Spectral Analysis for Stationary Solutions of the Cahn–Hilliard Equation in $\mathbb{R}^d$

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Abstract

We consider the spectrum associated with three types of bounded stationary solutions for the Cahn–Hilliard equation on $\mathbb{R}^d$, $d \geq 2$: radial solutions, saddle solutions (only for $d = 2$), and planar periodic solutions. In particular, we establish spectral instability for each type of solution. The important case of multiply periodic solutions does not fit into the framework of our approach, and we do not consider it here.

1 Introduction

We consider the Cahn–Hilliard equation on $\mathbb{R}^d$, $d \geq 2$,

$$u_t = \Delta(F'(u) - \Delta u), \tag{1.1}$$

where throughout the analysis we will make the following standard assumption on F:

(H) $F \in C^4(\mathbb{R})$ has a double-well form: there exist real numbers $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5$ so that F is strictly decreasing on $(-\infty, \alpha_1)$ and (α_3, α_5) and strictly increasing on (α_1, α_3) and $(\alpha_5, +\infty)$, and additionally F is concave up on $(-\infty, \alpha_2) \cup (\alpha_4, +\infty)$ and concave down on (α_2, α_4).

We observe at the outset that for each F satisfying assumptions (H), there exists a unique pair of values u_1 and u_2 (the *binodal* values) so that $F'(u_1) = \frac{F(u_2)-F(u_1)}{u_2-u_1} = F'(u_2)$ and the line passing through $(u_1, F(u_1))$ and $(u_2, F(u_2))$ lies entirely on or below F.

For a general discussion of the Cahn–Hilliard equation, its history and some of its applications, see the review in J. W. Cahn's 1967 Institute of Metals Lecture, printed as [7]. For an overview of results on unbounded domains $\mathbb{R}^d$ see the series of papers [17, 18, 19, 20], which will be referred to throughout. Our interest in bounded stationary solutions, and in particular with the spectrum associated with such solutions, is motivated in part by a suggestion of Langer's, described in [22] (p. 71) as follows: "1. A decomposing alloy, at least during the late stages of coarsening, spends most of its time in configurations which are very nearly stationary solutions of the generalized diffusion equation. 2. The rate of decay of one of these stationary solutions is determined primarily by thermal fluctuations." According to this point of view, we expect solutions of (1.1) to evolve as follows: at each time t, $u(t,x)$ will be a perturbation of a stationary solution, with the perturbation determined by thermal fluctuations in the material. In the event that $u(t,x)$ is a perturbation of an unstable stationary solution, the rate at which $u(t,x)$ moves away from this solution will be determined by these thermal fluctuations and by the leading eigenvalue of the linear operator obtained upon linearization of (1.1) about the stationary solution.

For the Cahn–Hilliard equation in one space dimension ($d = 1$) there are precisely three types of bounded non-constant stationary solutions: periodic solutions, pulse-type *reversal* solutions, and monotonic transition waves. As shown in [17, 19] these can be classified according to their stability properties as follows: each reversal solution is unstable with a positive real eigenvalue, while each transition wave is spectrally and non-linearly (phase-asymptotically) stable. The periodic solutions are spectrally unstable to general perturbations, with a positive real eigenvalue, but appear to be stable to perturbations with the same period as the wave.

In the case $d \geq 2$, each of these one-dimensional waves can be regarded as a planar solution, and in addition there are several more complicated stationary solutions. For example, in all cases $d \geq 2$ there exist radial solutions $\bar{u}(r)$, $r = |x|$, that satisfy $\bar{u}'(0) = 0$, $\bar{u}'(r) < 0$, for $r > 0$, and $\lim_{r\to\infty} = u_\infty$ (see [5] and Section 2 of the current paper). Moreover, in the case $d = 2$ there exist *saddle* solutions which (after an appropriate change of variables; see below) have infimum u_1 and supremum u_2 (the binodal values) and which have the same sign as xy (see [10, 31] and Section 2 of the current paper), doubly periodic solutions with rectangular nodal domains [12, 21], and doubly periodic solutions with non-rectangular nodal domains [12, 25]. (This list is not intended to be exhaustive.) Regarding stability of these solutions, planar reversals inherit spectral instability from the case $d = 1$, while it has been shown in [18, 20] that planar transition fronts are both spectrally and nonlinearly stable. Generally speaking, planar periodic waves also inherit spectral instability from the case $d = 1$, though not to perturbations with the same period as the wave.

In the current paper, we consider the spectrum associated with radial solutions, saddle solutions, and with planar periodic solutions under perturbations periodic in the direction of the wave, with the same period as the wave. In particular, we establish spectral instability for each type of solution. The important case of multiply periodic solutions does not fit into the framework of our approach, and we do not consider it here.

We note for convenience that for any linear function $G(u) = Au + B$, we can replace $F(u)$ in (1.1) with $H(u) = F(u) - G(u)$. If we take

$$G(u) = \frac{F(u_2) - F(u_1)}{u_2 - u_1} u + F(u_h) - \frac{F(u_2) - F(u_1)}{u_2 - u_1} u_h,$$

where u_h is the unique value for which both $F''(u_h) < 0$ and $F'(u_h) = (F(u_2) - F(u_1))/(u_2 - u_1)$, then $H(u)$ has a local maximum $H = 0$ and local minima at the binodal values $H(u_1) =$

$H(u_2)$. Finally, replacing u with $u + u_h$, we can shift H so that the local maximum is located at $u = 0$. We will refer to a double well function $F(u)$ for which the local maximum occurs at $u = 0$ and with equivalent local minima as *standard form.*

Upon linearization of (1.1) about a bounded stationary solution $\bar{u}$, and dropping higher order terms, we obtain the linear perturbation equation

$$v_t = Lv = \Delta H v, \tag{1.2}$$

where

$$H := -\Delta + F''(\bar{u}). \tag{1.3}$$

The eigenvalue problem associated with (1.2) is

$$\Delta H \phi = \lambda \phi. \tag{1.4}$$

It is important in what follows that the Cahn–Hilliard equation can be regarded as an H^{-1} (constrained) gradient flow associated with the energy functional

$$E(u) = \int_\Omega F(u) + \frac{1}{2}|\nabla u|^2 dx. \tag{1.5}$$

(See, for example, [11] for a development of (1.1) as a contrained gradient flow.) More precisely, (1.1) can be written in the form

$$u_t = \Delta \frac{\delta E}{\delta u},$$

and stationary solutions can be viewed either as critical points of the energy ($\frac{\delta E}{\delta u} = 0$) or as critical points associated with an appropriate Lagrange multiplier ($\frac{\delta E}{\delta u} = c$, for some constant c). In either case, it is natural to consider the second variational derivative of $E(u)$ evaluated at such stationary solutions; that is, to consider the operator $H := \frac{\delta^2 E}{\delta u^2}(\bar{u})$ (equivalent to H as defined above). These considerations lead to a natural definition of variational stability, taken from [8]:

Definition 1.1. *We will say that a bounded stationary solution $\bar{u}(x)$ of (1.1) is variationally stable if*

$$\langle H\phi, \phi \rangle := \int_{\mathbb{R}^d} \phi H \phi dx \geq 0$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$ (i.e., the space of infinitely differentiable functions with compact support). If $\bar{u}(x)$ is not variationally stable then we say it is variationally unstable.

We note that while our source for this terminology was [8], the importance of $\langle H\phi, \phi \rangle$ in analyzing the spectrum for (1.4) has been observed in [1, 3, 22] and others.

