Spectral Analysis of Planar Transition Fronts for the Cahn–Hilliard Equation

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Abstract

We consider the spectrum of the linear operator that arises upon linearization of the Cahn–Hilliard equation in dimensions $d \geq 2$ about a planar transition front (a solution that depends on only one distinguished space variable and that has different values at $\pm\infty$). In previous work the author has established conditions on this spectrum under which such planar transition fronts are asymptotically stable, and we verify here that those conditions hold for all such waves arising in a general form of the Cahn–Hilliard equation.

1 Introduction

We consider the Cahn–Hilliard equation on $\mathbb{R}^d$, $d \geq 2$,

$$u_t = \nabla \cdot \{M(u)\nabla(F'(u) - \kappa \triangle u)\}, \tag{1.1}$$

where $\kappa > 0$ is assumed constant, and throughout the analysis we will make the following assumptions:

(H0) $M \in C^2(\mathbb{R})$ and $F \in C^4(\mathbb{R})$.

(H1) F has a double-well form: there exist real numbers $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5$ so that F is strictly decreasing on $(-\infty, \alpha_1)$ and (α_3, α_5) and strictly increasing on (α_1, α_3) and $(\alpha_5, +\infty)$, and additionally F is concave up on $(-\infty, \alpha_2) \cup (\alpha_4, +\infty)$ and concave down on (α_2, α_4).

We note that for each F satisfying assumptions (H0)–(H1), there exists a unique pair of values u_1 and u_2 (the *binodal* values) so that $F'(u_1) = \frac{[F]}{[u]} = F'(u_2)$ and the line passing through $(u_1, F(u_1))$ and $(u_2, F(u_2))$ lies entirely on or below F. (Here, $[u] = (u_2 - u_1)$ and $[F] = F(u_2) - F(u_1)$.) We assume additionally:

(H2) For $u \in [u_1, u_2]$, $M(u) \geq m_0 > 0$.

For a discussion of the physicality of (1.1) and of the assumptions (H0)–(H2) the reader is referred to the discussion in [7] and more generally to the references therein. We note here that for any linear function $G(u) = Au + B$, we can replace $F(u)$ in (1.1) without loss of generality with $F(u) - G(u)$. If we take

$$G(u) = \frac{[F]}{[u]}u + F(u_h) - \frac{[F]}{[u]}u_h,$$

where u_h is the unique value for which both $F''(u_h) < 0$ and $F'(u_h) = [F]/[u]$, then F can be assumed to be 0 at its local maximum and to have local minima at the binodal values $F(u_1) = F(u_2)$. Finally, replacing u with $u - u_h$, we can shift F so that the local maximum is located at $u = 0$.

Definition 1. *We will say that a double well function $F(u)$ for which the local maximum is 0 and occurs at $u = 0$ and for which the local minima u_1 and u_2 satisfy $F(u_1) = F(u_2)$ is in standard form.*

Our first result regards the existence of planar transition front solutions $\bar{u}(x_1)$ to (1.1); that is, the existence of solutions $\bar{u}(x_1)$ that satisfy the asymptotic relationship $\bar{u}(\pm\infty) = u_\pm$, where $u_- \neq u_+$ and both values are bounded. This theorem is an immediate consequence of Theorem 1.1 of [7].

Theorem 1.1 (Planar wave existence.)**.** *For equation (1.1), under conditions (H0)–(H2), there exist two planar transition front solutions $\bar{u}(x_1)$ and $\bar{u}(-x_1)$, both of which are strictly monotonic, and both of which approach the binodal values $u_1 < u_2$:*

$$\lim_{x_1 \to -\infty} \bar{u}(x_1) = u_1; \qquad \lim_{x_1 \to +\infty} \bar{u}(x_1) = u_2.$$

Moreover, if $M(u) > 0$ for all $u \in \mathbb{R}$, then these are the only two transition front solutions for the associated F.

Upon linearization of (1.1) about $\bar{u}(x_1)$, we obtain the linear equation

$$v_t = Lv := \nabla \cdot \{M(\bar{u})\nabla(F''(\bar{u})v - \kappa\triangle v)\}, \tag{1.2}$$

with associated eigenvalue problem

$$L\varphi = \lambda\varphi. \tag{1.3}$$

(The nonlinear terms dropped off in this linearization will not be considered here, but details of the nonlinear analysis can be found in [8].) Observing that the coefficients of L depend only on the distinguished variable x_1, we take a Fourier transform in the transverse coordinates $\tilde{x} := (x_2, x_3, ..., x_d)$ (scaling the transform as $\int_{\mathbb{R}^{d-1}} e^{-i\xi\cdot\tilde{x}}$). In this way, we obtain the transformed operator

$$L_\xi := -D_\xi H_\xi, \tag{1.4}$$

where

$$D_\xi \phi := -(M(\bar{u})\phi')' + |\xi|^2 M(\bar{u})\phi \tag{1.5}$$

and

$$H_\xi \phi := -\kappa\phi'' + F''(\bar{u})\phi + \kappa|\xi|^2\phi. \tag{1.6}$$

Here, L_ξ is clearly a sectorial operator, and it's essential spectrum can be characterized by its asymptotic behavior. That is, the essential spectrum of L_ξ lies on or to the left of the pair of contours described by

$$\lambda_{\text{ess}}(l) := -|\xi|^2 M(u_\pm)F''(u_\pm) - \kappa|\xi|^4 M(u_\pm) - M(u_\pm)(F''(u_\pm) + 2\kappa|\xi|^2)l^2 - \kappa M(u_\pm)l^4, \tag{1.7}$$

with $l \in [0, \infty)$. For $\xi \in \mathbb{R}^{d-1}$, the essential spectrum is clearly confined to the negative real axis. We will also be interested in complexifications of ξ, with suitably small complex part, and we note that in this case the essential spectrum can move away from the negative real axis. In order for λ to be a point eigenvalue for L_ξ there must exist some $\phi(x_1; \lambda, \xi) \in L^2(\mathbb{R})$ such that

$$L_\xi \phi = \lambda \phi. \tag{1.8}$$

Letting $\phi_1^-(x_1; \lambda, \xi)$ and $\phi_2^-(x_1; \lambda, \xi)$ denote the two linearly independent asymptotically decaying solutions at $-\infty$ of (1.8) (for λ away from essential spectrum), and $\phi_1^+(x_1; \lambda, \xi)$ and $\phi_2^+(x_1; \lambda, \xi)$ similarly the two linearly independent asymptotically decaying solutions at $+\infty$ (this decomposition is established in Lemma 3.1 of [8]), we note that the eigenfunction $\phi(x_1; \lambda, \xi)$ must be a linear combination of $\phi_1^-(x_1; \lambda, \xi)$ and $\phi_2^-(x_1; \lambda, \xi)$ and also of $\phi_1^+(x_1; \lambda, \xi)$ and $\phi_2^+(x_1; \lambda, \xi)$. In this way, we only have an eigenvalue if there is linear dependence among these four solutions; that is, if $W(\phi_1^-, \phi_2^-, \phi_1^+, \phi_2^+) = 0$, where W is the standard Wronskian

$$W(\phi_1^-, \phi_2^-, \phi_1^+, \phi_2^+) = \det \begin{pmatrix} \phi_1^- & \phi_2^- & \phi_1^+ & \phi_2^+ \\ \phi_1^{-\prime} & \phi_2^{-\prime} & \phi_1^{+\prime} & \phi_2^{+\prime} \\ \phi_1^{-\prime\prime} & \phi_2^{-\prime\prime} & \phi_1^{+\prime\prime} & \phi_2^{+\prime\prime} \\ \phi_1^{-\prime\prime\prime} & \phi_2^{-\prime\prime\prime} & \phi_1^{+\prime\prime\prime} & \phi_2^{+\prime\prime\prime} \end{pmatrix}.$$

Loosely following [1, 5, 11], we define the Evans function as

$$D(\lambda, \xi) := W(\phi_1^-, \phi_2^-, \phi_1^+, \phi_2^+)\Big|_{x=0}. \tag{1.9}$$

As observed in [8] (see also [12, 13]), this Evans function is not analytic in a neighborhood of $(\lambda, \xi) = (0, 0)$. Consequently, we will find it convenient to work with the variables

$$\begin{aligned} r &:= |\xi|^2 \\ \rho_\pm &= \frac{\sqrt{\lambda + b_\pm r + c_\pm r^2}}{b_\pm + 2c_\pm r}, \end{aligned} \tag{1.10}$$

where $b_\pm = M(u_\pm)F''(u_\pm)$ and $c_\pm = \kappa M(u_\pm)$. The advantage of these variables is that if (with a slight abuse of notation) we re-define the Evans function as

$$D(r,\rho_-,\rho_+) = W(\phi_1^-(x_1;r,\rho_-),\phi_2^-(x_1;r,\rho_-),\phi_1^+(x_1;r,\rho_+),\phi_2^+(x_1;r,\rho_+))\Big|_{x_1=0},$$

then there is a neighborhood of $(r,\rho_-,\rho_+) = (0,0,0)$ in which D is analytic in each of its arguments.

