SHIFT INVARIANCE OF THE OCCUPATION TIME OF THE BROWNIAN BRIDGE PROCESS

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Abstract. In this note the distribution for the occupation time of a one-dimensional Brownian bridge process on any Lebesgue measurable set between the initial and final states of the bridge is shown to be invariant under translation and reflection, so long as the translation or reflection also lies between the initial and final states of the bridge. The proof employs only the strong Markov property and elementary symmetry properties of the Brownian bridge process.

Let W_s denote a standard, one-dimensional Wiener process. A *Brownian bridge* from a to b on the interval $[0,t]$, denoted $B^{a\to b}_{[0,t]}(s)$, can be defined as (see [KS])

$$B^{a\to b}_{[0,t]}(s) := a(1-\frac{s}{t}) + b\frac{s}{t} + (W_s - \frac{s}{t}W_t); \quad 0\le s\le t. \tag{1}$$

Note that the Brownian bridge process is a standard, one-dimensional Wiener process constrained to particular starting and ending points. An important aspect of this process, which we employ below, is the symmetry between going from a to b or from b to a.

The Feynman–Kac formula for a solution to the Fokker–Planck equation leads naturally to a study of the *occupation time* of such a process. In particular, for the

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equation

$$u_t = \frac{1}{2}u_{xx} - I_A(x)u; \quad u(0,x) = \delta_y(x),$$

where A is some Lebesgue measureable set, we have the solution (see [O])

$$u(t,x) = E\Big[e^{-\int_0^t I_A(X_s)ds}\delta_y(X_t)\Big],$$

where X_t satisfies the stochastic differential equation

$$dX_t = dW_t; \quad X_0 = x. \tag{2}$$

The delta function constrains X_t to satisfy $X_t = y$, leading to

$$u(t,x) = E\Big[e^{-\int_0^t I_A(B^{a\to b}_{[0,t]}(s))ds}p(t,x,y)\Big],$$

where $p(t,x,y)$ is the Wiener transition function.

The integral

$$\int_0^t I_A(B^{x\to y}_{[0,t]}(s))ds$$

is the occupation time of the Brownian bridge $B^{a\to b}_{[0,t]}(s)$ in the set A.

Proposition 1. *For any Lebesgue measurable set, A, such that $A \subset [a,b]$, the distribution of the occupation time, $\int_0^t I_A(B^{a\to b}_{[0,t]}(s))ds$, of $B^{a\to b}_{[0,t]}(s)$ in A is invariant under translations and reflections, so long as the translation or reflection also lies in $[a,b]$.*

Proof. For simplicity, we first carry out the proof in the case of an interval. Let $a \le c < d \le b$. We show that the distribution of time spent in the interval $[c,d]$ is the same as the distribution of time spent in the interval $[b-(d-c),b]$, that is, that this distribution depends only on the length of the interval $[c,d]$.

We observe first that this assertion holds trivially for the case $d = b$, and that by symmetry it must also hold for $c = a$. Now, let T_c denote the random variable or *hitting time* representing the first time $B_{[0,t]}^{a \to b}(s)$ equals c—an event that occurs with probability one, since $a \le c < b$. (See Figure 1, below.) We remark that the distribution of T_c depends on a, b and t.

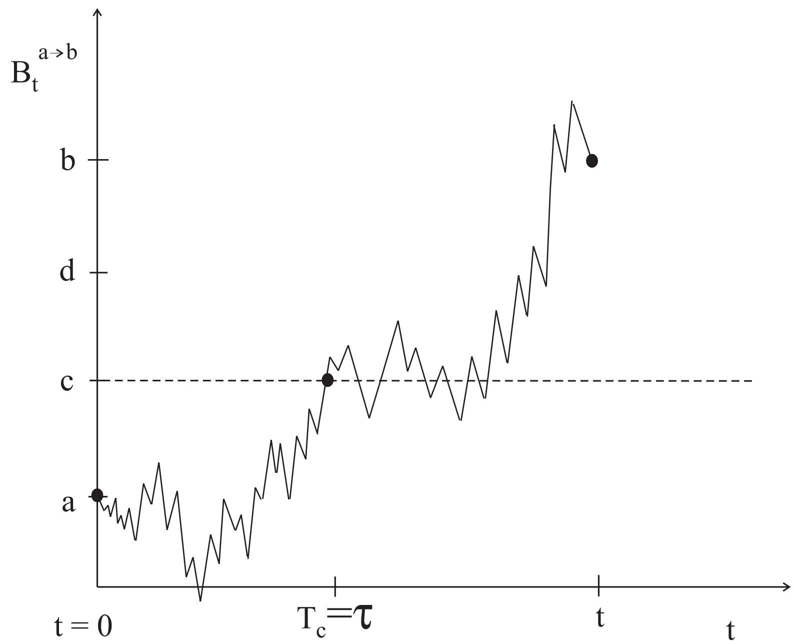

Figure 1. The Brownian bridge process.

