TROPICAL VARIETIES FOR EXPONENTIAL SUMS

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In memory of Joel Zinn (March 16, 1946 – December 5, 2018), beloved friend and brilliant colleague.

Abstract. We study the complexity of approximating complex zero sets of certain n-variate exponential sums. We show that the real part, R, of such a zero set can be approximated by the $(n-1)$-dimensional skeleton, T, of a polyhedral subdivision of $\mathbb{R}^n$. In particular, we give an explicit upper bound on the Hausdorff distance: $\Delta(R,T)=O\big(t^{3.5}/\delta\big)$, where t and δ are respectively the number of terms and the minimal spacing of the frequencies of g. On the side of computational complexity, we show that even the $n=2$ case of the membership problem for R is undecidable in the Blum-Shub-Smale model over $\mathbb{R}$, whereas membership and distance queries for our polyhedral approximation T can be decided in polynomial-time for any fixed n.

1. Introduction

We study zero sets of exponential sums of the form $g(z) := \sum_{j=1}^{t} e^{a_j\cdot z+\beta_j}$ where $z\in\mathbb{C}^n$, $a_j\in\mathbb{R}^n$, the a_j are pair-wise distinct, $\beta_j\in\mathbb{C}$, and $a_j\cdot z$ denotes the usual Euclidean inner product in $\mathbb{C}^n$. We call g an *n-variate exponential t-sum*, a_j a *frequency* of g, $\{a_1,\ldots,a_t\}$ the *spectrum* of g, and $\delta(g):=\min_{p\neq q}|a_p-a_q|$ the *minimal spacing of the frequencies of g*. (Throughout this paper, we use $|\cdot|$ for the standard ℓ_2-norm on $\mathbb{C}^N$ for any $N\in\mathbb{N}$.) We also call the β_j the *coefficients* of g. One can think of g as an analogue of a polynomial with real exponents, and hope to use algebraic intuition to derive new metric results in the broader setting of exponential sums. We shall do so by combining results on random projections with some new extensions of classical univariate polynomial bounds.

Exponential sums appear across pure and applied mathematics. For instance, exponential sums (in the form above) occur in the calculation of 3-manifold invariants (see, e.g., [McM00, Appendix A] and [Had16]), and have been studied from the point of view of Diophantine Geometry, Model Theory, and Computational Algebra, (see, e.g., [Ric83, MW96, Wil96, Zil02, AMW08, KZ14, SY14, HP16]). Also, the non-lattice Dirichlet polynomials appearing in the study of fractal strings [LV06] are a special case of the exponential sums we consider here. An application to radar antennae [FH95, HAGY08] — finding the directions of a set of unknown signals — reduces to finding the zeroes of a univariate exponential sum, with frequencies depending on the location of the sensors of the antenna. Approximating roots of multivariate exponential sums is also a fundamental computational problem in Geometric Programming [DPZ67, Chi05, BKVH07].

For any analytic function g on $\mathbb{C}^n$ let $Z(g)$ denote the set of complex zeroes of g. Also, for any $W\subseteq\mathbb{C}^n$, we define its *real part* to be $\mathbf{Re}(W):=\{(\mathbf{Re}(z_1),\ldots,\mathbf{Re}(z_n))\mid(z_1,\ldots,z_n)\in W\}$. One can wonder if exact computation with the roots of exponential sums is possible using only field operations and comparisons over $\mathbb{R}$, or if approximation is truly necessary. Exact computation turns out to be intractable, relative to a standard computational model (the *BSS model over* $\mathbb{R}$ [BCSS98]), already in the special case of two variables and three terms.

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Theorem 1.1. *Determining, for arbitrary input $r_1, r_2 \in \mathbb{R}$, whether (r_1, r_2) lies in $\mathbf{Re}(Z(1-e^{z_1}-e^{z_2}))$ is undecidable*[1] *in the BSS model over $\mathbb{R}$.*

We prove Theorem 1.1 in Section 3.1. There are certainly tractable special cases of the preceding problem, such as when the $r_i = \log s_i$ for some positive rational s_i (see, e.g., [The02, TdW15] and [AKNR18, Thm. 1.9]). Similarly, the famous Lindemann-Weierstrass Theorem tells us that $e^{r_1}+e^{r_2}$ is transcendental when $r_1, r_2 \in \mathbb{R}$ are distinct and algebraic. However, checking whether $e^{r_1}+e^{r_2} \stackrel{?}{\in} \mathbb{Q}(r_1, r_2)$ for *arbitrary* distinct transcendental $r_1, r_2 \in \mathbb{R}$ — using only *finitely* many rational operations and inequality checks in $\mathbb{Q}(r_1, r_2)$ — is already an open question. Theorem 1.1 thus highlights the need for approximation if one wants to work with roots of exponential sums in complete generality.

A natural question then is whether one can *efficiently* approximate the zero set of an exponential sum. For instance, can we at least decide — perhaps within polynomial-time — whether a given point is close to the real part of the zero set of an exponential sum? Our main algorithmic and quantitative results (Theorems 1.9 and 1.10) show that this is indeed the case, at least in a coarse sense: We derive a polyhedral structure that can be considered as a first-order approximation to the real part of the zero set, so that higher-order numerical iterative methods can be deployed when higher precision is needed in a specific application.

Clearly, $Z(g)$ is empty when $t=1$. That polyhedra arise from the real parts of zero sets of exponential sums is most easily seen in the special case of $t=2$ terms: Since $\left|\pm e^{\beta}\right| = e^{\mathbf{Re}(\beta)}$, the equality $e^{a_1\cdot z+\beta_1}+e^{a_2\cdot z+\beta_2}=0$ implies $e^{a_1\cdot \mathbf{Re}(z)+\mathbf{Re}(\beta_1)} = e^{a_2\cdot \mathbf{Re}(z)+\mathbf{Re}(\beta_2)}$, and we thus obtain the following basic fact after taking logarithms:

Proposition 1.2. *If $g(z)=e^{a_1\cdot z+\beta_1}+e^{a_2\cdot z+\beta_2}$ for some distinct $a_1, a_2 \in \mathbb{R}^n$, and $\beta_1, \beta_2 \in \mathbb{C}$, then $\mathbf{Re}(Z(g))$ is the affine hyperplane $\{u \in \mathbb{R}^n \mid (a_1-a_2)\cdot u + \mathbf{Re}(\beta_1-\beta_2)=0\}$.* ■

Before stating our main metric results in arbitrary dimension, it will be useful to observe some of the intricacies present already in the univariate case.

