LOCAL TRACKING AND STABILITY FOR DEGENERATE VISCOUS SHOCK WAVES

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December 12, 2001

Abstract. We study the pointwise behavior of perturbed degenerate (sonic) shock waves for scalar conservation laws with non-constant diffusion. Building on the pointwise Green's function approach of [ZH], we extend the linear analysis to an equation with non-integrable coefficients. In lieu of working with the integrated equation, we employ a tracking mechanism that we expect will allow degenerate waves to be incorporated into the general framework for nondegenerate systems [ZH].

1. Introduction

We consider the scalar viscous conservation law

$$\begin{aligned} u_t + f(u)_x &= (b(u)u_x)_x; \quad u, x, f \in \mathbb{R}, t \in \mathbb{R}_+, \\ u(0,x) &= u_0(x), \end{aligned} \tag{1.1}$$

where $u_0(\pm\infty) = u_\pm$, $b, f \in C^2(\mathbb{R})$. In particular, we will study the stability of degenerate, or sonic, shock solutions to (1.1); that is, solutions of the form $\bar{u}(x-st)$ which satisfy the Rankine–Hugoniot condition

$$s(u_+ - u_-) = f(u_+) - f(u_-),$$

as well as the degenerate condition that $f'(u_+) = s < f'(u_-)$ (or symmetrically $f'(u_+) < s = f'(u_-)$). Without loss of generality, we may take $s = 0$ and thus $f'(u_+) = 0 < f'(u_-)$. With $\bar{u}(x)$ thus defined, we make our final assumption on (1.1), that $b(\bar{u}(x)) \geq b_0 > 0$. For a brief discussion of previous work and applications of the analysis, the reader is referred to [H.3].

It is well known that solutions, $u(t,x)$, of viscous conservation laws initialized by $u(0,x)$ near $\bar{u}(x)$ will not generally approach $\bar{u}(x)$, but rather will approach a translate of $\bar{u}(x)$ determined uniquely by the mass of $u(0,x)-\bar{u}(x)$, measured by the integral $\int_{\mathbb{R}} u(0,x)-\bar{u}(x)dx$. In the case of Lax and degenerate waves arising in single equations, and for systems under the additional constraint $\int_{\mathbb{R}} u(0,x)-\bar{u}(x)dx = 0$, convergence to this asymptotic translate can be successfully studied [G]. Indeed, such an analysis for degenerate waves has been carried out in [H.3]. In general, however, mass propagated away from the shock complicates the picture, and a more suitable approach is to track the shock in time. Our primary goal here is to show that the stability analysis of degenerate viscous shock waves can be incorporated

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Research supported by the NSF under grant No. DMS-9804390 and by the Department of Energy under DOE grant DE-FG02-88ER25053.

into the local tracking framework of [HZ.1, ZH]. It is through this framework, then, that we expect to extend these results to systems.

The fundamental difficulty in carrying out this program was pointed out in [H.3]. The local tracking method of [ZH] made extensive use of the analyticity of the Evans function at $\lambda = 0$ (or its analyticity on a Riemann surface, see [GZ, KR]). It was pointed out in [H.3] that in the case of degenerate viscous shock waves, the Evans function is certainly not analytic in a neighborhood of $\lambda = 0$, and does not appear to admit analytic extension to a Riemann manifold. We surmount this obstacle here by dividing the Evans function (plus related objects) into two pieces: one analytic in a neighborhood of the origin and one sufficiently small. It would appear that our decomposition is accurate enough to provide sharp decay rates in all L^p norms.

Following [HZ.1], our method of study will be to let $u(t,x)$ denote a second solution of (1.1) and to consider the perturbation $v(t,x) := u(t, x + \delta(t)) - \bar{u}(x)$, where the shift $\delta(t)$ will be chosen in such a way that $u(t, x + \delta(t))$ will remain *near* $\bar{u}(x)$ at each time t (*near* in a sense discussed below). In this manner, we will always compare $u(t,x)$ with the shape of $\bar{u}(x)$ rather than its position. As mentioned above, a reasonable first choice for $\delta(t)$ is the asymptotically-selected translate $l = \delta(\infty)$. In addition to the limited applicability of this choice (as mentioned above), we observe that even when applicable, it will typically be rather poor for intermediate times t. The time-asymptotic location takes into account perturbation mass (measured as $\int v$) that is still far from the shock layer, hence has not yet had a chance to interact with it. This gives an overestimate of the shock shift of the order of magnitude of the mass remaining in the far field, with resulting L^∞ perturbation error of the same order. The L^∞ distance from a "correctly" located shock would be, rather, of the order of the oscillation in the far field, typically much smaller than the mass. The same considerations hold for any norm L^p, $p > 1$.

The challenge, then, lies in determining this "correct" local shift $\delta(t)$. One could go about this in a number of ways, for example through best L^p fit, but such an estimate is not directly related to the underlying dynamics of the problem and so does not seem entirely satisfactory (though see the linear analysis of [G.1], based on the flux transform). A more natural approach from the point of view of the time evolution nature of the problem would be to "window" the shock, considering only perturbation mass that has already arrived within a vicinity of the shock layer close enough to affect the location. We will see below that this windowing is essentially the outcome of our approach; however, we begin from a different, more technical direction, based on a detailed desciption of the Green's function for a certain linear equation. When filtered through the variation of constants (Duhamel) representation of the nonlinear solution, the estimates yield the most advantageous choice of shock location *from the point of view of this analysis*. The result can be recognized after the fact as a mathematically precise version of the windowing approach described above.

Prior to defining $\delta(t)$ exactly, we recall the framework of the pointwise Green's function approach taken in [HZ.1, ZH]. Substituting $u(t, x + \delta(t)) = \bar{u}(x) + v(t,x)$ into (1.1) we arrive at the linearized equation

$$v_t + (a(x)v)_x - (b(x)v_x)_x = Q(v) + \dot{\delta}(t)(\bar{u}_x(x) + v_x), \tag{1.2}$$

where $a(x) := f'(\bar{u}(x)) - b'(\bar{u}(x))\bar{u}_x(x)$, $b(x) = b(\bar{u}(x))$, and $Q(v) = \mathbf{O}(v^2) + \mathbf{O}(vv_x)$ is a smooth function of its arguments. Allowing $G(t,x;y)$ to represent the Green's function for $v_t = Lv, Lv := (b(x)v_x)_x - (a(x)v)_x$, we have the integral equation

$$v(t,x) = \int_{-\infty}^{+\infty} G(t,x;y)v_0(y)dy + \int_0^t \int_{-\infty}^{+\infty} G(t-s,x;y)\Big[Q(v(s,y))_y + \dot{\delta}(s)(\bar{u}_y(y) + v_y(s,y))\Big]dyds.$$

Using the fact that $\bar{u}_x$ is an eigenfunction of the linearized eigenvalue equation (for $\lambda = 0$) we see that $e^{Lt}\bar{u}_x = \bar{u}_x$, so that (after integration by parts on the nonlinear term)

$$v(t,x) = \int_{-\infty}^{+\infty} G(t,x;y)v_0(y)dy + \delta(t)\bar{u}_x(x) - \int_0^t \int_{-\infty}^{+\infty} G_y(t-s,x;y)\Big[Q(v(s,y)) + \dot{\delta}(s)v(s,y)\Big]dyds,$$

where we will see below that sharp estimates on $G(t,x;y)$ will lead us to the choice

$$\delta(t) = -P\int_{-a_-t}^{a_-\sqrt{t}} v_0(y)dy.$$

Letting

$$\tilde{G}(t,x;y) := G(t,x;y) - P\bar{u}_x(x)I_{\{-a_-t\le y\le a_-\sqrt{t}\}},$$

we have

$$v(t,x) = \int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} G_y(t-s,x;y)\Big[Q(v(s,y)) + \dot{\delta}(s)v(s,y)\Big]dyds. \tag{1.3}$$

Our first theorem consists of pointwise estimates on $G(t,x;y)$ and rests upon the following observations (see Figure 4.1):

(1) The essential spectrum of L lies on and to the left of a parabola passing through the origin and opening into the negative real complex plane. We will denote this parabola by Γ_e and represent it through

$$\lambda_e(k) = -k^2 - ia_-k,$$

where $a_- := \lim_{x\to-\infty} a(x)$.

(2) The point spectrum of L lies to the left of a parabola Γ_d defined through

$$\lambda_d(k) = -d_2k^2 - id_1k - d_0,$$

where $d_0, d_1, d_2 > 0$ will be chosen sufficiently small during the analysis. We will denote this contour Γ_d.

Theorem 1.1. *Let $f''(u_+) \neq 0$ for $f(u)$ as in (1.1) (first order degeneracy). For some constants $C, M, T > 1$ and $\eta > 0$, depending on $a(x) = f'(\bar{u}(x)) - b'(\bar{u}(x))\bar{u}_x(x)$, $b(x) = b(\bar{u}(x))$, and the spectrum of L, the Green's function, $G(t,x;y)$, for $v_t = Lv$ satisfies the following estimates.*

(I) For $|x-y| \geq Kt$, K sufficiently large, and also for $t \leq T$, all x, y (n = 0,1)

$$\partial_y^n G(t,x;y) = \mathbf{O}(t^{-\frac{n+1}{2}})e^{-\frac{(x-y)^2}{Mt}}.$$

(II) For $|x-y| \leq Kt$, K as above, $t \geq T$
(i) $y \leq x \leq 0$

$$\begin{aligned} G(t,x;y) &= \mathbf{O}(t^{-1/2})e^{-\frac{(x-y-a_- t)^2}{Mt}} + P\bar{u}_x(x) I_{\{|x-y| \leq a_- t\}} \\ &+ \mathbf{O}(e^{-\eta|x|})e^{-\frac{(x-y-a_- t)^2}{Mt}} + \mathbf{O}(e^{-\eta|x|})\Big(|x-y-a_- t| + 1\Big)^{-1/2} I_{\{|x-y| \leq a_- t\}}, \end{aligned}$$

$$\begin{aligned} G_y(t,x;y) &= \mathbf{O}(t^{-1})e^{-\frac{(x-y-a_- t)^2}{Mt}} + \mathbf{O}(e^{-\eta|x|})\mathbf{O}(t^{-1/2})e^{-\frac{(x-y-a_- t)^2}{Mt}} \\ &+ \mathbf{O}(e^{-\eta|x|})\Big(|x-y-a_- t| + 1\Big)^{-1} I_{\{|x-y| \leq a_- t\}}. \end{aligned}$$

(ii) $x \leq y \leq 0$

$$\begin{aligned} G(t,x;y) &= \mathbf{O}(t^{-1/2})e^{-\frac{(x-y-a_- t)^2}{Mt}} + P\bar{u}_x(x) I_{\{|x-y| \leq a_- t\}} \\ &+ \mathbf{O}(e^{-\eta|x|})e^{-\frac{(x-y-a_- t)^2}{Mt}} + \mathbf{O}(t^{-1/2})\mathbf{O}(e^{-\eta|x|})e^{-\frac{(x-y)^2}{Mt}} I_{\{|x-y| \leq a_- t\}}, \end{aligned}$$

$$\begin{aligned} G_y(t,x;y) &= \mathbf{O}(t^{-1})e^{-\frac{(x-y-a_- t)^2}{Mt}} + \mathbf{O}(e^{-\eta|x|})\mathbf{O}(t^{-1/2})e^{-\frac{(x-y-a_- t)^2}{Mt}} \\ &+ \mathbf{O}(t^{-1})\mathbf{O}(e^{-\eta|x|})e^{-\frac{(x-y)^2}{Mt}} I_{\{|x-y| \leq a_- t\}}. \end{aligned}$$