Our main concern in studying radial and saddle solutions to (1.1) will consist in showing that variational instability implies spectral instability, defined as follows:

Definition 1.2. *We will say that $\bar{u}$ is spectrally unstable as a solution to (1.1) if (1.4) admits a positive $L^2(\mathbb{R}^d)$ eigenvalue.*

We note that it is immediate that if $\bar{u}(x)$ is variationally stable then it is spectrally stable as a solution of the Allen–Cahn equation $u_t = \Delta u - F'(u)$ (i.e., the gradient flow of E in L^2). It is the implication regarding $\bar{u}(x)$ as a stationary solution of (1.1) that requires justification. In the case that (1.1) is specified on a bounded domain Ω, with natural boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

the relationship between variational stability and spectral stability has been studied in [3]. (The authors point out in [3] that the essential feature of this connection was exploited by Langer in [22].) In particular, our analysis of radial solutions here is closely related to the analysis of spike layer solutions in [4]. (Spike layer solutions are approximately constant in most of Ω, with several spikes at locations either in the interior of Ω or on its boundary. Each individual spike can be constructed as an approximate scaling of some radial solution of the problem on $\mathbb{R}^d$. See, for example, [2, 4, 6] and the references therein.) Generally speaking, our approach is quite similar to that of [3] in that we we show that the spectral min–max principle can be used to relate eigenvalues of the fourth-order problem (1.4) to the action of the second-order Hamiltonian-type operator H. We note two primary differences, however: first, in connecting the problem (1.4) with H we must consider the map $(-\Delta)^{1/2}$ and its inverse, and this is generally quite delicate in unbounded domains. For example, we note in Section 3 that for $\phi \in C_c^\infty(\mathbb{R}^2)$ the Hardy–Littlewood–Sobolev estimate does not imply $(-\Delta)^{-1/2}\phi \in L^2(\mathbb{R}^2)$, which would be natural for the analysis. Second, since mass is naturally conserved for Ω bounded (i.e., $\int_\Omega u(x,t)dx = c$ for all $t \geq 0$), the eigenfunctions in that case have zero mass,

$$\int_\Omega \phi(x)dx = 0.$$

In the case $\Omega = \mathbb{R}^d$ the concept of mass is ill-defined, and $L^2(\mathbb{R}^d)$ eigenfunctions of a fixed sign naturally arise. Indeed, it's clear from our analysis that the eigenfunctions associated with the unstable eigenvalues identified here for both radial and saddle solutions have fixed signs. Heuristically, our point of view is that when we consider the problem on $\mathbb{R}^d$ we are effectively taking a region in the bounded-domain problem that is far from the boundary and magnifying it (i.e., regarding it as the inner solution for an appropriate perturbation expansion). From this point of view it isn't natural to restrict our eigenspace to zero-mass eigenfunctions. On the other hand, the question of stability in this restricted question is quite interesting.

Our second interest in this paper is with planar periodic solutions $\bar{u}(x_1)$, and more precisely with perturbations of such solutions that are periodic in x_1 with the same period as $\bar{u}(x_1)$. It is known from [19] that for $d = 1$ $\bar{u}(x)$ is generally unstable as a solution to (1.1), and this instability is inherited by the planar solution $\bar{u}(x_1)$. On the other hand, $\bar{u}(x)$ appears stable to perturbations that have the same period as the wave (this appearance is suggested both by a positive stability index, consistent with stability, and by numerical evidence), and so it is natural to ask whether $\bar{u}(x_1)$ is stable to transverse perturbations; that is, to perturbations with the same period as $\bar{u}(x_1)$ but otherwise relatively general. The natural tools for such an analysis are the standard Floquet theory and the Evans function, the latter of which we adapt from work of Gardner [13, 14, 15] and of Oh and Zumbrun [26, 27]. In keeping with the analysis in [20] of the spectrum associated with planar transition fronts, we work in this case with the more general Cahn–Hilliard equation

$$u_t = \nabla \cdot \{M(u)\nabla(-\kappa\Delta u + F'(u))\}, \tag{1.7}$$

for which we make assumption (H) from above and add the following:

(Hp) $M \in C^2(\mathbb{R})$ and $M(u) \geq M_0 > 0$ for all $u \in [u_1, u_2]$, where u_1 and u_2 are the binodal values.

Upon linearization of (1.7) about about a planar stationary solution $\bar{u}(x_1)$ (assumed to solve $-\kappa\Delta\bar{u} + F'(\bar{u}) = c$ for some constant c), we obtain the linear equation

$$v_t = \tilde{L}v := \nabla \cdot \{M(\bar{u})\nabla(F''(\bar{u})v - \kappa\triangle v)\}, \tag{1.8}$$

with associated eigenvalue problem

$$\tilde{L}\varphi = \lambda\varphi. \tag{1.9}$$

(Our use of $\tilde{}$ serves only to distinguish $\tilde{L}$ from the previously defined linearized operator.) Observing that the coefficients of $\tilde{L}$ depend only on the distinguished variable x_1, we take a Fourier transform in the transverse coordinates $\tilde{x} := (x_2, x_3, ..., x_d)$ (scaling the transform as $(2\pi)^{(1-d)/2}\int_{\mathbb{R}^{d-1}} e^{-i\xi\cdot\tilde{x}}d\tilde{x}$). If we adopt the notation $r = |\xi|^2$, and for simplicity let L_r denote the transformed linear operator, we obtain the transformed eigenvalue problem

$$L_r\phi = \lambda\phi, \tag{1.10}$$

where $L_r = -D_rH_r$, with

$$D_r\phi := -(M(\bar{u})\phi')' + rM(\bar{u})\phi \tag{1.11}$$

and

$$H_r\phi := -\kappa\phi'' + F''(\bar{u})\phi + \kappa r\phi. \tag{1.12}$$

Specializing now to periodic solutions, we proceed by looking for Floquet solutions of the form

$$\phi(x_1) = e^{i\omega x_1}p(x_1), \tag{1.13}$$

where $p(x_1)$ is periodic with the same period as $\bar{u}(x_1)$ and $\omega \in \mathbb{R}$. Generally speaking, we would search for values λ for which there exists some $\omega \in \mathbb{R}$ so that the eigenvalue problem

$$L_\omega p = \lambda p; \quad p^{(k)}(0) = p^{(k)}(X), k = 0, 1, 2, 3, \tag{1.14}$$

where

$$L_\omega := e^{-i\omega x_1} L_r e^{i\omega x_1}. \tag{1.15}$$

has a solution. It is already known, however, from [19] that for a given planar periodic solution $\bar{u}(x_1)$ there exists some $\omega \in \mathbb{R}$ that corresponds with a positive real eigenvalue. Here, rather, we are interested in a specific class of perturbations, those with the same period as $\bar{u}(x_1)$ in x_1. As discussed in [26] (the discussion following Remark 3.3), values $\omega = \frac{2\pi}{n}$, $n = 1, 2, \dots$ correspond with perturbations that have period nX in x_1, where X is the period of $\bar{u}(x_1)$. The case $n = 1$, equivalent to $\omega = 0$, corresponds with perturbations with period X in x_1. In this case, the Floquet eigenvalue problem becomes

$$\begin{aligned} -D_r H_r p &= \lambda p, \\ p^{(k)}(0) &= p^{(k)}(X); \quad k = 0, 1, 2, 3. \end{aligned} \tag{1.16}$$

We will be concerned with the following question: Given that $\bar{u}(x_1)$ is spectrally stable to periodic perturbations in the case $d = 1$, is it spectrally stable to perturbations in $d \geq 2$ that are periodic in the direction x_1? Our answer to this is negative.

Theorem 1.1. *For equation (1.1), under conditions (H) and (Hp), let u_3 and u_4 denote real numbers fixed between the binodal values such that $u_1 < u_3 < u_4 < u_2$, and such that*

$$F'(u_3) > \frac{F(u_4) - F(u_3)}{(u_4 - u_3)} > F'(u_4).$$

Then there exists a planar periodic solution to (1.1) with minimum value u_3 and maximum value u_4. Under the additional assumption $M(u) \geq m_0 > 0$ for all $u \in \mathbb{R}$, this categorizes all possible periodic solutions to (1.1).

Suppose, in addition to the above conditions, that $F^{(4)}(u) > 0$ for all $u \in \mathbb{R}$. Then each of these planar periodic solutions is spectrally unstable in the following senses: (1) (General perturbations) There exists some $\omega \in \mathbb{R}$ so that the eigenvalue problem (1.14) with $r = 0$ has a real positive eigenvalue; and (2) (Periodic perturbations) In the case $\omega = 0$ there exists some value $r \geq 0$ so that the eigenvalue problem (1.16) has a real positive eigenvalue.

Remark 1.1. *We observe that the restriction $F^{(4)}(y) > 0$ is mild in this case and trivially covers, for example, the case in which F is a fourth order polynomial (in double-well form). Also, it will be clear from the analysis in Section 4 that this is not a necessary condition, and functions that fail to satisfy it can be tested with techniques similar to those developed here.*

2 Radial and Saddle Solutions

In this section we review known results regarding the existence and structure of radial and saddle solutions, and we also discuss what is known about the variational stability of such solutions.

2.1 Radial Solutions

We will state an existence result for radial solutions of the PDE

$$-\Delta u + F'(u) = c, \tag{2.1}$$

where c is some constant value. Letting $r = |x|$, we have that radial solutions $\bar{u}(r)$ of (2.1) must satisfy the ODE

$$-\bar{u}'' - \frac{d-1}{r}\bar{u}' = c - F'(u), \tag{2.2}$$

where we recall that d is the dimension of the space. We will look for monotonic solutions to the boundary value problem with $\bar{u}(0) = u_4$ and $\lim_{r\to\infty} \bar{u}(r) = u_3$, for some finite values u_3 and u_4 that remain to be selected. Clearly, if such a solution exists, we must have $c = F'(u_3)$. The following proposition is an immediate consequence of Theorem I.1 from [5]. We recall that u_1 and u_2 are the binodal values and we let α_2 and α_4 denote the spinodal values, which satisfy

$$(\alpha_2, \alpha_4) = \{u : F''(u) < 0\}. \tag{2.3}$$

(These are the same values for α_2 and α_4 specified in (H).)