We are now in a position to state the main result of the paper.

Theorem 1.2 (Spectral stability)**.** *Suppose $\bar{u}(x_1)$ is a planar transition front solution to (1.1) and suppose (H0)–(H2) hold, and additionally that F is in standard form, with $\bar{u}(x_1)$ shifted so that $\bar{u}(0) = 0$. Then the eigenvalues of the operator L_ξ, and equivalently the zeros of the Evans function $D(\lambda,\xi)$ satisfy the following:*

1. There is a neighborhood V of the origin in complex ξ-space and a value $\delta > 0$ so that for all $\xi \in V$ there exists an $L^2(\mathbb{R})$ eigenvalue $\lambda_*(\xi)$ of L_ξ that lies on the curve described by the relations $D(\lambda_*(\xi),\xi) = 0$, $\lambda(0) = 0$ and is contained in the disk $|\lambda| < \delta$. Moreover, for $\xi \in V$, $\lambda_*(\xi)$ is the only $L^2(\mathbb{R})$ eigenvalue of L_ξ in this disk, and $\lambda_*(\xi)$ satisfies

$$\lambda_*(\xi) = -\lambda_3|\xi|^3 + \mathbf{O}(|\xi|^4), \tag{1.11}$$

where

$$\lambda_3 = \frac{\sqrt{2\kappa}(M(u_-)+M(u_+))}{[u]^2}\int_{\min(u_-,u_+)}^{\min(u_-,u_+)}\sqrt{F(x)-F(u_-)}dx. \tag{1.12}$$

2. Outside the neighborhood described in Condition (1) (i.e., outside this region described by $\xi \in V$ and $|\lambda| < \delta$), and for $\xi = \xi_R + i\xi_I$, with $|\xi_I|$ sufficiently small, the point spectrum (i.e., $L^2(\mathbb{R})$ spectrum) of L_ξ is contained to the left of a wedge in the λ complex plane described by

$$Re\ \lambda = -c_1\Big(|\xi_R|^4 - C_2|\xi_I|^4 + |Im\ \lambda|\Big),$$

where c_1 and C_2 are both postive constants.

In the case of (1.1) with $F(u) = \frac{1}{8}u^4 - \frac{1}{4}u^2$, and $M(u) \equiv 1$, spectral conditions (1) and (2) have been shown to hold in [12] (Lemma 1.3; see also [13]). These conditions have also been established in [16], aside from one small gap in the analysis (see the final paragraph in the first column of p. 806). Arguments based on perturbation methods appear in [3, 10]). For the case $M \equiv 1$, the precise formulation (1.12) agrees with (2.14) of [16] and (27) of [3].

Combined with the nonlinear analysis of [8], we can conclude the following theorem on the stability of planar transition front solutions.

Theorem 1.3. *Suppose $\bar{u}(x_1)$ is a planar wave solution to (1.1) and suppose (H0)–(H2) hold. Then for Hölder continuous initial perturbations $(u(0,x) - \bar{u}(x)) \in C^{0+\gamma}(\mathbb{R}^d)$, $\gamma > 0$, with*

$$\|u(0,x) - \bar{u}(x)\|_{L^1_{\tilde{x}}} \leq E_0(1+|x_1|)^{-3/2}, \tag{1.13}$$

for some E_0 sufficiently small, there exists a function $\delta(t,\tilde{x})$ so that

$$\|u(t,x)-\bar{u}(x_1-\delta(t,\tilde{x}))\|_{L^p_{\tilde{x}}} \leq CE_0\Big[(1+t)^{-\frac{d-1}{2}(1-\frac{1}{p})}+(1+t)^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{1}{6}+\sigma}h_d(t)\Big]\Theta(t,x_1),$$

with

$$\|\partial^{\beta}_{\tilde{x}}\delta(t,\tilde{x})\|_{L^p_{\tilde{x}}}\leq CE_0(1+t)^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{|\tilde{\alpha}|}{3}+\sigma},$$

where $|\beta|\leq 1$,

$$\Theta(t,x_1)=(1+t)^{-1/2}e^{-\frac{x_1^2}{Lt}}+(1+|x_1|+\sqrt{t})^{-\frac{3}{2}},$$

and

$$h_d(t)=\begin{cases}\ln t & d=2\\ 1 & d\geq 3,\end{cases}$$

where $\sigma=0$ for $d=2$, and the estimates are valid for any $\sigma>0$ in the cases $d\geq 3$. Moreover, we have the derivative estimates

$$\|u_{x_1}(t,x)-\bar{u}'(x_1-\delta(t,\tilde{x}))\|_{L^p_{\tilde{x}}} \leq CE_0t^{-1/4}\Big[(1+t)^{-\frac{d-1}{3}(1-\frac{1}{p})+\frac{1}{12}+\sigma}\Theta(t,x_1)+(1+t)^{-\frac{d-1}{3}(1-\frac{1}{p})-\frac{5}{12}}h_d(t)e^{-\eta|x_1|}\Big]$$

and for $k=2,3,...,d$,

$$\|\partial_{x_k}\Big(u(t,x)-\bar{u}(x_1-\delta(t,\tilde{x}))\Big)\|_{L^p_{\tilde{x}}}\leq CE_0t^{-1/4}(1+t)^{-\frac{d-1}{3}(1-\frac{1}{p})+\frac{1}{12}}\Theta(t,x_1).$$

In the remainder of the paper, we give a proof of Theorem 1.2. The proof is divided into two parts, corresponding with conclusions (1) and (2) of our theorem. In particular, in Section 2, we employ a straightforward min–max estimate to establish estimates on the spectrum of L_ξ for $\xi\in\mathbb{R}^{d-1}$, while in Section 3, we analyze the behavior of the leading eigenvalue $\lambda_*(\xi)$. Theorem 1.2 is established by a straightforward continuation argument for complex values of ξ.

2 The Min–Max Principle Estimates

In this section, we employ the min–max principle argument of [12, 13] (Sections 1.2 and 4 respectively) to establish estimates on the spectrum of L_ξ for $\xi\in\mathbb{R}^{d-1}$. The following lemma is a generalization of Lemma 1.3 of [12].

Lemma 2.1. *Suppose $\bar{u}(x_1)$ is a planar wave solution to (1.1) and suppose (H0)–(H2) hold. For $\xi\in\mathbb{R}^{d-1}$, the point spectrum for the operator L_ξ satisfies the following:*

1. The point spectrum lies entirely on the real axis and is bounded to the left of $-\kappa m_0|\xi|^4$, where $m_0=\min_{u\in[u_1,u_2]}M(u)$.

2. For $|\xi| \le \delta$, some $\delta > 0$ sufficiently small, the leading eigenvalue of L_ξ satisfies

$$\lambda_*(|\xi|) \ge -c_1 |\xi|^3,$$

for some constant $c_1 > 0$, and the remainder of the point spectrum for L_ξ lies to the left of $-c_2|\xi|^2$, for some constant $c_2 > 0$.

Proof. We begin by observing that for $\xi = 0$ the eigenvalue problem (1.8) reduces to

$$L_0 \phi := (M(\bar{u})(H_0 \phi)')' = \lambda \phi; \quad H_0 := F''(\bar{u}) - \kappa \partial_{xx}, \tag{2.1}$$

which is precisely the equation studied in [7] in the context of the Cahn–Hilliard equation in one space dimension. In particular, it was shown in [7] that the spectrum (point and essential) of L_0 is contained in the negative real axis, and that the leading eigenvalue is at $\lambda = 0$.