1.1. Clustering of Real Parts in One Variable.

The simple sum $e^{z_1}-1$ shows that the imaginary part $\mathbf{Im}(Z(g))$ can be infinite already in the univariate case, unlike the polynomial setting. A more subtle phenomenon, however, is that $\mathbf{Re}(Z(g))$ need not even be closed.

Proposition 1.3. *$X := \mathbf{Re}\left(Z\left(e^{\sqrt{2}z_1}+e^{\sqrt{3}z_1}+e^{\sqrt{5}z_1}\right)\right)$ is countably infinite, contained in the open interval $\left(-\frac{\log 2}{\sqrt{3}-\sqrt{2}}, \frac{\log 2}{\sqrt{3}-\sqrt{2}}\right)$ ($\subset (-2.181, 2.181)$), and dense in the open interval $(-1.06, 1.06)$. In particular, X does not contain all its limit points.*

We prove Proposition 1.3 in Section 2. Another subtlety behind $\mathbf{Re}(Z(g))$ is that finding its points in the special case where $n=1$ and the spectrum of g lies in $\mathbb{Z}$ is the same as finding the logarithms of the absolute values of the complex roots of a polynomial. In particular, just deciding $0 \stackrel{?}{\in} \mathbf{Re}(Z(g))$ in this special case is already **NP**-hard [Pla84].

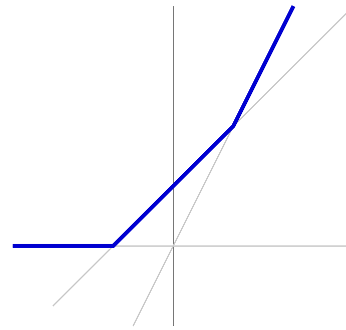
A natural trick we will soon justify is that we can predict real parts by examining pairs of terms of g with large absolute value, in order to locally reduce to the two-term case:

Definition 1.4. *Let us define, for any n-variate exponential t-sum g, with $t \geq 2$, its* tropical variety *as*

$$\mathrm{Trop}(g) := \mathbf{Re}\left(\left\{z \in \mathbb{C}^n \ : \ \max_j \left|e^{a_j\cdot z+\beta_j}\right| \text{ is attained at at least two distinct } j\right\}\right). \qquad \diamond$$

[1][Poo14] provides an excellent survey on undecidability, in the classical Turing model, geared toward non-experts in complexity theory.

The calculation preceding Proposition 1.2 in fact yields $\mathrm{Trop}(g) = \mathbf{Re}(Z(g))$ when $t=2$.



More generally, among many other equivalent characterizations, $\mathrm{Trop}(g)$ can also be defined as the set of points at which the piece-wise linear function $\mathcal{N}_g : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $\mathcal{N}_g(u) := \max_j \{a_j \cdot u + \mathbf{Re}(\beta_j)\}$ is non-differentiable. So, for $n=1$, the graph of $\mathcal{N}_g$ is concave upward, with at most $t-1$ "corners," and thus $\mathrm{Trop}(g)$ consists of at most $t-1$ points. For instance, $g(z_1) := (e^{z_1}+1)^2$ implies that $\mathcal{N}_g(u) = \max\{0, u+\log 2, 2u\}$ (with graph drawn to the right) and thus $\mathrm{Trop}(g) = \{\pm \log 2\}$.

Computing $\mathrm{Trop}(g)$ when $n=1$ is thus no harder than computing a convex hull in $\mathbb{R}^2$, and $\mathbf{Re}(Z(g))$ turns out to always accumulate predictably near $\mathrm{Trop}(g)$. In what follows, we use $\#S$ for the cardinality of a set S.

Theorem 1.5. *Suppose g is any univariate exponential t-sum with spectrum $\{a_1, \ldots, a_t\} \subset \mathbb{R}$, minimal frequency spacing $\delta(g) := \min_{p \neq q} |a_p - a_q|$, and $t \geq 3$. Let $s := \#\mathrm{Trop}(g)$, $u_{\min} := \min \mathrm{Trop}(g)$, $u_{\max} := \max \mathrm{Trop}(g)$, and let U_g be the union of open intervals*

$$\left(u_{\min} - \frac{\log 2}{\delta(g)}, u_{\max} + \frac{\log 2}{\delta(g)}\right) \cap \bigcup_{u \in \mathrm{Trop}(g)} \left(u - \frac{\log 3}{\delta(g)}, u + \frac{\log 3}{\delta(g)}\right).$$

Then $1 \leq s \leq t-1$ and:

(1) $\mathbf{Re}(Z(g)) \subset U_g$.

(2) $\mathbf{Re}(Z(g))$ has at least one point in each connected component of U_g.

(3) For any $u \in \mathrm{Trop}(g)$ there is a root $\zeta \in \mathbb{C}$ of g with

$$|u - \mathbf{Re}(\zeta)| < \frac{(\log 9)s - \log\frac{9}{2}}{\delta(g)} \leq \frac{(\log 9)t - \log\frac{81}{2}}{\delta(g)} < (2.2t - 3.7)/\delta(g).$$

We prove Theorem 1.5 in Section 2.2. The constants $\log 2$ and $\log 3$ in the definition of the neighborhood U_g above are in fact optimal:

Lemma 1.6. *(See, e.g.,* [AKNR18, Cor. 2.3(c) & Lemma 2.5]*.) Consider any real $\delta > 0$, any integer $t \geq 2$, and the exponential sums*

$$g_{1,t}(z_1) := e^{(t-1)\delta z_1} - e^{(t-2)\delta z_1} - \cdots - e^0 \quad , \quad g_{2,t}(z_1) := g_{1,t}(-z_1) \quad , \quad \textit{and}$$

$$g_{3,t}(z_1) := 1 + e^{\delta z_1} + \cdots + e^{(t-1)\delta z_1} - e^{t\delta z_1} + e^{(t+1)\delta z_1 - 1\cdot\log 9} + \cdots + e^{(t+t)\delta z_1 - t\log 9}.$$

Then we have:

(1) $\mathrm{Trop}(g_{1,t}) = \mathrm{Trop}(g_{2,t}) = \{0\}$ but $\mathbf{Re}(Z(g_{1,t}))$ (resp. $\mathbf{Re}(Z(g_{2,t}))$) contains points strictly increasing (resp. strictly decreasing) toward a limit of $\frac{\log 2}{\delta}$ (resp. $-\frac{\log 2}{\delta}$) as $t \longrightarrow \infty$.