Proposition 2.1. *Given any value $u_3 \in (u_1, \alpha_2)$, let ζ_0 be the unique value so that $u_3 + \zeta_0 < u_2$ and*

$$F'(u_3) = \frac{F(u_3 + \zeta_0) - F(u_3)}{\zeta_0},$$

and let β be the unique value so that $\beta > u_2$ and

$$F'(u_3) = F'(u_3 + \beta).$$

Then there exists $u_4 \in (\zeta_0, \beta)$ so that there is a monotonic ($\bar{u}'(r) < 0, r > 0$) radial solution $\bar{u} \in C^2(\mathbb{R}_+)$ to the ODE

$$\begin{aligned}
-\bar{u}'' - \frac{d-1}{r}\bar{u}' &= F'(u_3) - F'(\bar{u}), \\
\bar{u}(0) = u_4; &\quad \bar{u}'(0) = 0 \\
\lim_{r\to\infty} \bar{u}(r) &= u_3.
\end{aligned}$$

Moreover, there exist constants $C > 0$ and $\eta > 0$ so that $u(x) = \bar{u}(|x|)$ satisfies

$$|D^\alpha u(x)| \le Ce^{-\eta|x|},$$

for all multiindices $|\alpha| \le 2$.

If we write $v = u - u_3$ and set

$$g(v) := F'(u_3) - F'(u_3 + v), \tag{2.4}$$

then (2.1) becomes

$$\Delta v + g(v) = 0, \tag{2.5}$$

and g satisfies the following assumptions, taken along with labeling from [6]:

(g1) There exists $b > 0$ so that $g(0) = g(b) = 0$, $g(v) < 0$ for $v \in (0, b)$, $g'(0) < 0$ and $g'(b) > 0$.

(g2) There exists $\theta > b$ so that $g(u) > 0$ in $(b, \theta]$ and $\int_0^\theta g(v)dv = 0$.

(g3B) There exists some $c > \theta$ so that $g(v) > 0$ for all $v \in (b, c)$ and $g(v) < 0$ for all $v > c$.

We can conclude, then, from Theorem 5.4 of [6] that the linear operator $H = -\Delta + F''(\bar{u})$ has a simple negative principal eigenvalue with a corresponding eigenfunction ϕ of constant sign. In particular, for this eigenfunction

$$\langle H\phi, \phi\rangle < 0,$$

and so we see that the radial solutions guaranteed by Proposition 2.1 are all variationally unstable in the sense of Definition 1.1.

2.2 Saddle Solutions

In this section we take $F(u)$ to be in standard form, and additionally we require that F be an even function. In this case the binodal values u_1, u_2 satisfy $u_1 = -u_2$, and by a choice of scale we can take $u_2 = 1$. Consequently, this scaled choice of $F'(u)$ satisfies the following conditions from [10]:

(DFP1) $F' \in C^2[0, 1]$ is odd and $F'(0) = F'(\pm 1)$. Also, $F''(0) < 0$ and $F''(1) > 0$.

(DFP2) The mapping $u \mapsto \frac{F'(u)}{u}$ is strictly increasing on $(0, 1)$.

Under these conditions it is shown in [10] that there exists a unique solution $u \in C^2(\mathbb{R}^2)$ of

$$-\Delta u + F'(u) = 0,$$

so that $|u(x)| \leq 1$ for all $x \in \mathbb{R}^2$, and u has the same sign as the product x_1x_2. Such a solution is referred to as a *saddle* solution. It is shown in [16] that the second condition (DFP2) can be dropped, and for convenience we summarize this latter result as Proposition 2.2 (it appears as Proposition 3.1 in [16]).

Proposition 2.2. *Let (DFP1) hold. Then there exists a saddle solution to the equation*

$$-\Delta u + (F'(u) - F'(-1)) = 0,$$

and more precisely

$$\begin{aligned} u(x_1, x_2) &= -u(x_1, -x_2) = -u(-x_1, x_2) \\ sgn(x) &= x_1x_2, \end{aligned} \tag{2.6}$$

for all $x \in \mathbb{R}^2$.

We observe that since $F'(u)$ appears only under differentiation in (1.1) the saddle solutions guaranteed in Proposition 2.2 are certainly saddle solutions for (1.1). Regarding the variational stability of saddle solutions, we have the following result from [30]. (This summarizes Lemmas 3.4 and 3.6 and Corollary 3.5 from [30].)

Proposition 2.3. *Let (DFP1)–(DFP2) hold, and let $\bar{u}(x)$ denote the saddle solution guaranteed by Proposition 2.2. Then the operator $H = -\Delta + F''(\bar{u})$ has a strictly negative eigenvalue with corresponding eigenfunction $\phi(x)$ of constant sign. Moreover, there exist constants $C > 0$ and $\eta > 0$ so that*

$$|D^\alpha \phi(x)| \leq C e^{-\eta |x|},$$

for $|\alpha| \leq 2$ and all $x \in \mathbb{R}^2$.

We note that it follows immediately from Proposition 2.3 that under the conditions (DFP1)–(DFP2) the saddle solutions guaranteed by Proposition 2.2 are variationally unstable in the sense of Definition 1.1.

3 Instability of Radial and Saddle Solutions

In order to work with a self-adjoint operator, we will proceed by considering the eigenvalue problem

$$\mathcal{L}\varphi := (-\Delta)^{1/2} H (-\Delta)^{1/2} \varphi = -\lambda \varphi. \tag{3.1}$$

where we recall

$$H := -\Delta + F''(\bar{u}). \tag{3.2}$$

We remark at the outset that a much more general analysis along these lines has been carried out in the case of bounded domains by Bates and Fife [3].

Remark 3.1. *We note that if $\varphi \in H^4(\mathbb{R}^d)$ is an eigenfunction for $\mathcal{L}$, with eigenvalue λ then $\varphi \in H^k(\mathbb{R}^d)$ for $k = 5, 6, ...$ by bootstrapping and Sobolev interpolation. Setting, then, $\phi = (-\Delta)^{1/2}\varphi$, we can conclude that ϕ is an $H^4(\mathbb{R}^d)$ eigenfunction for $L = \Delta H$ with eigenvalue λ.*

Lemma 3.1. *Suppose $V(x)$ and all its first and second order partial derivatives are bounded on $\mathbb{R}^d$. Then the operator $\mathcal{T} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined by*

$$\mathcal{T} := (-\Delta)^{1/2}(-\Delta + V(x))(-\Delta)^{1/2}$$

is self-adjoint (densely defined on $H^4(\mathbb{R})$) and bounded below.

Proof. We will apply the Kato–Rellich Theorem (see [28], p. 162), which can be stated in the following form, useful here: *Suppose A is a self-adjoint linear operator on a Hilbert space X, densely defined on $\mathcal{D}(A)$, and that B is a symmetric linear operator on the same Hilbert* *space, densely defined on $\mathcal{D}(B)$. If $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and there exist values $a < 1$ and $b \in \mathbb{R}$ so that*

$$\|B\phi\|_X^2 \leq a\|A\phi\|_X^2 + b\|\phi\|_X^2, \tag{3.3}$$

then the operator $A + B$ is self-adjoint in X, densely defined on $\mathcal{D}(A)$. Moreover, if A is bounded below, then so is $A + B$.

