Spectral Analysis for Transition Front Solutions in Multidimensional Cahn-Hilliard Systems

Peter Howard

September 16, 2013

Abstract

We consider the spectrum associated with the linear operator obtained when a Cahn-Hilliard system on $\mathbb{R}^n$ is linearized about a planar transition front solution. In the case of single Cahn-Hilliard equations on $\mathbb{R}^n$, it's known that under general physical conditions the leading eigenvalue moves into the negative real half plane at a rate $|\xi|^3$, where ξ is the Fourier transform variable corresponding with components transverse to the wave. Moveover, it has recently been verified that for single equations this spectral behavior implies nonlinear stability. In the current analysis, we establish that the same cubic rate law holds for a broad range of multidimensional Cahn-Hilliard systems. The analysis of nonlinear stability will be carried out separately.

1 Introduction

We consider Cahn-Hilliard systems on $\mathbb{R}^n$,

$$\frac{\partial u_j}{\partial t} = \nabla \cdot \Big\{ \sum_{k=1}^{m} M_{jk}(u) \nabla \Big((-\Gamma \Delta u)_k + F_{u_k}(u) \Big) \Big\}, \tag{1.1}$$

for $j = 1, 2, \dots, m$. Here, $F : \mathbb{R}^n \to \mathbb{R}$, and Γ and M are $m \times m$ matrices. For notational convenience, we will often use the tensor form

$$u_t = \nabla \cdot \Big\{ M(u) D_x \Big(-\Gamma \Delta u + D_u F \Big) \Big\}, \tag{1.2}$$

where the operator D is a Jacobian operator with respect to the designated variable, as described, for example, in [6], and since the expression in brackets is an $m \times m$ matrix we interpret the divergence as a vector (see, e.g., [7]).

For convenient reference, we collect some assumptions that will be made throughout the analysis.

(H1) (Assumptions on Γ and M) Γ and M denote constant, symmetric, positive definite matrices.

(H2) (Assumptions on F) $F \in C^4(\mathbb{R}^m)$, and F has at least two distinct local minimizers at which $D_u^2 F(u)$ is positive definite and (by subtracting an appropriate hyperplane from F if necessary) we can take F to be zero. We denote this class of values

$$\mathcal{M} := \{u \in \mathbb{R}^m : F(u) = 0, D_u F(u) = 0, D_u^2 F(u) \text{ is positive definite}\},$$

where $D_u^2 F(u)$ denotes Hessian matrix.

(H3) (Transition front existence and structure) There exists a transition front solution to (1.1) $\bar{u} \in C^4(\mathbb{R})$ so that

$$-\Gamma \bar{u}_{xx} + D_u F(\bar{u}) = 0, \tag{1.3}$$

with $\bar{u}(\pm\infty) = u_\pm$, $u_\pm \in \mathcal{M}$.

(H4) (Endstate Assumptions) We set $B_\pm := D_u^2 F(u_\pm)$ (a symmetric, positive definite matrix) and assume one of the following holds: (H4a) the matrices $MB_\pm$ have distinct eigenvalues, as do the matrices $\Gamma^{-1} B_\pm$; *or* (H4b) one or more of these matrices has a repeated eigenvalue, but the solutions μ of

$$\det\Big(-\mu^4 M\Gamma + \mu^2 M B_\pm - \lambda I \Big) = 0$$

can be strictly divided into two cases: if $\mu(0) \neq 0$ then $\mu(\lambda)$ is analytic in λ for $|\lambda|$ sufficiently small, while if $\mu(0) = 0$ $\mu(\lambda)$ can be written as $\mu(\lambda) = \sqrt{\lambda} h(\lambda)$, where h is analytic in λ for $|\lambda|$ sufficiently small.

Regarding (H1), a more natural set of assumptions to make on the matrix M is as follows:

(H1)' Γ is as in (H1); $M \in C^2(\mathbb{R}^m)$; M is uniformly positive definite along the wave; i.e., there exists $\theta > 0$ so that for all $\xi \in \mathbb{R}^m$ and all $x \in \mathbb{R}$ we have

$$\xi^{tr} M(\bar{u}(x)) \xi \geq \theta |\xi|^2;$$

and $M_\pm$ are symmetric. Under this condition the matrix M in (H4) must be replaced with the matrices $M_\pm$, and the condition described in (H4) must hold for both.

In fact, there are only two places in our analysis in which we use the more restrictive (H1), and so we will carry out most of our calculations using only (H1)'. Indeed, our more precise requirement on $\bar{M}$ is that if $G(x_1, y_1; |\xi|)$ denotes the Green's function for the operator

$$D_\xi = -\partial_{x_1} \bar{M}(x_1) \partial_{x_1} + |\xi|^2 \bar{M}(x_1),$$

then

$$\lim_{|\xi| \to 0} |\xi| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle G(x_1, y_1; |\xi|) \bar{u}'(y_1), \bar{u}'(x_1) \rangle dy_1 dx_1 \geq c > 0,$$

for some constant $c > 0$. (Here $\langle \cdot, \cdot \rangle$ denotes standard Euclidean inner product.) This limit is easily verified if $\bar{M} = M =$ constant, since in this case

$$G(x_1, y_1; |\xi|) = \frac{1}{2|\xi|} e^{-|\xi||x_1 - y_1|} M^{-1}.$$

For (H2), we note that standard choices of the bulk free energy density F typically have precisely $m+1$ minimizers, and that the associated minima of F can all be placed at zero by subtracting a supporting hyperplane from F. Since our original system (1.1) is unchanged by this subtraction, we can take it without loss of generality.

Regarding (H3), we note that Alikakos and others have established that transition front solutions arise quite generally as minimizers of the energy

$$E(u) = \int_{-\infty}^{+\infty} F(u) + \frac{1}{2} \langle \Gamma u_{x_1}, u_{x_1} \rangle dx_1. \tag{1.4}$$

(See [1, 2, 25]).

Finally, (H4) is taken from [14], and we use it so that we can directly apply the results developed there. (In the current formulation we've simplified the statement slightly by noting that since Γ, $B_\pm$, and $M_\pm$ are all positive definite the matrices $M_\pm B_\pm$ and $\Gamma^{-1} B_\pm$ will be similar to symmetric matrices, and so $\mathbb{R}^m$ will always be spanned by the associated eigenvectors; e.g. $M_\pm B_\pm$ is similar to $M_\pm^{1/2} B_\pm M_\pm^{1/2}$). As noted in that reference, we can ensure that (H4) holds by taking arbitrarily small perturbations of the matrices M and Γ. Since we expect stability to be insensitive to such perturbations, we view this assumption as purely for technical convenience.

The system (1.1) is a standard model of certain phase separation processes, and its physicality is discussed in detail in [14] and the references cited there. Our interest in this analysis is to describe the spectrum associated with the linear operator obtained upon linearization of (1.1) about $\bar{u}(x_1)$ (more precisely, the spectrum of the operator obtained by taking a Fourier transform of this linear operator in the transverse variables $\tilde{x} = (x_2, x_3, \dots, x_n)$).