For $\xi \ne 0$, we proceed similarly as in the analyses of [4] and [12, 13] and write the eigenvalue problem (1.8) in the form

$$D_\xi H_\xi \phi = -\lambda \phi, \tag{2.2}$$

where D_ξ and H_ξ are (for $\xi \in \mathbb{R}^{d-1}$, $\xi \ne 0$) the positive self-adjoint operators defined in (1.5) and (1.6). Since D_ξ is positive and self-adjoint, it has a well-defined square root that is also self-adjoint, and we set $\varphi = D_\xi^{-1/2}\phi$. In this way, φ can be seen to solve the self-adjoint eigenvalue problem

$$\mathcal{L}_\xi := D_\xi^{1/2} H_\xi D_\xi^{1/2} \varphi = -\lambda \varphi. \tag{2.3}$$

If ϕ is an L^2 eigenfunction of (2.2) then φ is an L^2 eigenfunction of (2.3). (This follows from the observation that if ϕ is an L^2 eigenfunction of (2.2) then it must decay at exponential rate, and this exponential decay is inherited through $D_\xi^{-1/2}$ by φ.) In practice, we can readily compute $D_\xi^{-1/2}$ by methods very similar to those of the nonlinear analysis of [8]. That is, we can compute $D_\xi^{-1/2}$ as the operator-valued Cauchy integral

$$D_\xi^{-1/2} = \frac{1}{2\pi i} \int_\Gamma \lambda^{-1/2} (\lambda I - D_\xi)^{-1} d\lambda, \tag{2.4}$$

where the resolvent operator $(\lambda I - D_\xi)^{-1}$ can be computed in terms of the Green's function $g(x, y; \lambda, \xi)$ for the operator $(\lambda I - D_\xi)$. That is, if g solves the Green's function equation $(\lambda I - D_\xi) g = \delta(x - y)$, where δ denotes a standard Dirac delta function, then

$$(\lambda I - D_\xi)^{-1} \phi = \int_{-\infty}^{+\infty} g(x, y; \lambda, \xi) \phi(y) dy. \tag{2.5}$$

The methods of [6], extended from the analyses of [9, 17], are sufficient for establishing the required estimates on g.

Letting now $\langle \cdot, \cdot \rangle$ denote an inner product on $L^2(\mathbb{R})$, we have

$$\langle \varphi, \mathcal{L}\varphi \rangle = \langle D_\xi^{1/2}\varphi, H_\xi D_\xi^{1/2}\varphi \rangle = \langle D_\xi^{1/2}\varphi, H_0 D_\xi^{1/2}\varphi \rangle + \kappa|\xi|^2 \langle D_\xi^{1/2}\varphi, D_\xi^{1/2}\varphi \rangle,$$

where H_0 is known from [7] to be a positive operator. Since $\mathcal{L}$ is a self-adjoint operator, bounded from below, the min–max principle (see e.g. [14], Theorem XIII.1) gives that the leading H^2 eigenvalue $-\lambda_1(\xi)$ satisfies

$$\begin{aligned} -\lambda_1(\xi) &= \inf_{\varphi \in H^2 \setminus \{0\}} \frac{\langle \varphi, \mathcal{L}\varphi \rangle}{\langle \varphi, \varphi \rangle} = \inf_{\varphi \in H^2 \setminus \{0\}} \Big[\frac{\langle D_\xi^{1/2}\varphi, H_0 D_\xi^{1/2}\varphi \rangle}{\langle \varphi, \varphi \rangle} + \kappa|\xi|^2 \frac{\langle D_\xi \varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} \Big] \\ &= \inf_{\varphi \in H^2 \setminus \{0\}} \Big[\frac{\langle D_\xi^{1/2}\varphi, H_0 D_\xi^{1/2}\varphi \rangle}{\langle \varphi, \varphi \rangle} + \kappa|\xi|^2 \frac{\langle M(\bar{u})\varphi', \varphi' \rangle}{\langle \varphi, \varphi \rangle} + \kappa|\xi|^4 \frac{\langle M(\bar{u})\varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} \Big] \geq \kappa m_0 |\xi|^4. \end{aligned} \tag{2.6}$$

On the other hand, for $|\xi|$ small, we recall $H_0 \bar{u}_{x_1} = 0$, so that if we choose $\varphi(x) = D_\xi^{-1/2} \bar{u}_{x_1}$, we have

$$-\lambda_1(\xi) \leq \kappa|\xi|^2 \frac{\langle \bar{u}_{x_1}, \bar{u}_{x_1} \rangle}{\langle D_\xi^{-1} \bar{u}_{x_1}, \bar{u}_{x_1} \rangle}, \tag{2.7}$$

where

$$D_\xi^{-1} \bar{u}_{x_1} = \int_{-\infty}^{+\infty} g(x_1, y_1; 0, \xi) \bar{u}_{y_1}(y_1) dy_1.$$

From this representation, it is straightforward to see that the monotonicity of $\bar{u}(x_1)$ insures that

$$\langle D_\xi^{-1} \bar{u}_{x_1}, \bar{u}_{x_1} \rangle \geq \gamma_0 |\xi|^{-1},$$

for some constant $\gamma_0 > 0$. The first part of the second assertion of the lemma follows immediately.

For the second part of the second assertion of the lemma, we observe that according to the min–max principle, the second eigenvalue of L_ξ satisfies

$$-\lambda_2 = \sup_{v \in H^2} \inf_{\substack{\varphi \in H^2 \setminus \{0\} \\ \langle \varphi, v \rangle = 0}} \frac{\langle \mathcal{L}_\xi \varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} \geq \sup_{v \in H^2} \inf_{\substack{\varphi \in H^2 \setminus \{0\} \\ \langle \varphi, v \rangle = 0}} \frac{\langle H_0 D_\xi^{1/2}\varphi, D_\xi^{1/2}\varphi \rangle}{\langle \varphi, \varphi \rangle}. \tag{2.8}$$

If we now set $\psi = D_\xi^{1/2}\varphi$, then this last expression is equivalent to

$$\begin{aligned} \sup_{v \in H^2} \inf_{\substack{\psi \in H^1 \setminus \{0\} \\ \langle D_\xi^{-1/2}\psi, v \rangle = 0}} \frac{\langle H_0 \psi, \psi \rangle}{\langle D_\xi^{-1/2}\psi, D_\xi^{-1/2}\psi \rangle} &= \sup_{v \in H^2} \inf_{\substack{\psi \in H^1 \setminus \{0\} \\ \langle \psi, D_\xi^{1/2} v \rangle = 0}} \frac{\langle H_0 \psi, \psi \rangle}{\langle D_\xi^{-1/2}\psi, D_\xi^{-1/2}\psi \rangle} \\ &= \sup_{w \in H^1} \inf_{\substack{\psi \in H^1 \setminus \{0\} \\ \langle \psi, w \rangle = 0}} \frac{\langle H_0 \psi, \psi \rangle}{\langle D_\xi^{-1/2}\psi, D_\xi^{-1/2}\psi \rangle}. \end{aligned} \tag{2.9}$$

We can now obtain a lower bound on $-\lambda_2$ by taking a particular choice of w. In particular, it is observed in [7] (see also Section 5 of [2] and Section 2 of [15]) that for $\psi \in H^1(\mathbb{R})\backslash\{0\}$, with additionally $\langle \psi, \bar{u}_{x_1} \rangle = 0$, there holds $\langle H_0 \psi, \psi \rangle \geq \gamma \langle \psi, \psi \rangle$, where $\gamma > 0$. Accordingly, we choose $w = \bar{u}_{x_1}(x_1)$ and obtain the inequality

$$
\begin{aligned}
-\lambda_2 &\geq \inf_{\substack{\psi \in H^1\backslash\{0\} \\ \langle \psi, \bar{u}_{x_1} \rangle = 0}} \frac{\langle H_0 \psi, \psi \rangle}{\langle D_\xi^{-1/2} \psi, D_\xi^{-1/2} \psi \rangle} = \inf_{\substack{\psi \in H^1\backslash\{0\} \\ \langle \psi, \bar{u}_{x_1} \rangle = 0}} \frac{\langle H_0 \psi, \psi \rangle}{\langle D_\xi^{-1} \psi, \psi \rangle} \\
&= \inf_{\substack{\psi \in H^1\backslash\{0\} \\ \langle \psi, \bar{u}_{x_1} \rangle = 0}} \frac{\langle H_0 \psi, \psi \rangle}{\langle \psi, \psi \rangle} \cdot \frac{\langle \psi, \psi \rangle}{\langle D_\xi^{-1} \psi, \psi \rangle} \geq \gamma \inf_{\substack{\psi \in H^1\backslash\{0\} \\ \langle \psi, \bar{u}_{x_1} \rangle = 0}} \frac{\langle \psi, \psi \rangle}{\langle D_\xi^{-1} \psi, \psi \rangle}.
\end{aligned}
$$