(iii) $x \leq 0 \leq y$

$$G(t,x;y) = \mathbf{O}(t^{-1/2})\mathbf{O}_1(|y|)\mathbf{O}(e^{-\eta|x|})e^{-\frac{y^2}{Mt}} + P\bar{u}_x(x) I_{\{|x| \leq a_- t\} \cap \{|y| \leq a_- \sqrt{t}\}},$$

$$G_y(t,x;y) = \mathbf{O}(t^{-1})\mathbf{O}_1(|y|)\mathbf{O}(e^{-\eta|x|})e^{-\frac{y^2}{Mt}}$$

(iv) $y \leq 0 \leq x$

$$\begin{aligned} G(t,x;y) &= \mathbf{O}(t^{-1/4})\mathbf{O}_1(|x|^{-1})e^{-\frac{(x-y-a_- t)^2}{Mt}} I_{\{|y| \geq a_- t\}} \\ &+ \mathbf{O}_1(|x|^{-2})e^{-\frac{(x-y-a_- t)^2}{Mt}} I_{\{|y| \geq a_- t\}} \\ &+ \mathbf{O}_1(|x|^{-1})\Big(y + a_- t - \frac{3b_- x^2}{2a_-^2 b_0 t^2} y\Big)^{-1/2} e^{-\frac{x^2}{Mt}} I_{\{\{|y| \leq a_- t\} \cap \{x \geq 1\}\}} \\ &+ \Big(y + a_- t - \frac{3b_-}{2a_-^2 b_0 t} y\Big)^{-1/2} I_{\{\{|y| \leq a_- t\} \cap \{x \leq 1\}\}} + P\bar{u}_x(x) I_{\{|y| \leq a_- t\} \cap \{|x| \leq a_- \sqrt{t}\}}, \end{aligned}$$

$$G_y(t,x;y) = \mathbf{O}(t^{-1/2})\mathbf{O}_1(|x|^{-2})e^{-\frac{(x-y-a_-t)^2}{Mt}}I_{\{|y|\geq a_-t\}}$$
$$+\mathbf{O}_1(|x|^{-2})\Big(y+a_-t-\frac{3b_-x^2}{2a_-^2b_0t^2}y\Big)^{-1}e^{-\frac{x^2}{Mt}}I_{\{|y|\leq a_-t\}\cap\{x\geq 1\}\}}$$
$$+\mathbf{O}(1)\Big(y+a_-t-\frac{3b_-}{2a_-^2t}y\Big)^{-1}I_{\{\{|y|\leq a_-t\}\cap\{x\leq 1\}\}},$$

(v) $0\leq y\leq x$

$$G(t,x;y) = \mathbf{O}(t^{-1/2})\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}} + P\bar{u}_x(x)I_{\{|x-y|\leq a_-\sqrt{t}\}},$$

$$G_y(t,x;y) = \mathbf{O}(t^{-1})\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}}$$

(vi) $0\leq x\leq y$

$$G(t,x;y) = \mathbf{O}(t^{-1/2})\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}} + P\bar{u}_x(x)I_{\{|x-y|\leq a_-\sqrt{t}\}}$$

$$G_y(t,x;y) = \mathbf{O}(t^{-1})\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}}$$

where $\wedge$ *denotes minimum and estimates of form* $\mathbf{O}_1(f(\cdot))$ *satisfy* $\mathbf{O}_1(f(\cdot))\leq Cf(1+\cdot)$ *(allowing numerous expressions that would otherwise extend over two lines to be completed on one).*

A detailed discussion of estimates of the form of those from Theorem 1.1 appears in [H.1]. We mention here only that the estimates on G, G_y for Cases (i)–(iv) are not assumed sharp, and should be compared with the more natural estimates of [H.1, ZH] (though it certainly is not asserted that these hold in the present case). The difficulty in obtaining sharp Green's function estimates in these cases centers around our inability to extend contours through the negative real axis, and also upon our inability to expand functions of $\sqrt{\lambda}$ in a Taylor series about the crucial point $\lambda = 0$. (Indeed, the Evans function is in some sense even worse than $\sqrt{\lambda}$, as it appears to admit no analytic extension to a Riemann manifold. See Lemma 3.1.) As the analysis is essentially dictated by the purely degenerate case, $x, y\geq 0$, however, these estimates do not effect our final result. A critical refinement over the analysis of [H.3] is the elimination of a number of $\log t$ terms. We shall see that these arise naturally as (sharp!) $\log\lambda$ terms in the ODE analysis (of Lemma 3.1), but subtly cancel in the estimation of G_λ (see Lemma 3.4). This observation allows us to drop the $\log t$ behavior in the estimates of [H.3], making them sharp. Finally, we point out that these estimates could be sharpened by the refined analysis of Zumbrun [Z]. Again, however, we find that since the degenerate-side estimates dominate and are sharp, nothing new is gained. (Actually, sharper estimates on the non-degenerate side could allow one to consider initial data with slower decay on that side, but this does not strike me as a critical point.)

We turn now to the critical task of choosing $\delta(t)$. First, we observe that in each case of Theorem 1.1 $G(t,x;y)$ has one term that does not decay in time (called the *excited term*, following [ZH]). For example, for $y\leq x\leq 0$, the excited term is $E(t,x;y) = P\bar{u}_x(x)I_{\{|x-y|\leq a_-t\}}$, with $E(\infty,x;y) = P\bar{u}_x(x)$. Comparing this observation with our definition $v(t,x) := u(t,x+\delta(t)) - \bar{u}(x)$ and our integral representation (1.3), we see that this non-decaying mass is directly connected to our shift from the stationary shock: mass that fails to decay in time forces $u(t,x)$ toward a translate of $\bar{u}(x)$ rather than $\bar{u}(x)$ itself. Accordingly, we will choose $\delta(t)$ so as to annihilate the mass accumulating at the shock layer. At first glance, this suggests we take $\delta(t)$ so that

$$\int_{-\infty}^{+\infty} E(t,x;y)v_0(y)dy + \delta(t)\bar{u}_x(x) = 0.$$

(In the case of Lax and degenerate waves, G_y does not contain any terms that do not decay in time, simplifying the analysis to its linear part. For a fully nonlinear version, see [HZ.1].) Observing that $E(t,x;y) = P\bar{u}_x(x)I_{\{|x-y|\le e(t)\}}$, where $e(t)$ represents an expression in t dependent upon the case of x, y from Theorem 1.1, we make a choice that insures that $\delta(t)$ will not depend on x; namely,

$$\int_{-\infty}^{+\infty} P\bar{u}_x(x)I_{\{|y|\le e(t)\}}v_0(y)dy + \delta(t)\bar{u}_x(x) = 0,$$

which becomes

$$\delta(t) = -P\int_{-a_-t}^{a_-\sqrt{t}} v_0(y)dy,$$

where the upper limit of $\sqrt{t}$ is indicative of the degeneracy. Finally, we mention that as persuasive as this motivating argument may or may not be, the wisdom of this choice of $\delta(t)$ will ultimately be determined by its efficacy in our estimates on $v(t,x)$ (see Theorem 1.2).

Before stating our main theorem we make the following definitions.

Definition 1.1. *(Class of initial data) Denote by Δ_r the space of functions $d(\cdot)\ge 0$ such that $d(x)\le C(1+|x|)^{-r}, r>1$. Denote by $D(\cdot)$ the asymptotically decaying antiderivative of $d(\cdot)$,*

$$D(x) := \begin{cases} \int_{-\infty}^{x} d(y)dy, & x<0, \\ \int_{x}^{+\infty} d(y)dy, & x\ge 0. \end{cases}$$

Our rate of decay in time will essentially be $D(\sqrt{t})$; hence, we define

$$\tilde{D}(t) := \begin{cases} D(\sqrt{t})(\sim (1+t)^{\frac{1-r}{2}}), & 1<r<2 \\ (1+t)^{-1/2}\log(2+t), & r=2 \\ (1+t)^{-1/2}, & r>2. \end{cases}$$

Definition 1.2. *(Orbital stability) We say that a standing wave solution $\bar{u}(x)$ to (1.1) is orbitally stable in norm $\|\cdot\|$ if there exists an $\epsilon>0$ and a translate of $\bar{u}$, say $\bar{u}_l = \bar{u}(x-l)$, such that if another solution, u, to (1.1) satisfies $\|u(0,x)-\bar{u}_l(x)\|\le\epsilon$, then $\|u(t,x)-\bar{u}_l(x)\|$ decays to zero in time.*

We now state the main result of the paper, from which orbital stability follows in every L^p norm.

Theorem 1.2. *Suppose $\bar{u}(x)$ is a degenerate standing wave solution to (1.1) with $f''(u_+) \neq 0$ (first order degeneracy). For a second solution to (1.1), $u(t,x)$, with initial data, $u_0(x)$, suppose $u_0(x) - \bar{u}(x) \in \mathcal{A}^r_\zeta$, with*

$$\mathcal{A}^r_\zeta := \{v_0(x) : |v_0(x)| \leq \zeta d(x), \text{some } d \in \Delta_r\},$$

ζ sufficiently small and $r > 1$. In the event that $b(u)$ in (1.1) is non-constant, suppose additionally that $v_0(x) \in C^{0+\alpha}$. Then we obtain the estimates

(I) $x \leq 0$

$$|u(t,x) - \bar{u}(x)| \leq C\zeta\Big[e^{-\frac{\eta}{2}|x|}\tilde{D}(t) + d(x - a_- t)\Big],$$

(II) $x \geq 0$

$$|u(t,x)-\bar{u}(x)| \leq C\zeta\Big[(1+x)^{-1}e^{-\frac{x^2}{2Mt}}\tilde{D}(t)+t^{-1}\wedge(1+x)^{-2}+(1+x)^{-r}\wedge t^{-\frac{1}{2}}(1+x)^{1-r}\Big],$$

where η and M are as in Theorem 1.1 and $\wedge$ represents minimum.

Remark. We observe that the maximal rate of decay in time is slower here than for the case of [H.3], in which the initial perturbation was taken to have zero mass. This is in interesting contrast with the case of compressive waves, for which local tracking yields faster decay [HZ.2]. In this respect, degenerate waves are similar to undercompressive waves: the assumption of zero mass perturbations is inherently stronger. (Much more so, of course, for undercompressive waves, for which the assumption of zero mass perturbations is entirely unphysical.) The main point is that for perturbations from degenerate waves, the lack of transport leaves a decay rate determined by diffusion. For zero-mass perturbations, we are justified in thinking of diffusion both above and below the profile, providing the maximal (double) rate t^{-1}.

Theorem 1.2 provides the following immediate corollary on stability. (See [H.3] for comments on the proof.)

Corollary 1.3. *(Nonlinear stability) Under the assumptions of Theorem 1.2 and with $u_0(x) - \bar{u}(x) \in \mathcal{A}^r_\zeta$ as there, we have*

$$\|u(t,x) - \bar{u}(x)\|_{L^p} \leq C\tilde{D}(t), \quad p > 1,$$

and

$$\|u(t,x) - \bar{u}(x)\|_{L^1} \leq C\tilde{D}(t)t^\epsilon,$$

for $\epsilon > 0$ arbitrarily small when $1 < r \leq 2$, and for t^ϵ replaced by $\log t$ when $r > 2$.

2. Structure of Degenerate Shock Waves

We begin by collecting some observations regarding the behavior of degenerate shock waves. The proofs are similar to those of the analogous propositions in [H.3] and are omitted. For definiteness we will assume throughout that $f'(u_-) > 0$ and $f'(u_+) = 0$.