Accordingly, we write

$$\mathcal{T} = (-\Delta)^2 + (-\Delta)^{1/2}V(x)(-\Delta)^{1/2},$$

and make the identifications $A = (-\Delta)^2$ and $B = (-\Delta)^{1/2}V(x)(-\Delta)^{1/2}$. The following facts are straightforward: $(-\Delta)^2 : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is self-adjoint, densely defined on $H^4(\mathbb{R}^d)$, and bounded below. Similarly, $(-\Delta)^{1/2}V(x)(-\Delta)^{1/2}$ is densely defined (in $L^2(\mathbb{R}^d)$) on $H^2(\mathbb{R}^d)$ and symmetric. Since $H^4(\mathbb{R}^d) \subset H^2(\mathbb{R}^d)$, we need only establish (3.3). Toward this, we compute as follows: For $\phi \in H^4(\mathbb{R}^d)$,

$$\begin{aligned}
&\|(-\Delta)^{1/2}V(x)(-\Delta)^{1/2}\phi\|_{L^2}^2 = \langle(-\Delta)^{1/2}V(x)(-\Delta)^{1/2}\phi, (-\Delta)^{1/2}V(x)(-\Delta)^{1/2}\phi\rangle \\
&\quad = \langle V(x)(-\Delta)^{1/2}\phi, (-\Delta)V(x)(-\Delta)^{1/2}\phi\rangle \leq \|V(-\Delta)^{1/2}\phi\|_{L^2}\|(-\Delta)V(x)(-\Delta)^{1/2}\phi\|_{L^2} \\
&\quad \leq \frac{1}{2}\|V(-\Delta)^{1/2}\phi\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)V(x)(-\Delta)^{1/2}\phi\|_{L^2}^2 \\
&\quad = \frac{1}{2}\|V(-\Delta)^{1/2}\phi\|_{L^2}^2 + \frac{1}{2}\|V(-\Delta)^{3/2}\phi + (-\Delta V)(-\Delta)^{1/2}\phi - 2(\nabla V)\cdot\nabla(-\Delta)^{1/2}\phi\|_{L^2}^2 \\
&\quad \leq C_1\|(-\Delta)^{1/2}\phi\|_{L^2}^2 + C_2\|(-\Delta)^{3/2}\phi\|_{L^2}^2 + C_3\||\nabla(-\Delta)^{1/2}\phi|\|_{L^2}^2.
\end{aligned}$$

By Plancherel isometry, we can take a Fourier transform inside each of these L^2 norms to obtain the estimate

$$\|(-\Delta)^{1/2}V(x)(-\Delta)^{1/2}\phi\|_{L^2}^2 \leq C\|(|\xi| + |\xi|^3)\hat{\phi}\|_{L^2}^2. \tag{3.4}$$

Likewise,

$$a\|(-\Delta)^2\phi\|_{L^2}^2 + b\|\phi\|_{L^2}^2 = a\||\xi|^4\hat{\phi}\|_{L^2}^2 + b\|\hat{\phi}\|_{L^2}^2. \tag{3.5}$$

Estimate (3.3) now follows from the observation that for any $\epsilon > 0$ there exists a C_ϵ so that $|\xi| + |\xi|^3 \leq (\epsilon|\xi|^4 + C_\epsilon)$. □

Lemma 3.1 justifies applying the min–max principle (see, for example, Theorem XIII.1 of [29]) to the operator $\mathcal{L}$, so long as $F''(\bar{u}(x))$ is a potential satisfying the assumptions on $V(x)$. For F in the double-well form described in (H) this requires only boundedness for $\bar{u}(x)$ and its first and second derivatives.

Lemma 3.2. *Suppose $V(x)$ is as specified in Lemma 3.1, and additionally that there exists a function $\phi \in C_c^\infty(\mathbb{R}^d)$ so that*

$$\langle H\phi, \phi\rangle < 0, \tag{3.6}$$

where $H = -\Delta + V(x)$. Then for $\mathcal{T} = (-\Delta)^{1/2}H(-\Delta)^{1/2}$, we have

$$\inf_{\varphi \in H^4(\mathbb{R}^d)\setminus\{0\}} \frac{\langle \mathcal{T}\varphi, \varphi\rangle}{\langle\varphi, \varphi\rangle} < 0.$$

Proof. According to Lemma 3.1 the operator $\mathcal{T}$ is self-adjoint on $L^2(\mathbb{R}^d)$ (densely defined on $H^4(\mathbb{R}^d)$), and bounded below. Consequently the min–max principle applies, and we have that the lowest eigenvalue of $\mathcal{T}$ (denoted $-\lambda$) satisfies

$$-\lambda = \inf_{\varphi \in H^4 \setminus \{0\}} \frac{\langle \mathcal{T}\varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} = \inf_{\varphi \in H^4 \setminus \{0\}} \frac{\langle H(-\Delta)^{1/2}\varphi, (-\Delta)^{1/2}\varphi \rangle}{\langle \varphi, \varphi \rangle}. \tag{3.7}$$

We proceed now by setting $\varphi = (-\Delta)^{-1/2}\phi$, where the assumption $\phi \in C_c^\infty(\mathbb{R}^d)$ justifies representation by the Riesz potential

$$\varphi(x) = (-\Delta)^{-1/2}\phi = c_n \int_{\mathbb{R}^d} \frac{\phi(y)}{|x-y|} dy, \tag{3.8}$$

where $c_n = \Gamma(\frac{n-1}{2})/(2\pi^{(n+1)/2})$. (See, for example, [24])

Case (i), $d \geq 3$. In the event that $d \geq 3$, the Hardy–Littlewood–Sobolev estimate (see, for example, [32]) ensures that $\varphi \in L^2(\mathbb{R}^d)$. By construction, $(-\Delta)^{1/2}\varphi \in C_c^\infty(\mathbb{R}^d)$, and so in fact $\varphi \in H^k(\mathbb{R}^d)$ for any $0 \leq k < \infty$. This establishes, then, by direct substitution of φ into the Rayleigh quotient that

$$\inf_{\varphi \in H^4 \setminus \{0\}} \frac{\langle H(-\Delta)^{1/2}\varphi, (-\Delta)^{1/2}\varphi \rangle}{\langle \varphi, \varphi \rangle} = -\gamma, \tag{3.9}$$

for some $\gamma > 0$.

Case (ii), $d = 2$. The case $d = 2$ is complicated by the fact that the Hardy–Littlewood–Sobolev estimate implies only that $(-\Delta)^{-1/2}\phi \in L^p(\mathbb{R}^2)$ for all $p > 2$, which isn't sufficient for the argument used in the case $d \geq 3$. On the other hand, since ϕ has compact support, we have $(-\Delta)^{-1/2}\phi = \mathbf{O}(|x|^{-1})$, as $|x| \to \infty$. In this case, rather than substituting a particular test function into the Rayleigh quotient, we proceed by considering a sequence of functions. To this end, we set $\varphi = (-\Delta)^{-1/2}\phi$ and consider the sequence of $C_c^\infty(\mathbb{R}^2)$ functions $\varphi_j(x) = \rho_j(x)\varphi$, $j = 1, 2, ...$, where $\rho_j(x) = \rho(x/j)$, and $\rho(x) \in C^\infty(\mathbb{R}^2)$ is a standard cut-off function that is 1 for $|x| \leq K$ (some fixed $K > 0$) and is only supported on a disk of radius $2K$ centered at the origin.

We have, then

$$\langle \mathcal{T}\varphi_j, \varphi_j \rangle = \langle (-\Delta)\varphi_j, (-\Delta)\varphi_j \rangle + \langle V(-\Delta)^{1/2}\varphi_j, (-\Delta)^{1/2}\varphi_j \rangle,$$

where the operator $(-\Delta)^{1/2}$ can be understood as a principal value integral

$$(-\Delta)^{1/2}\varphi_j(x) = -\frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} \frac{\varphi_j(x) - \varphi_j(y)}{|x-y|^3} dy.$$

(See, for example, the recent study [9] or the standard reference [23].) Computing directly, we find that

$$\lim_{j \to \infty} \langle \mathcal{T}\varphi_j, \varphi_j \rangle = \langle \mathcal{T}\varphi, \varphi \rangle,$$

and it follows that we can choose an integer J sufficiently large so that $\langle \mathcal{T}\varphi_j, \varphi_j \rangle < 0$ for all $j \geq J$. Since $\varphi_J \in H^4(\mathbb{R}^2)$ this completes the proof. $\square$

Proposition 3.1. *Let (H) hold, and let $\bar{u}(x)$ denote a radial stationary solution as described in Proposition 2.1. If $\bar{u}(x)$ is variationally unstable then it is spectrally unstable to (1.1) with a positive real eigenvalue.*

Proof. It is immediate from Lemma 3.2 and our definition of variational instability that if $\bar{u}(x)$ is variationally unstable then the lowest value in the spectrum of $\mathcal{L}$, denoted $-\lambda$, satisfies

$$-\lambda_1 = \inf_{\varphi \in H^4 \setminus \{0\}} \frac{\langle \mathcal{L}\varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} < 0.$$

It only remains to show that $-\lambda_1$ is an eigenvalue and not the bottom of the essential spectrum.