Finally, one can observe either from the asymptotic behavior of $g(x, y; 0, \xi)$ or from spectral considerations that

$$
\frac{\langle D_\xi^{-1} \psi, \psi \rangle}{\langle \psi, \psi \rangle} \leq \frac{C}{|\xi|^2},
$$

from which we conclude $-\lambda_2 \geq c|\xi|^2$ for some constant c. This establishes the second half of Part (2) of Lemma 2.1, completing the proof. □

3 The Evans Function

In this section, we analyze the Evans function as defined in (1.9). For this analysis, it will be convenient to write the operator L_ξ in the expanded form

$$
\begin{aligned}
L_\xi \phi = &-(c(x_1)\phi_{x_1 x_1 x_1})_{x_1} + (b(x_1)\phi_{x_1})_{x_1} - (a(x_1)\phi)_{x_1} \\
&+ |\xi|^2 \Big[(c(x_1)\phi_{x_1})_{x_1} + c(x_1)\phi_{x_1 x_1} \Big] - \Big[|\xi|^2 b(x_1) + |\xi|^4 c(x_1) \Big] \phi,
\end{aligned} \tag{3.1}
$$

where

$$
\begin{aligned}
b(x_1) &= M(\bar{u}(x_1)) F''(\bar{u}(x_1)) \\
c(x_1) &= \nu M(\bar{u}(x_1)) \\
a(x_1) &= -M(\bar{u}(x_1)) F'''(\bar{u}(x_1)) \bar{u}_{x_1}.
\end{aligned} \tag{3.2}
$$

According to hypotheses (H0) and (H1), we have that $a, b, c \in C^1(\mathbb{R})$, and for $k = 0, 1$

$$
|\partial_{x_1}^k a(x_1)| = \mathbf{O}(e^{-\alpha|x_1|}); \quad |\partial_{x_1}^k (b(x_1) - b_\pm)| = \mathbf{O}(e^{-\alpha|x_1|}); \quad |\partial_{x_1}^k (c(x_1) - c_\pm)| = \mathbf{O}(e^{-\alpha|x_1|}), \tag{3.3}
$$

as $x_1 \to \pm\infty$, where $\alpha > 0$ and $\pm$ denote the asymptotic limits as $x_1 \to \pm\infty$. We can now write our eigenvalue problem (1.8) as a first order system

$$
W' = \mathbb{A}(x_1; \lambda, \xi) W, \tag{3.4}
$$

where

$$\mathbb{A}(x_1;\lambda,\xi) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\lambda + a'(x_1) + |\xi|^2 b(x_1) + |\xi|^4 c(x_1)}{c(x_1)} & +\frac{b'(x_1) - a(x_1) + |\xi|^2 c'(x_1)}{c(x_1)} & \frac{b(x_1) + 2|\xi|^2 c(x_1)}{c(x_1)} & -\frac{c'(x_1)}{c(x_1)} \end{pmatrix}.$$

Under assumptions (H0) and (H1), $\mathbb{A}(x_1;\lambda,\xi)$ has the asymptotic behavior

$$\mathbb{A}(x_1;\lambda,\xi) = \begin{cases} \mathbb{A}_-(\lambda,\xi) + \mathbb{E}(x_1;\lambda,\xi), & x_1 < 0 \\ \mathbb{A}_+(\lambda,\xi) + \mathbb{E}(x_1;\lambda,\xi), & x_1 > 0, \end{cases}$$

where

$$\mathbb{A}_\pm(\lambda,\xi) := \lim_{x_1 \to \pm\infty} \mathbb{A}(x_1;\lambda,\xi) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\lambda + b_\pm |\xi|^2 + c_\pm |\xi|^4}{c_\pm} & 0 & \frac{b_\pm + 2|\xi|^2 c_\pm}{c_\pm} & 0 \end{pmatrix}, \tag{3.5}$$

and for $|\lambda|$ and $|\xi|$ both bounded $\mathbb{E}(x_1;\lambda,\xi) = \mathbf{O}(e^{-\alpha|x_1|})$. The eigenvalues of the matrices $\mathbb{A}_\pm(\lambda,\xi)$, denoted here by $\mu_\pm$ satisfy

$$c_\pm \mu_\pm^4 - (b_\pm + 2|\xi|^2 c_\pm)\mu_\pm^2 + (\lambda + b_\pm |\xi|^2 + c_\pm |\xi|^4) = 0, \tag{3.6}$$

or equivalently one of

$$\mu_\pm^2 = \frac{(b_\pm + 2|\xi|^2 c_\pm) - \sqrt{b_\pm^2 - 4c_\pm \lambda}}{2c_\pm},$$
$$\mu_\pm^2 = \frac{(b_\pm + 2|\xi|^2 c_\pm) + \sqrt{b_\pm^2 - 4c_\pm \lambda}}{2c_\pm}.$$

In terms of the variables (1.10), we can write these eigenvalues as

$$\begin{aligned} \mu_1^\pm &= -\sqrt{(\frac{b_\pm}{2c_\pm} + r)}\sqrt{1 + \sqrt{1 - 4c_\pm \rho_\pm^2}} \\ \mu_2^\pm &= -\sqrt{(\frac{b_\pm}{2c_\pm} + r)}\frac{2\sqrt{c_\pm}\rho_\pm}{\sqrt{1 + \sqrt{1 - 4c_\pm \rho_\pm^2}}} \\ \mu_3^\pm &= +\sqrt{(\frac{b_\pm}{2c_\pm} + r)}\frac{2\sqrt{c_\pm}\rho_\pm}{\sqrt{1 + \sqrt{1 - 4c_\pm \rho_\pm^2}}} \\ \mu_4^\pm &= +\sqrt{(\frac{b_\pm}{2c_\pm} + r)}\sqrt{1 + \sqrt{1 - 4c_\pm \rho_\pm^2}}, \end{aligned} \tag{3.7}$$

where the slow eigenvalues $\mu_2^\pm$ and $\mu_3^\pm$ have been written in a form from which analyticity in r and $\rho_\pm$ is apparent. (See the discussion of [6] just above Lemma 2.1. This development follows closely the notation of [6]; the reader is also referred to the almost identical development of [12], p. 11 and [13], p. 20, in which r is replaced by k^2 and $\rho_\pm$ is replaced by $i\tau$.)

We are now in a position to state a lemma from [8] regarding the asymptotic behavior of a choice of bases for the eigenvalue problem (1.8).