(2) $\mathrm{Trop}(g_{3,t}) = \left\{0, \frac{\log 9}{\delta}\right\}$ and $\mathbf{Re}(Z(g_{3,t})) \cap \left[-\frac{\log 3}{\delta}, \frac{\log 3}{\delta}\right]$ is empty. However, for any $\varepsilon > 0$, there is a $t \in \mathbb{N}$ such that $\mathbf{Re}(Z(g_{3,t})) \cap \left(\frac{\log 3}{\delta} - \varepsilon, \frac{\log 3}{\delta} + \varepsilon\right)$ is non-empty. ■

We note that [AKNR18, Cor. 2.3(c) & Lemma 2.5], while phrased in terms of univariate polynomials $f(x_1)$, directly yield Assertions (1) and (2) above upon substituting $x_1 = e^{\delta z_1}$.

The clustering of $\mathbf{Re}(Z(g))$ about $\mathrm{Trop}(g)$ persists in higher dimension.

1.2. **Efficiently Finding Clusters of Real Parts in Arbitrary Dimension.** Our definition of tropical variety generalizes an earlier version defined just for polynomials: When the spectrum of g lies in $\mathbb{Z}^n$, one can associate to our exponential sum g the Laurent polynomial $f(x) := \sum_{j=1}^t e^{\beta_j} x^{a_j} \in \mathbb{C}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$. Recall that the *amoeba* of f is the set

$$\mathrm{Amoeba}(f) := \{(\log|x_1|, \ldots, \log|x_n|) \mid f(x_1, \ldots, x_n) = 0; x_1, \ldots, x_n \in \mathbb{C}^*\}.$$

It is then clear that, under these restrictions, $\mathbf{Re}(Z(g))=\text{Amoeba}(f)$. *Tropical geometry* (see, e.g., [Vir01, PR04, EKL06, Pay09, IMS09, BR10, ABF13, MS15, AKNR18]) enables algebraic varieties over various complete algebraically closed fields (such as $\mathbb{C}$, $\mathbb{C}\langle\langle t\rangle\rangle$, or $\mathbb{C}_p$, to name a few) to be approached polyhedrally. In our notation here, defining $\text{Trop}(f):=\text{Trop}(g)$ results in the *Archimedean* tropical variety of f, whose metric aspects were studied in [AKNR18]. This kind of tropical variety over $\mathbb{C}$ can be traced back to 1893 work of Hadamard revealing how to polyhedrally approximate products of norms of complex roots of univariate polynomials [Had93].

Our $\text{Trop}(\cdot)$ here is thus a small first step toward extending tropical methods from polynomial functions to certain exponential sums. It should be noted that the theory of $\mathcal{A}$-discriminants [GKZ94] now also has a generalization to exponential sums [RR18], and these generalizations have led to sharper bounds in real fewnomial theory [FNR19].

Recall that the *affine span* of a point set $S\subset\mathbb{R}^n$ is the smallest affine subspace of $\mathbb{R}^n$ containing S. Via polyhedral duality (see, e.g., [Grü03, Zie95, dLRS10]), an immediate consequence of our characterization of $\text{Trop}(g)$ via the graph of $\mathcal{N}_g$ is the following fact:

Proposition 1.7. *Let d be the dimension of the affine span of the spectrum of a real n-variate exponential t-sum g. Then $\text{Trop}(g)$ is a polyhedral complex of pure dimension $n-1$, and is connected when $d\geq 2$.* ■

Definition 1.8. *For any n-variate exponential t-sum g, let $\Sigma(\text{Trop}(g))$ denote the polyhedral complex whose cells are exactly the (possibly improper) faces of the closures of the connected components of $\mathbb{R}^n\setminus\text{Trop}(g)$.* ⋄

We can now make precise how easy $\text{Trop}(g)$ is to work with algorithmically. In the theorem below, the underlying computational model is the *BSS model over* $\mathbb{R}$ [BCSS98], and the *input size* of a point in $\mathbb{R}^n$ (resp. an n-variate t-nomial g) is defined to be n (resp. $(n+1)t$), i.e., we merely measure the input size as the number of real numbers fed into a BSS machine.

Theorem 1.9. *Suppose n is fixed. Then there is a polynomial-time algorithm that, for any input $r\in\mathbb{R}^n$ and n-variate exponential t-sum g, outputs the closure — described as an explicit intersection of $O(t^2)$ half-spaces — of the unique cell σ_r of $\Sigma(\text{Trop}(g))$ containing r.*

We prove Theorem 1.9 in Section 3.2.

By applying the standard formula for point-hyperplane distance, and the well-known efficient algorithms for approximating square-roots (see, e.g., [BB88]), Theorem 1.9 implies that we can also efficiently check membership in any ε-neighborhood about $\text{Trop}(g)$. Our complexity bound above, combined with our final main result below, tells us that membership in a neighborhood of $\text{Trop}(g)$ is a tractable and potentially useful relaxation of the problem of deciding membership in $\mathbf{Re}(Z(g))$.

Theorem 1.10. *Let $t\geq 3$ and let g be any n-variate exponential t-sum with spectrum $S:=\{a_1,\ldots,a_t\}\subset\mathbb{R}^n$, minimal frequency spacing $\delta(g):=\min_{p\neq q}|a_p-a_q|$, and d the dimension of the affine span of S. Then $d\leq\min\{n,t-1\}$ and:*

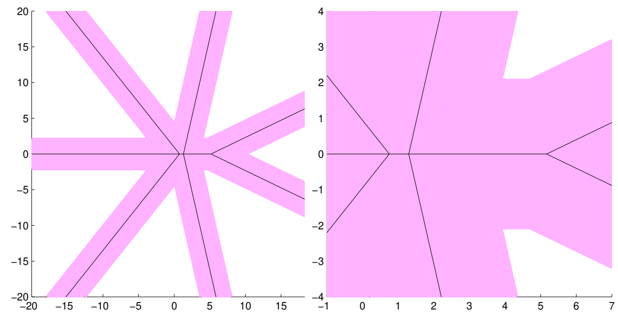
(1) If $t=d+1$ then $\text{Trop}(g)\subseteq\mathbf{Re}(Z(g))$.