First, we observe that the essential spectrum of $H = -\Delta + F''(\bar{u}(x))$ is determined in the radial case by the essential spectrum of the $|r| \to \infty$ asymptotic operator $H_\infty = -\Delta + F''(u_3)$, which satisfies $\sigma_{\text{ess}}(H_\infty) \subseteq [F''(u_3), \infty)$, where $F''(u_3) > 0$ for any radial solution as specified in Proposition 2.1. Now let $\mathcal{S}$ denote the subspace spanned by the H^2 eigenfunctions of H and note

$$\inf_{\phi \in \mathcal{S}^\perp \setminus \{0\}} \frac{\langle H\phi, \phi \rangle}{\langle \phi, \phi \rangle} = F''(u_3). \tag{3.10}$$

Denote the eigenfunctions of H ϕ_1, ϕ_2, etc. and note that each ϕ_j is an $H^2(\mathbb{R}^d)$ eigenfunction and so by bootstrapping and Sobolev interpolation is an H^k eigenfunction for $k = 2, 3, \dots$. Setting now $\varphi_j = (-\Delta)^{1/2}\phi_j$ for each j, we consider the Rayleigh quotient

$$\begin{aligned}
\sup_{\{\psi_1, \psi_2, \dots\} \in H^4} \inf_{\substack{\varphi \in H^4 \setminus \{0\} \\ \langle \varphi, \psi_j \rangle = 0 \forall j}} \frac{\langle \mathcal{L}\varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} &\geq \inf_{\substack{\varphi \in H^4 \setminus \{0\} \\ \langle \varphi, (-\Delta)^{1/2}\phi_j \rangle = 0 \forall j}} \frac{\langle \mathcal{L}\varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} \\
&= \inf_{\substack{\varphi \in H^4 \setminus \{0\} \\ \langle (-\Delta)^{1/2}\varphi, \phi_j \rangle = 0 \forall j}} \frac{\langle \mathcal{L}\varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} = \inf_{\substack{\varphi \in H^4 \setminus \{0\} \\ \langle (-\Delta)^{1/2}\varphi, \phi_j \rangle = 0 \forall j}} \frac{\langle H(-\Delta)^{1/2}\varphi, (-\Delta)^{1/2}\varphi \rangle}{\langle \varphi, \varphi \rangle} > 0,
\end{aligned} \tag{3.11}$$

where this final inequality follows from (3.10) and the observation that $(-\Delta)^{1/2}\phi \in \mathcal{S}^\perp$. □

Proposition 3.2. *For $x \in \mathbb{R}^2$, let $\bar{u}(x)$ be a saddle solution of (1.1)–(H) as described in Proposition 2.2. Then $\bar{u}(x)$ is spectrally unstable as a solution of (1.1) with a positive real eigenvalue.*

Proof. The proof is almost identical to that of Proposition 3.1, and we remark only on the difference, which involves the essential spectrum of $H = -\Delta + F''(\bar{u})$. In the case of saddle solutions Schatzman has shown that

$$\sigma_{\text{ess}}(H) \subseteq [0, +\infty)$$

(see Corollary 3.3 of [30]; more precisely, Schatzman's analysis is carried out under less general conditions than those described above from [31], but this part of Schatzman's argument generalizes immediately). In light of this the final inequality in (3.11) is not strict in this case. □

4 Planar Periodic Solutions

In this section we consider the case of planar periodic solutions, and more precisely the following question: Given that a periodic solution $\bar{u}(x)$ is spectrally stable in $d = 1$ to perturbations with the same period as $\bar{u}(x)$, is the corresponding planar periodic solution $\bar{u}(x_1)$ (taken now as a solution to (1.1) with $d \geq 2$) stable to perturbations that are periodic in x_1 with the same period as $\bar{u}(x_1)$ but otherwise relatively general. This analysis follows closely the development of [19, 20]; in particular, [19] specifies a number of references from which ideas were taken.

We note at the outset that the planar periodic waves $\bar{u}(x_1)$ of Theorem 1.1 satisfy the ODE

$$\kappa \bar{u}'' = F'(\bar{u}) - \frac{F(u_4) - F(u_3)}{u_4 - u_3}, \tag{4.1}$$

and that without loss of generality we will shift $\bar{u}(x_1)$ so that $\bar{u}''(0) = 0$. This clearly ensures

$$F'(\bar{u}(0)) = \frac{F(u_4) - F(u_3)}{u_4 - u_3}, \tag{4.2}$$

where we recall from Theorem 1.1 that u_3 is the minimum value obtained by $\bar{u}(x_1)$ and u_4 is the maximum value obtained by $\bar{u}(x_1)$.

Following [19] (and the references cited there), we construct the monodromy (or Floquet) matrix

$$M(\lambda; X) = \begin{pmatrix} \phi_1(X;\lambda) & \phi_2(X;\lambda) & \phi_3(X;\lambda) & \phi_4(X;\lambda) \\ (b\phi_1)'(X;\lambda) & (b\phi_2)'(X;\lambda) & (b\phi_3)'(X;\lambda) & (b\phi_4)'(X;\lambda) \\ \phi_1''(X;\lambda) & \phi_2''(X;\lambda) & \phi_3''(X;\lambda) & \phi_4''(X;\lambda) \\ \phi_1'''(X;\lambda) & \phi_2'''(X;\lambda) & \phi_3'''(X;\lambda) & \phi_4'''(X;\lambda) \end{pmatrix}, \tag{4.3}$$

where the $\{\phi_j\}_{j=1}^4$ form a basis of solutions to (1.16) (dependence on r is suppressed for notational brevity), initialized by $\phi_j^{(k-1)}(0;\lambda) = \delta_j^k$ for $k = 1, 3, 4$, with $(b\phi_j)'(0;\lambda) = \delta_2^j$, where $b(x_1) := M(\bar{u}(x_1))F''(\bar{u}(x_1))$ and δ_j^k denotes a standard Kronecker delta function. As discussed in [19] eigenvalues of the operator $-D_r H_r$ (see (1.11) and (1.12)) correspond with zeros of the Evans function

$$E(\lambda, r) = \det(M(\lambda, X) - I) = \det \begin{pmatrix} [\phi_1] & [\phi_2] & [\phi_3] & [\phi_4] \\ [(b\phi_1)'] & [(b\phi_2)'] & [(b\phi_3)'] & [(b\phi_4)'] \\ [\phi_1''] & [\phi_2''] & [\phi_3''] & [\phi_4''] \\ [\phi_1'''] & [\phi_2'''] & [\phi_3'''] & [\phi_4'''] \end{pmatrix}, \tag{4.4}$$

where our notation is $[f] = f(X) - f(0)$. Upon integration of (1.16) over the interval $[0, X]$, and rearrangement of terms, we obtain

$$[{\phi_k}'''] = \frac{1}{\kappa}[(b\phi_k)'] + r[{\phi_k}'] - \frac{\lambda}{\kappa M_0}\int_0^X \phi_k dx_1 - \frac{r}{\kappa M_0}\int_0^X M(\bar{u}) H_r \phi_k dx_1, \tag{4.5}$$

where $M_0 = M(\bar{u}(X)) = M(\bar{u}(0))$. Substituting (4.5) into (4.4) for each k we obtain

$$E(\lambda, r) = -\frac{1}{\kappa M_0} \det \begin{pmatrix} [\phi_1] & [\phi_2] & [\phi_3] & [\phi_4] \\ [(b\phi_1)'] & [(b\phi_2)'] & [(b\phi_3)'] & [(b\phi_4)'] \\ [\phi_1''] & [\phi_2''] & [\phi_3''] & [\phi_4''] \\ \mathcal{I}\phi_1 & \mathcal{I}\phi_2 & \mathcal{I}\phi_3 & \mathcal{I}\phi_4 \end{pmatrix}, \tag{4.6}$$

where for bevity of notation we have defined the integral operator

$$\mathcal{I}(\lambda, r)\phi_k = \lambda \int_0^X \phi_k dx_1 + r \int_0^X M(\bar{u}) H_r \phi_k dx_1. \tag{4.7}$$

In what follows, we will understand the spectrum of $-D_r H_r$ (for sufficiently small values of $|\lambda|$ and $|r|$) through careful consideration of the zeros of $E(\lambda, r)$. More precisely, we proceed by constructing terms in the Taylor series

$$E(\lambda, r) = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} (D^\alpha E)(0,0) \lambda^{\alpha_1} r^{\alpha_2}, \tag{4.8}$$

where α denotes a standard multiindex for $d = 2$. In what follows we will find that $E(0,0) = E_\lambda(0,0) = E_r(0,0) = 0$, but that $E_{\lambda\lambda}(0,0)$ and $E_{rr}(0,0)$ are generally not zero, so that for sufficiently small values of $|\lambda|$ and r, $E(\lambda, r)$ is determined to order by the polynomial

$$P(\lambda, r) = \frac{1}{2} E_{\lambda\lambda}(0,0)\lambda^2 + E_{\lambda r}(0,0)\lambda r + \frac{1}{2} E_{rr}(0,0) r^2, \tag{4.9}$$

with zeros satisfying

$$\lambda = \frac{-E_{\lambda r} r \pm \sqrt{(E_{\lambda r})^2 r^2 - E_{\lambda\lambda} E_{rr} r^2}}{E_{\lambda\lambda}}, \tag{4.10}$$

where each derivative is evaluated at $(\lambda, r) = (0,0)$. We see immediately from (4.10) that we will have an eigenvalue with positive real part so long as $E_{\lambda\lambda}(0,0)E_{rr}(0,0) < 0$. That is, in this case, we can fix a sufficiently small value of r and show that as λ increases from 0 $E(\lambda, r)$ changes sign.