Lemma 3.1. *For the eigenvalue problem (1.8), with L_ξ as defined in (3.1) assume $a, b, c \in C^1(\mathbb{R})$, $c(x_1) \geq c_0 > 0$, and additionally that (3.3) holds with finite values $b_\pm > 0$ and $c_\pm > 0$. Then for some $\bar{\alpha} > 0$ and $k = 0, 1, 2, 3$, we have the following estimates on a choice of linearly independent solutions of (1.8). For $|\lambda| + |\xi|^2 \leq \delta$, some $\delta > 0$ sufficiently small, there holds:*

(i) For $x_1 \leq 0$

$$\begin{aligned}
\partial_{x_1}^k \phi_1^-(x_1; \lambda, \xi) &= e^{\mu_3^-(\lambda,\xi)x_1}(\mu_3^-(\lambda, \xi)^k + \mathbf{O}(e^{-\bar{\alpha}|x_1|})) \\
\partial_{x_1}^k \phi_2^-(x_1; \lambda, \xi) &= e^{\mu_4^-(\lambda,\xi)x_1}(\mu_4^-(\lambda, \xi)^k + \mathbf{O}(e^{-\bar{\alpha}|x_1|})) \\
\partial_{x_1}^k \psi_1^-(x_1; \lambda, \xi) &= e^{\mu_1^-(\lambda,\xi)x_1}(\mu_1^-(\lambda, \xi)^k + \mathbf{O}(e^{-\bar{\alpha}|x_1|})) \\
\partial_{x_1}^k \psi_2^-(x_1; \lambda, \xi) &= \frac{1}{\mu_2^-(\lambda, \xi)}\Big(\mu_2^-(\lambda, \xi)^k e^{\mu_2^-(\lambda,\xi)x_1} - \mu_3^-(\lambda, \xi)^k e^{\mu_3^-(\lambda,\xi)x_1}\Big) + \mathbf{O}(e^{-\bar{\alpha}|x_1|}).
\end{aligned}$$

(ii) For $x_1 \geq 0$

$$\begin{aligned}
\partial_{x_1}^k \phi_1^+(x_1; \lambda, \xi) &= e^{\mu_1^+(\lambda,\xi)x_1}(\mu_1^+(\lambda, \xi)^k + \mathbf{O}(e^{-\bar{\alpha}|x_1|})) \\
\partial_{x_1}^k \phi_2^+(x_1; \lambda, \xi) &= e^{\mu_2^+(\lambda,\xi)x_1}(\mu_2^+(\lambda, \xi)^k + \mathbf{O}(e^{-\bar{\alpha}|x_1|})) \\
\partial_{x_1}^k \psi_1^+(x_1; \lambda, \xi) &= \frac{1}{\mu_3^+(\lambda, \xi)}\Big(\mu_3^+(\lambda, \xi)^k e^{\mu_3^+(\lambda,\xi)x_1} - \mu_2^+(\lambda, \xi)^k e^{\mu_2^+(\lambda,\xi)x_1}\Big) + \mathbf{O}(e^{-\bar{\alpha}|x_1|}) \\
\partial_{x_1}^k \psi_2^+(x_1; \lambda, \xi) &= e^{\mu_4^+(\lambda,\xi)x_1}(\mu_4^+(\lambda, \xi)^k + \mathbf{O}(e^{-\bar{\alpha}|x_1|})).
\end{aligned}$$

We next state a technical lemma describing the behavior of the operator

$$T := -c(x_1)\partial^3_{x_1x_1x_1} + b(x_1)\partial^2_{x_1} - a(x_1)\partial_{x_1} \tag{3.8}$$

(the integrated operator associated with L_0) when acting on derivatives of the $\phi_k^\pm$ with respect to the parameters r and $\rho_\pm$.

Lemma 3.2. *Under the assumptions of Theorem 1.2, for the $\phi_k^\pm$ as in Lemma 3.1, and for T as defined in (3.8), we have the following relations, where for notational brevity we have suppressed that the left hand side is evaluated in every case at the parameter values*

$(r, \rho_-, \rho_+) = (0, 0, 0)$:

$$
\begin{aligned}
(i) \quad & T\frac{\partial \phi_1^-}{\partial \rho_-}(x_1) = b_-^{3/2} \\
(ii) \quad & T\frac{\partial \phi_2^+}{\partial \rho_+}(x_1) = -b_+^{3/2} \\
(iii) \quad & T\frac{\partial^2 \phi_2^-}{\partial \rho_-^2}(x_1) = 2b_-^2(\bar{u}(x_1) - u_-) \\
(iv) \quad & T\frac{\partial^2 \phi_1^+}{\partial \rho_+^2}(x_1) = -2b_+^2(u_+ - \bar{u}(x_1)) \\
(v) \quad & T\frac{\partial \phi_2^-}{\partial r}(x_1) = -b_-(\bar{u}(x_1) - u_-) - c(x_1)\bar{u}_{x_1x_1} \\
(vi) \quad & T\frac{\partial \phi_1^+}{\partial r}(x_1) = b_+(u_+ - \bar{u}(x_1)) - c(x_1)\bar{u}_{x_1x_1} \\
(vii) \quad & T\Big(\frac{\partial \phi_2^-}{\partial r} - \frac{\partial \phi_1^+}{\partial r}\Big)(x_1) = -[bu] + [b]\bar{u}(x_1).
\end{aligned}
$$

Remark on the Proof. Though Lemma 3.2 is not stated in this useful form in [8], it is proven in the course of the proof of Lemma 3.4 of that reference.

We next state a lemma regarding the Wronskian of (various combinations of) the $\phi_k^\pm$ for parameter values $(r, \rho_-, \rho_+) = (0, 0, 0)$.

Lemma 3.3. *Under the assumptions of Theorem 1.2, and for the $\phi_k^\pm$ as in Lemma 3.1 we have the following relations, where evaluation of the left hand side is taken at the parameter values $(r, \rho_-, \rho_+) = (0, 0, 0)$:*

$$
\begin{aligned}
&(i)\, W(\phi_1^-, \bar{u}_{x_1})(x_1) = \frac{F''(u_-)}{\kappa}(\bar{u}(x_1) - u_-) \\
&(ii)\, W(\bar{u}_{x_1}, \phi_2^+)(x_1) = \frac{F''(u_+)}{\kappa}(u_+ - \bar{u}(x_1) \\
&(iii)\, W(\phi_1^{-\prime}, \bar{u}_{x_1x_1})(x_1) = -\frac{F''(u_-)F''(u_+)}{\kappa^2}[u] + \mathbf{O}(e^{-\eta|x_1|}), x_1 \to +\infty \\
&(iv)\, W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(x_1) = -\frac{F''(u_-)F''(u_+)}{\kappa^2}[u] \\
&(v)\, W(\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(x_1) = -M(u_-)^{1/2}\frac{F''(u_-)^{3/2}F''(u_+)}{\kappa^2}[u]x_1 + \mathbf{O}(e^{-\eta|x_1|}), x_1 \to -\infty \\
&(vi)\, W(\partial_{\rho_+}\phi_2^+, \phi_1^-, \bar{u}_{x_1})(x_1) = M(u_+)^{1/2}\frac{F''(u_-)F''(u_+)^{3/2}}{\kappa^2}[u]x_1 + \mathbf{O}(e^{-\eta|x_1|}), x_1 \to +\infty.
\end{aligned}
$$

Proof. The proof of this lemma is quite similar to the proof of Theorem 1.1 of [7]. We begin by more fully characterizing the functions $\phi_k^\pm(x_1, \lambda, \xi)$ at the point $(r, \rho_-, \rho_+) = (0, 0, 0)$ (i.e.,

$(\lambda, \xi) = (0,0)$). At these parameter values, each of the $\phi_k^\pm$ satisfies the equation

$$\Big(M(\bar{u})(F''(\bar{u})\phi - \kappa\phi'')'\Big)' = 0, \tag{3.9}$$

which can be integrated and divided by M so that

$$(F''(\bar{u})\phi - \kappa\phi'')' = 0. \tag{3.10}$$

The solutions of (3.10) can be entirely characterized in terms of the three linearly independent solutions $\bar{u}_{x_1}$, $\phi_A(x_1) := \bar{u}_{x_1}\int_0^{x_1}\frac{dy}{\bar{u}_y(y)^2}$, and $\phi_B(x_1) := \bar{u}_{x_1}\int_0^{x_1}\frac{\bar{u}(y)}{\bar{u}_y(y)^2}dy$. In particular, for $(\lambda, \xi) = (0,0)$ we can write

$$\begin{aligned}\phi_1^-(x_1) &= \alpha_1\phi_A(x_1) + \alpha_2\phi_B(x_1)\\ \phi_2^+(x_1) &= \beta_1\phi_A(x_1) + \beta_2\phi_B(x_1),\end{aligned} \tag{3.11}$$

where by choice we can take the coefficient of $\bar{u}_{x_1}(x_1)$ to be 0 (since any correction at this level can be absorbed by the exponentially decaying error estimates of Lemma 3.1). Observing now that $\phi_1^-(x_1)$ remains bounded as $x_1 \to -\infty$ and that $\phi_2^+(x_1)$ remains bounded as $x_1 \to +\infty$, we can conclude the relations $\alpha_1 = -u_-\alpha_2$ and similarly $\beta_1 = -u_+\beta_2$. We have, then

$$\begin{aligned}\phi_1^-(x_1) &= \alpha_2\Big(-u_-\phi_A(x_1) + \phi_B(x_1)\Big) = \alpha_2\bar{u}_{x_1}(x_1)\int_0^{x_1}\frac{\bar{u}(y) - u_-}{\bar{u}_y(y)^2}dy\\ \phi_2^+(x_1) &= \beta_2\Big(-u_+\phi_A(x_1) + \phi_B(x_1)\Big) = \beta_2\bar{u}_{x_1}(x_1)\int_0^{x_1}\frac{\bar{u}(y) - u_+}{\bar{u}_y(y)^2}dy.\end{aligned} \tag{3.12}$$