(2) If $t\geq d+1$ then (a) $\displaystyle\sup_{r\in\mathbf{Re}(Z(g))}\inf_{u\in\text{Trop}(g)}|r-u|\leq\frac{\log(t-1)}{\delta(g)}$ *and*

$$\text{(b)}\ \sup_{u\in\text{Trop}(g)}\inf_{r\in\mathbf{Re}(Z(g))}|r-u|\leq\sqrt{e}dt^2\left((\log 9)t-\log\frac{81}{2}\right)\Big/\delta(g).$$

(3) *The bound from Assertion (2a) is optimal in the following sense: If* $\delta>0$ *and* $\varphi(z)$ *is defined as* $1+e^{\delta z_1}+\cdots+e^{\delta z_{t-1}}$ *and* $r:=-\log(t-1)(1,\ldots,1)/\delta\in\mathbb{R}^{t-1}$, *then* $\mathbf{Re}(Z(\varphi))\ni r$ *and* $\inf\limits_{u\,\in\,\mathrm{Trop}(g)}|r-u|=(\log(t-1))/\delta$.

Example 1.11. *When* g *is the 2-variate exponential 7-sum* $\sum_{j=0}^{6}\binom{7}{j}e^{\cos(2\pi j/7)z_1+\sin(2\pi j/7)z_2}$, *Theorem 1.10 tells us that every point of* $\mathbf{Re}(Z(g))$ *lies within distance* $\log(6)/\sqrt{(1-\cos(2\pi/7))^2+\sin(2\pi/7)^2}<2.065$ *of some point of* $\mathrm{Trop}(g)$. *To the right, we can see* $\mathrm{Trop}(g)$ *as the black piecewise linear curve drawn on the right, along with the stated neighborhood of* $\mathrm{Trop}(g)$ *containing* $\mathbf{Re}(Z(g))$.



We prove Theorem 1.10 in Section 4. Prior to our work, there have been many fundamental results on the geometric and topological structure of the zero loci of exponential sums, e.g., [Mor73, Kaz81, Kho91, Fav01, Sil08, Sop08, Sil12, Ale13, MSV13]. However, to the best of our knowledge, our results are the first to give an efficient approximation to all of $\mathbf{Re}(Z(g))$ with explicit distance bounds.[2] Recently, Forsgård has found a bound complementary to Assertion (2a) of Theorem 1.10 that is tighter when the number of terms is exponential in the dimension. We rephrase his bound [For16, Thms. 1.2 & 1.3] into our notation below:

Forsgård's Theorem. *Following the notation of Theorem 1.10,*

$$\sup_{r\,\in\,\mathbf{Re}(Z(g))}\ \inf_{u\,\in\,\mathrm{Trop}(g)}|r-u|\leq\frac{2n\sqrt{n}\log(2+\sqrt{3})}{\delta(g)}.$$

In particular, if the spectrum of g *lies in* $\mathbb{Z}^n$, *then the upper bound can be further improved to* $n\log(2+\sqrt{3})$. ■

For instance, for arbitrary real spectra, Forsgård's bound improves Assertion (2a) of our Theorem 1.10 when $t>1+e^{2.634n\sqrt{n}}$.

One can also view the polyhedral structure in Theorem 1.10 as a limit shape of a parametric family of real parts of complex zero sets. Recall that, given any subsets $U,V\subseteq\mathbb{R}^n$, their *Hausdorff distance* is $\Delta(U,V):=\max\left\{\sup\limits_{u\in U}\inf\limits_{v\in V}|u-v|,\sup\limits_{v\in V}\inf\limits_{u\in U}|u-v|\right\}$.

Corollary 1.12. *For any exponential sum* $g(z):=\sum_{j=1}^{t}e^{a_j\cdot z+\beta_j}$ *we define a parametric family of exponential sums via* $g_s(z):=\sum_{j=1}^{t}e^{a_j\cdot z+s\cdot\beta_j}$ *for any* $s>0$. *We then have* $\Delta\big(\frac{1}{s}\mathbf{Re}(Z(g_s)),\mathrm{Trop}(g)\big)\longrightarrow 0$ *as* $s\longrightarrow\infty$.

Proof: First observe that $\mathrm{Trop}(g_s)=s\mathrm{Trop}(g)$ by definition. Applying Theorem 1.10 we then obtain $\Delta(\mathbf{Re}(Z(g_s)),\mathrm{Trop}(g_s))\leq\sqrt{en}t^2(2.2t-3.7)/\delta(g)$. So then, $\Delta(\mathbf{Re}(Z(g_s)),\mathrm{Trop}(g_s))=s\Delta\big(\frac{1}{s}\mathbf{Re}(Z(g_s)),\mathrm{Trop}(g)\big)$ and thus $\Delta\big(\frac{1}{s}\mathbf{Re}(Z(g_s)),\mathrm{Trop}(g)\big)\leq\sqrt{en}t^2(2.2t-3.7)/(\delta(g)s)$. ■

Corollary 1.12 can be thought of as an exponential sum analogue of *Maslov dequantization*. The latter is a process by which one can obtain a (non-Archimedean) tropical variety as a limit of (complex) polynomial amoebae (see, e.g., [Vir01]).

[2] A preliminary version of Theorem 1.10 appeared in our December 2014 Math ArXiV preprint `1412.4423` and was presented by the first author at MEGA 2015 (June 16, University of Trento).

Let us now see a key ingredient, possibly of independent interest, behind the proof of our main multivariate metric bound.

1.3. **Careful Projection to Reduce to the Univariate Case.** Much of the recent literature on random projections aims toward creating random matrices whose corresponding linear maps are "nearly" isometries. The approach is to create a random projection matrix on a geometric object of interest, and the rank of the matrix is ultimately controlled by the statistical dimension of the geometric object [Ver17]. For our proof of Theorem 1.10, we'll need a projection of rank 1 that distorts distances only slightly. Since most of the random matrix literature focusses on asymptotic behavior in high dimensions, we'll use a folkloric result stated as Lemma 1.13 below.

Let $G_{n,k}$ be the Grassmanian of k-dimensional subspaces of $\mathbb{R}^n$, equipped with its unique rotation-invariant Haar probability measure $\mu_{n,k}$.