Before stating a lemma regarding the values of these second derivatives, we review some observations made in [19] in the case $d = 1$ (which corresponds here to $r = 0$). First, for $(\lambda, r) = (0,0)$ we have the second-order equations for the ϕ_k,

$$\begin{aligned}
&\kappa\phi_1'' - b(x_1)\phi_1 = -b(0); \quad \phi_1(0) = 1, (b\phi_1)'(0) = 0 \\
&\kappa\phi_2'' - b(x_1)\phi_2 = -M_0 \int_0^{x_1} \frac{dy}{M(\bar{u}(y))}; \quad \phi_2(0) = 0, (b\phi_2)'(0) = 1 \\
&\kappa\phi_3'' - b(x_1)\phi_3 = \kappa; \quad \phi_3(0) = 0, (b\phi_3)'(0) = 0 \\
&\kappa\phi_4'' - b(x_1)\phi_4 = \kappa M_0 \int_0^{x_1} \frac{dy}{M(\bar{u}(y))}; \quad \phi_4(0) = 0, (b\phi_4)'(0) = 0.
\end{aligned} \tag{4.11}$$

In addition to these relations, our analysis will make use of two important combinations, $m(x_1) := \kappa\phi_1(x_1;0) + b(0)\phi_3(x_1;0)$ and $w(x_1) = \kappa\phi_2(x_1;0) + \phi_4(x_1;0)$, which respectively satisfy

$$\begin{aligned} \kappa m'' - b(x_1)m = 0; &\quad m(0) = \kappa, m'(0) = -\kappa\frac{b'(0)}{b(0)} \\ \kappa w'' - b(x_1)w = 0; &\quad w(0) = 0, w'(0) = \frac{\kappa}{b(0)}. \end{aligned} \tag{4.12}$$

By a standard variation of parameters representation, we can now understand each of the ϕ_k in terms of two linearly independent solutions to the homogeneous problem

$$\kappa\phi'' - b(x_1)\phi = 0. \tag{4.13}$$

As is clear from (4.1), one solution to this equation is $\bar{u}'(x_1)$, while the second can be written in terms of $\bar{u}'(x_1)$ by reduction of order:

$$\psi(x_1) = \begin{cases} \bar{u}'(x_1)\int_0^{x_1}\frac{dy}{\bar{u}'(y)^2} & 0 \le x_1 \le X_1 \\ \bar{u}'(x_1)\int_{2X_1}^{x}\frac{dy}{\bar{u}'(y)^2} + \frac{2K_1}{\bar{u}''(X_1)}\bar{u}'(x_1) & X_1 \le x_1 \le X_2 \\ -\bar{u}'(x_1)\int_{x_1}^{X}\frac{dy}{\bar{u}'(y)^2} + 2\bar{u}'(x_1)\Big(\frac{\bar{u}''(X_1)K_2 - \bar{u}''(X_2)K_1}{\bar{u}''(X_1)\bar{u}''(X_2)}\Big) & X_2 \le x_1 \le X \end{cases}, \tag{4.14}$$

where $X_1 < X_2$ denote the two values for which $\bar{u}'(X_k) = 0$, and

$$\begin{aligned} K_1 &:= \frac{1}{\bar{u}'(0)} + \int_0^{X_1}\frac{\bar{u}''(X_1) - \bar{u}''(x)}{\bar{u}'(x)^2}dx \\ K_2 &:= -\frac{1}{\bar{u}'(0)} + \int_{2X_1}^{X_2}\frac{\bar{u}''(X_2) - \bar{u}''(x)}{\bar{u}'(x)^2}dx + 2K_1\frac{\bar{u}''(X_2)}{\bar{u}''(X_1)}, \end{aligned}$$

both of which are well defined. We note in particular, that $\bar{u}'(x_1)$ and $\psi(x_1)$ are the solutions of (4.13) with initial conditions $\bar{u}'(0) > 0, \bar{u}''(0) = 0$ (the second by our choice of shift) and $\psi(0) = 0, \psi'(0) = 1/\bar{u}'(0) > 0$, and consequently $W(\bar{u}'(x_1), \psi(x_1)) \equiv 1$, where W denotes a standard Wronskian.

With this notation established we can state our lemma regarding derivatives of $E(\lambda, r)$, evaluated at $(0,0)$.

Lemma 4.1. *For $E(\lambda, r)$ as specified in (4.4) we have*

$$\begin{aligned} &E(0,0) = E_\lambda(0,0) = E_r(0,0) = 0 \\ &sgnE_{\lambda\lambda}(0,0) = -sgn\Big[\psi(X)\int_0^X \psi(y)(\bar{u}(y) - \bar{u}(0))dy + \frac{\bar{u}'(X)}{2}\Big(\int_0^X \psi(y)dy\Big)^2\Big] \\ &sgnE_{rr}(0,0) = sgn\psi(X). \end{aligned} \tag{4.15}$$

Proof. First, by variation of parameters, we have

$$\phi_3(x_1;0) = -\bar{u}'(x_1)\int_0^{x_1}\psi(y)dy + \psi(x_1)(\bar{u}(x_1)-\bar{u}(0)), \tag{4.16}$$

from which differentiation reveals the useful relation $[\phi_3'] = 0$. Likewise, we can show

$$[\phi_2'] = \frac{\psi'(X)M_0}{\kappa}\int_0^X \frac{\bar{u}(y)-\bar{u}(0)}{M(\bar{u}(y))}dy. \tag{4.17}$$

Additionally, we observe that $w(x_1) = \frac{\kappa\bar{u}'(0)}{b(0)}\psi(x_1)$, from which we find $[w'] = 0$. (To verify the relation $w(x_1) = \frac{\kappa\bar{u}'(0)}{b(0)}\psi(x_1)$ compare the IVP solved by $w(x_1)$ with the IVP solved by $\psi(x_1)$.) Since $m(x_1)$ and $w(x_1)$ constitute a complete basis for (4.13), we have

$$\begin{aligned}\bar{u}'(x_1) &= \frac{\bar{u}'(0)}{\kappa}m(x_1) + \frac{b'(0)\bar{u}'(0)}{\kappa}w(x_1)\\ &= \bar{u}'(0)\phi_1(x_1;0) + b'(0)\bar{u}'(0)\phi_2(x_1;0) + \frac{b(0)}{\kappa}\bar{u}'(0)\phi_3(x_1;0) + \frac{b'(0)\bar{u}'(0)}{\kappa}\phi_4(x_1;0).\end{aligned} \tag{4.18}$$

We can immediately conclude the linear dependencies

$$\begin{aligned}&\int\phi_1 + b'(0)\int\phi_2 + \frac{b(0)}{\kappa}\int\phi_3 + \frac{b'(0)}{\kappa}\int\phi_4 = 0\\ &[\phi_1^{(k)}] + b'(0)[\phi_2^{(k)}] + \frac{b(0)}{\kappa}[\phi_3^{(k)}] + \frac{b'(0)}{\kappa}[\phi_4^{(k)}] = 0,\end{aligned} \tag{4.19}$$

for differentiation up to any order $k = 0, 1, 2, ...$ For notational brevity we have adopted the convention of [26, 27],

$$\int f := \int_0^X f(y)dy.$$

We are now in a position to begin evaluating the $(D^\alpha E)(0,0)$. First, since $\mathcal{I}(0,0) = 0$, we clearly have $E(0,0) = 0$. Upon setting $r = 0$ in (4.6) we reduce the problem to the $d = 1$ analysis considered in [19], from which we have $E_\lambda(0,0) = 0$ and

$$\begin{aligned}E_{\lambda\lambda}(0,0) &= 2\frac{b(X)\psi'(X)}{\bar{u}'(0)\kappa^3}\det\begin{pmatrix}\int\phi_3 & \int w\\ [\phi_3] & [w]\end{pmatrix}\\ &\times\Big[\Big(\int_0^X\frac{\bar{u}(x)-\bar{u}(0)}{M(\bar{u}(x))}dx\Big)^2 - \int_0^X\frac{dx}{M(\bar{u}(x))}\int_0^X\frac{(\bar{u}(x)-\bar{u}(0))^2}{M(\bar{u}(x))}dx\Big].\end{aligned} \tag{4.20}$$