Proposition 2.1. *Suppose $f(u)$ and $u_\pm$ in (1.1) satisfy the Rankine–Hugoniot condition ($s = 0$, without loss of generality), $f, b \in C^{k+1}(\mathbb{R}), k \geq 1$, and $f'(u_+) = f''(u_+) = \cdots = f^{(k)}(u_+) = 0$, with $f^{(k+1)}(u_+) \neq 0$. Suppose also that Oleĭnik's entropy condition holds:*

$$f(u) - f(u_\pm) \begin{cases} < 0, & u_+ < u < u_-, \\ > 0, & u_- < u < u_+. \end{cases}$$

Then there exists a traveling wave solution $\bar{u}(x)$ of (1.1) so that $\bar{u}(\pm\infty) = u_\pm$, unique up to a shift. Moreover, we have

$$|\bar{u}(x) - u_-| = \mathbf{O}(e^{-\alpha|x|}),\ x \leq 0, \quad \textit{and} \quad |\bar{u}(x) - u_+| = \mathbf{O}(|x|^{-1/k}),\ x \geq 0.$$

Proposition 2.2. *Under the hypotheses of Proposition 2.1, for $a(x) := f'(\bar{u}(x)) - b'(\bar{u}(x))\bar{u}_x$, $b(x) := b(\bar{u}(x))$, $n = 0, 1$, we have (for any order of degeneracy)*

$$\begin{aligned} &(i)\ \Big|\frac{\partial^n}{\partial x^n}(a(x) - a_-)\Big| = \mathbf{O}(e^{-\alpha|x|}), \quad x \leq 0 \\ &(ii)\ \Big|\frac{\partial^n}{\partial x^n} a(x)\Big| = \mathbf{O}(|x|^{-1-n}), \quad x \geq 0. \end{aligned}$$

Moreover, in the case of degeneracy of order 1,

$$(iii)\ \gamma_+(x) := \frac{2\bar{u}_x}{\bar{u} - u_+} - \frac{\bar{u}_{xx}}{\bar{u}_x} = \mathbf{O}(|x|^{-2}).$$

3. ODE Estimates

Equations of type (1.2) readily lend themselves to study through the behavior of solutions of the eigenvalue ODE

$$Lv = \lambda v, \tag{3.1}$$

where we recall that L represents the linear operator $Lv := (b(x)v_x)_x - (a(x)v)_x$. Following [ZH], our approach will be to solve the associated Green's function equation

$$(L - \lambda)v = -\delta_y(x). \tag{3.2}$$

If we let $R(\lambda) := (\lambda I - L)^{-1}$ denote the resolvent of L, then (3.2) is solved by the Green's function

$$G_\lambda(x, y) = R(\lambda)\delta_y(x)$$

wherever $R(\lambda)$ is defined.

The computation of $G_\lambda(x, y)$ is standard (see [CH], for example) in terms of decaying solutions of (3.1). Our notation will be to let $\phi^\pm$ denote the (unique) decay modes of (3.1) at $\pm\infty$, and $\psi^\pm$ a choice of (nonunique) growth solutions at $\pm\infty$. We can directly compute the asymptotic growth and decay rates of ϕ and ψ from (3.1) by noting that at $\pm\infty$ (3.1) becomes

$$b_\pm v_{xx} - a_\pm v_x - \lambda v = 0,$$

where $a(-\infty) = a_- > 0$ and $a(+\infty) = a_+ = 0$, so that solutions of the form $v \sim e^{\mu x}$ give $b_\pm \mu^2 - a_\pm \mu - \lambda = 0$, which can readily be solved for the $-\infty$ modes

$$\mu_1^-(\lambda) = \frac{a_- - \sqrt{a_-^2 + 4\lambda b_-}}{2b_-}; \quad \mu_2^-(\lambda) = \frac{a_- + \sqrt{a_-^2 + 4\lambda b_-}}{2b_-},$$

and the $+\infty$ modes

$$\mu_1^+(x,\lambda) = -\int_0^x \sqrt{\lambda/b(s)}ds; \quad \mu_2^+(x,\lambda) = +\int_0^x \sqrt{\lambda/b(s)}ds,$$

(The algebraic decay of $b(\bar{u}(x))$ to $b(u_+)$ necessitates our keeping the x dependence in μ_1^+, μ_2^+.) We immediately see that a crucial aspect of the analysis is that while the modes at $-\infty$ are analytic in a neighborhood of the origin, the modes at $+\infty$ are not.

In terms of the above notation, the Green's function $G_\lambda(x,y)$ for (3.1) takes the form

$$G_\lambda(x,y) = \begin{cases} \frac{\phi^+(x)\phi^-(y)}{b(y)W_\lambda(y)}, & x \geq y, \\ \frac{\phi^+(y)\phi^-(x)}{b(y)W_\lambda(y)}, & x \leq y, \end{cases}$$

where $W_\lambda(y)$ denotes the usual Wronskian,

$$W_\lambda(y) = \phi^+(y)\phi^{-\prime}(y) - \phi^{+\prime}(y)\phi^-(y),$$

and consequently satisifies Abel's equation,

$$\partial_y W_\lambda(y) = \Big(\frac{a(y)}{b(y)} - \frac{b'(y)}{b(y)}\Big)W_\lambda(y).$$

The Evans function of [AGJ] is here $W_\lambda(0)$.

Finally, we will achieve the desired estimates on $G(t,x;y)$ from Dunford's Integral (the resolvent formula for the semigroup, or simply the Laplace transform under certain conditions) [Y], which gives

$$G(t,x;y) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} G_\lambda(x,y)d\lambda,$$

where Γ is a contour enclosing the entire spectrum of L (possibly passing through the point at ∞).

The analysis depends upon an extremely detailed understanding of $\phi^\pm$ and $\psi^\pm$. We have the following lemma.

Lemma 3.1. *Under the assumptions of Theorem 1.1 and for some constant M_s, we have the following estimates on the growth and decay modes ($\psi^\pm$ and $\phi^\pm$) of (3.1).*
(i) ($x \leq 0$) For all $|\lambda| \leq M_s$ and to the right of Γ_d,

$$\phi^-(x) = e^{-\mu_1^- x}(\bar{u}(x) - u_-)\Big(-\mu_1^- + \frac{\bar{u}_x(x)}{\bar{u}(x) - u_-} + e_2^-(x,\lambda) + \frac{\bar{u}_x(x)}{\bar{u}(x) - u_-}e_1^-(x,\lambda)\Big),$$

$$\phi^{-\,\prime}(x) = \frac{a(x)}{b(x)}\phi^-(x) + e^{-\mu_1^- x}(\bar{u}(x) - u_-)b(x)^{-1}(\lambda + \lambda e_1^-(x,\lambda)),$$

$$\frac{\partial^k}{\partial x^k}\psi^-(x) = e^{\mu_1^- x}((\mu_1^-)^n + \mathbf{O}(e^{-\alpha|x|})).$$

(ii) $(x \geq 0)$ For all $|\lambda| \leq M_s$, to the right of Γ_d and off the negative real axis,

$$\phi^+(x) = e^{-\int_0^x \sqrt{\lambda/b(s)}ds}(\bar{u}(x) - u_+)$$
$$\times\Big(-\sqrt{\lambda/b(x)} + \frac{\bar{u}_x(x)}{\bar{u}(x) - u_+} + e_2^+(x,\lambda) + \frac{\bar{u}_x(x)}{\bar{u}(x) - u_+}e_1^+(x,\lambda)\Big),$$

$$\phi^{+\,\prime}(x) = \frac{a(x)}{b(x)}\phi^+(x) + b(x)^{-1}e^{-\int_0^x \sqrt{\lambda/b(s)}ds}(\bar{u}(x) - u_+)(\lambda + \lambda e_1^+(x,\lambda)),$$

$$\psi^+(x) = e^{\int_0^x \sqrt{\lambda/b(s)}ds}(\bar{u}(x) - u_+)$$
$$\times\Big(\sqrt{\lambda/b(x)} + \frac{\bar{u}_x(x)}{\bar{u}(x) - u_+} + \tilde{e}_2^+(x,\lambda) + \frac{\bar{u}_x(x)}{\bar{u}(x) - u_+}\tilde{e}_1^+(x,\lambda)\Big),$$

$$\psi^{+\,\prime}(x) = \frac{a(x)}{b(x)}\psi^+(x) + b(x)^{-1}e^{\int_0^x \sqrt{\lambda/b(s)}ds}(\bar{u}(x) - u_+)(\lambda + \lambda\tilde{e}_1^+(x,\lambda)),$$

where

$$e_1^-(x,\lambda), e_2^-(x,\lambda) = \mathbf{O}(\lambda)\mathbf{O}(e^{-\eta|x|}),$$

while ($\wedge = \min$)

$$e_1^+(x,\lambda), \tilde{e}_1^+(x,\lambda) = \mathbf{O}(\sqrt{\lambda}\log\lambda) \wedge \mathbf{O}_1(|x|^{-1}),$$

and

$$e_2^+(x,\lambda), \tilde{e}_2^+(x,\lambda) = \mathbf{O}(\sqrt{\lambda})\mathbf{O}_1(|x|^{-1}).$$

Moreover, for $x\sqrt{\lambda/b_0} < 1$, we will require the extended representations,

$$e_1^+(x,\lambda) = e_1^+(0,\lambda) + \int_0^x \sqrt{\lambda/b(s)}ds + \int_0^x \frac{\sqrt{\lambda}\bar{u}_s}{(\bar{u}(s) - u_+)^2}ds + \mathbf{O}(\lambda\log\lambda),$$

$$e_2^+(x,\lambda) = \sqrt{\lambda/b(x)} + \frac{\sqrt{\lambda}\bar{u}_x}{(\bar{u}(x) - u_+)^2} + \mathbf{O}(\lambda),$$

and similarly,

$$\tilde{e}_1^+(x,\lambda) = \tilde{e}_1^+(0,\lambda) - \int_0^x \sqrt{\lambda/b(s)}ds - \int_0^x \frac{\sqrt{\lambda}\bar{u}_s}{(\bar{u}(s) - u_+)^2}ds + \mathbf{O}(\lambda\log\lambda),$$

$$\tilde{e}_2^+(x,\lambda) = -\sqrt{\lambda/b(x)} - \frac{\sqrt{\lambda}\bar{u}_x}{(\bar{u}(x) - u_+)^2} + \mathbf{O}(\lambda).$$

Remark. The odd form of the error estimate $e_1^+(x,\lambda)$ is a consequence of the fact that equations of form $v_{xx} + (\kappa/x)v_x = \lambda v$—which essentially govern ϕ and

ψ, since $a(x) \sim x^{-1}$—have the property that they can be scaled to depend only upon $\xi := \sqrt{\lambda}x$, leading to trade-off of $\lambda \to 0$ decay versus $x \to \infty$ blow-up, or vice versa. Though the $\log \lambda$ terms here are sharp—not simply a consequence of the analysis—we shall find that they cancel with one another as the analysis proceeds (see Lemma 3.3).

Proof. As these ODE estimates are slightly esoteric, we make some comments on their proof. We begin with the ODE

$$(b(x)v_x)_x - (a(x)v)_x = \lambda v. \tag{3.3}$$

If $w(x)$ satisfies the integrated equation

$$b(x)w_{xx} - a(x)w_x = \lambda w, \tag{3.4}$$

then it is straightforward to verify that $v(x) := w_x(x)$ satisfies (3.3). It will be convenient to obtain estimates on the growth and decay modes of (3.3) by obtaining estimates on the growth and decay modes of (3.4) and differentiating. We remark, however, that this use of the integrated equation requires no assumption on the mass of the initial perturbation.