In the full nonlinear analysis (carried out elsewhere), we will introduce a shift function $\delta(\tilde{x}, t)$ (to be chosen during the nonlinear analysis), and define a perturbation variable

$$v(x,t) := u(x,t) - \bar{u}(x_1 - \delta(\tilde{x}, t)). \tag{1.5}$$

Upon substitution of (1.5) into (1.1) we obtain the perturbation equation

$$(\partial_t - L)v = (\partial_t - L)(\delta \bar{u}'(x_1)) + \nabla \cdot Q, \tag{1.6}$$

where

$$Lv := \nabla \cdot \Big\{ \bar{M}(x_1) D_x \Big(- \Gamma \Delta v + \bar{B}(x_1) v \Big) \Big\}, \tag{1.7}$$

with

$$\begin{aligned} \bar{M}(x_1) &:= M(\bar{u}(x_1)) \\ \bar{B}(x_1) &:= D_u^2 F(\bar{u}(x_1)), \end{aligned} \tag{1.8}$$

and Q is a collection of nonlinear terms that won't play a role in the current analysis. Here, $D_u^2 F$ denotes Hessian matrix.

The eigenvalue problem for L can be expressed as $L\phi = \lambda\phi$, and we take the Fourier transform of this equation in the transverse variable $\tilde{x}$, using the scaling

$$\hat{\phi}(x_1, \xi) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\tilde{x}\cdot\xi}\phi(x)d\tilde{x}. \tag{1.9}$$

The eigenvalue problem transforms to

$$L_\xi\hat{\phi} = -D_\xi H_\xi\hat{\phi} = \lambda\hat{\phi}, \tag{1.10}$$

where

$$\begin{aligned} D_\xi &:= -\partial_{x_1}\bar{M}(x_1)\partial_{x_1} + |\xi|^2\bar{M}(x_1) \\ H_\xi &:= -\Gamma\partial^2_{x_1x_1} + \bar{B}(x_1) + |\xi|^2\Gamma. \end{aligned} \tag{1.11}$$

We note that under our current assumptions D_ξ and H_ξ are both self-adjoint (though of course L_ξ is not). For convenient reference, we collect here a set of conditions on (1.10) that follow from our assumptions (H1)-(H4).

(C1) Same as (H1).

(C2) $\bar{B} \in C^2(\mathbb{R})$ is symmetric; there exists a constant $\alpha_B > 0$ so that

$$\partial^j_{x_1}(\bar{B}(x_1) - B_\pm) = \mathbf{O}(e^{-\alpha_B|x_1|}), \quad x_1 \to \pm\infty,$$

for $j = 0, 1, 2$; $B_\pm$ are both positive definite matrices. Here, $\mathbf{O}(\cdot)$ denotes standard "big-O" notation.

(C3) Same as (H4).

In the case that (H1) is replaced by (H1)' we have accordingly

(C1)' Γ is as in (C1); $\bar{M} \in C^2(\mathbb{R})$; there exists a constant $\alpha_M > 0$ so that

$$\partial^j_{x_1}(\bar{M}(x_1) - M_\pm) = \mathbf{O}(e^{-\alpha_M|x_1|}), \quad x_1 \to \pm\infty,$$

for $j = 0, 1, 2$; $\bar{M}(x_1)$ is uniformly positive definite on $\mathbb{R}$.

As a test case for clarification of the discussion, we'll consider (1.1) with $m = 2$, $\Gamma = I$, $M(u) = I$, and

$$F(u_1, u_2) = u_1^2u_2^2 + u_1^2(1 - u_1 - u_2)^2 + u_2^2(1 - u_1 - u_2)^2. \tag{1.12}$$

The associated wave $\bar{u}(x_1)$ is depicted in Figure 1 of [14], and it is shown in that reference that this wave is stable as a solution to (1.1) with $n = 1$.

Before stating our main result, we clarify our terminology for the spectrum of L_ξ (which follows [13]; see particularly the appendix to Chapter 5).

Definition 1.1. *We define the point spectrum of L_ξ, denoted $\sigma_{pt}(L_\xi)$, as the set*

$$\sigma_{pt}(L_\xi) = \{\lambda \in \mathbb{C} : L_\xi \phi = \lambda \phi \text{ for some } \phi \in H^2(\mathbb{R})\}.$$

We define the essential spectrum of L_ξ, denoted $\sigma_{ess}(L_\xi)$, as the values in $\mathbb{C}$ that are not in the resolvent set of L_ξ and are not isolated eigenvalues of finite multiplicity.

We note that $\sigma(L_\xi) = \sigma_{pt}(L_\xi) \cup \sigma_{ess}(L_\xi)$, but the sets $\sigma_{pt}(L_\xi)$ and $\sigma_{ess}(L_\xi)$ are not necessarily disjoint. We will see that the spectrum of L_ξ is confined to the real line (though L_ξ is not self-adjoint), and is bounded above. We will refer to the largest (right-most) eigenvalue of L_ξ as its *leading* eigenvalue, and we will denote this eigenvalue $\lambda_*(\xi)$.

Remark 1.1. *Another commonly used definition of the point and essential spectra can be formulated in terms of whether or not $L_\xi - \lambda I$ is a Fredholm operator with index 0. (See, e.g., [22].) For the current analysis, the only difference between these definitions is that Henry's places $\lambda_*(0) = 0$ in both the point and the essential spectrum, while Kapitula and Promislow's definition places it only in essential spectrum.*

The assumptions for the main theorem of this paper are all straightforward, except for a condition associated with the stability of $\bar{u}$ with respect to (1.1) in $\mathbb{R}$. That is, since $\bar{u}$ is a function of only one variable, it can be viewed as a stationary solution for a Cahn-Hilliard system on $\mathbb{R}$,

$$u_t = \Big\{ M(u)(-\Gamma u_{xx} + D_u F)_x \Big\}_x. \tag{1.13}$$

In [14], the authors identify a spectral stability criterion for $\bar{u}$ as a solution of (1.13), and verify that it is satisfied for certain example systems. In [15, 16], the authors establish that this spectral condition is sufficient to imply nonlinear stability for $\bar{u}$ as a solution of (1.13).

Although we will postpone our full discussion of this condition until Section 4, we will denote it $(\mathcal{D})$ in the statement of our theorem, and we note here that it is ultimately a transversality condition in the following sense. When (1.3) (in (H3)) is written as a first order autonomous ODE system, our condition ensures that $\bar{u}$ arises as a transverse connection either from the m-dimensional unstable linearized subspace for u_-, denoted $\mathcal{U}^-$, to the m-dimensional stable linearized subspace for u_+, denoted $\mathcal{S}^+$, or (by isotropy) vice versa. (We recall that since our ambient manifold is $\mathbb{R}^{2m}$, the intersection of $\mathcal{U}^-$ and $\mathcal{S}^+$ is referred to as transverse if at each point of intersection the tangent spaces associated with $\mathcal{U}^-$ and $\mathcal{S}^+$ generate $\mathbb{R}^{2m}$. In particular, in this setting a transverse connection is one in which the the intersection of these two manifolds has dimension 1; i.e., our solution manifold will comprise shifts of $\bar{u}$.)