Lemma 1.13. *(See, e.g.,* [MS00, Fact 3.2(c)] *and* [MP00, Lemma 6]*.) Let* $k \in \{1, \ldots, n-1\}$, $x \in \mathbb{R}^n$, *and* $\varepsilon \leq \frac{1}{\sqrt{e}}$. *Then*

$$\mu_{n,k}\left(\left\{F \in G_{n,k} \;\middle|\; |P_F(x)| \leq \varepsilon\sqrt{\frac{k}{n}} \cdot |x| \right\}\right) \leq \left(\sqrt{e}\varepsilon\right)^k,$$

where P_F *is the surjective orthogonal projection mapping* $\mathbb{R}^n$ *onto* F. ■

(See also [Vem04, Ver17] for more beautiful results on the theory and applications of random projections.) A simple consequence of Lemma 1.13 is the following existential result.

Proposition 1.14. *Let* $\gamma > 0$ *and* $x_1, \ldots, x_N \in \mathbb{R}^n$ *be such that* $|x_i - x_j| \geq \gamma$ *for all distinct* i, j. *Then, following the notation of Lemma 1.13, there is an* $F \in G_{n,k}$ *such that* $|P_F(x_i) - P_F(x_j)| \geq \sqrt{\frac{k}{en}} \cdot \frac{\gamma}{N^{2/k}}$ *for all distinct* i, j.

Proof: Let $z_{\{i,j\}} := |x_i - x_j|$. Then our assumption becomes $z_{\{i,j\}} \geq \gamma$ for all distinct i, j, and there are no more than $N(N-1)/2$ such pairs $\{i, j\}$. By Lemma 1.13 we have, for any fixed $\{i, j\}$, that $|P_F(z_{\{i,j\}})| \geq \varepsilon\sqrt{\frac{k}{n}} z_{\{i,j\}}$ with probability at least $1 - (\sqrt{e}\varepsilon)^k$. So the union bound for probabilities implies that, for all distinct i, j, we have $|P_F(x_i) - P_F(x_j)| \geq \varepsilon\gamma\sqrt{\frac{k}{n}}$ with probability at least $1 - \frac{N(N-1)}{2}(\sqrt{e}\varepsilon)^k$. So our desired F exists when $\varepsilon = \frac{1}{\sqrt{e}N^{2/k}}$ and we are done. ■

We now prove our main results.

2. Extending Classical Univariate Bounds to Exponential Sums: Proving Theorem 1.5

The following simple quantitative bound on exponential sums will prove quite useful. In what follows, we let $[j] := \{1, \ldots, j\}$.

Proposition 2.1. *Suppose* $t \geq 3$ *and* $g(z_1) := \sum_{j=1}^{t} e^{a_j z_1 + \beta_j}$ *satisfies* $a_1 < \cdots < a_t$ *and* $\beta_j \in \mathbb{C}$ *for all* j. *Suppose further that* $u \in \mathrm{Trop}(g)$, ℓ *is the largest index such that*

$\left|e^{a_\ell u+\beta_\ell}\right| = \max_{j\in[t]}\left|e^{a_j u+\beta_j}\right|$, *and we set* $\delta_\ell := \min\limits_{p,q\,\in\,[\ell]\&p\neq q}|a_p - a_q|$. *Then for any* $N\in\mathbb{N}$ *and* $z_1\in\left[u+\frac{\log(N+1)}{\delta_\ell},\infty\right)\times\mathbb{R}$ *we have* $\left|\sum\limits_{j=1}^{\ell-1} e^{a_j z_1+\beta_j}\right| < \frac{1}{N}\left|e^{a_\ell z_1+\beta_\ell}\right|$.

Proof: First note that $2\leq\ell\leq t$ by construction. Let $b_j:=\mathbf{Re}(\beta_j)$, $r:=\mathbf{Re}(z_1)$, and observe

$$\left|\sum_{j=1}^{\ell-1} e^{a_j z_1+\beta_j}\right| \leq \sum_{j=1}^{\ell-1}\left|e^{a_j z_1+\beta_j}\right| = \sum_{j=1}^{\ell-1} e^{a_j r+b_j} = \sum_{j=1}^{\ell-1} e^{a_j(r-u)+a_j u+b_j}.$$

Now, since $a_{j+1}-a_j\geq\delta_\ell$ for all $j\in\{1,\ldots,\ell-1\}$, we obtain $a_j\leq a_\ell-(\ell-j)\delta_\ell$. So for $r>u$ we have $\left|\sum\limits_{j=1}^{\ell-1} e^{a_j z_1+b_j}\right| \leq \sum\limits_{j=1}^{\ell-1} e^{(a_\ell-(\ell-j)\delta_\ell)(r-u)+a_j u+b_j} \leq \sum\limits_{j=1}^{\ell-1} e^{(a_\ell-(\ell-j)\delta_\ell)(r-u)+a_\ell u+b_\ell}$, and thus

$$\begin{aligned}
\left|\sum_{j=1}^{\ell-1} e^{a_j z_1+b_j}\right| &\leq e^{(a_\ell-(\ell-1)\delta_\ell)(r-u)+a_\ell u+b_\ell}\sum_{j=1}^{\ell-1} e^{(j-1)\delta_\ell(r-u)}\\
&= e^{(a_\ell-(\ell-1)\delta_\ell)(r-u)+a_\ell u+b_\ell}\left(\frac{e^{(\ell-1)\delta_\ell(r-u)}-1}{e^{\delta_\ell(r-u)}-1}\right)\\
&< e^{(a_\ell-(\ell-1)\delta_\ell)(r-u)+a_\ell u+b_\ell}\left(\frac{e^{(\ell-1)\delta_\ell(r-u)}}{e^{\delta_\ell(r-u)}-1}\right) = \frac{e^{a_\ell r+b_\ell}}{e^{\delta_\ell(r-u)}-1}.
\end{aligned}$$

To prove our desired inequality, it thus clearly suffices to enforce $e^{(r-u)\delta_\ell}-1\geq N$. The last inequality clearly holds for all $r\geq u+\frac{\log(N+1)}{\delta_\ell}$, so we are done. ∎

It is then easy to prove that the largest (resp. smallest) point of $\mathbf{Re}(Z(g))$ can't be too much larger (resp. smaller) than the largest (resp. smallest) point of $\mathrm{Trop}(g)$. Put another way, we can give an explicit vertical strip containing all the complex roots of g.

Corollary 2.2. *Suppose* g *is a univariate exponential* t*-sum with real spectrum and minimal frequency spacing* $\delta(g):=\min_{p\neq q}|a_p-a_q|$, $u_{\min}:=\min\mathrm{Trop}(g)$, *and* $u_{\max}:=\max\mathrm{Trop}(g)$. *Then* $\mathbf{Re}(Z(g))$ *is contained in the open interval* $\left(u_{\min}-\frac{\log 2}{\delta(g)}, u_{\max}+\frac{\log 2}{\delta(g)}\right)$.