Proceeding as in [H.3] we make the change of variables $w(x) = (\bar{u}(x) - u_+)u(x)$, natural in this context since the eigenfunction at $\lambda = 0$ of (3.4) is $\bar{u}(x)$. We obtain

$$\begin{aligned} b(x)(\bar{u}(x) - u_+)u_{xx} + 2b(x)\bar{u}_x u_x + b(x)\bar{u}_{xx}u& \\ - a(x)(\bar{u}(x) - u_+)u_x - a(x)\bar{u}_x u &= \lambda(\bar{u} - u_+)u. \end{aligned}$$

Observing that $b(x)\bar{u}_{xx} - a(x)\bar{u}_x = 0$ and dividing by $b(x)(\bar{u}(x) - u_+)$, we obtain

$$u_{xx} + \gamma_+(x)u_x = \lambda u; \quad \gamma_+(x) := \frac{2\bar{u}_x}{\bar{u} - u_+} - \frac{\bar{u}_{xx}}{\bar{u}_x}. \tag{3.5}$$

Letting $U_1(x) = u(x)$, $U_2(x) = u_x(x)$, we write (3.5) as a system. In order to investigate solutions of (3.5) that decay at $+\infty$ we look for solutions of the from $U(x) = e^{-\int_0^x \sqrt{\lambda/b(s)}ds} Z(x)$, so that

$$Z'(x) = A(x,\lambda)Z(x) + E(x)Z(x),$$

where

$$A(x,\lambda) = \begin{pmatrix} \sqrt{\lambda/b(x)} & 1 \\ \lambda/b(x) & \sqrt{\lambda/b(x)} \end{pmatrix}; \quad \text{and} \quad E(x) = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma_+(x) \end{pmatrix}.$$

We make the diagonalizing change of variables $Z(x) = P(x)W(x)$, where $P(x,\lambda)$ represents the matrix of eigenvectors,

$$P(x,\lambda) = \begin{pmatrix} 1 & 1 \\ -\sqrt{\lambda/b(x)} & \sqrt{\lambda/b(x)} \end{pmatrix}; \quad \text{and} \quad P(x,\lambda)^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{\lambda/b(x)}} \\ \frac{1}{2} & \frac{1}{2\sqrt{\lambda/b(x)}} \end{pmatrix}.$$

A page or so of matrix algebra brings us to the ODE for $W(x,\lambda)$,

$$W'(x) = D(\lambda)W(x) + \tilde{E}(x,\lambda)W(x), \tag{3.6}$$

where

$$D(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & 2\sqrt{\lambda/b(x)} \end{pmatrix}; \quad \text{and} \quad \tilde{E}(x,\lambda) = \frac{\tilde{\gamma}_+(x)}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$\tilde{\gamma}_+(x) = \gamma_+(x) - \frac{b'(x)}{2b(x)}.$$

Equation (3.6) takes the integral form

$$\begin{aligned} W_1(x) &= 1 - \int_x^{+\infty} \frac{1}{2}\tilde{\gamma}(\xi)(-W_1(\xi) + W_2(\xi))d\xi \\ W_2(x) &= -\int_x^{+\infty} \frac{1}{2}\tilde{\gamma}(\xi)(W_1(\xi) - W_2(\xi))e^{2\int_\xi^x \sqrt{\lambda/b(s)}ds}d\xi. \end{aligned} \tag{3.7}$$

We observe that $\tilde{\gamma}_+(\xi) = \mathbf{O}(|\xi|^{-2})$ is sufficient for establishing the validity of (3.7) for $W \in L^\infty[N,+\infty)$, N sufficiently large. Indeed, we obtain estimates of the form $W_1(x) = 1+\mathbf{O}(|x|^{-1})$ and $W_2(x) = \mathbf{O}(|x|^{-1})$. Observing that for the case $2x \geq \sqrt{\lambda}$ these estimates give decay in λ as well, we shall proceed in the case $2x \leq \sqrt{\lambda}$. Here, we will establish the estimates of Lemma 3.1 through iteration. Beginning with the large x solution, $(W_1(+\infty), W_2(+\infty)) = (1,0)$, we find

$$\begin{aligned} W_1(x) &= 1 + \frac{1}{2}\int_x^{x+\frac{1}{2\sqrt{\lambda/b_0}}} \tilde{\gamma}(\xi)d\xi \\ &\quad + \frac{1}{2}\int_x^{x+\frac{1}{2\sqrt{\lambda/b_0}}} \int_\xi^{\xi+\frac{1}{2\sqrt{\lambda/b_0}}} \tilde{\gamma}_+(s)ds\tilde{\gamma}_+(\xi)d\xi + \cdots + \mathbf{O}(\sqrt{\lambda}) \\ W_2(x) &= -\frac{1}{2}\int_x^{x+\frac{1}{2\sqrt{\lambda/b_0}}} \tilde{\gamma}(\xi)d\xi \\ &\quad - \frac{1}{2}\int_x^{x+\frac{1}{2\sqrt{\lambda/b_0}}} \int_\xi^{\xi+\frac{1}{2\sqrt{\lambda/b_0}}} \tilde{\gamma}_+(s)ds\tilde{\gamma}_+(\xi)d\xi + \cdots + \mathbf{O}(\sqrt{\lambda}). \end{aligned}$$

Hence, $-W_1(x) + W_2(x) = -1 - \varphi_0(x) + \mathbf{O}(\sqrt{\lambda})$, where

$$\varphi_0(x) = \int_x^\infty \tilde{\gamma}(\xi)d\xi + \int_x^\infty \int_\xi^\infty \tilde{\gamma}_+(s)ds\tilde{\gamma}_+(\xi)d\xi + \ldots$$

satisfies

$$\varphi_0(x) = -1 - \frac{\bar{u}_x(x)\sqrt{b(x)}}{(\bar{u}(x) - u_+)^2}.$$

Recalling our transformation $Z(x) = P(x)W(x)$, we have

$$Z_2(x) = \sqrt{\lambda/b(x)}(-W_1 + W_2) = -\sqrt{\lambda/b(x)}(1 + \varphi_0(x) + \mathbf{O}(\sqrt{\lambda})). \tag{3.8}$$

Writing

$$\begin{aligned} Z_1(x) &= 1 + e_1^+(x,\lambda), \\ Z_2(x) &= -\sqrt{\lambda/b(x)} + e_2^+(x,\lambda), \end{aligned}$$

we have that $e_2^+(x,\lambda)$ satisfies the claimed estimate. We recover u, u_x through

$$u(x) = e^{-\int_0^x \sqrt{\lambda/b(s)}ds}(1 + e_1^+(x,\lambda)),$$
$$u_x(x) = e^{-\int_0^x \sqrt{\lambda/b(s)}ds}(-\sqrt{\lambda/b(x)} + e_2^+(x,\lambda)),$$

and set $u_x = u_x$ to obtain the relation, $\partial_x e_1^+(x,\lambda) - \sqrt{\lambda/b(x)}e_1^+(x,\lambda) = e_2^+(x,\lambda)$, so that

$$e_1^+(x,\lambda) = e_1^+(0,\lambda) + \int_0^x e^{\int_\xi^x \sqrt{\lambda/b(s)}ds} e_2^+(\xi,\lambda)d\xi,$$

from which the estimate on $e_1^+(x,\lambda)$ is immediate. The large x estimate $e_1^+(x,\lambda) = \mathbf{O}(\sqrt{\lambda}\log\lambda) \wedge \mathbf{O}_1(|x|^{-1})$ can be established as in [H.3]. Checking that $\phi^\pm, \psi^\pm$ have the claimed form is now routine.

Remark. Since our integral equations for the W_i do not necessarily hold for $x = 0$, it is critical to observe that $e_1^+(x,\lambda), e_2^+(x,\lambda)$ are bounded here by standard ODE continuation. Decay in $\sqrt{\lambda}$ is clear from (3.8). Estimating the growth modes is slightly more delicate, but can be carried out as in [ZH], with M (as there) chosen carefully as a function of $\sqrt{\lambda}$. □

Before applying Lemma 3.1 toward some non-trivial observations regarding the Wronskian and various expansion coefficients, we state without proof the following large $|\lambda|$ ODE estimates, which may be proved as in [H.1, ZH].

Lemma 3.2. *For $|\lambda| \geq M_l$ and to the right of Γ_d, we have ($k = 0, 1$)*

$$\frac{\partial^k}{\partial x^k}\phi^\pm(x) = (\mp\sqrt{\lambda})^k K_\pm(x)(1 + \mathbf{O}(|\lambda|^{-1/2})),$$

where $x \in \mathbb{R}$ and $K_\pm(x)$ is bounded in λ.

A critical feature of the degenerate shock case is that while $\lambda = 0$ lies in both the point and essential spectrum, as in the Lax case, it is also a branch point of the Evans function. In the following key lemma, we prove that while the Wronskian is not analytic at zero, its behavior remains $\mathbf{O}(\lambda)$, as in the Lax case.

Lemma 3.3. *For λ sufficiently small, to the right of Γ_d, and off the negative real axis, we have*

$$\begin{aligned} W_0(\lambda) = \frac{\lambda}{b(0)}\Big[&\bar{u}_x(0)(u_+ - u_-)(1 + e_1^+(0,\lambda)) \\ &+ (\bar{u}(0) - u_+)(\bar{u}(0) - u_-)(-\sqrt{\lambda/b(0)} + e_2^+(0,\lambda)) + \mathbf{O}(\lambda)\Big]. \end{aligned}$$

Moreover,

(I) For the scattering coefficients $A(\lambda)$, $B(\lambda)$; $\phi^-(x) = A(\lambda)\phi^+(x) + B(\lambda)\psi^+(x)$, $x \geq 0$ we have

$$\begin{aligned} (\frac{2\lambda^{3/2}\bar{u}_x(0)}{b(0)^{3/2}} + \mathbf{O}(\lambda^2))A(\lambda) = -\frac{\lambda}{b(0)}\Big[&\bar{u}_x(0)(u_+ - u_-)(1 + \tilde{e}_1^+(0,\lambda)) \\ &+ (\bar{u}(0) - u_+)(\bar{u}(0) - u_-)(\sqrt{\lambda/b(0)} + \tilde{e}_2^+(0,\lambda)) + \mathbf{O}(\lambda)\Big], \end{aligned}$$

$$\begin{aligned} (\frac{2\lambda^{3/2}\bar{u}_x(0)}{b(0)^{3/2}} + \mathbf{O}(\lambda^2)B(\lambda) = \frac{\lambda}{b(0)}\Big[&\bar{u}_x(0)(u_+ - u_-)(1 + \tilde{e}_1^+(0,\lambda)) \\ &+ (\bar{u}(0) - u_+)(\bar{u}(0) - u_-)(\sqrt{\lambda/b(0)} + \tilde{e}_2^+(0,\lambda)) + \mathbf{O}(\lambda)\Big], \end{aligned}$$

from which we obtain

$$\begin{aligned}
&\mathit{1.}\ \Big(\frac{2\lambda^{3/2}\bar{u}_x(0)}{b(0)^{3/2}} + \mathbf{O}(\lambda^2)\Big)(A(\lambda)+B(\lambda))\\
&\quad = \frac{\lambda}{b(0)}\Big[\bar{u}_x(0)(u_+ - u_-)(e_1^+(0,\lambda) - \tilde{e}_1^+(0,\lambda))\\
&\qquad + 2(\bar{u}(0)-u_+)(\bar{u}(0)-u_-)\frac{\sqrt{\lambda}\bar{u}_x(0)}{(\bar{u}(0)-u_+)^2} + \mathbf{O}(\lambda)\Big],\\
&\mathit{2.}\ A(\lambda)\phi^+(x) + B(\lambda)\psi^+(x) = e^{\int_0^x \sqrt{\lambda/b(s)}ds}\Big[P\bar{u}_x(x) + \mathbf{O}(\sqrt{\lambda}) \wedge \mathbf{O}_1(|x|^{-1})\Big],\\
&\mathit{3.}\ A(\lambda)(\phi^{+\,\prime}(x) - \frac{a(x)}{b(x)}\phi^+(x)) + B(\lambda)(\psi^{+\,\prime}(x) - \frac{a(x)}{b(x)}\psi^+(x))\\
&\quad = e^{\int_0^x \sqrt{\lambda/b(s)}ds}\sqrt{\lambda}[\mathbf{O}(\sqrt{\lambda}) \wedge \mathbf{O}_1(|y|^{-1})].
\end{aligned}$$