Theorem 1.1. *Let Assumptions (H1)-(H4) hold, as well as Condition $(\mathcal{D})$, and assume that $\bar{u}$ minimizes the energy (1.4). The spectrum of the operator L_ξ satisfies the following:*

1. The spectrum $\sigma(L_\xi)$ lies entirely on $\mathbb{R}$.

2. The essential spectrum of L_ξ lies in the union of the two intervals

$$(-\infty, -m_\pm b_\pm |\xi|^2 - m_\pm \gamma |\xi|^4],$$

where $m_\pm$, $b_\pm$, and γ respectively denote the smallest eigenvalues of $M_\pm$, $B_\pm$, and Γ.

3. There exists a constant $\theta_0 > 0$ so that the point spectrum of L_ξ is confined to the interval $(-\infty, -\theta_0|\xi|^4]$.

4. There exists a constant $r > 0$ sufficiently small so that for $|\xi| < r$ the leading eigenvalue of L_ξ, denoted $\lambda_*(\xi)$, satisfies

$$\lambda_*(\xi) = -c_3|\xi|^3(1 + \mathbf{o}(|\xi|)),$$

where

$$c_3 = 4\frac{\int_{-\infty}^{+\infty} F(\bar{u}(x_1))dx_1}{\langle \bar{M}^{-1}[u], [u]\rangle} > 0,$$

and $\mathbf{o}(\cdot)$ denotes standard "little-O" notation. Here, $\langle\cdot,\cdot\rangle$ denotes Euclidean inner product. Moreover, for any $0 < |\xi_0| < r$, there exists $0 < r_0 < r$ sufficiently small so that $\lambda_*(\xi)$ is analytic on $|\xi - \xi_0| < r_0$.

5. The constant $r > 0$ from Part 4 can be taken sufficiently small so that there exists a constant $\theta_1 > 0$ so that for $|\xi| < r$ the set $\sigma_{pt}(L_\xi)\backslash\{\lambda_*(\xi)\}$ is confined to the interval $(-\infty, -\theta_1|\xi|^2]$.

The main observations summarized in Theorem 1.1 are as follows: The spectrum of L_ξ lies entirely in the stable (i.e., negative-real) half-plane, and indeed the leading eigenvalue moves into the stable half-plane like $|\xi|^3$. Moreover, the remainder of the spectrum (both point and essential) separates from $\lambda_*(\xi)$ by moving into the stable half-plane at the faster rate $|\xi|^2$ (faster for $|\xi|$ small).

Outline of the paper. In Section 2 we develop some preliminary findings regarding the structure of $\bar{u}$ and the essential spectrum of L_ξ, along with some useful estimates. In Section 3 we combine energy and minimax methods to locate parts of the point spectrum of L_ξ. In Section 4 we connect the current analysis to the analysis of (1.13) in [14, 15, 16], and finally in Section 5 we complete the proof of Theorem 1.1 by establishing our claimed asymptotic expression for $\lambda_*(\xi)$.

2 Preliminaries

We begin by observing that if $\bar{u}(x_1)$ is a solution to (1.1) then it will satisfy

$$\Big(M(\bar{u})(-\Gamma\bar{u}'' + D_u F(\bar{u}))'\Big)' = 0, \tag{2.1}$$

where prime denotes differentiation with respect to x_1. In this way, $\bar{u}$ is also a stationary solution for a related Cahn-Hilliard systems on $\mathbb{R}$, given as (1.13). Upon integrating (2.1) on

$(-\infty, x_1]$ and noting that $\bar{u}'(x_1) \to 0$ as $x_1 \to -\infty$ (and similarly for higher order derivatives; also, of course, as $x_1 \to +\infty$), we obtain

$$M(\bar{u})(-\Gamma\bar{u}'' + D_u F(\bar{u}))' = 0.$$

Since M is invertible, we have

$$-\Gamma\bar{u}'' + D_u F(\bar{u}) = c \tag{2.2}$$

for some (vector) constant c, and finally by (H2) we have $D_u F(u_-) = 0$, giving $c = 0$ (in the limit as $x_1 \to -\infty$). This gives the form expressed in (H3).

If we now take a derivative of (2.2) with respect to x_1 we obtain

$$-\Gamma\bar{u}''' + D_u^2 F(\bar{u})\bar{u}' = 0,$$

which gives the useful relation

$$H_0\bar{u}' = 0. \tag{2.3}$$

We see from (2.3) that $L_0\bar{u}' = 0$, and so certainly $\lambda = 0$ is an eigenvalue of L_0 (well known to arise from shift invariance).

We turn now to some observations regarding the spectrum of L_ξ. We begin by collecting some observations about the operators H_ξ and D_ξ.

Lemma 2.1. *Let assumptions (H1)'-(H3) hold (not necessarily (H1) or (H4)). Then there exists a constant $\theta_M > 0$, depending only on $\bar{M}(x_1)$, and a constant $\theta_\Gamma > 0$ depending on Γ so that for all $\phi \in H^1(\mathbb{R})$, $w \in \mathbb{R}^m$:*

$$\begin{aligned}
\langle D_\xi\phi, \phi\rangle_2 &\geq \theta_M\Big(\|\phi_{x_1}\|^2 + |\xi|^2\|\phi\|^2\Big)\\
\langle \Gamma w, w\rangle &\geq \theta_\Gamma |w|^2\\
\langle H_0\phi, \phi\rangle_2 &\geq 0.
\end{aligned}$$

If, in addition, we assume Condition $(\mathcal{D})$ then there exists a constant $\theta_H > 0$ so that

$$\langle H_0\psi, \psi\rangle_2 \geq \theta_H\|\psi\|^2$$

for all ψ satisfying $\langle \psi, \bar{u}'\rangle_2 = 0$. Here, $\langle\cdot,\cdot\rangle$ denotes standard Euclidean inner product, $\langle\cdot,\cdot\rangle_2$ denotes L^2 inner product, $|\cdot|$ denotes Euclidean norm, and $\|\cdot\|$ denotes L^2 norm.

Proof. The first two inequalities are immediate from the definitions. The third follows directly from assumption (H3), in which it is assumed that $\bar{u}$ is a minimizer of the energy (1.4), and the additional inequality follows from Condition $(\mathcal{D})$, which implies that $\lambda = 0$ is an isolated eigenvalue of H_0 with geometric multiplicity 1. For more details, see [14]. $\square$

We see from the first inequality in Lemma 2.1 that D_ξ is a positive definite self-adjoint operator for all $\xi \in \mathbb{R}^{n-1}\backslash\{0\}$, and it follows that $D_\xi^{1/2}$ is a positive definite self-adjoint operator. Moreover, for $\xi \neq 0$ we have that D_ξ^{-1} is a bounded, positive definite self-adjoint operator. (See, for example, [3].) Indeed, a straightforward calculation based on Green's function techniques can be used to verify that

$$D_\xi^{-1}\phi = \int_{-\infty}^{+\infty} G(x_1, y_1; |\xi|)\phi(y_1)dy_1, \tag{2.4}$$

where

$$|G(x_1, y_1; |\xi|)| \leq \frac{C}{|\xi|} e^{-|\xi||x_1-y_1|}.$$

In the case that $\bar{M}$ is constant, we have the more precise expression

$$G(x_1, y_1; |\xi|) = \frac{1}{2|\xi|} e^{-|\xi||x_1-y_1|} \bar{M}^{-1}. \tag{2.5}$$

For $\xi \neq 0$, we set $\varphi := D_\xi^{-1/2}\phi$, so that $L_\xi \phi = \lambda\phi$ becomes

$$\mathcal{L}_\xi \varphi = -D_\xi^{1/2} H_\xi D_\xi^{1/2} \varphi = \lambda\varphi. \tag{2.6}$$

(For notational convenience, the hat associated with our Fourier transform has been dropped off for this discussion.) Since $\xi \neq 0$ a straightforward bootstrapping argument can be employed to verify that if $\varphi \in H^2(\mathbb{R})$ then φ will be in $H^3(\mathbb{R})$ (and higher regularity spaces as well), and so $\phi \in H^2(\mathbb{R})$. (This is discussed at length in the scalar setting in [4, 12].) In this way, we see that for $\xi \neq 0$ the point spectrum of L_ξ is precisely the same as the point spectrum of $\mathcal{L}_\xi$.