According to the normalization chosen in Lemma 3.1, we find $\alpha_2 = -F''(u_-)/\kappa$ and $\beta_2 = -F''(u_+)/\kappa$. Given these exact expressions for $\phi_1^-(x_1;0,0)$ and $\phi_2^+(x_1;0,0)$, the first four results of Lemma 3.3 become straightforward calculations.

For result (v), we begin by observing that the estimates of Lemma 3.1, along with analyticity in the variables (r, ρ_-, ρ_+), give the asymptotic relations

$$\frac{d^k}{dx_1^k}\frac{\partial\phi_1^-}{\partial\rho_-}(x_1;0,0) = \frac{d^k}{dx_1^k}(\sqrt{b_-}x_1) + \mathbf{O}(e^{-\eta|x_1|}), \quad x_1 \to -\infty,$$

for $k = 0,1,2$. We have, then,

$$W(\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(x_1) = \det\begin{pmatrix}\sqrt{b_-}x_1 & \bar{u}_{x_1} & \phi_2^+(x_1)\\ \sqrt{b_-} & \bar{u}_{x_1x_1} & \phi_2^{+\prime}(x_1)\\ 0 & \bar{u}_{x_1x_1x_1} & \phi_2^{+\prime\prime}(x_1)\end{pmatrix} + \mathbf{O}(e^{-\eta|x_1|}), \quad x_1 \to -\infty,$$

and the result follows from an exact calculation involving (3.12).

The proof of (vi) is almost precisely the same as that of (v). □

We are now in a position to prove the main technical lemma of the section. This significantly improves Lemma 3.4 of [8].

Lemma 3.4. *Under the assumptions of Theorem 1.2, for the $\phi_k^\pm$ as in Lemma 3.1, there exists a neighborhood V of $(r, \rho_-, \rho_+) = (0,0,0)$ such that the Evans function is analytic in V. Moreover, if (without loss of generality) we specify the choice*

$$\phi_1^+(x_1; 0, 0) = \bar{u}_{x_1}(x_1) = \phi_2^-(x_1; 0, 0), \tag{3.13}$$

there holds

$$D(r, \rho_-, \rho_+) = D(0,0,0) + \sum_{k=1}^{\infty} \frac{1}{k!}(r\partial_{r'} + \rho_- \partial_{\rho'_-} + \rho_+ \partial_{\rho'_+})^k D(r', \rho'_-, \rho'_+)\Big|_{(r', \rho'_-, \rho'_+) = (0,0,0)}, \tag{3.14}$$

with

$$\begin{aligned} D(0,0,0) = \frac{\partial D}{\partial \rho_\pm}(0,0,0) &= \frac{\partial D}{\partial \rho_- \partial \rho_+}(0,0,0) = 0; \\ \frac{\partial D}{\partial r}(0,0,0) &= -\frac{F''(u_-)F''(u_+)}{\kappa^2 c(0)}[bu][u] \\ \frac{\partial^2 D}{\partial \rho_\pm \partial \rho_\pm}(0,0,0) &= \pm \frac{F''(u_-)F''(u_+)}{\kappa^2 c(0)} 2[u] b_\pm^2 u_\pm. \end{aligned} \tag{3.15}$$

In addition, we have the combination

$$\begin{aligned} A :=& \frac{1}{\sqrt{b_-}} D_{r\rho_-} + \frac{1}{\sqrt{b_+}} D_{r\rho_+} + \frac{1}{6 b_-^{3/2}} D_{\rho_-\rho_-\rho_-} + \frac{1}{6 b_+^{3/2}} D_{\rho_+\rho_+\rho_+} \\ &+ \frac{1}{2 b_- \sqrt{b_+}} D_{\rho_-\rho_-\rho_+} + \frac{1}{2 b_+ \sqrt{b_-}} D_{\rho_-\rho_+\rho_+} \\ &= \frac{F''(u_-)F''(u_+)}{\kappa^2 M(0)} (M(u_-) + M(u_+)) \int_{-\infty}^{+\infty} \bar{u}_{x_1}(x_1)^2 dx_1, \end{aligned} \tag{3.16}$$

where the entire right hand side is evaluated at $(r, \rho_-, \rho_+) = (0,0,0)$.

Proof. First, the statement regarding $D(0,0,0)$, $\frac{\partial D}{\partial \rho_\pm}(0,0,0)$, $\frac{\partial D}{\partial \rho_- \partial \rho_+}(0,0,0)$ is taken directly from Lemma 3.4 of [8]. For $\frac{\partial D}{\partial r}(0,0,0)$ (for which the current claim improves on the result of Lemma 3.4 of [8]), we begin by observing that a straightforward calculation gives

$$\frac{\partial D}{\partial r}(0,0,0) = W(\phi_1^-, \partial_r(\phi_2^- - \phi_1^+), \bar{u}_{x_1}, \phi_2^+)\Big|_{x_1 = 0},$$

where evaluation of the right hand side at $(r, \rho_-, \rho_+) = (0,0,0)$ is suppressed for notational convenience. (More details on the first steps of the analysis are given in the proof of Lemma

3.4 of [8].) We have now

$$W(\phi_1^-, \partial_r(\phi_2^- - \phi_1^+), \bar{u}_{x_1}, \phi_2^+)(x_1) = \det \begin{pmatrix} \phi_1^- & \partial_r(\phi_2^- - \phi_1^+) & \bar{u}_{x_1} & \phi_2^+ \\ \phi_1^{-\prime} & \partial_r(\phi_2^- - \phi_1^+)' & \bar{u}_{x_1x_1} & \phi_2^{+\prime} \\ \phi_1^{-\prime\prime} & \partial_r(\phi_2^- - \phi_1^+)'' & \bar{u}_{x_1x_1x_1} & \phi_2^{+\prime\prime} \\ \phi_1^{-\prime\prime\prime} & \partial_r(\phi_2^- - \phi_1^+)''' & \bar{u}_{x_1x_1x_1x_1} & \phi_2^{+\prime\prime\prime} \end{pmatrix}$$
$$= \det \begin{pmatrix} \phi_1^- & \partial_r(\phi_2^- - \phi_1^+) & \bar{u}_{x_1} & \phi_2^+ \\ \phi_1^{-\prime} & \partial_r(\phi_2^- - \phi_1^+)' & \bar{u}_{x_1x_1} & \phi_2^{+\prime} \\ \phi_1^{-\prime\prime} & \partial_r(\phi_2^- - \phi_1^+)'' & \bar{u}_{x_1x_1x_1} & \phi_2^{+\prime\prime} \\ 0 & \frac{[bu]-[b]\bar{u}(x_1)}{c(x_1)} & 0 & 0 \end{pmatrix},$$

where this final equality is a consequence of the observation made previously that for $(r, \rho_-, \rho_+) = (0,0,0)$, ϕ_1^-, $\bar{u}_{x_1}$, and ϕ_2^+ are all solutions of (3.10), while $(\phi_2^- - \phi_1^+)$ satisfies Lemma 3.2 (vii). This establishes the equality

$$W(\phi_1^-, \partial_r(\phi_2^- - \phi_1^+), \bar{u}_{x_1}, \phi_2^+)(x_1) = \frac{[bu] - [b]\bar{u}(x_1)}{c(x_1)} W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(x_1),$$

where $W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(x_1)$ is a Wronskian associated with (3.10) and is consequently constant as a function of x_1. In light of this, we have

$$W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(0) = \lim_{x_1 \to +\infty} W(\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(x_1) = \lim_{x_1 \to +\infty} \begin{pmatrix} \phi_1^- & \bar{u}_{x_1} & 1 \\ \phi_1^{-\prime} & \bar{u}_{x_1x_1} & 0 \\ \phi_1^{-\prime\prime} & \bar{u}_{x_1x_1x_1} & 0 \end{pmatrix}, \qquad (3.17)$$

where in this last equality we have observed that derivatives of $\phi_2^+(x_1)$ decay at exponential rate as $x_1 \to +\infty$, and that this decay, along with the exponential decay of $\bar{u}_{x_1}$ insures that there is no contribution from $\phi_2^{+\prime}(x_1)$ and $\phi_2^{+\prime\prime}(x_1)$. The result on $\frac{\partial D}{\partial r}(0,0,0)$ now follows immediately from Lemma 3.3 (iii).