(II) For $x \le 0$, $\phi^+(x) = E(\lambda)\phi^-(x) + F(\lambda)\psi^-(x)$, with $F(\lambda)/W_0(\lambda)$ bounded and analytic, and

$$\frac{E(\lambda)}{W_0(\lambda)} = \frac{P}{\lambda} + \mathbf{O}(|\lambda|^{-1/2}).$$

Proof. The proof of Lemma 3.3 follows directly from Lemma 3.1 and tedious algebraic manipulation. In order to indicate the ideas, we develop two results. First, the estimates on $A(\lambda)$, $B(\lambda)$, and $W_0(\lambda)$ are similar. We have

$$\begin{aligned}
W_0(\lambda) &= \phi^+(0)\phi^{-\,\prime}(0) - \phi^{+\,\prime}(0)\phi^-(0)\\
&= \phi^+(0)\Big(\frac{a(0)}{b(0)}\phi^-(0) + (\bar{u}(0)-u_-)(\frac{\lambda}{b(0)} + \frac{\lambda}{b(0)}e_1^-(0,\lambda))\Big)\\
&\quad - \phi^-(0)\Big(\frac{a(0)}{b(0)}\phi^+(0) + (\bar{u}(0)-u_-)(\frac{\lambda}{b(0)} + \frac{\lambda}{b(0)}e_1^+(0,\lambda))\Big)\\
&= (\bar{u}(0)-u_+)(\bar{u}(0)-u_-)\\
&\times\Big(-\sqrt{\frac{\lambda}{b(0)}} + \frac{\bar{u}_x(0)}{\bar{u}(0)-u_+} + e_2^+(0,\lambda) + \frac{\bar{u}_x(0)}{\bar{u}(0)-u_+}e_1^+(0,\lambda)\Big)\Big(\frac{\lambda}{b(0)} + \frac{\lambda}{b(0)}e_1^-(0,\lambda)\Big)\\
&- \Big(-\mu_1^- + \frac{\bar{u}_x(0)}{\bar{u}(0)-u_-} + e_2^-(0,\lambda) + \frac{\bar{u}_x(0)}{\bar{u}(0)-u_-}e_1^-(0,\lambda)\Big)\Big(\frac{\lambda}{b(0)} + \frac{\lambda}{b(0)}e_1^+(0,\lambda)\Big)\\
&= \frac{\lambda}{b(0)}\Big[\bar{u}_x(0)(u_+ - u_-)(1 + e_1^+(0,\lambda))\\
&\quad - (\bar{u}(0)-u_+)(\bar{u}(0)-u_-)(\sqrt{\lambda/b(0)} - e_2^+(0,\lambda)) + \mathbf{O}(\lambda)\Big].
\end{aligned}$$

The cancellation estimates involving $A(\lambda)$ and $B(\lambda)$ are quite delicate. We prove only the less involved, derivative estimate (I.3). From Lemma 3.1 and our estimates on $A(\lambda)$, $B(\lambda)$, we have

$$\begin{aligned}
&A(\lambda)(\phi^{+\prime}(x) - \frac{a(x)}{b(x)}\phi^+(x)) + B(\lambda)(\psi^{+\prime}(x) - \frac{a(x)}{b(x)}\psi^+(x)) \\
&\quad = A(\lambda)e^{-\int_0^x \sqrt{\lambda/b(s)}ds}(\bar{u}(x) - u_+)b(x)^{-1}(\lambda + \lambda e_1^+(x,\lambda)) \\
&\qquad + B(\lambda)e^{\int_0^x \sqrt{\lambda/b(s)}ds}(\bar{u}(x) - u_+)b(x)^{-1}(\lambda + \lambda\tilde{e}_1^+(x,\lambda)) \\
&\quad = e^{\int_0^x \sqrt{\lambda/b(s)}ds}\lambda(\bar{u}(x) - u_+)b(x)^{-1}\Big[A(\lambda)e^{-2\int_0^x \sqrt{\lambda/b(s)}ds} + B(\lambda) \\
&\qquad + A(\lambda)e^{-2\int_0^x \sqrt{\lambda/b(s)}ds}e_1^+(x,\lambda) + B(\lambda)\tilde{e}_1^+(x,\lambda)\Big].
\end{aligned}$$

In the event that $2x\sqrt{\lambda/b_0} \geq 1$, we have $x^{-1} \leq 2\sqrt{\lambda/b_0}$ and consequently an estimate by

$$e^{\int_0^x \sqrt{\lambda/b(s)}ds}\lambda[\mathbf{O}(1) \wedge \mathbf{O}(|\lambda|^{-1/2}x^{-1})],$$

depending upon how we choose to employ $\bar{u}(x) - u_+ = \mathbf{O}_1(|x|^{-1})$. On the other hand, for $2x\sqrt{\lambda/b_0} \leq 1$, we have $2\int_0^x \sqrt{\lambda/b(s)}ds \leq 1$, and we can expand the exponent in brackets to give

$$\begin{aligned}
&\Big[A(\lambda)e^{-2\int_0^x \sqrt{\lambda/b(s)}ds} + B(\lambda) + A(\lambda)e^{-2\int_0^x \sqrt{\lambda/b(s)}ds}e_1^+(x,\lambda) + B(\lambda)\tilde{e}_1^+(x,\lambda)\Big] \\
&\quad = \Big[A(\lambda) + B(\lambda) + A(\lambda)e_1^+(0,\lambda) + B(\lambda)\tilde{e}_1^+(0,\lambda) + \mathbf{O}_1(|x|)\Big].
\end{aligned}$$

The critical computation becomes

$$\begin{aligned}
&A(\lambda) + B(\lambda) + A(\lambda)e_1^+(0,\lambda) + B(\lambda)\tilde{e}_1^+(0,\lambda) \\
&= \Big(\frac{2\lambda^{3/2}\bar{u}_x(0)}{b(0)^{3/2}} + \mathbf{O}(\lambda^2)\Big)^{-1}\Big[\frac{\lambda}{b(0)}\bar{u}_x(0)(u_+ - u_-)(e_1^+(0,\lambda) - \tilde{e}_1^+(0,\lambda)) \\
&- \frac{\lambda}{b(0)}\bar{u}_x(0)(u_+ - u_-)e_1^+(0,\lambda) + \frac{\lambda}{b(0)}\bar{u}_x(0)(u_+ - u_-)\tilde{e}_1^+(0,\lambda) + \mathbf{O}(\lambda^{3/2})\Big] = \mathbf{O}(1).
\end{aligned}$$

This final calculation is an example of precisely the cancellation that went unobserved in [H.3]. The other estimates are like these, only more so. □

We now develop estimates on the ODE Green's function $G_\lambda(x,y)$. Through the estimates of Lemmas 3.1–3.3 these can be obtained with varying levels of precision. For brevity, we will state estimates here in a form convenient for the later analysis.

Lemma 3.4. *(Small $|\lambda|$) Under the assumptions of Theorem 1.1 and for $|\lambda| \leq M_s$, we have the following estimates, for which terms containing $\mathbf{O}_a$ are analytic to the right of Γ_d, while the remaining terms are analytic to the right of Γ_d and away from the negative real axis.*

(i) $y \leq x \leq 0$

$$G_\lambda(x,y) = \mathbf{O}_a(1)e^{\mu_1^-(x-y)} + \frac{P\bar{u}_x(x)}{\lambda}e^{-\mu_1^-(x+y)} + \mathbf{O}(|\lambda|^{-1/2})\mathbf{O}(e^{-\eta|x|})e^{\mu_1^-(x-y)},$$

$$\partial_y G_\lambda(x,y) = \mathbf{O}_a(\lambda)e^{\mu_1^-(x-y)} + \mathbf{O}(e^{-\eta|x|})e^{-\mu_1^-(x+y)},$$

(ii) $x \le y \le 0$

$$G_\lambda(x,y) = \mathbf{O}_a(1) e^{\mu_2^-(x-y)} + \frac{P\bar{u}_x(x)}{\lambda} e^{-\mu_1^-(x+y)} + \mathbf{O}(|\lambda|^{-1/2}) e^{\mu_2^- x - \mu_1^- y},$$

$$\partial_y G_\lambda(x,y) = \mathbf{O}_a(1) e^{\mu_2^-(x-y)} + \mathbf{O}(e^{-\eta|x|}) e^{-\mu_1^-(x+y)}$$

(iii) $x \le 0 \le y$

$$\begin{aligned} G_\lambda(x,y) &= \mathbf{O}(|\lambda|^{-1/2}) \mathbf{O}(e^{-\eta|x|}) \mathbf{O}_1(|y|) e^{-\mu_1^- x - \int_0^y \sqrt{\lambda/b(s)}ds} \\ &+ \frac{P\bar{u}_x(x)}{\lambda} e^{-\mu_1^- x - \int_0^y \sqrt{\lambda/b(s)}ds}, \\ \partial_y G_\lambda(x,y) &= \mathbf{O}(e^{-\eta|x|}) \mathbf{O}_1(|y|) e^{-\mu_1^- x - \int_0^y \sqrt{\lambda/b(s)}ds}, \end{aligned}$$

(iv) $y \le 0 \le x$

$$\begin{aligned} G_\lambda(x,y) &= \mathbf{O}(|\lambda|^{-1/2}) \mathbf{O}_1(|x|^{-1}) e^{-\int_0^x \sqrt{\lambda/b(s)}ds - \mu_1^- y} \\ &+ \frac{P\bar{u}_x(x)}{\lambda} e^{-\int_0^x \sqrt{\lambda/b(s)}ds - \mu_1^- y}, \end{aligned}$$

$$\begin{aligned} \partial_y G_\lambda(x,y) &= \mathbf{O}_1(|x|^{-2}) e^{-\int_0^x \sqrt{\lambda/b(s)}ds - \mu_1^- y} \\ &+ \mathbf{O}(|\sqrt{\lambda}|) \mathbf{O}_1(|x|^{-1}) e^{-\int_0^x \sqrt{\lambda/b(s)}ds - \mu_1^- y}, \end{aligned}$$

(v) $0 \le y \le x$

$$\begin{aligned} G_\lambda(x,y) &= \mathbf{O}(|\lambda|^{-1/2}) \mathbf{O}_1(|x|^{-1}) e^{-\int_y^x \sqrt{\lambda/b(s)}ds} + \frac{P\bar{u}_x(x)}{\lambda} e^{-\int_y^x \sqrt{\lambda/b(s)}ds} \\ &+ \Big[\mathbf{O}(|\lambda|^{1/2}) \wedge \mathbf{O}_1(|y|^{-1})\Big] \mathbf{O}(|\lambda|^{-1/2}) \mathbf{O}_1(|x|^{-1}) \mathbf{O}_1(|y|^2) e^{-\int_y^x \sqrt{\lambda/b(s)}ds} \\ &+ \Big[\mathbf{O}(|\lambda|^{1/2}) \wedge \mathbf{O}_1(|y|^{-1})\Big] \mathbf{O}(|\lambda|^{-1}) \mathbf{O}_1(|x|^{-2}) \mathbf{O}_1(|y|^2) e^{-\int_y^x \sqrt{\lambda/b(s)}ds}, \end{aligned}$$

$$\begin{aligned} \partial_y G_\lambda(x,y) &= \Big[\mathbf{O}(|\lambda|^{1/2}) \wedge \mathbf{O}_1(|y|^{-1})\Big] \mathbf{O}(|\lambda|^{-1/2}) \mathbf{O}_1(|x|^{-2}) \mathbf{O}_1(|y|^2) e^{-\int_y^x \sqrt{\lambda/b(s)}ds} \\ &+ \Big[\mathbf{O}(|\lambda|^{1/2}) \wedge \mathbf{O}_1|y|^{-1})\Big] \mathbf{O}_1(|x|^{-1}) \mathbf{O}_1(|y|^2) e^{-\int_y^x \sqrt{\lambda/b(s)}ds}, \end{aligned}$$