One of the advantages of working with $\mathcal{L}_\xi$ is that it is a self-adjoint operator; this calculation is the analogue in this multidimensional setting of working with the integrated equation for a single equation (see, for example, [11]). Indeed, we see immediately from self-adjointness that the spectrum of L_ξ must be real-valued. Furthermore, it is straightforward to check that $\mathcal{L}_\xi$ is bounded above (see (3.1), below). We can conclude that the quantity

$$\inf_{\substack{\varphi \in H^2 \\ \varphi \neq 0}} \frac{\langle -\mathcal{L}_\xi \varphi, \varphi \rangle_2}{\langle \varphi, \varphi \rangle_2}$$

corresponds either with the largest eigenvalue of $\mathcal{L}_\xi$ or with the boundary of essential spectrum. (Here, and for the remainder of our minimax argument, our primary general reference is [23], and our references for applying the minimax framework in the Cahn-Hilliard setting are [12, 20, 21, 24].) In subsequent calculations, we'll show that this minimum gives a value larger than the upper limit of essential spectrum, and we can conclude the existence of a leading eigenvalue.

For the essential spectrum of L_ξ, we have from [13] (see particularly the appendix to Chapter 5) that it is identified by the asymptotic operators obtained by taking $x_1 \to \pm\infty$. These asymptotic operators are

$$L_\xi^\pm \phi := -M_\pm \Gamma \phi'''' + (M_\pm B_\pm + 2|\xi|^2 M_\pm \Gamma)\phi'' - (|\xi|^2 M_\pm B_\pm + |\xi|^4 M_\pm \Gamma)\phi.$$

The essential spectrum is determined by solutions to the eigenvalue problems $L_\xi^\pm \phi = \lambda\phi$ of the form $\phi(x_1; |\xi|) = e^{ikx_1} v(|\xi|)$, where $v \in \mathbb{R}^m \backslash \{0\}$ does not depend on x_1, and $k \in \mathbb{R}$.

Upon substitution, we obtain the eigenvalue problem

$$\Big\{ - k^4 M_\pm \Gamma - k^2 (M_\pm B_\pm + 2|\xi|^2 M_\pm \Gamma) - (|\xi|^2 M_\pm B_\pm + |\xi|^4 M_\pm \Gamma) \Big\} v = \lambda v.$$

We multiply this equation by $M_\pm^{-1}$ (on the left), and take an inner product with v to find

$$\lambda \langle M_\pm^{-1} v, v\rangle = -k^4 \langle \Gamma v, v\rangle - k^2 \langle B_\pm v, v\rangle - 2k^2 |\xi|^2 \langle \Gamma v, v\rangle - |\xi|^2 \langle B_\pm v, v\rangle - |\xi|^4 \langle \Gamma v, v\rangle.$$

Since the essential spectrum is described by letting k run over $\mathbb{R}$, we see that

$$\begin{aligned} \lambda_{ess} &\le -\frac{|\xi|^2 \langle B_\pm v, v\rangle + |\xi|^4 \langle \Gamma v, v\rangle}{\langle M_\pm^{-1} v, v\rangle} \\ &\le -m_\pm b_\pm |\xi|^2 - m_\pm \gamma |\xi|^4, \end{aligned}$$

where $m_\pm$, $b_\pm$, and γ respectively denote the smallest eigenvalues of $M_\pm$, $B_\pm$, and Γ, and in obtaining this inequality we have used our assumption that $M_\pm$, $B_\pm$, and Γ are all positive definite.

3 Minimax Estimates

In this section, we analyze the point spectrum for the operator $\mathcal{L}_\xi$, specified in (2.6), taking advantage of the fact that $\mathcal{L}_\xi$ is self-adjoint. To begin, we assume (2.6) has an eigenvalue λ with associated eigenvector $\varphi \in H^2(\mathbb{R})$, and we take an L^2 inner product of (2.6) with φ to obtain

$$\begin{aligned} \lambda \|\varphi\|^2 &= -\langle D_\xi^{1/2} H_\xi D_\xi^{1/2} \varphi, \varphi\rangle_2 = -\langle H_\xi D_\xi^{1/2} \varphi, D_\xi^{1/2} \varphi\rangle_2 \\ &= -\langle H_0 D_\xi^{1/2} \varphi, D_\xi^{1/2} \varphi\rangle_2 - |\xi|^2 \langle \Gamma D_\xi^{1/2} \varphi, D_\xi^{1/2} \varphi\rangle_2 \\ &\le -|\xi|^2 \theta_\Gamma \langle D_\xi^{1/2} \varphi, D_\xi^{1/2} \varphi\rangle_2 = -|\xi|^2 \theta_\Gamma \langle D_\xi \varphi, \varphi\rangle_2 \\ &\le -|\xi|^2 \theta_\Gamma \theta_M \Big(\|\varphi'\|^2 + |\xi|^2 \|\varphi\|^2 \Big), \end{aligned} \tag{3.1}$$

where we have used Lemma 2.1. We obtain the inequality

$$\lambda \le -\theta_\Gamma \theta_M |\xi|^4 - \theta_\Gamma \theta_M \frac{\|\varphi'\|^2}{\|\varphi\|^2} |\xi|^2. \tag{3.2}$$

We conclude from this that there exists a constant $\theta_0 > 0$ so that $\lambda \le -\theta_0 |\xi|^4$. For large values of $|\xi|$ this is exactly as expected, but for small values of $|\xi|$ we see that this inequality allows eigenvalues to move very slowly into the stable half-plane—as would be expected in a problem with only fourth order regularization.