Our earlier Lemma 1.6 tell us that the $\log 2$ in Corollary 2.2 can not be replaced by any smaller constant. While the polynomial analogue of Corollary 2.2 goes back to work of Cauchy, Birkhoff, and Fujiwara pre-dating 1916 (see [RS02, pp. 243–249, particularly bound 8.1.11 on pg. 247] and [Fuj16] for further background) we were unable to find an explicit bound for exponential sums like Corollary 2.2 in the literature. So we supply a proof below.

Proof of Corollary 2.2: Replacing z_1 by its negative, it clearly suffices to prove $\mathbf{Re}(Z(g))\subset\left(-\infty,u_{\max}+\frac{\log 2}{\delta(g)}\right)$. Writing $g(z_1)=\sum_{j=1}^{t} e^{a_j z_1+b_j}$ with $a_1<\cdots<a_t$, let ζ denote any root of g, $r:=\mathbf{Re}(\zeta)$, and $\beta_j:=\mathbf{Re}(b_j)$ for all j. Since we must have $\sum_{j=1}^{t-1} e^{a_j\zeta+b_j} = -e^{a_t\zeta+b_t}$, taking absolute values implies that $\left|\sum_{j=1}^{t-1} e^{a_j\zeta+b_j}\right| = \left|e^{a_t\zeta+b_t}\right|$. However, this equality is contradicted by Proposition 2.1 for $\mathbf{Re}(z_1)\geq u_{\max}+\frac{\log 2}{\delta(g)}$. So we are done. ∎

Proposition 1.3 will then be a simple consequence of Corollary 2.2 and the following special case of a fundamental result of Moreno.

Theorem 2.3. *(Special case of* [Mor73, Main Theorem, pg. 73]*.) Suppose* $1, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ *are linearly independent over* $\mathbb{Q}$*,* $g(z_1) := e^{\alpha_1 z_1} + e^{\alpha_2 z_1} + e^{\alpha_3 z_1}$*,* $\sigma \in \mathbb{R}$*, and the inequalities* $|e^{\alpha_i \sigma}| \leq \sum_{j\in\{1,2,3\}\setminus\{i\}} |e^{\alpha_j \sigma}|$ *hold for all* $i\in\{1,2,3\}$*. Then* σ *is a limit point of* $\mathbf{Re}(Z(g))$*.* ■

Proof of Proposition 1.3: Let $g(z_1) := e^{\sqrt{2}z_1} + e^{\sqrt{3}z_1} + e^{\sqrt{5}z_1}$. Clearly then, $\sqrt{3}-\sqrt{2} < \sqrt{5}-\sqrt{3}$, $\mathrm{Trop}(g)=\{0\}$, and thus Corollary 2.2 immediately implies the containment $X \subseteq \left(-\frac{\log 2}{\sqrt{3}-\sqrt{2}}, \frac{\log 2}{\sqrt{3}-\sqrt{2}}\right)$. Furthermore, since g is an analytic function, its zeroes are isolated, and thus must be countable in number [Ahl79].

Now note that $e^{\sqrt{5}u} > e^{\sqrt{3}u} > e^{\sqrt{2}u}$ for $u > 0$, and this ordering is reversed for $u < 0$. Furthermore, the same orderings apply to the corresponding derivatives. An elementary calculation then reveals that the hypothesis for Theorem 2.3 is satisfied at any σ in the open interval $(-1.06, 1.06)$. So we are done. ■

Our next result isolates vertical strips where no roots of g can lie.

Corollary 2.4. *Suppose* $g(z_1) := \sum_{j=1}^{t} e^{a_j z_1+\beta_j}$ *satisfies* $a_1 < \cdots < a_t$*,* $\beta_j \in \mathbb{C}$ *for all* j*,* $\delta(g) := \min_{p\neq q} |a_p - a_q|$*, and that* u_1 *and* u_2 *are* consecutive *points of* $\mathrm{Trop}(g)$ *satisfying* $u_2 \geq u_1 + \frac{\log 9}{\delta(g)}$*. Then the vertical strip* $\left[u_1 + \frac{\log 3}{\delta(g)}, u_2 - \frac{\log 3}{\delta(g)}\right] \times \mathbb{R}$ *contains* no *roots of* g*.*

Proof: First note that $t\geq 3$ since $\#\mathrm{Trop}(g)\geq 2$. Let ℓ be the unique index such that
$$\left|e^{a_\ell u_1+\beta_\ell}\right| = \max_{j\in[t]} \left|e^{a_j u_1+\beta_j}\right| \text{ and } \left|e^{a_\ell u_2+\beta_\ell}\right| = \max_{j\in[t]} \left|e^{a_j u_2+\beta_j}\right|.$$
There is a unique such index because, by the definition of $\mathrm{Trop}(g)$, the point $(a_\ell, \mathbf{Re}(\beta_\ell))$ lies at the intersection of two lines: One line goes through a pair of distinct points of the form $(a_i, \mathbf{Re}(\beta_i))$ with $\left|e^{a_i u_1+\beta_i}\right| = \max_{j\in[t]} \left|e^{a_j u_1+\beta_j}\right|$, while the other goes through a pair of distinct points of the form $(a_k, \mathbf{Re}(\beta_k))$ with $\left|e^{a_k u_2+\beta_k}\right| = \max_{j\in[t]} \left|e^{a_j u_2+\beta_j}\right|$.

By Proposition 2.1, we have $\left|\sum_{j=1}^{\ell-1} e^{a_j z_1+\beta_j}\right| < \frac{1}{2}\left|e^{a_\ell z_1+\beta_\ell}\right|$ for all $z_1 \in \left[u_1 + \frac{\log 3}{\delta(g)}, \infty\right)$ and, employing the change of variables $z_1 \mapsto -z_1$, we obtain $\left|\sum_{j=\ell+1}^{t} e^{a_j z_1+\beta_j}\right| < \frac{1}{2}\left|e^{a_\ell z_1+\beta_\ell}\right|$ for all $z_1 \in \left(-\infty, u_2 - \frac{\log 3}{\delta(g)}\right]$. So $\left|\sum_{j\neq\ell} e^{a_j z_1+\beta_j}\right| < \left|e^{a_\ell z_1+\beta_\ell}\right|$ in the stated vertical strip, and this inequality clearly contradicts the existence of a root of g in $\left[u_1 + \frac{\log 3}{\delta(g)}, u_2 - \frac{\log 3}{\delta(g)}\right] \times \mathbb{R}$. ■

An immediate consequence of Corollary 2.4 is that the roots of g always lie in the union of open vertical strips $\bigcup_{u\in\mathrm{Trop}(g)} \left(u - \frac{\log 3}{\delta(g)}, u + \frac{\log 3}{\delta(g)}\right) \times \mathbb{R}$. It will in fact be the case that each connected component of this union contains roots of g as well. To prove this, we will need some refined integral estimates.