Here, $b(X) < 0$, $\psi'(X) = 1/\bar{u}'(X) > 0$, and an application of the Cauchy–Schwartz inequality establishes that the quantity in square brackets is negative. We have, then, that

$$\mathrm{sgn}E_{\lambda\lambda}(0,0) = \mathrm{sgn}\det\begin{pmatrix}\int\phi_3 & \int w\\ [\phi_3] & [w]\end{pmatrix}.$$

According to (4.16) and the relation $w(X) = \frac{\kappa \bar{u}'(0)}{b(0)} \psi(X)$, we find

$$\operatorname{sgn} E_{\lambda\lambda}(0,0) = -\operatorname{sgn}\Big[\psi(X) \int_0^X \psi(y)(\bar{u}(y) - \bar{u}(0))dy + \frac{\bar{u}'(X)}{2}\Big(\int_0^X \psi(y)dy\Big)^2\Big].$$

Next, we set $\lambda = 0$ in (4.6) and focus on $E(0,r)$. For the first derivative, we find

$$E_r(0,0) = -\frac{1}{\kappa M_0} \det \begin{pmatrix} [\phi_1] & [\phi_2] & [\phi_3] & [\phi_4] \\ [(b\phi_1)'] & [(b\phi_2)'] & [(b\phi_3)'] & [(b\phi_4)'] \\ [\phi_1''] & [\phi_2''] & [\phi_3''] & [\phi_4''] \\ \int MH_0\phi_1 & \int MH_0\phi_2 & \int MH_0\phi_3 & \int MH_0\phi_4 \end{pmatrix}. \tag{4.21}$$

Observing that $H_0\bar{u}'(x_1) = 0$, we see from (4.18) that

$$\int MH_0\phi_1 + b'(0)\int MH_0\phi_2 + \frac{b(0)}{\kappa}\int MH_0\phi_3 + \frac{b'(0)}{\kappa}\int MH_0\phi_4 = 0,$$

from which we immediately conclude $E_r(0,0) = 0$.

We proceed now with the calculation of $E_{rr}(0,0)$. We have

$$E_{rr}(0,0) = -\frac{2}{\kappa M_0}\frac{\partial}{\partial r} \det \begin{pmatrix} [\phi_1] & [\phi_2] & [\phi_3] & [\phi_4] \\ [(b\phi_1)'] & [(b\phi_2)'] & [(b\phi_3)'] & [(b\phi_4)'] \\ [\phi_1''] & [\phi_2''] & [\phi_3''] & [\phi_4''] \\ \int MH_r\phi_1 & \int MH_r\phi_2 & \int MH_r\phi_3 & \int MH_r\phi_4 \end{pmatrix}\Big|_{r=0}. \tag{4.22}$$

The determinant derivative can be expanded as a sum of four derivatives, each with a derivative on exactly one column. If we then use (4.19), to replace all occurrences of ϕ_1 except those differentiated with respect to r, we find

$$E_{rr}(0,0) = \frac{2}{\kappa M_0 \bar{u}'(0)} \det \begin{pmatrix} \int M(H_0q + \kappa\bar{u}') & \int MH_0\phi_2 & \int MH_0\phi_3 & \int MH_0\phi_4 \\ [q] & [\phi_2] & [\phi_3] & [\phi_4] \\ [(bq)'] & [(b\phi_2)'] & [(b\phi_3)'] & [(b\phi_4)'] \\ [q''] & [\phi_2''] & [\phi_3''] & [\phi_4''] \end{pmatrix}, \tag{4.23}$$

where

$$q(x_1) = \bar{u}'(0)\partial_r\phi_1 + b'(0)\bar{u}'(0)\partial_r\phi_2 + \frac{b(0)\bar{u}'(0)}{\kappa}\partial_r\phi_3 + \frac{b'(0)\bar{u}'(0)}{\kappa}\partial_r\phi_4, \tag{4.24}$$

where $\partial_r\phi_k = (\partial_r\phi_k)(x_1;0,0)$. In order to understand $q(x_1)$, we differentiate the equation $-D_rH_r\phi_k = 0$ with respect to r and evaluate the result at $r = 0$ to obtain

$$(M(\bar{u})(H_0\partial_r\phi_k)')' = -\kappa(M(\bar{u})\phi_k')'M(\bar{u})H_0\phi_k.$$

If we now sum over k we find

$$\begin{aligned} -D_0H_0q &= -\kappa(M(\bar{u})\bar{u}'')', \\ q^{(k)}(0) &= 0; \quad k = 0,1,2,3, \end{aligned} \tag{4.25}$$

where the initial condition arises because the initial conditions of the ϕ_k are independent of r. Integrating (4.25), we can add to (4.11) the relation

$$-\kappa q'' + b(x_1)q = -\kappa(\bar{u}'(x_1) - \bar{u}'(0)), \tag{4.26}$$

and the resulting statement $q''(X) = \frac{1}{\kappa}b(X)q$. Combining (4.26) with (4.23) and (4.11) we obtain

$$E_{rr}(0,0) = \frac{2}{\kappa M_0 \bar{u}'(0)} \det \begin{pmatrix} \int M(H_0 q + \kappa \bar{u}') & \int M H_0 \phi_2 & \int M H_0 \phi_3 & \int M H_0 \phi_4 \\ [q] & [\phi_2] & [\phi_3] & [\phi_4] \\ [(bq)'] & [(b\phi_2)'] & [(b\phi_3)'] & [(b\phi_4)'] \\ 0 & -\frac{M_0}{\kappa}\int_0^X \frac{dx_1}{M(\bar{u}(x_1))} & 0 & M_0 \int_0^X \frac{dx_1}{M(\bar{u}(x_1))} \end{pmatrix}, \tag{4.27}$$

which can be expanded as

$$E_{rr}(0,0) = \frac{2}{\kappa^2 \bar{u}'(0)} \int_0^X \frac{dx_1}{M(\bar{u}(x_1))} \det \begin{pmatrix} \int M(H_0 q + \kappa \bar{u}') & \int M H_0 w & \int M H_0 \phi_3 \\ [q] & [w] & [\phi_3] \\ b(X)[q'] & b(X)[w'] & b(X)[\phi_3'] \end{pmatrix}. \tag{4.28}$$

We now find appropriate expressions for $q'(X)$, $w'(X)$, and $\phi_3'(X)$. For q, we solve (4.26) by variation of parameters to obtain

$$q(x_1) = -\bar{u}'(x_1)\int_0^{x_1} \psi(y)(\bar{u}'(y) - \bar{u}'(0))dy + \psi(x_1)\int_0^{x_1} \bar{u}'(y)(\bar{u}'(y) - \bar{u}'(0))dy, \tag{4.29}$$

so that

$$q'(X) = \psi'(X)\int_0^X \bar{u}'(y)^2 dy,$$

where we have used the choice $\bar{u}''(X) = 0$. Likewise, we have $[w'] = 0$, and according to (4.16) $\phi_3'(X) = 0$. Combining these observations, we conclude

$$\begin{aligned} E_{rr}(0,0) &= \frac{2}{\kappa^2 \bar{u}'(0)} \int_0^X \frac{dx_1}{M(\bar{u}(x_1))} \det \begin{pmatrix} \int M(H_0 q + \kappa \bar{u}'(x_1)) & \int M H_0 w & \int M H_0 \phi_3 \\ [q] & [w] & [\phi_3] \\ b(X)\psi'(X)\int_0^X \bar{u}'(y)^2 dy & 0 & 0 \end{pmatrix} \\ &= \frac{2b(X)\psi'(X)}{\kappa^2 \bar{u}'(0)} \int_0^X \frac{dy}{M(\bar{u}(y))} \int_0^X \bar{u}'(y)^2 dy \det \begin{pmatrix} \int M H_0 w & \int M H_0 \phi_3 \\ w(X) & \phi_3(X) \end{pmatrix}. \end{aligned} \tag{4.30}$$

In order to simplify this determinant, we recall from (4.11) that $H_0\phi_3 = -\kappa$ and $H_0 w = 0$. Finally, from (4.16) and the following discussion $\phi_3(X) = -\bar{u}'(X)\int_0^X \psi(y)dy$, while $w(X) = \frac{\kappa \bar{u}'(0)}{b(X)}\psi(X)$. In summary, we have

$$E_{rr}(0,0) = 2\psi'(X)\psi(X)\int_0^X \frac{dy}{M(\bar{u}(y))} \int_0^X \bar{u}'(y)^2 dy \int_0^X M(\bar{u}(y))dy. \tag{4.31}$$

Recalling now that $\psi'(X) = 1/\bar{u}'(X) > 0$, we find

$$\text{sgn}E_{rr}(0,0) = \text{sgn}\psi(X). \tag{4.32}$$

This completes the proof of Lemma 4.1. □

Remark 4.1. *It is clear from the relation*

$$sgnE_{rr}(0,0) = sgn\psi(X)$$

that for $E_{\lambda\lambda}(0,0) > 0$ stability is determined by the sign of $\psi(X)$, where ψ solves the ODE

$$\kappa\psi'' - b\psi = 0; \quad \psi(0) = 0, \psi'(0) = 1/\bar{u}'(0) > 0.$$

This condition can readily be checked numerically.