In order to be a bit more precise, we take advantage of self-adjointedness and apply the minimax formulation. Letting λ_* denote the leading eigenvalue, we compute

$$\begin{aligned}
-\lambda_*(\xi) &= \inf_{\substack{\varphi\in H^2\\ \varphi\neq 0}} \frac{\langle -\mathcal{L}_\xi \varphi, \varphi\rangle_2}{\langle \varphi,\varphi\rangle_2} \\
&= \inf_{\substack{\varphi\in H^2\\ \varphi\neq 0}} \Big\{ \frac{\langle H_0 D_\xi^{1/2}\varphi, D_\xi^{1/2}\varphi\rangle_2}{\langle \varphi,\varphi\rangle_2} + |\xi|^2 \frac{\langle \Gamma D_\xi^{1/2}\varphi, D_\xi^{1/2}\varphi\rangle_2}{\langle \varphi,\varphi\rangle_2} \Big\}.
\end{aligned} \tag{3.3}$$

We now make the choice $\varphi = D_\xi^{-1/2}\bar{u}'$, so that $D_\xi^{1/2}\varphi = \bar{u}'$, and we recall that $H_0\bar{u}' = 0$. We find

$$-\lambda_* \le |\xi|^2 \frac{\langle \Gamma \bar{u}', \bar{u}'\rangle_2}{\langle D_\xi^{-1/2}\bar{u}', D_\xi^{-1/2}\bar{u}'\rangle_2} = |\xi|^2 \frac{\langle \Gamma \bar{u}', \bar{u}'\rangle_2}{\langle D_\xi^{-1}\bar{u}', \bar{u}'\rangle_2}. \tag{3.4}$$

In order to get an upper bound on the right-hand side, we need to obtain a lower bound on the denominator. Using our Green's function formulation (2.4) we can write

$$\langle D_\xi^{-1}\bar{u}', \bar{u}'\rangle_2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \langle G(x_1, y_1; |\xi|)\bar{u}'(y_1), \bar{u}'(x_1)\rangle dy_1 dx_1.$$

In the case that $\bar{M}$ is constant, we have

$$\langle D_\xi^{-1}\bar{u}', \bar{u}'\rangle_2 = \frac{1}{2|\xi|}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-|\xi||x_1-y_1|}\langle \bar{M}^{-1}\bar{u}'(y_1), \bar{u}'(x_1)\rangle dy_1 dx_1.$$

Recalling that $\bar{u}'$ decays at exponential rate, we easily verify the limit

$$\begin{aligned}
\lim_{|\xi|\to 0} &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-|\xi||x_1-y_1|}\langle \bar{M}^{-1}\bar{u}'(y_1), \bar{u}'(x_1)\rangle dy_1 dx_1 \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \langle \bar{M}^{-1}\bar{u}'(y_1), \bar{u}'(x_1)\rangle dy_1 dx_1 \\
&= \langle \bar{M}^{-1}[u], [u]\rangle \ge \frac{1}{\bar{m}}|[u]|^2,
\end{aligned}$$

where $\bar{m}$ denotes the largest eigenvalue of $\bar{M}$, and $[\cdot]$ denotes the jump in our function, so that $[u] = (u_+ - u_-)$. We can conclude

$$\langle D_\xi^{-1}\bar{u}', \bar{u}'\rangle_2 \ge \frac{|[u]|^2}{2\bar{m}|\xi|}.$$

Returning to (3.4) we see that

$$\lambda_*(\xi) \ge -c_0|\xi|^3,$$

for some appropriate constant $c_0 > 0$. Later, we'll use a perturbation argument to be more precise about λ_*, but that argument will require that λ_* be separated from the remaining spectrum. We already see that λ_* is separated from the essential spectrum, and our next goal is to verify that it also separates from any additional elements of point spectrum.

For the remaining eigenvalues, we let λ_2 denote the first eigenvalue below λ_* and write

$$-\lambda_2 = \sup_{v \in H^2} \inf_{\substack{\varphi \in H^2\{0\} \\ \langle \varphi, v \rangle = 0}} \frac{\langle -\mathcal{L}_\xi \varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} \geq \sup_{v \in H^2} \inf_{\substack{\varphi \in H^2\{0\} \\ \langle \varphi, v \rangle = 0}} \frac{\langle H_0 D_\xi^{1/2} \varphi, D_\xi^{1/2} \varphi \rangle}{\langle \varphi, \varphi \rangle}, \tag{3.5}$$

as in (3.3). Now set $\psi := D_\xi^{1/2} \varphi$ (i.e., $\psi = D_\xi \phi$), giving

$$\begin{aligned} -\lambda_2 &= \sup_{v \in H^2} \inf_{\substack{\psi \in H^1 \setminus \{0\} \\ \langle D_\xi^{-1/2} \psi, v \rangle = 0}} \frac{\langle H_0 \psi, \psi \rangle}{\langle D_\xi^{-1/2} \psi, D_\xi^{-1/2} \psi \rangle} = \sup_{v \in H^2} \inf_{\substack{\psi \in H^1 \setminus \{0\} \\ \langle \psi, D_\xi^{-1/2} v \rangle = 0}} \frac{\langle H_0 \psi, \psi \rangle}{\langle D_\xi^{-1} \psi, \psi \rangle} \\ &= \sup_{w \in H^1} \inf_{\substack{\psi \in H^1 \setminus \{0\} \\ \langle \psi, w \rangle = 0}} \frac{\langle H_0 \psi, \psi \rangle}{\langle D_\xi^{-1} \psi, \psi \rangle}. \end{aligned} \tag{3.6}$$

We take $w = \bar{u}'$, so that

$$-\lambda_2 \geq \inf_{\substack{\psi \in H^1 \setminus \{0\} \\ \langle \psi, \bar{u}' \rangle = 0}} \frac{\langle H_0 \psi, \psi \rangle}{\langle D_\xi^{-1} \psi, \psi \rangle}.$$

It follows from our transversality assumption (H2) that the null space of H_0 is spanned by $\bar{u}'$, and so for any ψ orthogonal to $\bar{u}'$ we must have $\langle H_0 \psi, \psi \rangle \geq \theta_H \|\psi\|^2$ for some $\theta_H > 0$, as stated in Lemma 2.1. Also, we have

$$0 \leq \langle D_\xi^{-1} \psi, \psi \rangle \leq \|D_\xi^{-1} \psi\| \|\psi\| \leq \frac{C_0}{|\xi|^2} \|\psi\|^2,$$

for some constant $C_0 > 0$, where in obtaining this inequality we have used our Green's function representation (2.4). We see that

$$-\lambda_2 \geq \frac{\gamma_0}{C_0} |\xi|^2 \Rightarrow \lambda_2 \leq -\frac{\gamma_0}{C_0} |\xi|^2$$

We conclude that $\lambda_*(\xi)$ is separated from the rest of the point spectrum, which moves more rapidly into the stable half-plane.

4 The Eigenvalue $\lambda_*(0) = 0$

We will complete our analysis of the spectrum of L_ξ by carrying out a perturbation argument for small $|\xi|$, starting with $\lambda_*(0) = 0$. The case $\xi = 0$ (i.e., the analysis of L_0) was considered in [14, 15, 16], and we proceed now by connecting our current analysis to the analysis there. To begin, we consider the asymptotic form for our eigenvalue problem (1.10),

$$-M_\pm \Gamma \phi'''' + (M_\pm B_\pm + 2|\xi|^2 M_\pm \Gamma) \phi'' - (\lambda I + |\xi|^2 M_\pm B_\pm + |\xi|^4 M_\pm \Gamma) \phi = 0. \tag{4.1}$$

That is, these two equations are obtained by taking $x_1 \to \pm\infty$ in (1.10).