2.1. Winding Numbers and Rectangles Around Tropical Points.

It will be useful to first observe a basic fact on winding numbers along line segments.

Proposition 2.5. *Suppose* $I \subset \mathbb{C}$ *is any (compact) line segment and* g *and* h *are functions analytic on a neighborhood of* I *with* $|h(z)| < |g(z)|$ *for all* $z\in I$*. Then* $\left|\mathbf{Im}\left(\int_I \frac{g'+h'}{g+h}dz - \int_I \frac{g'}{g}dz\right)\right| < \pi$*.*

Proof: The quantity $V_1 := \mathbf{Im}\left(\int_I \frac{g'}{g}dz\right)$ (resp. $V_2 := \mathbf{Im}\left(\int_I \frac{g'+h'}{g+h}dz\right)$) is nothing more than the variation of the argument of g (resp. $g+h$) along the segment I. Since I is compact, $|g|$ and $|g+h|$ are bounded away from 0 on I by construction. So we can lift the paths $g(I)$

and $(g+h)(I)$ (in $\mathbb{C}^*$) to the universal covering space induced by the extended logarithm function. Clearly then, V_1 (resp. V_2) is simply a difference of values of $\mathbf{Im}(\mathrm{Log}(g))$ (resp. $\mathbf{Im}(\mathrm{Log}(g+h))$), evaluated at the endpoints I, where different branches of Log may be used at each endpoint. In particular, at any $z\in I$, our assumptions on $|g|$ and $|h|$ clearly imply that $g(z)+h(z)$ and $g(z)$ both lie in the open half-plane normal (as a vector in $\mathbb{R}^2$) to $g(z)$. So $|\mathbf{Im}(\mathrm{Log}(g(z)+h(z)))-\mathbf{Im}(\mathrm{Log}(g(z)))|<\frac{\pi}{2}$ at each of the two endpoints of I, and thus $|V_1-V_2|<\frac{\pi}{2}+\frac{\pi}{2}=\pi$. ■

We will also need the following technical fact on the total variation of the imaginary part of g'/g along horizontal line segments.

Theorem 2.6. [Voo79, Thm. 2] *Let $g(z_1):=\sum_{j=1}^t e^{a_jz_1+\beta_j}$ with $a_1<\cdots<a_t$ and $\beta_j\in\mathbb{C}$ for all j. Also let $u,v\in\mathbb{R}$ with $g(u)g(v)\neq 0$ and define N to be the number of roots of g on the closed interval $[u,v]$. Then* $\displaystyle\int_u^v\left|\mathbf{Im}\left(\frac{g'(z)}{g(z)}\right)\right|dz+N\pi\leq(t-1)\pi$. ■

Note that since the β_j are allowed to be complex, the bound above continues to hold if we integrate over any horizontal line segment in $\mathbb{C}$. Voorhoeve proved earlier in [Voo76, Lemma 1] that, for any f meromorphic on an interval $[u,v]\subset\mathbb{R}$, the function $\mathbf{Im}\left(\frac{f'}{f}\right)$ is analytic on $[u,v]$, save for a finite set of removable singularities. So the integral above is well-defined even if g vanishes in the open interval (u,v). [Voo79, Thm. 2] in fact gives a sharper upper bound depending on the imaginary parts of the differences of the β_j, but we will only need the weaker bound stated above. See also [Voo77] for an elegant and fascinating development of root counts for univariate exponential polynomials in various regions.

We now state our final key root count behind Theorem 1.5.

Lemma 2.7. *Let $g(z_1):=\sum_{j=1}^t e^{a_jz_1+\beta_j}$ with $t\geq 3$, $a_1<\cdots<a_t$, $\beta_j\in\mathbb{C}$ for all j, and let $\delta(g):=\min_{p\neq q}|a_p-a_q|$, $u_{\min}:=\min\mathrm{Trop}(g)$, and $u_{\max}:=\max\mathrm{Trop}(g)$. Let U_g be the union of open intervals $\left(u_{\min}-\frac{\log 2}{\delta(g)},u_{\max}+\frac{\log 2}{\delta(g)}\right)\cap\bigcup_{u\in\mathrm{Trop}(g)}\left(u-\frac{\log 3}{\delta(g)},u+\frac{\log 3}{\delta(g)}\right)$. Let Γ be any connected component of U_g and let p (resp. q) be the minimal (resp. maximal) index such that $\left|e^{a_p\cdot u+\beta_p}\right|=\max_j\left|e^{a_j\cdot u+\beta_j}\right|$ (resp. $\left|e^{a_q\cdot u+\beta_q}\right|=\max_j\left|e^{a_j\cdot u+\beta_j}\right|$) for some $u\in\Gamma$. Then $q>p$ and g has at least one root in the rectangle $\Gamma\times\left[0,\frac{2(t+1)\pi}{\delta(g)}\right]$.*

Proof of Lemma 2.7: That $q>p$ follows easily from the definition of $\mathrm{Trop}(g)$: $\mathrm{Trop}(g)\cap\Gamma$ is non-empty by construction, and if $u\in\Gamma\setminus\mathrm{Trop}(g)$ then $\max_j\left|e^{a_j\cdot u+\beta_j}\right|$ is attained exactly once. Furthermore, at least two terms of g must be maximized in norm at any $u\in\mathrm{Trop}(g)\cap\Gamma$, and p (resp. q) must be no larger (resp. no smaller) than the index of any such term.