Next, working under the additional restriction $F^{(4)}(y) > 0$ for all $y \in [u_1, u_2]$ we establish a sign on the term $\int_0^X \psi(x)(\bar{u}(x) - \bar{u}(0))dx$.

Lemma 4.2. *Under the assumptions of Theorem 1.1, and under the additional restriction $F^{(4)}(y) > 0$, we have*

$$\int_0^X \psi(x)(\bar{u}(x) - \bar{u}(0))dx > 0. \tag{4.33}$$

Proof. Proceeding by a direct calculation using (4.14), and taking advantage of the symmetry of $\bar{u}(x_1)$ around X_1 for $x_1 \in [0, 2X_1]$ and around X_2 for $x_1 \in [2X_1, X]$, we find

$$\begin{aligned}
\int_0^X \psi(x)(\bar{u}(x) - \bar{u}(0))dx = &\int_0^{X_1} \frac{(\bar{u}(X_1) - \bar{u}(0))^2 - (\bar{u}(x) - \bar{u}(0))^2}{\bar{u}'(x)^2} dx \\
&+ \int_{2X_1}^{X_2} \frac{(\bar{u}(X_2) - \bar{u}(0))^2 - (\bar{u}(x) - \bar{u}(0))^2}{\bar{u}'(x)^2} dx \\
&+ c_1 \frac{(\bar{u}(X_2) - \bar{u}(0))^2 - (\bar{u}(X_1) - \bar{u}(0))^2}{2} + c_2 \frac{(\bar{u}(X_2) - \bar{u}(0))^2}{2},
\end{aligned} \tag{4.34}$$

where c_1 and c_2 are given by

$$c_1 = \frac{2K_1}{\bar{u}''(X_1)}; \quad c_2 = c_1 - \frac{2K_2}{\bar{u}''(X_2)}. \tag{4.35}$$

For notational brevity, we define

$$G(y) := (F(y) - \frac{[F]}{[u]} y) - (F(u_2) - \frac{[F]}{[u]} u_2), \tag{4.36}$$

for which we observe the important relations $G'(u_0) = 0$ (from (4.2), with $\bar{u}(0) = u_0$), $G'(u_2) < 0 < G'(u_1)$ (from the inequality in Theorem 1.1), $\bar{u}''(x_1) = \kappa^{-1}G'(\bar{u})$, and $\bar{u}'(x_1)^2 = \frac{2}{\kappa}G(\bar{u})$ (the final two both from (4.1)).

In this new notation, K_1 and K_2 become

$$K_1 = \sqrt{\frac{\kappa}{2}}\Big[\frac{1}{\sqrt{G(u_0)}} + \frac{1}{2}\int_{u_0}^{u_2}\frac{G'(u_2) - G'(y)}{G(y)^{3/2}}dy\Big], \tag{4.37}$$

and

$$K_2 = \sqrt{\frac{\kappa}{2}}\Big[-\frac{1}{\sqrt{G(u_0)}} + \frac{1}{2}\int_{u_1}^{u_0}\frac{G'(u_1) - G'(y)}{G(y)^{3/2}}dy\Big] + 2K_1\frac{\bar{u}''(X_2)}{\bar{u}''(X_1)}. \tag{4.38}$$

Upon substitution of these expressions into (4.34), we obtain

$$\begin{aligned}
&\int_0^X \psi(x)(\bar{u}(x) - \bar{u}(0))dx \\
&\quad = \Big(\frac{\kappa}{2}\Big)^{3/2}\Big\{\int_{u_1}^{u_0}\frac{(u_1-u_0)^2 - (y-u_0)^2}{G(y)^{3/2}} + \int_{u_0}^{u_2}\frac{(u_2-u_0)^2 - (y-u_0)^2}{G(y)^{3/2}}\Big\} \\
&\quad - \frac{\kappa^{3/2}(u_2-u_0)^2}{\sqrt{2}G'(u_2)}\Big[\frac{1}{\sqrt{G(u_0)}} + \frac{1}{2}\int_{u_0}^{u_2}\frac{G'(u_2) - G'(y)}{G(y)^{3/2}}dy\Big] \\
&\quad - \frac{\kappa^{3/2}(u_1-u_0)^2}{\sqrt{2}G'(u_1)}\Big[-\frac{1}{\sqrt{G(u_0)}} + \frac{1}{2}\int_{u_1}^{u_0}\frac{G'(u_1) - G'(y)}{G(y)^{3/2}}dy\Big].
\end{aligned} \tag{4.39}$$

Combining the integrals, we find

$$\begin{aligned}
\int_0^X \psi(x)(\bar{u}(x) - \bar{u}(0))dx &= \Big(\frac{\kappa}{2}\Big)^{3/2}\frac{1}{G'(u_1)}\int_{u_1}^{u_0}\frac{G'(y)(u_1-u_0)^2 - G'(u_1)(y-u_0)^2}{G(y)^{3/2}}dy \\
&+ \Big(\frac{\kappa}{2}\Big)^{3/2}\frac{1}{G'(u_2)}\int_{u_0}^{u_2}\frac{G'(y)(u_2-u_0)^2 - G'(u_2)(y-u_0)^2}{G(y)^{3/2}}dy \\
&+ \frac{\kappa^{3/2}}{\sqrt{2G(u_0)}}\Big[\frac{(u_1-u_0)^2}{G'(u_1)} - \frac{(u_2-u_0)^2}{G'(u_2)}\Big].
\end{aligned} \tag{4.40}$$

Recalling the inequality $G'(u_2) < 0 < G'(u_1)$, we see that the summand on the third line of (4.40) is clearly positive. In order to understand the signs of the two integrals, we focus (with a change of sign since $G'(u_2) < 0$) on the numerator of the second

$$\Phi(y) := G'(u_2)(y-u_0)^2 - G'(y)(u_2-u_0)^2, \quad u \in [u_0, u_2],$$

for which we have $\Phi(u_0) = \Phi(u_2) = 0$. Moreover, $\Phi'(u_0) > 0$, and so all that remains to show is that $\Phi(y)$ does not cross 0 for $y \in (u_0, u_2)$. In order to see this, we observe that by our condition $F^{(4)}(y) > 0$, the function

$$\Phi''(y) = 2G'(u_2) - G'''(y)(u_2-u_0)^2$$

has precisely one zero, and so Φ has only one change in concavity. If $\Phi(y)$ has a zero in (u_0, u_2), one concavity change will be required so that Φ returns to 0 at $y = u_2$, while a second will be required since $\Psi(y)$ goes to $-\infty$ as $y \to +\infty$. A similar argument holds for the first integral on the right-hand side of (4.40), and this concludes the proof of Lemma 4.2. □

As discussed in [19], in the event that $\bar{u}(x_1)$ is stable in $d = 1$ to perturbations with the same period as $\bar{u}$, we must have $E_{\lambda\lambda}(0, 0) > 0$. (This follows from an analysis of the stability index for this problem.) We close this section by establishing that if $\bar{u}(x_1)$ is such a wave then it is unstable to perturbations that have the same period as $\bar{u}(x_1)$ in the x_1 direction but are otherwise relatively general.

Proposition 4.1. *Let the assumptions of Lemma 4.2 hold and suppose $E_{\lambda\lambda}(0, 0) > 0$. Then there exists a positive real λ and a function $p(x_1)$ so that p is a solution of (1.16).*

Proof. First, we have seen in the discussion immediately following (4.10) that an eigenvalue with positive real part is guaranteed by the condition $E_{\lambda\lambda}(0, 0)E_{rr}(0, 0) < 0$. In the event that $E_{\lambda\lambda}(0, 0) > 0$, this condition reduces to the requirement that $E_{rr}(0, 0) < 0$, which by Lemma 4.1 reduces to the requirement $\psi(X) < 0$.

On the other hand, we observe from our expression for the sign of $E_{\lambda\lambda}(0, 0)$ in Lemma 4.1 that $E_{\lambda\lambda}(0, 0)$ can only be positive if

$$\psi(X) \int_0^X \psi(y)(\bar{u}(y) - \bar{u}(0))dy < 0. \tag{4.41}$$

In addition, we know from Lemma 4.2 that $\int_0^X \psi(y)(\bar{u}(y) - \bar{u}(0))dy > 0$, and consequently $\psi(X) < 0$.

This completes the proof. □

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