If we search for solutions of the form $\phi(x) = e^{\mu x_1} r$, where μ is a scalar constant and $r \in \mathbb{C}^m$ is a constant vector we obtain the associated eigenvalue problem

$$\Big\{ -\mu^4 M_\pm \Gamma + \mu^2 (M_\pm B_\pm + 2|\xi|^2 M_\pm \Gamma) - (\lambda I + |\xi|^2 M_\pm B_\pm + |\xi|^4 M_\pm \Gamma) \Big\} r = 0. \tag{4.2}$$

Since our system is $m \times m$, and our equation is fourth order in μ, we expect $4m$ values of μ.

When $(\lambda, |\xi|) = (0,0)$, our equation reduces to

$$(-\mu^4 M_\pm \Gamma + \mu^2 M_\pm B_\pm) r = 0,$$

so that

$$\mu^2 M_\pm \Gamma (\Gamma^{-1} B_\pm - \mu^2 I) r = 0.$$

Since Γ and $B_\pm$ are positive definite, the eigenvalues of $\Gamma^{-1} B_\pm$ are positive (see, for example, [14]). If we denote these eigenvalues $\{\nu_j^\pm\}_{j=1}^m$ then m of our growth rates satisfy $\mu(0,0) = -\sqrt{\nu_j^\pm}$ for some j, while m satisfy $\mu(0,0) = +\sqrt{\nu_j^\pm}$. Since the corresponding solutions $\phi(x) = e^{\mu x_1} r$ will grow or decay at exponential rate for $|\lambda| + |\xi|^2$ sufficiently small, we refer to these as the *fast* rates. The remaining $2m$ decay rates satisfy $\mu(0,0) = 0$, and we refer to these as the *slow* rates.

In [14], the authors find that under our assumptions the asymptotically decaying solutions of $L_0 \phi = \lambda \phi$ can be expressed relative to solutions of the asymptotic problems given above. Before stating a result from [14], we set some notation, consistent with the notation given there. First, as in [14] we set

$$\sigma(\Gamma^{-1} B_\pm) = \{\nu_j^\pm\}_{j=1}^m$$
$$\sigma(M_\pm B_\pm) = \{\beta_j^\pm\}_{j=1}^m,$$

ordering the eigenvalues so that $i < j$ implies both $\nu_i^\pm \le \nu_j^\pm$ and $\beta_i^\pm \le \beta_j^\pm$. As verified in [14], if we additionally order the rates $\mu_j^\pm(\lambda, 0)$ in ascending order (by real parts), then for $|\lambda|$ sufficiently small we can characterize them for $j = 1, \dots, m$ as

$$\begin{aligned}
\mu_j^\pm(\lambda,0) &= -\sqrt{\nu_{m+1-j}^\pm} + \mathbf{O}(|\lambda|) \\
\mu_{m+j}^\pm(\lambda,0) &= -\sqrt{\frac{\lambda}{\beta_j^\pm}} + \mathbf{O}(|\lambda|^{3/2}), \\
\mu_{2m+j}^\pm(\lambda,0) &= \sqrt{\frac{\lambda}{\beta_{m+1-j}^\pm}} + \mathbf{O}(|\lambda|^{3/2}), \\
\mu_{3m+j}^\pm(\lambda,0) &= \sqrt{\nu_j^\pm} + \mathbf{O}(|\lambda|).
\end{aligned} \tag{4.3}$$

Indeed, under assumption (H4) we can be sure that the $\{\mu_i^\pm\}_{i=1}^{4m}$ are all analytic functions of the variable $\rho = \sqrt{\lambda}$. (The rates $\{\mu_i^\pm\}_{i=1}^m$ and $\{\mu_i^\pm\}_{i=3m+1}^{4m}$ are analytic in λ.) Finally, we'll let $r_j^\pm(\lambda, 0)$ denote the eigenvector of (4.2) associated with the rate $\mu_j^\pm(\lambda, 0)$.

We now restate a lemma from [14], keeping in mind that all relevant quantities are evaluated at $\xi = 0$.

Lemma 4.1. *Under Conditions (C1)'–(C3) (not necessarily (C1)), there exist values $\eta > 0$ and $r > 0$ so that for a choice of linearly independent solutions of the eigenvalue problem (1.10), we have the following estimates, uniformly in the set $\{\lambda : \lambda \in B(0,r), Arg\lambda \neq \pi\}$:*

(I) For $x_1 \leq 0$, $k = 0, 1, 2, 3$, and $j = 1, \dots, 2m$ we have:

$$\partial_{x_1}^k \phi_j^-(x_1;\lambda) = e^{\mu_{2m+j}^-(\lambda)x_1}\Big(\mu_{2m+j}^-(\lambda)^k r_{2m+j}^-(\lambda) + \mathbf{O}(e^{-\eta|x_1|})\Big);$$

(II) For $x_1 \geq 0$, $k = 0, 1, 2, 3$, and $j = 1, \dots, 2m$ we have:

$$\partial_{x_1}^k \phi_j^+(x_1;\lambda) = e^{\mu_j^+(\lambda)x_1}\Big(\mu_j^+(\lambda)^k r_j^+(\lambda) + \mathbf{O}(e^{-\eta|x_1|})\Big);$$

Now, if ϕ is an eigenfunction of L_0 then it must be a linear combination of the $\{\phi_j^-\}_{j=2m+1}^{4m}$ (because it must decay as $x_1 \to -\infty$), and it must also be a linear combination of the $\{\phi_j^+\}_{j=1}^{2m}$ (because it must decay as $x_1 \to +\infty$). We will check such linear dependence with an appropriate Wronskian, and in preparation for this it will be convenient to set some notation.

Definition 4.1. *Suppose $\{\phi_j\}_{j=1}^N$ denote N vectors, each of length $M \leq N$, and suppose $N/M = l$, where l is an integer. Then we set the Wronskian notation*

$$W(\phi_1, \phi_2, \dots, \phi_N) := \det \begin{pmatrix} \phi_1 & \phi_2 & \dots & \phi_N \\ \phi_1{}' & \phi_2{}' & \dots & \phi_N{}' \\ \vdots & \vdots & \vdots & \vdots \\ \phi_1{}^{(l)} & \phi_2{}^{(l)} & \dots & \phi_N{}^{(l)} \end{pmatrix}. \tag{4.4}$$

We will define a Wronskian that might appropriately be regarded as an Evans function for the case $\xi = 0$. We set

$$D(\lambda, 0) = W(\underbrace{\phi_1^+, \dots, \phi_m^+}_{\text{fast}}, \overbrace{\phi_{m+1}^+, \dots, \phi_{2m}^+}^{\text{slow}}, \overbrace{\phi_1^-, \dots, \phi_m^-}^{\text{slow}}, \underbrace{\phi_{m+1}^-, \dots, \phi_{2m}^-}_{\text{fast}}).$$

Now, $\bar{u}'$ satisfies $L_0\bar{u}' = 0$ and decays at exponential rate as $x_1 \to \pm\infty$, so $\bar{u}'$ must be a linear combination of the solutions $\{\phi_j^+(x_1;0,0)\}_{j=1}^m$ and also must be a linear combination of the solutions $\{\phi_j^-(x_1;0,0)\}_{j=m+1}^{2m}$. If we let J^- index the element of $\{\phi_j^-(x_1;0,0)\}_{j=m+1}^{2m}$ with slowest rate of decay appearing in this linear combination, and similarly for J^+, we can write

$$\phi_{J^-}^-(x_1;0,0) = \bar{u}'(x_1) = \phi_{J^+}^+(x_1;0,0), \tag{4.5}$$

where we've possibly incorporated some faster decaying terms into the exponential errors in $\phi^-_{J^-}$ and $\phi^+_{J^+}$. In our example (1.12), we find that $\bar{u}'$ can be regarded as a connection between ϕ^-_3 and ϕ^+_2, so $J_- = 3$ and $J^+ = 2$.