We condition on T_c and (intuitively) consider the random variable

$$B_\tau := \left[\int_0^t I_{[c,d]}(B_{[0,t]}^{a \to b}(s))ds \middle| T_c = \tau \right]$$

for all $\tau \in [0, t]$. For $z \in \mathbb{R}$ we can write the distribution function for the occupation time on $[c, d]$ in terms of B_τ as

(3)

$$Pr\left[\int_0^t I_{[c,d]}(B_{[0,t]}^{a \to b}(s))ds \le z \right] = \int_0^t Pr\left[\int_0^t I_{[c,d]}(B_{[0,t]}^{a \to b}(s))ds \le z \middle| T_c = \tau \right] d\mu_{T_c}(\tau),$$

where $\mu_{T_c}(\tau)$ is the probability measure associated with the random variable T_c.

Let $B^{c\to b}_{[\tau,t]}(s)$ represent the Brownian bridge on the interval $[\tau,t]$ that goes from c to b. Since $d\le b$ we have $c\le b-(d-c)$, so that the amount of time spent by $B^{c\to b}_{[\tau,t]}(s)$ in either of the intervals $[c,d]$ or $[b-(d-c),b]$ is exactly the time spent by B_τ in that interval. But, by the symmetry discussed above, the distribution of time spent by $B^{c\to b}_{[\tau,t]}(s)$ in $[c,d]$ is the same as the distribution of time spent by $B^{c\to b}_{[\tau,t]}(s)$ in $[b-(d-c),b]$. Hence, we can make the following computation, which begins with the right-hand side of (3).

$$\begin{aligned}
&\int_0^t Pr\Big[\int_0^t I_{[c,d]}(B^{a\to b}_{[0,t]}(s))ds\le z\Big|T_c=\tau\Big]d\mu_{T_c}(\tau)\\
&\quad=\int_0^t Pr\Big[\int_\tau^t I_{[c,d]}(B^{c\to b}_{[\tau,t]}(s))ds\le z\Big]d\mu_{T_c}(\tau)\\
&\quad=\int_0^t Pr\Big[\int_\tau^t I_{[b-(d-c),b]}(B^{c\to b}_{[\tau,t]}(s))ds\le z\Big]d\mu_{T_c}(\tau)\\
&\quad=\int_0^t Pr\Big[\int_0^t I_{[b-(d-c),b]}(B^{a\to b}_{[0,t]}(s))ds\le z\Big|T_c=\tau\Big]d\mu_{T_c}(\tau)\\
&\quad=Pr\Big[\int_0^t I_{[b-(d-c),b]}(B^{a\to b}_{[0,t]}(s))ds\le z\Big].
\end{aligned}$$

We note that the first and third equalities follow from the strong Markov property of the Brownian bridge process (see [FPY]). This completes the proof in the case of intervals, where reflections are equivalent to translations.

The assertion regarding Lebesgue measurable sets can be obtained similarly. Let A^d_c represent a Lebesgue measurable set with endpoints c and d. Then

$$\int_\tau^t I_{A^d_c}(B^{c\to b}_{[\tau,t]}(s))ds \stackrel{d}{=} \int_\tau^t I_{\tilde{A}^b_{b-(d-c)}}(B^{c\to b}_{[\tau,t]}(s))ds,$$

by symmetry, where $\tilde{A}^b_{b-(d-c)}$ is the reflection of A^d_c with upper endpoint b and $\stackrel{d}{=}$ denotes equality in distribution.

Using this observation, we compute as before

$$\begin{aligned}
&Pr\Big[\int_0^t I_{A_c^d}(B_{[0,t]}^{a\to b}(s))ds \le z\Big] \\
&= \int_0^t Pr\Big[\int_0^t I_{A_c^d}(B_{[0,t]}^{a\to b}(s))ds \le z\Big| T_c = \tau\Big] d\mu_{T_c}(\tau) \\
&= \int_0^t Pr\Big[\int_\tau^t I_{A_c^d}(B_{[\tau,t]}^{c\to b}(s))ds \le z\Big] d\mu_{T_c}(\tau) \\
&= \int_0^t Pr\Big[\int_\tau^t I_{\tilde{A}^b_{b-(d-c)}}(B_{[\tau,t]}^{c\to b}(s))ds \le z\Big] d\mu_{T_c}(\tau) \\
&= \int_0^t Pr\Big[\int_0^t I_{\tilde{A}^b_{b-(d-c)}}(B_{[0,t]}^{a\to b}(s))ds \le z\Big| T_c = \tau\Big] d\mu_{T_c}(\tau) \\
&= \int_0^t Pr\Big[\int_0^t I_{A_a^{a+(d-c)}}(B_{[0,t]}^{a\to b}(s))ds \le z\Big| T_c = \tau\Big] d\mu_{T_c}(\tau) \\
&= Pr\Big[\int_0^t I_{A_a^{a+(d-c)}}(B_{[0,t]}^{a\to b}(s))ds \le z\Big],
\end{aligned}$$

where $A_a^{a+(d-c)}$ is the reflection of $\tilde{A}^b_{b-(d-c)}$ (thus a translation of A_c^d) with lower endpoint a, and again we have employed the strong Markov property.

This yields the result for translations. The result regarding reflections follows similarly. □

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