The expressions for $\frac{\partial^2 D}{\partial \rho_\pm \partial \rho_\pm}(0,0,0)$ can be derived with a calculation almost identical to the one employed above for $\frac{\partial D}{\partial r}(0,0,0)$, and we omit it.

The relation (3.16) is obtained by a tedious calculation in which we find expressions for each of the derivatives involved. Since these derivations are all similar, we will include the full analysis only for $D_{r\rho_-}(0,0,0)$ (though for completeness, we will state the individual expression for each). Similarly as in our study of $D_r(0,0,0)$, our starting point is the relation

$$D_{r\rho_-}(0,0,0) = W(\partial_{\rho_-}\phi_1^-, \partial_r(\phi_2^- - \phi_1^+), \bar{u}_{x_1}, \phi_2^+)(0),$$

where

$$W(\partial_{\rho_-}\phi_1^-, \partial_r(\phi_2^- - \phi_1^+), \bar{u}_{x_1}, \phi_2^+)(x_1) = \det \begin{pmatrix} \partial_{\rho_-}\phi_1^- & \partial_r(\phi_2^- - \phi_1^+) & \bar{u}_{x_1} & \phi_2^+ \\ \partial_{\rho_-}\phi_1^{-\prime} & \partial_r(\phi_2^- - \phi_1^+)' & \bar{u}_{x_1x_1} & \phi_2^{+\prime} \\ \partial_{\rho_-}\phi_1^{-\prime\prime} & \partial_r(\phi_2^- - \phi_1^+)'' & \bar{u}_{x_1x_1x_1} & \phi_2^{+\prime\prime} \\ -\frac{b_-^{3/2}}{c(x_1)} & \frac{[bu]-[b]\bar{u}(x_1)}{c(x_1)} & 0 & 0 \end{pmatrix},$$

and where this final equality is a consequence of previous observations and also Lemma 3.2 (i). This final determinant can be written as

$$\frac{b_-^{3/2}}{c(x_1)}W(\partial_r(\phi_2^- - \phi_1^+), \bar{u}_{x_1}, \phi_2^+)(x_1) + \frac{[bu]-[b]\bar{u}(x_1)}{c(x_1)}W(\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(x_1).$$

We now separately analyze each of the Wronskians in this last expression, beginning with $W(\partial_r(\phi_2^- - \phi_1^+), \bar{u}_{x_1}, \phi_2^+)(x_1)$, which we further subdivide as

$$W(\partial_r(\phi_2^- - \phi_1^+), \bar{u}_{x_1}, \phi_2^+)(x_1) = W(\partial_r\phi_2^-, \bar{u}_{x_1}, \phi_2^+)(x_1) - W(\partial_r\phi_1^+, \bar{u}_{x_1}, \phi_2^+)(x_1).$$

For the first of these last two Wronskians, we observe

$$\lim_{x_1\to-\infty} W(\partial_r\phi_2^-, \bar{u}_{x_1}, \phi_2^+)(x_1) = 0,$$

so that

$$W(\partial_r\phi_2^-, \bar{u}_{x_1}, \phi_2^+)(x_1) = \int_{-\infty}^{x_1}\frac{d}{dx_1}W(\partial_r\phi_2^-, \bar{u}_{x_1}, \phi_2^+)(x_1),$$

where

$$\frac{d}{dx_1}W(\partial_r\phi_2^-, \bar{u}_{x_1}, \phi_2^+)(x_1) = \det\begin{pmatrix}\partial_r\phi_2^- & \bar{u}_{x_1} & \phi_2^+ \\ \partial_r\phi_2^{-\prime} & \bar{u}_{x_1x_1} & \phi_2^{+\prime} \\ \partial_r\phi_2^{-\prime\prime\prime} & \bar{u}_{x_1x_1x_1x_1} & \phi_2^{+\prime\prime\prime}\end{pmatrix}$$
$$= \det\begin{pmatrix}\partial_r\phi_2^- & \bar{u}_{x_1} & \phi_2^+ \\ \partial_r\phi_2^{-\prime} & \bar{u}_{x_1x_1} & \phi_2^{+\prime} \\ \bar{u}_{x_1x_1} + \frac{b_-(\bar{u}(x_1)-u_-)}{c(x_1)} & 0 & 0\end{pmatrix} = \Big[\bar{u}_{x_1x_1} + \frac{b_-(\bar{u}(x_1)-u_-)}{c(x_1)}\Big]\frac{F''(u_+)}{\kappa}(u_+ - \bar{u}(x_1)), \tag{3.18}$$

where for the last equality we have used Lemma 3.3 (ii). We conclude that

$$W(\partial_r\phi_2^-, \bar{u}_{x_1}, \phi_2^+)(0) = \int_{-\infty}^0 \Big[\bar{u}_{x_1x_1} + \frac{b_-(\bar{u}(y)-u_-)}{c(y)}\Big]\frac{F''(u_+)}{\kappa}(u_+ - \bar{u}(y))dy. \tag{3.19}$$

Proceeding similarly, we can show

$$W(\partial_r\phi_1^+, \bar{u}_{x_1}, \phi_2^+)(0) = -\int_0^{+\infty} \Big[\bar{u}_{x_1x_1} - \frac{b_+(u_+-\bar{u}(y))}{c(y)}\Big]\frac{F''(u_+)}{\kappa}(u_+ - \bar{u}(y))dy. \tag{3.20}$$

We next consider the Wronskian $W(\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(x_1)$, for which one final aspect of the analysis arises. Recalling from Lemma 3.3 (vi) that we understand this Wronskian as x_1 approaches $-\infty$, we write

$$W(\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(x_1) = W(\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(\bar{x}_1) + \int_{\bar{x}_1}^{x_1}\frac{d}{dy}W(\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(y)dy,$$

where $\bar{x}_1$ can be any value in $(-\infty, x_1]$. Proceeding similarly as in (3.18), we can show that

$$\frac{d}{dx_1} W(\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(x_1) = -\frac{b_-^{3/2}}{c(x_1)} \frac{F''(u_+)}{\kappa}(u_+ - \bar{u}(x_1)).$$

If we combine this last expression with Lemma 3.3 (vi), we have

$$W(\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(0) = \int_{\bar{x}_1}^0 b_-^{1/2}[u]\frac{F''(u_-)F''(u_+)}{\kappa^2} - \frac{b_-^{3/2}F''(u_+)}{\kappa c(x_1)}(u_+ - \bar{u}(x_1))dx_1.$$

Taking a limit now as $\bar{x}_1 \to -\infty$, we conclude

$$W(\partial_{\rho_-}\phi_1^-, \bar{u}_{x_1}, \phi_2^+)(0) = \int_{-\infty}^0 b_-^{1/2}[u]\frac{F''(u_-)F''(u_+)}{\kappa^2} - \frac{b_-^{3/2}F''(u_+)}{\kappa c(x_1)}(u_+ - \bar{u}(x_1))dx_1. \quad (3.21)$$