Clearly, then, we have

$$D(0,0) = 0. \tag{4.6}$$

In addition to (4.6), we need to understand derivatives of the Evans function at $(0,0)$. Since $D(\lambda, 0)$ depends on analytically on $\sqrt{\lambda}$, it is convenient to define an analytic function by setting $\rho = \sqrt{\lambda}$ and

$$D_a(\rho) := D(\lambda, 0).$$

(Analyticity of D_a is straightforward and established in [14].)

As discussed in [14], the condition for stability of $\bar{u}$ as a solution of (1.13) is

$$\frac{d^{m+1} D_a}{d\rho^{m+1}}\Big|_{\rho=0} \neq 0. \tag{$\mathcal{D}$}$$

Moreover, this condition can only hold if the leading eigenvalue $\lambda = 0$ has geometric multiplicity 1, as discussed in the paragraph leading into Theorem 1.1. (As verified in [14], we have

$$\frac{dD_a}{d\rho}\Big|_{\rho=0} = \frac{d^2 D_a}{d\rho^2}\Big|_{\rho=0} = 0. = \cdots = \frac{d^m D_a}{d\rho^m}\Big|_{\rho=0} = 0,$$

so D_a, in this sense at least, does not record a corresponding algebraic multiplicity.) In [14], condition ($\mathcal{D}$) is verified for certain examples; for the current analysis we will assume it to hold.

5 The Leading Eigenvalue $\lambda_*(\xi)$

In this section we combine the results of [14] with the approach of [24] to establish that the leading eigenvalue $\lambda_*(\xi)$ of L_ξ satisfies the relation

$$\lambda_*(\xi) = -c_3 |\xi|^3 \Big(1 + \mathbf{o}(|\xi|)\Big), \tag{5.1}$$

where $\mathbf{o}(\cdot)$ denotes standard "little-O" notation. We would like to proceed by a perturbation argument, starting with the eigenvalue $\lambda_*(0) = 0$ discussed in the previous section, but we must keep in mind that $\lambda_*(0) = 0$ is embedded in essential spectrum, and so standard perturbation theory such as described in Chapter VII of [19] does not guarantee that $\lambda_*(\xi)$ is analytic (though see Remark 5.1, just below). As outlined below, this difficulty can be overcome using an argument of [24], and we will obtain the expression (5.1) for a constant $c_3 > 0$ that will be identified in the analysis.

Remark 5.1. *We note that given any $\xi_0 \in \mathbb{R}^{n-1} \backslash \{0\}$, with $|\xi_0|$ sufficiently small, the leading eigenvalue $\lambda_*(\xi)$ will be analytic for ξ sufficiently close to ξ_0. This follows from standard perturbation theory. See, for example, [19], Chapter VII.*

Recalling that $H_0\bar{u}' = 0$, we obtain the relation

$$H_\xi \bar{u}' = |\xi|^2 \Gamma \bar{u}'. \tag{5.2}$$

For $|\xi| \neq 0$, the operator D_ξ is invertible, so we can express (1.10) as

$$-H_\xi \phi = \lambda D_\xi^{-1} \phi,$$

and so for the leading eigenvalue we have

$$-H_\xi \phi_* = \lambda_* D_\xi^{-1} \phi_*,$$

where ϕ_* denotes the eigenfunction of L_ξ associated with $\lambda_*(\xi)$.

We take an inner product of this last equation with $\bar{u}'$ and observe that H_ξ is self-adjoint to find

$$-\langle \phi_*, H_\xi \bar{u}' \rangle_2 = \lambda_*(\xi) \langle D_\xi^{-1} \phi_*, \bar{u}' \rangle_2.$$

Using (5.2), we see that

$$-\langle \phi_*, |\xi|^2 \Gamma \bar{u}' \rangle_2 = \lambda_*(\xi) \langle D_\xi^{-1} \phi_*, \bar{u}' \rangle_2,$$

and so

$$\lambda_*(\xi) = -|\xi|^2 \frac{\langle \phi_*, \Gamma \bar{u}' \rangle_2}{\langle D_\xi^{-1} \phi_*, \bar{u}' \rangle_2}. \tag{5.3}$$

In the case that $\bar{M}$ is constant, the Green's function for D_ξ is given by (2.5), and we have

$$\langle D_\xi^{-1} \phi_*, \bar{u}' \rangle_2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2|\xi|} e^{-|\xi||x_1 - y_1|} \langle \bar{M}^{-1} \phi_*(y_1), \bar{u}'(x_1) \rangle dy_1 dx_1.$$

We have, then,

$$\lambda_*(\xi) = -2|\xi|^3 \frac{\langle \phi_*, \Gamma \bar{u}' \rangle_2}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-|\xi||x_1 - y_1|} \langle \bar{M}^{-1} \phi_*(y_1), \bar{u}'(x_1) \rangle dy_1 dx_1}. \tag{5.4}$$

Noting that when $\xi = 0$, $\phi_* = \bar{u}'$ we see that if an appropriate perturbation argument can be justified we'll have

$$\begin{aligned} \lambda_*(\xi) &= -2|\xi|^3 \frac{\langle \bar{u}', \Gamma \bar{u}' \rangle_2}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle \bar{M}^{-1} \bar{u}'(y_1), \bar{u}'(x_1) \rangle dy_1 dx_1} \Big(1 + \mathbf{o}(|\xi|)\Big) \\ &= -2|\xi|^3 \frac{\langle \bar{u}', \Gamma \bar{u}' \rangle_2}{\langle \bar{M}^{-1}[u], [u] \rangle} \Big(1 + \mathbf{o}(|\xi|)\Big), \end{aligned} \tag{5.5}$$

which is simply a straightforward generalization of equation (2.14) from [24]. Here, $[u] := u_+ - u_-$.

Under our current assumption that $\bar{M}$ is constant the perturbation argument of [24] (based primarily on the standard reference [19]) carries over with only trivial modifications. Here, we will mention only the key points, referring the reader to pp. 805-807 of [24] for details.

The major difficulty we encounter with such a perturbation argument arises from the fact that $\lambda_*(0) = 0$ is an eigenvalue of L_0, embedded in essential spectrum (which for $\xi = 0$ is $(-\infty, 0]$), as described in Theorem 1.1 and computed in Section 2. In [24], the authors proceed by observing that $D_\xi H_0 \bar{u}' = 0$, so the operator $-D_\xi H_0$ has leading eigenvalue at 0. However, the essential spectrum of $-D_\xi H_0$ is confined to (the union of) $(-\infty, -m_\pm b_\pm |\xi|^2]$, and so $\lambda = 0$ is separated as an eigenvalue of $-D_\xi H_0$. This justifies the application of regular perturbation theory, so long as there is a sufficient *gap* between the initial operator and its perturbation. (See [19] Chapter IV, Section 1.4 for a precise definition of *gap*.) In order to control this gap, the authors work with operators $-D_\xi \mathcal{H}_\xi$, with

$$H_\xi := -\Gamma \partial^2_{x_1 x_1} + \bar{B}(x_1) + \alpha |\xi|^2 \Gamma,$$

where α is initially taken small and then continuously varied to 1.