(vi) $0 \le x \le y$

$$\begin{aligned} G_\lambda(x,y) &= \mathbf{O}(|\lambda|^{-1/2}) \mathbf{O}_1(|x|^{-2}) \mathbf{O}_1(|y|) e^{-\int_x^y \sqrt{\lambda/b(s)}ds} + \frac{P\bar{u}_x(x)}{\lambda} e^{-\int_x^y \sqrt{\lambda/b(s)}ds} \\ &+ \Big[\mathbf{O}(|\lambda|^{1/2}) \wedge \mathbf{O}_1(|x|^{-1})\Big] \mathbf{O}(|\lambda|^{-1/2}) \mathbf{O}_1(|y|) e^{-\int_x^y \sqrt{\lambda/b(s)}ds} \\ &+ \Big[\mathbf{O}(|\lambda|^{1/2}) \wedge \mathbf{O}_1(|x|^{-1})\Big] \mathbf{O}(|\lambda|^{-1}) e^{-\int_x^y \sqrt{\lambda/b(s)}ds}, \end{aligned}$$

$$\begin{aligned} \partial_y G_\lambda(x,y) &= \mathbf{O}_1(|x|^{-2}) \mathbf{O}_1|y| e^{-\int_x^y \sqrt{\lambda/b(s)}ds} \\ &+ \Big[\mathbf{O}(|\lambda|^{1/2}) \wedge \mathbf{O}_1(|x|^{-1})\Big] \mathbf{O}_1(|y|) e^{-\int_x^y \sqrt{\lambda/b(s)}ds}. \end{aligned}$$

Proof. Employing Lemmas 3.1 and 3.3, the proof is similar to that of the analogous Lemma 3.4 of [H.3]. □

The final lemma of this section regards large $|\lambda|$ estimates on $G_\lambda(x,y)$. The proof is exactly that of Lemma 3.5 of [H.1] and is omitted.

Lemma 3.5. *For $|\lambda| \geq M_l$, some $M_l > 0$, and to the right of Γ_d, we have*

$$\frac{\partial^k}{\partial x^k} G_\lambda(x,y) = \mathbf{O}(|\lambda|^{\frac{k-1}{2}}) e^{-Re\frac{\sqrt{\lambda/b_0}}{2}|x-y|}.$$

4. Estimates on the time-propagating Green's function

We now employ the estimates of Lemmas 3.4 and 3.5 to derive estimates on the time-propagating Green's function $G(t,x;y)$. The analysis is governed by the observation that though we cannot extend the Evans function onto the negative real axis, we can still extend contours into the essential spectrum, provided they do not cross the negative real axis.

We begin with the observation that the large-$|\lambda|$, or small-time, analysis of [H.1] remains virtually unchanged. Recalling our relation

$$G(t,x;y) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} G_\lambda(x,y) d\lambda,$$

where Γ encircles the point spectrum of L, we have for all $|\lambda| \geq M_l$ integrals of the form

$$\int_\Gamma e^{\lambda t} \mathbf{O}(|\lambda|^{-1/2}) e^{-\frac{\sqrt{\lambda/b_0}}{2}|x-y|} d\lambda.$$

In the case $|x-y| \geq Kt$, some K sufficiently large, we proceed as in [H.1] with the contour, Γ_d^l, determined by

$$\sqrt{\lambda_l(k)} = \frac{|x-y|}{4\sqrt{b_0}t} + ik,$$

for λ_l to the right of Γ_d and Γ_d—along which exponential time decay is clear—otherwise (see Figure 4.1). We develop, then, an estimate of the form $t^{-1/2}e^{-\frac{|x-y|^2}{Mt}}$, where the only effect of x or y derivatives is that of increasing the algebraic $t \to 0$ blow-up by a power of $t^{-1/2}$. Similarly, the bounded-time Green's function estimates of [H.1] remain unchanged. We note that each contour we take in the following analysis will proceed similarly, following Γ_d out to the point at ∞. The contour Γ_d may be thought of as analogous to that which we would take were there a gap between essential spectrum and the imaginary axis.

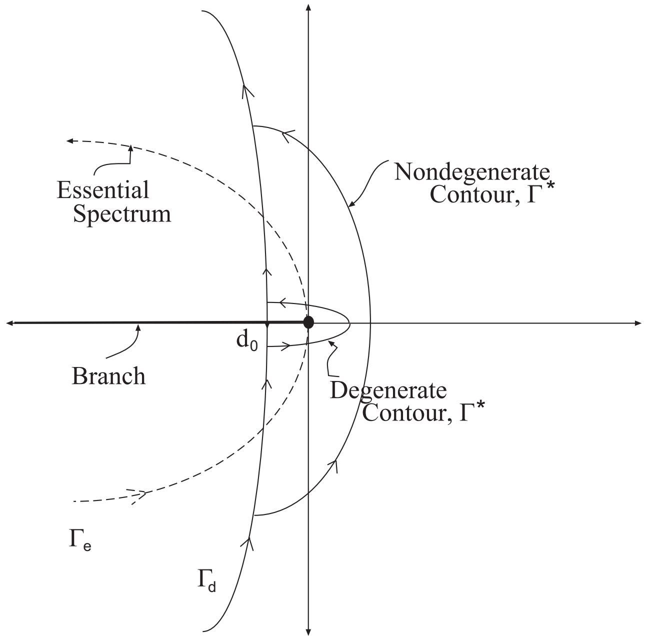

Figure 4.1. Contours Γ_d, Γ_e, Γ^*. In all cases, contours intersecting Γ_d follow it out to the point at ∞, avoiding possible point spectrum.

For $|x-y| \le Kt$ we divide the analysis into cases, similar to those of Lemma 3.4. We begin in the case $y \le x \le 0$, for which

$$G_\lambda(x,y) = \mathbf{O}_a(1)e^{\mu_1^-(x-y)} + \frac{P\bar{u}_x(x)}{\lambda}e^{-\mu_1^-(x+y)} + \mathbf{O}(|\lambda|^{-1/2})\mathbf{O}(e^{-\eta|x|})e^{\mu_1^-(x-y)}.$$

The first two terms are analytic to the right of Γ_d and may be analyzed as in [H.1, ZH]. We obtain, under integration, an estimate by

$$\mathbf{O}(t^{-1/2})e^{-\frac{(x-y-a_-t)^2}{Mt}} + \mathbf{O}(e^{-\eta|x|})e^{-\frac{(x-y-a_-t)^2}{Mt}} + P\bar{u}_x(x)I_{\{|x-y|\le a_-t\}}.$$

We mention that it is precisely these latter two terms that can be refined by Zumbrun's recent analysis [Z]. In particular, he shows that terms of the form $\lambda^{-1}e^{-\mu_1^+(x+y)}$ give rise to estimates

$$\text{errfn}(\frac{x+y-a_-t}{\sqrt{4b_-t}}) - \text{errfn}(\frac{x-y-a_-t}{\sqrt{4b_-t}});$$

hence, that $e^{-\eta|x|}e^{-\frac{(x-y-a_-t)^2}{Mt}}$ is too relaxed, by factor $t^{-1/2}$. We omit this refinement here, simply because it is the final term in $G_\lambda(x,y)$ that determines our rate of decay. This term has a branch point at $\lambda = 0$, and consequently we cannot carry a contour across the negative real axis. As in [H.3] we take the heat-equation-like contour described through

$$\sqrt{\lambda(k)} = t^{-1/2} + ik$$

for k such that λ lies to the right of Γ_d ($k \le k^*, say$), and Γ_d out to the point at ∞. (In Figure 4.1, this contour is denoted by Γ^*.) We obtain an additional estimate of

$$\mathbf{O}(e^{-\eta|x|})(1+|x-y-a_-t|)^{-1/2}I_{\{|x-y|\le a_-t\}}.$$

Since taking a derivative in y annihilates the pole at $\lambda = 0$, the estimate on G_y is easier. The case $x \le y \le 0$ is similar.

For $x \le 0 \le y$, we have

$$G_\lambda(x,y) = \mathbf{O}(|\lambda|^{-1/2})\mathbf{O}_1(|y|)\mathbf{O}(e^{-\eta|x|})e^{-\mu_1^- x - \int_0^y \sqrt{\lambda/b(s)}ds} + \frac{P\bar{u}_x}{\lambda}e^{-\mu_1^- x - \int_0^y \sqrt{\lambda/b(s)}ds}.$$

On the first term, we proceed as in [H.3], by taking the heat-equation-like contour defined through

$$\sqrt{\lambda(k)} = \frac{y}{2\sqrt{b_0}t} + ik,$$

so long as λ lies to the right of Γ_d, and Γ_d out to the point at ∞. We obtain an estimate of the form

$$\mathbf{O}(t^{-1/2})\mathbf{O}_1(|y|)\mathbf{O}(e^{-\eta|x|})e^{-\frac{y^2}{Mt}}. \tag{4.1}$$

The second term here is crucial, especially as its analysis—and the analyses of terms like it—mark the most fundamental difference between Section 4 of the current work and Section 4 of [H.3]. In particular, this term will yield a piece of $G(t,x;y)$ that does not decay in time. In the case of non-degenerate waves, such "projection" terms do not arise until the contour crosses the negative-real axis, at which point the contribution is picked out by Cauchy's integral formula [ZH]. Since our contour here never crosses the negative real axis it is not clear when our projection will arise. Below, we find that whereas such terms appear in the non-degenerate analysis with an indicator function $I_{\{|x-y|\le a_- t\}}$, the appropriate extension to degenerate waves involves (for $x \le 0 \le y$) the indicator $I_{\{|y|\le a_-\sqrt{t}\}\cap\{|x|\le a_- t\}}$.

We begin as with integration over the first expression, along the contour Γ_*. Observing that the exponential decay of $\bar{u}_x(x)$ dominates $e^{-\mu_1^- x}$ (in the portion of the essential spectrum Γ_* enters), we consider only

$$\begin{aligned}&\int_{\Gamma_*} \frac{e^{\lambda t - \int_0^y \sqrt{\lambda/b(x)}ds}}{\lambda}d\lambda \\ &= \int_{\Gamma_*\cap\Gamma_d} \frac{e^{\lambda t - \int_0^y \sqrt{\lambda/b(x)}ds}}{\lambda}d\lambda + \int_{\Gamma_*\backslash\Gamma_d} \frac{e^{\lambda t - \int_0^y \sqrt{\lambda/b(x)}ds}}{\lambda}d\lambda.\end{aligned}$$

Along the contour $\Gamma_* \cap \Gamma_d$, we obtain exponential decay in time, which will easily be subsumed into further estimates. Along $\Gamma_*\backslash\Gamma_d$, we have

$$\Big|\int_{\Gamma_*\backslash\Gamma_d} \frac{e^{\lambda t - \int_0^y \sqrt{\lambda/b(x)}ds}}{\lambda}d\lambda\Big| \le Ce^{-\frac{y^2}{4b_0 t}} \int_{-k_*}^{k_*} \frac{e^{-k^2 t}}{\sqrt{\frac{y^2}{4b_0 t^2} + k^2}}dk.$$

The critical point is that for $y = 0$ the integral on the right-hand side is undefined; hence, we must be wary of how far we push the analysis. We observe further, however, that when $y = 0$ the integrand on the left is analytic, allowing a contour that passes through the negative-real axis as in previous analyses. We take advantage of this in the following way. For $y \ge a_-\sqrt{t}$, the integrand on the right-hand side is clearly bounded, with no t-decay, giving a trivial estimate by $e^{-y^2/4b_0 t}$. But for $y \geq a_- \sqrt{t}$, we have

$$e^{-\frac{y^2}{4b_0 t}} \leq Cyt^{-1/2} e^{-\frac{y^2}{4b_0 t}},$$

which can be subsumed into (4.1). In this manner, we discover the appropriate domain for the indicator function associated with our excited term ($y \leq a_- t$).