Combining (3.19), (3.20), and (3.21), and recalling the definitions (3.2) we find

$$D_{r\rho_-}(0,0,0) = \frac{b_-^{1/2}M(u_-)F''(u_-)F''(u_+)}{\kappa^3 M(0)}\Big[\kappa\int_{-\infty}^{+\infty}\bar{u}_{x_1}(x_1)^2 dx_1 - b_+\int_0^{+\infty}\frac{(u_+ - \bar{u}(x_1))^2}{M(\bar{u}(x_1))}dx_1 + \int_{-\infty}^0 \frac{[bu][u]}{M(u_-)} - \frac{(b_+u_+ - b_-\bar{u}(x_1))(u_+ - \bar{u}(x_1))}{M(\bar{u}(x_1))}dx_1\Big].$$

Proceeding similarly, we can establish each of the following:

$$D_{r\rho_+}(0,0,0) = \frac{b_+^{1/2}M(u_+)F''(u_-)F''(u_+)}{\kappa^3 M(0)}\Big[\kappa\int_{-\infty}^{+\infty}\bar{u}_{x_1}(x_1)^2 dx_1 - b_-\int_{-\infty}^0\frac{(\bar{u}(x_1) - u_-)^2}{M(\bar{u}(x_1))}dx_1 + \int_0^{+\infty} \frac{[bu][u]}{M(u_+)} - \frac{(b_+\bar{u}(x_1) - b_-u_-)(\bar{u}(x_1) - u_-)}{M(\bar{u}(x_1))}dx_1\Big].$$

$$D_{\rho_-\rho_-\rho_-}(0,0,0) = \frac{6b_-^{5/2}M(u_-)F''(u_-)F''(u_+)}{\kappa^3 M(0)}\int_{-\infty}^0 \frac{[u]u_-}{M(u_-)} - \frac{(u_+ - \bar{u}(x_1))\bar{u}}{M(\bar{u}(x_1))}dx_1.$$

$$D_{\rho_+\rho_+\rho_+}(0,0,0) = -\frac{6b_+^{5/2}M(u_+)F''(u_-)F''(u_+)}{\kappa^3 M(0)}\int_0^{+\infty} \frac{[u]u_+}{M(u_+)} - \frac{(\bar{u}(x_1) - u_-)\bar{u}}{M(\bar{u}(x_1))}dx_1.$$

$$D_{\rho_-\rho_-\rho_+}(0,0,0) = \frac{2b_+^{1/2}b_-^2M(u_+)F''(u_-)F''(u_+)}{\kappa^3 M(0)}\Big[\int_{-\infty}^0 \frac{(\bar{u}(x_1) - u_-)^2}{M(\bar{u}(x_1))}dx_1 + \int_0^{+\infty}\frac{[u]u_-}{M(u_+)} - \frac{(\bar{u}(x_1) - u_-)u_-}{M(\bar{u}(x_1))}dx_1\Big].$$

$$D_{\rho_-\rho_+\rho_+}(0,0,0) = \frac{2b_-^{1/2}b_+^2M(u_-)F''(u_-)F''(u_+)}{\kappa^3 M(0)}\Big[\int_0^{+\infty} \frac{(u_+ - \bar{u}(x_1))^2}{M(\bar{u}(x_1))}dx_1 - \int_{-\infty}^0\frac{[u]u_+}{M(u_-)} - \frac{(u_+ - \bar{u}(x_1))u_+}{M(\bar{u}(x_1))}dx_1\Big].$$

The final claim of Lemma 3.4 can now be obtained by combining these expressions. □

We now state our main lemma on the behavior of $\lambda_*(\xi)$.

Lemma 3.5. *Under the assumptions of Theorem 1.2, there exists a neighborhood V of the origin in complex ξ-space and a value $\delta > 0$ so that for all $\xi \in V$ there exists an $L^2(\mathbb{R})$ eigenvalue $\lambda_*(\xi)$ of L_ξ that lies on the curve described by the relations $D(\lambda_*(\xi), \xi) = 0$, $\lambda(0) = 0$ and is contained in the disk $|\lambda| < \delta$. Moreover, for $\xi \in V$, $\lambda_*(\xi)$ is the only $L^2(\mathbb{R})$ eigenvalue of L_ξ in this disk, and $\lambda_*(\xi)$ satisfies*

$$\lambda_*(\xi) = -\lambda_3|\xi|^3 + \mathbf{O}(|\xi|^4), \tag{3.22}$$

where

$$\lambda_3 = \frac{\sqrt{2\kappa}(M(u_-) + M(u_+))}{[u]^2} \int_{\min(u_-,u_+)}^{\min(u_-,u_+)} \sqrt{F(x) - F(u_-)}dx. \tag{3.23}$$

Proof. First, we observe that since $|\xi|$ appears only with lower order terms in the eigenvalue problem $L_\xi \phi = \lambda \phi$, the existence of such a $\lambda_*(\xi)$ follows from standard perturbation techniques. In order to understand the precise form of $\lambda_*(\xi)$ and to verify its uniqueness, we observe that for $|\xi|$ sufficiently small $\lambda_*(\xi)$ must correspond with a zero of $D(r, \rho_-, \rho_+)$. Such zeros were analyzed in detail in [8] Lemma 3.5, where it was shown that

$$\lambda_*(\xi) = -\lambda_3|\xi|^3 + \mathbf{O}(|\xi|^4),$$

where

$$\lambda_3 = 2b_-^{3/2}\frac{A}{B},$$

where A is as defined in (3.16) and

$$B = \frac{1}{\sqrt{b_-}} D_{\rho_-\rho_-}(0,0,0) + \frac{b_-^{3/2}}{b_+^2} D_{\rho_+\rho_+}(0,0,0).$$

Combining these observations with Lemma 3.4 we conclude

$$\lambda_3 = \frac{\kappa}{[u]^2}(M(u_-) + M(u_+)) \int_{-\infty}^{+\infty} \bar{u}_{x_1}(x_1)^2 dx_1.$$

If $F(u)$ is in the standard form of Definition 1, then it is easy to see that

$$\bar{u}_{x_1}^2 = \frac{2}{\kappa}(F(\bar{u}) - F(u_-)),$$

and consequently

$$\lambda_3 = \frac{2}{[u]^2}(M(u_-) + M(u_+)) \int_{-\infty}^{+\infty} (F(\bar{u}(x_1)) - F(u_-)) dx_1.$$

For the case $u_- < u_+$, $\bar{u}(x_1)$ is a strictly increasing function of x_1, and we are justified in making the change of variables $y = \bar{u}(x_1)$, from which we conclude (3.23). Clearly, the same calculation works for $u_+ < u_-$, where in that case $\bar{u}(x_1)$ is a strictly decreasing function of x_1.

Finally, we note that there can be no other zeros of the Evans function for (r, ρ_-, ρ_+) in a neighborhood of $(0,0,0)$, giving uniqueness. $\square$

Proof of Theorem 1.2. First, for ξ in the described neighborhood V, Lemma 3.5 entirely characterizes the part of $\sigma_{\text{pt}}(L_\xi)$ that lies in $|\lambda| < \delta$. (Here, $\sigma_{\text{pt}}(L_\xi)$ denotes the point spectrum of L_ξ; i.e., the eigenvalues for which there corresponds an $L^2(\mathbb{R})$ eigenfunction.) Since $\xi \in \mathbb{R}^{d-1}$ implies (by Lemma 2.1) $\sigma_{\text{pt}}(L_\xi) \in \mathbb{R}_-$, we have that for $\xi \in \mathbb{R}^{d-1}$, $\sigma_{\text{pt}} \backslash \{\lambda_*(\xi)\} \leq -\delta$. For $|\xi|$ away from V, there exists some $\delta_1 > 0$ so that $|\xi| \geq \delta_1$. For $\xi \in \mathbb{R}^{d-1}$ the estimates of Lemma 2.1 insure that the spectrum of L_ξ is bounded to the left of $-\kappa m_0 \delta_1^4$, where $m_0 = \min_{u \in [u_1, u_2]} M(u)$. Regarding the complexification $\xi = \xi_R + i\xi_I$ now as a perturbation of ξ, Condition (2) of Theorem 1.2 follows from continuity of the spectrum of L_ξ and from the observation that there is a value $R > 0$ sufficiently large so that for $|\lambda| + |\xi|^2 \geq R$ the operator L_ξ has no eigenvalues (point spectrum). $\square$

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