We complete this section, and the proof of Theorem 1.1 by deriving the form of c_3 stated there. This is motivated by the scalar calculation carried out in [10]. To begin, we take an inner product of our equation $-\Gamma \bar{u}'' + F'(\bar{u}) = 0$ with $\bar{u}'$ to get

$$-\langle \Gamma \bar{u}'', \bar{u}' \rangle + \langle F'(\bar{u}), \bar{u}' \rangle = 0, \tag{5.6}$$

which can be rewritten as

$$\Big(-\frac{1}{2} \langle \Gamma \bar{u}', \bar{u}' \rangle + F(\bar{u}) \Big)' = 0. \tag{5.7}$$

Upon integrating once and evaluating the constant of integration by taking $x_1 \to -\infty$, we find that

$$\frac{1}{2} \langle \Gamma \bar{u}', \bar{u}' \rangle = F(\bar{u}), \tag{5.8}$$

where we have used our choice of setting $F(u_-) = 0$. We see that the multiplier of $-|\xi|^3$ in (5.5) is

$$c_3 = 4 \frac{\int_{-\infty}^{+\infty} F(\bar{u}(x_1)) dx_1}{\langle \bar{M}^{-1}[u], [u] \rangle}. \tag{5.9}$$

Working either with (5.5) or (5.9) we see that $c_3 > 0$, guaranteeing that λ_* indeed moves at cubic rate into the stable half-plane.

Finally, we note that in the case of a single equation, we can set $y = \bar{u}(x_1)$, and take advantage of the monotonicity of $\bar{u}(x_1)$ (see, e.g., [11]) to obtain

$$c_3 = \frac{2\bar{m}\sqrt{2\gamma}}{[u]^2} \int_{\min\{u_-, u_+\}}^{\max\{u_-, u_+\}} \sqrt{F(y)} dy,$$

which agrees with Theorem 1.2 of [10] in the case that $\bar{M}$ is constant. (Here, since Γ and $\bar{M}$ are scalars, they've been respectively denoted γ and $\bar{m}$.) The advantage of this formulation is that we need not generate the wave $\bar{u}$ in order to compute c_3.

Acknowledgments. The author gratefully acknowledges support from the National Science Foundation under grant DMS-0906370. This is a continuation of work the author began with Bongsuk Kwon, and the author is indebted to Dr. Kwon for several enlightening discussions.

References

[1] N. D. Alikakos, S. I. Betelu, and X. Chen, *Explicit stationary solutions in multiple well dynamics and non-uniqueness of interfacial energy densities*, Euro. Jnl of Applied Mathematics **17** (2006) 525–556.

[2] N. D. Alikakos and G. Fusco, *On the connection problem for potentials with several global minima*, Indiana U. Math. J. **57** (2008) 1871–1906.

[3] S. J. Bernau, *The square root of a positive self-adjoint operator*, J. Australian Mathematical Society, Feb. 1968, 17–36.

[4] P. W. Bates and P. C. Fife, *Spectral comparison principles for the Cahn-Hilliard and phase-field equations, and time scales for coarsening*, Physica D **43** (1990) 335-348.

[5] F. Boyer and C. Lapuerta, *Study of a three-component Cahn-Hilliard flow model*, ESAIM: Mathematical modeling and numerical analysis, **40** (2006) 653-687.

[6] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 19, AMS 1998.

[7] M. E. Gurtin, *An introduction to continuum mechanics*, Mathematics in Science and Engineering **158**, Academic Press 1981.

[8] P. Howard, *Asymptotic behavior near transition fronts for equations of generalized Cahn-Hilliard form*, Commun. Math. Phys. **269** (2007) 765–808.

[9] P. Howard, *Asymptotic behavior near planar transition fronts for equations of Cahn-Hilliard type*, Physica D **229** (2007) 123-165.

[10] P. Howard, *Spectral analysis of planar transition fronts for the Cahn-Hilliard equation*, J. Differential Equations **245** (2008) 594-615.

[11] P. Howard, *Spectral analysis of stationary solutions of the Cahn-Hilliard equation*, Advances in Differential Equations **14** (2009) 87-120.

[12] P. Howard, *Spectral analysis for stationary solutions of the Cahn-Hilliard equation in* $\mathbb{R}^d$, Communications in Partial Differential Equations **35** (2010) 590-612.

[13] D. Henry, *Geometric theory of semilinear parabolic systems*, Lecture notes in mathemtics **840** Springer-Verlag, Berlin, Heidelberg, New York 1981.

[14] P. Howard and B. Kwon, *Spectral analysis for transition front solutions in Cahn-Hilliard systems*, Discrete Contin. Dyn. Syst. **32** (2012) 125-166.

[15] P. Howard and B. Kwon, *Asymptotic stability analysis for transition wave solutions in Cahn-Hilliard systems*, Physica D **241** (2012) 1193–1222.

[16] P. Howard and B. Kwon, *Asymptotic L^p stability for transition fronts in Cahn-Hilliard systems*, J. Differential Equations **252** (2012) 5814-5831.

[17] P. Howard and C. Hu, *Nonlinear stability for multidimensional fourth order shock fronts*, Arch. Rational Mech. Anal. **181** (2006) 201–260.

[18] P. Howard and K. Zumbrun, *Stability of undercompressive shock profiles*, J. Differential Eqns. **225** (2006) 308-360.

[19] T. Kato, *Perturbation theory for linear operators*, 2nd ed. (Springer, Berlin, 1976).

[20] T. Korvola, A. Kupiainen, and J. Taskinen, *Anomalous scaling for three-dimensional Cahn-Hilliard fronts*, Comm. Pure Appl. Math. **LVIII** (2005) 1-39.

[21] T. Korvola, *Stability of Cahn-Hilliard fronts in three dimensions*, Doctoral dissertation, University of Helsinki, 2003.

[22] T. Kapitula and K. Promislow, *Spectral and dynamical stability of nonlinear waves*, Applied Mathematical Sciences **185**, Spring 2013.

[23] M. Reed and B. Simon, *Analysis of Operators IV*, Academic Press, New York, 1978.

[24] A. Shinozaki and Y. Oono, *Dispersion relation around the kink solution of the Cahn-Hilliard Equation*, Physical Review E **47** (1993) 804–811.

[25] V. Stefanopoulos, *Heteroclinic connections for multiple-well potentials: the anisotropic case*, Proc. Royal Soc. Edinburgh **138A** (2008) 1313-1330.

[26] K. Zumbrun and P. Howard, *Pointwise semigroup methods and stability of viscous shock waves*, Indiana U. Math. J. **47** (1998) 741–871. See also the errata for this paper: Indiana U. Math. J. **51** (2002) 1017–1021.