Focusing now on the case $y \leq a_- \sqrt{t}$, we note that the exponential decay $e^{-y^2/4b_0 t}$ can be dropped. We consider

$$\begin{aligned}
&\int_{\Gamma_* \setminus \Gamma_d} \frac{e^{\lambda t - \mu_1^- x - \int_0^y \sqrt{\lambda/b(x)}ds}}{\lambda} d\lambda \\
&= \int_{\Gamma_* \setminus \Gamma_d} \frac{e^{\lambda t - \mu_1^- x}}{\lambda} d\lambda + \int_{\Gamma_* \setminus \Gamma_d} \frac{e^{\lambda t - \mu_1^- x}(e^{-\int_0^y \sqrt{\lambda/b(s)}ds} - 1)}{\lambda} d\lambda.
\end{aligned} \tag{4.2}$$

The first of these two integrands is analytic, and we may employ Cauchy's integral formula to obtain an integral over $\Gamma_d \setminus \Gamma_*$ plus a residue. We obtain a term that decays at exponential rate in time, plus the indicator function $I_{\{0 \leq y \leq a_- \sqrt{t}\}}$ (arising simply because that is the case we happen to be in). Last, we turn our attention to the second integral of (4.2). Over the portion of $\Gamma_* \setminus \Gamma_d$ for which $\int_0^y \sqrt{\lambda/b(s)}ds \leq 1$, Taylor expansion of the exponent provides an estimate by

$$\int_{\Gamma_* \setminus \Gamma_d} \frac{e^{\Re \lambda t} \int_0^y \sqrt{1/b(s)}ds}{|\sqrt{\lambda}|} |d\lambda| \leq Ct^{-1/2}|y|.$$

Alternatively, over the portion of $\Gamma_* \setminus \Gamma_d$ for which $\int_0^y \sqrt{\lambda/b(s)}ds \geq 1$, we have $|\lambda|^{-1/2} \leq \int_0^y \sqrt{1/b(s)}ds$ and consequently an estimate by $Ct^{-1/2}|y|$. Each of these estimates can be subsumed into (4.1).

We turn now to the case $y \leq 0 \leq x$. Though not particularly critical, the analysis of this case is complicated by the transition in behavior as a kernel starting at y is swept through the shock layer toward x. In this case, we have

$$G_\lambda(x,y) = \mathbf{O}(|\lambda|^{-1/2})\mathbf{O}_1(|x|^{-1})e^{-\int_0^x \sqrt{\lambda/b(s)}ds - \mu_1^- y} + \frac{P\bar{u}_x(x)}{\lambda} e^{-\int_0^x \sqrt{\lambda/b(s)}ds - \mu_1^- y}.$$

We begin with the case $y \leq -a_- t$, for which the kernel has not yet arrived at the origin. Here, we follow [ZH] and work in a sufficiently small neighborhood of the origin, noting that our contour never crosses the negative real axis. In such a neighborhood, $\Re(-\int_0^x \sqrt{\lambda/b(s)}ds) \leq \Re\mu_1^-$, so that the previous analysis applies directly. We obtain the terms

$$\mathbf{O}(t^{-1/4})\mathbf{O}_1(|x|^{-1})e^{-\frac{(x-y-a_- t)^2}{Mt}} I_{\{y \leq -a_- t\}} + \mathbf{O}_1(|x|^{-2})e^{-\frac{(x-y-a_- t)^2}{Mt}} I_{\{y \leq -a_- t\}}.$$

These estimates are not assumed sharp, but they will not be a determining factor in the analysis.

For $-a_- t \leq y \leq 0$, we take the heat-equation-like contour defined through

$$\sqrt{\lambda(k)} = \frac{x}{2\sqrt{b_0}t} + ik$$

to the right of Γ_d and Γ_d out to the point at ∞ (denoted, as usual, by Γ_*). The critical action occurs in an ϵ-ball around the origin, where to the left of our essential spectrum boundary (Γ_e), we have $\Re\mu_1^- \geq 0$. Writing

$$\lambda(k) = \frac{x^2}{4b_0t^2} + ik\frac{x}{\sqrt{b_0 t}} - k^2; \quad \mu_1^-(k) = -\frac{1}{a_-}\lambda + \frac{b_-}{a_-^3}\lambda^2 + \mathbf{O}(\lambda^3),$$

we compute, as in [H.3],

$$\begin{aligned}
&\mathbf{O}_1(|x|^{-1})\int_{\Gamma_*}\mathbf{O}(|\lambda|^{-1/2})e^{\lambda t - \int_0^x \sqrt{\lambda/b(s)}ds - \mu_1^- y}d\lambda \\
&= \mathbf{O}_1(|x|^{-1})\Big(y + a_-t - \frac{3b_-x^2}{2a_-^2 b_0 t^2}y\Big)^{-1/2} e^{-\frac{x^2}{4b_0 t} - \frac{x^2}{4b_0t^2}|y|} I_{\{y\geq -a_-t\}}.
\end{aligned}$$

As in the case $x \leq 0 \leq y$, this analysis cannot be applied to integration over the residue term. (For $x = 0$ the estimated integrand is not integrable along this contour.) For $|x| \geq \epsilon\sqrt{t}$, however, some $\epsilon > 0$ suitably small, we obtain a subsumable estimate as before. In the case $|x| \leq \epsilon\sqrt{t}$, we no longer have exponential decay and may consider

$$\begin{aligned}
&\int_{\Gamma_*}\frac{e^{\lambda t - \int_0^x \sqrt{\lambda/b(s)} - \mu_1^- y}}{\lambda}d\lambda \\
&= \int_{\Gamma_*\cap\Gamma_d}\frac{e^{\lambda t - \int_0^x \sqrt{\lambda/b(s)} - \mu_1^- y}}{\lambda}d\lambda + \int_{\Gamma_*\backslash\Gamma_d}\frac{e^{\lambda t - \int_0^x \sqrt{\lambda/b(s)} - \mu_1^- y}}{\lambda}d\lambda.
\end{aligned}$$

Though a portion of $\Gamma_*\cap\Gamma_d$ may pass through the essential spectrum, where $\Re\mu_1^- \geq 0$, we find that for $y \geq -a_-t$ and $\lambda \in \Gamma_d$, $e^{\lambda t - \int_0^x \sqrt{\lambda/b(s)}ds - \mu_1^- y}$ decays at exponential rate in t (and hence y). Along the contour $\Gamma_*\backslash\Gamma_d$, we further divide the integral up as

$$\begin{aligned}
&\int_{\Gamma_*\backslash\Gamma_d}\frac{e^{\lambda t - \int_0^x \sqrt{\lambda/b(s)}ds - \mu_1^- y}}{\lambda}d\lambda \\
&= \int_{\Gamma_*\backslash\Gamma_d}\frac{e^{\lambda t - \mu_1^- y}}{\lambda}d\lambda + \int_{\Gamma_*\backslash\Gamma_d}\frac{e^{\lambda t - \mu_1^- y}(e^{-\int_0^x \sqrt{\lambda/b(s)}ds} - 1)}{\lambda}d\lambda.
\end{aligned}$$

The first of these two integrands is analytic, and we may employ Cauchy's integral formula as before to obtain integration over $\Gamma_d\backslash\Gamma_*$ plus a residue. We obtain a term that decays at exponential rate in time plus $I_{\{y\geq -a_-t\}\cap\{x\leq a_-\sqrt{t}\}}$, a natural consequence of our transition in behavior as we pass through the shock layer. The final integral can be subsumed as before into previous estimates.

Consider now the case $0 \leq y \leq x$, for which we have

$$\begin{aligned}
G_\lambda(x,y) &= \mathbf{O}(|\lambda|^{-1/2})\mathbf{O}_1(|x|^{-1})e^{-\int_y^x \sqrt{\lambda/b(s)}ds} + \frac{P\bar{u}_x(x)}{\lambda}e^{-\int_y^x \sqrt{\lambda/b(s)}ds} \\
&+ \Big[\mathbf{O}_1(|y|^{-1}) \wedge \mathbf{O}(|\lambda|^{1/2})\Big]\mathbf{O}(|\lambda|^{-1/2})\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|^2)e^{-\int_y^x \sqrt{\lambda/b(s)}ds} \\
&+ \Big[\mathbf{O}_1(|y|^{-1}) \wedge \mathbf{O}_1(|\lambda|^{1/2})\Big]\mathbf{O}(|\lambda|^{-1})\mathbf{O}_1(|x|^{-2})\mathbf{O}_1(|y|^2)e^{-\int_y^x \sqrt{\lambda/b(s)}ds}.
\end{aligned}$$

For integration over each of these, we proceed along the heat-equation-like contour defined through

$$\sqrt{\lambda(k)} = \frac{|x-y|}{2\sqrt{b_0}t} + ik$$

for λ to the right of Γ_d, and Γ_d out to the point at ∞. On the first, we obtain an estimate by $\mathbf{O}(t^{-1/2})\mathbf{O}_1(|x|^{-1})e^{-(x-y)^2/(Mt)}$, with analogous estimates on the third and fourth. For the critical second term, we divide the analysis into subcases, $x \geq \epsilon\sqrt{t}$ and $x \leq \epsilon\sqrt{t}$. For $x \geq \epsilon\sqrt{t}$, $\bar{u}_x(x)$ yields t-decay, and we arrive at an estimate that can be subsumed into those above. For $x \leq \epsilon t$, we write

$$\begin{aligned}
&\int_{\Gamma_*} \frac{e^{\lambda t - \int_y^x \sqrt{\lambda/b(s)}ds}}{\lambda} d\lambda \\
&= \int_{\Gamma_* \cap \Gamma_d} \frac{e^{\lambda t - \int_y^x \sqrt{\lambda/b(s)}ds}}{\lambda} d\lambda + \int_{\Gamma_* \setminus \Gamma_d} \frac{e^{\lambda t - \int_y^x \sqrt{\lambda/b(s)}ds}}{\lambda} d\lambda.
\end{aligned}$$

Since $\Re\sqrt{\lambda/b(s)} \geq 0$, along $\Gamma_* \cap \Gamma_d$ we have exponential decay in time. Along $\Gamma_* \setminus \Gamma_d$, we further divide the integral as

$$\begin{aligned}
&\int_{\Gamma_* \setminus \Gamma_d} \frac{e^{\lambda t - \int_y^x \sqrt{\lambda/b(s)}ds}}{\lambda} d\lambda \\
&= \int_{\Gamma_* \setminus \Gamma_d} \frac{e^{\lambda t}}{\lambda} d\lambda + \int_{\Gamma_* \setminus \Gamma_d} \frac{e^{\lambda t}(e^{-\int_y^x \sqrt{\lambda/b(s)}ds} - 1)}{\lambda} d\lambda.
\end{aligned}$$

The first of these two integrands is analytic, and we may employ Cauchy's integral formula to obtain an integral over $\Gamma_d \setminus \Gamma_*$ plus a constant residue. We obtain a subsumable term that decays at exponential rate plus $I_{\{x \leq \epsilon\sqrt{t}\}}$. As in the analysis of $x \leq 0 \leq y$, the second integral is bounded by $Ct^{-1/2}|x-y|$.

The analysis for $0 \leq x \leq y$ differs negligibly from that of the previous case and is omitted. Derivative estimates are also similar. □

5. Estimates on the perturbation

In this section we will prove Theorem 1.2 through a lemma similar to Lemma 1.5 of [ZH]. We have

Lemma 5.1. *Let C_1 and C_2 be constants and let $h_0(x), h(t,x) \geq 0$ satisfy the relations*

$$\int_{-\infty}^{+\infty} |\tilde{G}(t,x;y)| h_0(y) dy \leq C_1 h(t,x)$$

and

$$\int_0^t \int_{-\infty}^{+\infty} |G_y(t-s,x;y)| \Big[M h(s,y)^2 + |\dot{\delta}(s)| h(y,s) \Big] dy ds \leq C_1 h(t,x),$$

for all $t > 0, x \in \mathbb{R}$. If then $|v(0,x)| \leq \zeta_0 h_0(x)$ for some ζ_0 sufficiently small, then $|v(t,x)| \leq C_2 \zeta_0 h(t,x)$ for all $t > 0, x \in \mathbb{R}$.

