1^2}{Mt}}\\&+Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-1+\sigma}e^{-\eta|x_1-y_1|}+Ct^{-\frac{d-1}{3}(1-\frac{1}{p})-1+\sigma}e^{-\eta|x_1|}e^{-\frac{y_1^2}{Mt}}.\end{aligned}\tag{7.7}$$

For the first term on the right-hand side of (7.7), we have an estimate by

$$\begin{aligned} &C\int_0^{t/2}\int_{-\infty}^{+\infty}(t-s)^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{4}{3}+\sigma}e^{-\frac{(x_1-y_1)^2}{M(t-s)}}e^{-\eta|y_1|}s^{-3/4}(1+s)^{-\frac{d-1}{3}+\frac{5}{12}+2\sigma}dy_1ds \\ &+C\int_{t/2}^{t}\int_{-\infty}^{+\infty}(t-s)^{-\frac{4}{3}+\sigma}e^{-\frac{(x_1-y_1)^2}{M(t-s)}}e^{-\eta|y_1|}s^{-3/4}(1+s)^{-\frac{d-1}{3}(2-\frac{1}{p})+\frac{5}{12}+2\sigma}dy_1ds. \end{aligned} \tag{7.8}$$

Proceeding as with (7.5), we estimate this by

$$C_1\Big[t^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{4}{3}+\sigma}\max\{1,t^{-\frac{d-1}{3}+\frac{2}{3}+2\sigma}\}+t^{-\frac{d-1}{3}(2-\frac{1}{p})-\frac{1}{3}+2\sigma}\Big]\Big[e^{-\frac{x_1^2}{4Mt}}+e^{-\frac{\eta}{4}|x_1|}\Big].$$

For the critical case $d=2$, we obtain an estimate by

$$C_2t^{-\frac{1}{3}(2-\frac{1}{p})-\frac{2}{3}}\Big[e^{-\frac{x_1^2}{4Mt}}+e^{-\frac{\eta}{4}|x_1|}\Big].$$

In particular, we note that this analysis does not recover the linear estimate on v_{x_1}. For the second and third estimates of (7.7), we have precisely the same s decay as in (7.5), combined with exponential decay in $|x_1|$. Consequently, in the case $d=2$, we obtain an estimate by

$$C_1t^{-\frac{1}{3}(1-\frac{1}{p})-\frac{2}{3}}\ln t e^{-\frac{\eta}{4}|x_1|}.$$

We have presented full details for the calculations that determine the estimates of Theorem 1.2. The remaining estimates of Lemma 6.2 can be established similarly. □

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