Remark. Lemmas of this form have now been proven in a variety of contexts [L, LZ.1–2, ZH HZ.1]. For $b(\cdot) = const$, Lemma 5.1 applies with no smoothness assumption on initial data $v_0(x)$. For $b(\cdot)$ nonconstant, we must assume $v_0(x) \in C^{0+\alpha}$; i.e., Hölder continuous for some index $\alpha > 0$. (See, in particular, [ZH, Section 11].)

The following useful lemma was proven in [H.3].

Lemma 5.2. *For $M > 0, t > 0$, and $\alpha \in \mathbb{R}$, we have*

$$\int_0^{+\infty} (1+y)^\alpha e^{-\frac{y^2}{Mt}} dy \le C_\alpha \begin{cases} 1, & \alpha < -1, \\ \log(2+\sqrt{t}), & \alpha = -1, \\ (1+t)^{\frac{1}{2}+\frac{\alpha}{2}}, & \alpha > -1. \end{cases}$$

Proof of Theorem 1.2. In order to employ Lemma 5.1, we require an ansatz $h(t,x)$ for the behavior of $v(t,x)$. We write

$$h(t,x) = \begin{cases} h_-(t,x), & x \le 0 \\ h_+(t,x), & x \ge 0, \end{cases}$$

with

$$h_-(t,x) = d(x - a_- t) + e^{-\frac{\eta}{2}|x|}\tilde{D}(t),$$

η as in Theorem 1.1, and

$$h_+(t,x) = (1+x)^{-1} e^{-\frac{x^2}{Mt}}\tilde{D}(t) + (1+x)^{-r} \wedge t^{-1/2}(1+x)^{1-r} + (1+x)^{-2} \wedge t^{-1},$$

where we recall that

$$\tilde{D}(t) := \begin{cases} (1+t)^{\frac{1-r}{2}}, & 1 < r < 2 \\ (1+t)^{-1/2}\log(2+t), & r = 2 \\ (1+t)^{-1/2}, & r > 2. \end{cases}$$

Linear analysis for $\mathbf{x \le 0}$. The linear analysis will consist of integrals of the form

$$\int_{-\infty}^{+\infty} G(t,x;y) h_0(y) dy,$$

where $h_0(y) = d(y) \le C(1+|y|)^{1-r}, r > 1$. In the case $|x-y| \ge Kt$, this analysis is unchanged from that of previous work (see [H.1, ZH]) and yields estimates bounded by $h^{\pm}(t,x)$ for $x \gtreqless 0$. The sectorial nature of L insures that we have no difficulty as $t \to 0$, as for example, in the case of equations of odd order (see [HZ.1]).

For $|x-y| \le Kt$, we begin in the case $y \le x \le 0$, for which we have

$$\begin{aligned} G(t,x;y) &= \mathbf{O}(t^{-1/2}) e^{-\frac{(x-y-a_- t)^2}{Mt}} + P\bar{u}_x(x) I_{\{|x-y| \le a_- t\}} \\ &+ \mathbf{O}(e^{-\eta|x|}) e^{-\frac{(x-y-a_- t)^2}{Mt}} + \mathbf{O}(e^{-\eta|x|}) \Big(|x-y-a_- t| + 1 \Big)^{-1/2} I_{\{|x-y| \le a_- t\}}. \end{aligned}$$

Integrated against initial data, the first and third terms yield estimates

$$C[d(x-a_- t) + t^{1/2} e^{-\eta|x|} d(x - a_- t)].$$

For $P\bar{u}_x(x) I_{\{|x-y| \le a_- t\}}$, we have, taking into account our tracking,

$$P\bar{u}_x(x) \Big[\int_{x-a_- t}^{x} h_0(y) dy - \int_{-a_- t \wedge x}^{x} h_0(y) dy \Big].$$

For $|x| \geq \epsilon t$, we have exponential decay in both x and t; for $|x| \leq \epsilon t$, we have

$$\Big| P\bar{u}_x(x) \int_{x-a_-t}^{-a_-t} h_0(y) ds \Big| \leq C e^{-\frac{\eta}{2}|x|} d(-a_-t).$$

Finally, we have the dominant term,

$$e^{-\eta|x|} \int_{x-a_-t}^{x} (1 + |x - y - a_-t|)^{-1/2} h_0(y) dy \leq C e^{-\eta|x|} t^{-1/2}.$$

A similar estimate follows for $x \leq y \leq 0$.

The linear analysis for $x \leq 0$ is determined by its degenerate portion: $x \leq 0 \leq y$. Here, we have ($|x - y| \leq Kt$)

$$G(t, x; y) = \mathbf{O}(t^{-1/2}) \mathbf{O}(e^{-\eta|x|}) \mathbf{O}_1(|y|) e^{-\frac{y^2}{Mt}} + P\bar{u}_x(x) I_{\{|x| \leq a_-t\} \cap \{|y| \leq a_-\sqrt{t}\}}.$$

Integrating the first term against $(1+|y|)^{-r}$, we may apply Lemma 5.2 to obtain an estimate by $Ce^{-\eta|x|}\tilde{D}(t)$. For $|x| \geq \epsilon t$, the second term yields exponential decay in both space and time, while for $|x| \leq \epsilon t$, it cancels exactly with our tracking term.

Linear analysis for $\mathbf{x \geq 0}$. We turn now to the analysis for $x \geq 0$, which we begin with the case $y \leq 0 \leq x$, where we have

$$\begin{aligned} G(t, x; y) &= \mathbf{O}(t^{-1/4}) \mathbf{O}_1(|x|^{-1}) e^{-\frac{(x-y-a_-t)^2}{Mt}} I_{\{|y| \geq a_-t\}} \\ &+ \mathbf{O}_1(|x|^{-2}) e^{-\frac{(x-y-a_-t)^2}{Mt}} I_{\{|y| \geq a_-t\}} \\ &+ \mathbf{O}_1(|x|^{-1}) \Big(y + a_-t - \frac{3b_-}{2a_-^2 b_0} \frac{x^2}{t^2} y \Big)^{-1/2} e^{-\frac{x^2}{Mt}} I_{\{\{|y| \leq a_-t\} \cap \{x \geq 1\}\}} \\ &+ \bar{u}_x(x) P I_{\{|y| \leq a_-t\} \cap \{|x| \leq a_-\sqrt{t}\}}. \end{aligned}$$

Integrating over the first and second terms, we obtain an estimate of the form

$$C \Big[\mathbf{O}_1(|x|^{-1}) e^{-\frac{x^2}{2Mt}} t^{1/4} d(t) + \mathbf{O}_1(|x|^{-2}) e^{-\frac{x^2}{2Mt}} t^{1/2} d(t) \Big],$$

both of which will later be subsumed into sharper estimates. For the third expression, we compute,

$$\begin{aligned} &|x|^{-1} e^{-\frac{x^2}{Mt}} \int_{-a_-t}^{0} (y + a_-t - \frac{3b_-x^2}{2a_-^2 b_0 t^2} y)^{-1/2} d(y) dy \\ &\leq |x|^{-1} e^{-\frac{x^2}{Mt}} d(t) \int_{-a_-t}^{-\frac{a_-t}{2}} (y + a_-t - \frac{3b_-x^2}{2a_-^2 b_0 t^2} y)^{-1/2} d(y) dy \\ &+ |x|^{-1} e^{-\frac{x^2}{Mt}} \int_{-\frac{a_-t}{2}}^{0} t^{-1/2} d(y) dy \\ &\leq |x|^{-1} e^{-\frac{x^2}{2Mt}} t^{1/2} d(t) + |x|^{-1} t^{-1/2} e^{-\frac{x^2}{Mt}}. \end{aligned}$$

For $|x| \le a_-\sqrt{t}$, we obtain no contribution from the last term. For $|x| \ge a_-\sqrt{t}$, we have only $\bar{u}_x(x)\delta(t)$,

$$P\bar{u}_x(x)\int_{-a_-t}^{0} d(y)dy \le (1+x)^{-2} \wedge t^{-1}.$$

For $0 \le x, y$, the analysis is considerably shortened by the observation that no final t-decay on the perturbation is lost if we use the umbrella estimate

$$G(t,x;y) = \mathbf{O}(t^{-1/2})\mathbf{O}_1(|x|^{-1})\mathbf{O}_1(|y|)e^{-\frac{(x-y)^2}{Mt}} + P\bar{u}_x(x)I_{\{|x-y|\le a_-\sqrt{t}\}}.$$

Integrating over the first term, we have,

$$\begin{aligned}
&t^{-1/2}|x|^{-1}\int_0^{\frac{x}{4}}(1+y)^{1-r}e^{-\frac{(x-y)^2}{Mt}}dy + t^{-1/2}|x|^{-1}\int_{\frac{x}{4}}^{2x}(1+y)^{1-r}e^{-\frac{(x-y)^2}{Mt}}dy \\
&+ t^{-1/2}|x|^{-1}\int_{\frac{x}{2}}^{+\infty}(1+y)^{1-r}e^{-\frac{(x-y)^2}{Mt}}dy \\
&\le C(1+x)^{-1}\tilde{D}(t)e^{-\frac{x^2}{2Mt}} + t^{-1/2}(1+x)^{1-r} \wedge (1+x)^{-r}.
\end{aligned}$$

For the second term, we have

$$P\bar{u}_x(x)\Big[\int_{(x-a_-\sqrt{t})\vee 0}^{x+a_-\sqrt{t}} h_0(y)dy - \int_0^{a_-\sqrt{t}} h_0(y)dy\Big].$$

For $x \ge a_-\sqrt{t}$, integrability of $v_0(y)$ yields an estimate by $(1+x)^{-2} \wedge t^{-1}$. For $x \le a_-\sqrt{t}$, we have

$$P\bar{u}_x(x)\int_{a_-\sqrt{t}}^{x+a_-\sqrt{t}} v_0(y)dy \le C\Big[(1+x)^{-2}(1+\sqrt{t})^{1-r} \wedge (1+x)^{-1}(1+\sqrt{t})^{-r}\Big].$$

Nonlinear Analysis. Following Lemma 5.1, the nonlinear analysis involves estimating integrals of the form

$$\int_0^t\int_{-\infty}^{+\infty}|G_y(t-s,x;y)|\Big[Mh(s,y)^2 + |\dot{\delta}(s)||h(s,y)\Big]dyds.$$

This calculation is almost precisely the same as the nonlinear analysis of [H.4], in which we estimated integrals of the form

$$\int_0^t\int_{-\infty}^{+\infty}|\mathcal{G}_x(t-s,x;y)|Mh(s,y)^2dyds,$$

where $\mathcal{G}$ represents the Green's function for the integrated equation $w_t + a(x)w_x = b(x)w_{xx}$. Writing $w = \mathcal{G} * w_0$ and $v = G * v_0$, and considering $v = w_x$, we have $v = \mathcal{G}_x * w_0$. But additionally, $v = G * v_0 = G * (w_0)_y = G_y * w_0$, so that $\mathcal{G}_x$ and G_y satisfy identical estimates (also clear from direct comparison). Hence, the new calculation is quite redundant, and we omit it. $\square$

Acknowledgements

I would like to express my appreciation to Kevin Zumbrun for suggesting the problem and for offering numerous useful suggestions as I worked on it. Finally, this research was supported in part by the NSF under Grant No. DMS-9804390 and in part by the Department of Energy under the DOE grant DE-FG02-88ER25053.

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