Now let $\gamma_{\inf}:=\inf\Gamma$ and $\gamma_{\sup}:=\sup\Gamma$. Since g is analytic, the Argument Principle (see, e.g., [Ahl79]) tells us that the number of roots in our rectangle in question is exactly

$$A:=\frac{1}{2\pi\sqrt{-1}}\int_{I_-\cup I_+\cup J_-\cup J_+}\frac{g'(z)}{g(z)}dz$$

where I_- (resp. I_+, J_-, J_+) is the oriented line segment from

$$\left(\gamma_{\inf},\frac{2(t+1)\pi}{\delta(g)}\right)\text{ (resp. }(\gamma_{\sup},0),\ (\gamma_{\inf},0),\ \left(\gamma_{\sup},\frac{2(t+1)\pi}{\delta(g)}\right))$$

to

$$(\gamma_{\inf},0)\text{ (resp. }\left(\gamma_{\sup},\frac{2(t+1)\pi}{\delta(g)}\right),\ (\gamma_{\sup},0),\ \left(\gamma_{\inf},\frac{2(t+1)\pi}{\delta(g)}\right)),$$

assuming no root of g lies on $I_- \cup I_+ \cup J_- \cup J_+$. By Corollaries 2.2 and 2.4, there can be no roots of g on $I_- \cup I_+$. So let assume temporarily that there are no roots of g on $J_- \cup J_+$.

By construction, any point of $\mathrm{Trop}(g) \cap \Gamma$ is at least distance $\frac{\log 9}{\delta(g)}$ from any point of $\mathrm{Trop}(g) \setminus \Gamma$. So Proposition 2.1 tells us that when $p>1$ we have:

$$\frac{1}{2}\left|e^{a_p\left(\gamma_{\inf}+v\sqrt{-1}\right)+\beta_p}\right| > \left|\sum_{j=1}^{p-1} e^{a_j\left(\gamma_{\inf}+v\sqrt{-1}\right)+\beta_j}\right| \quad \text{and} \quad \frac{1}{2}\left|e^{a_p\left(\gamma_{\inf}+v\sqrt{-1}\right)+\beta_p}\right| > \left|\sum_{j=p+1}^{t} e^{a_j\left(\gamma_{\inf}+v\sqrt{-1}\right)+\beta_j}\right|$$

for all $v \in \mathbb{R}$. So then, $\left|e^{a_p\left(\gamma_{\inf}+v\sqrt{-1}\right)+\beta_p}\right| > \left|\sum_{j\neq p} e^{a_j\left(\gamma_{\inf}+v\sqrt{-1}\right)+\beta_j}\right|$. (When $p=1$ Proposition 2.1 yields the same conclusion in just one step.) So we can apply Proposition 2.5 and deduce that $\left|\mathbf{Im}\left(\int_{I_-} \frac{g'(z)}{g(z)}dz - \int_{I_-} \frac{(e^{a_p z+\beta_p})'}{e^{a_p z+\beta_p}}dz\right)\right| < \pi$. So then, since $\int_{I_-} \frac{(e^{a_p z+\beta_p})'}{e^{a_p z+\beta_p}}dz = \int_{I_-} a_p dz = \frac{-2\pi\sqrt{-1}(t+1)a_p}{\delta(g)}$, we clearly obtain

$$\left|\mathbf{Im}\left(\int_{I_-} \frac{g'(z)}{g(z)}dz\right) - \frac{-2\pi\sqrt{-1}(t+1)a_p}{\delta(g)}\right| < \pi. \tag{1}$$

An almost identical argument (applying Propositions 2.1 and 2.5 again, but with the term $\left|e^{a_q(\gamma_{\sup}+v\sqrt{-1})+\beta_q}\right|$ dominating instead) then yields

$$\left|\mathbf{Im}\left(\int_{I_+} \frac{g'(z)}{g(z)}dz\right) - \frac{2\pi\sqrt{-1}(t+1)a_q}{\delta(g)}\right| < \pi. \tag{2}$$

So now we need only prove sufficiently sharp estimates on $\mathbf{Im}\left(\int_{J_\pm} \frac{g'(z)}{g(z)}dz\right)$. Toward this end, observe that Theorem 2.6 implies directly that $\displaystyle\int_{J_\pm} \left|\mathbf{Im}\left(\frac{g'(z)}{g(z)}\right)\right| dz \leq (t-1)\pi$. So combining with our estimates (1) and (2), and the additivity of integration, we obtain $\left|A - \dfrac{(a_q-a_p)(t+1)}{\delta(g)}\right| < t$, in the special case where no roots of g lie on $J_- \cup J_+$.

To address the case where a root of g lies on $J_- \cup J_+$, note that the analyticity of g implies that the roots of g are a discrete subset of $\mathbb{C}$. So we can find arbitrarily small $\eta>0$ with the boundary of the slightly stretched rectangle $\Gamma \times \left[-\eta, \frac{2(t+1)\pi}{\delta(g)}+\eta\right]$ not intersecting any roots of g, and define a similar normalized integral implementing the Argument Principle, which we'll call A_η, over the new contour. By the special case of our lemma already proved, we have $\left|A_\eta - \dfrac{(a_q-a_p)\left(t+1+\frac{\eta\delta(g)}{\pi}\right)}{\delta(g)}\right| < t$. Let n_Γ be the number of roots of g in the rectangle $\Gamma \times \left[0, \frac{2\pi t}{\delta(g)}\right]$. Since $A_\eta = n_\Gamma$ for η sufficiently small, we obtain $\left|n_\Gamma - \dfrac{(a_q-a_p)(t+1)}{\delta(g)}\right| \leq t$. So $n_\Gamma \geq \frac{(a_q-a_p)(t+1)}{\delta(g)} - t \geq \frac{(t+1)\delta(g)}{\delta(g)} - t = 1$, and g thus indeed has at least one root in $\Gamma \times \left[0, \frac{2(t+1)\pi}{\delta(g)}\right]$. ■

2.2. **The Proof of Theorem 1.5:** First note that the graph of $\mathcal{N}_g$ is the lower hull of an intersection P of exactly t half-planes with edges of distinct slopes. So the polyhedron P has at most t edges, at most $t-1$ vertices, at least one vertex (since $t \geq 2$), and thus the graph of $\mathcal{N}_g$ has at most $t-1$ corners since corners correspond to vertices of P. We thus obtain $s \in \{1, \ldots, t-1\}$.

Assertion (1) on the containment $\mathbf{Re}(Z(g)) \subset U_g$ is immediate from Corollaries 2.2 and 2.4.

Assertion (2) on each connected component of U_g containing at least one point from $\mathbf{Re}(Z(g))$ is immediate from Lemma 2.7.

To prove Assertion (3), we must show that near every point $u \in \mathrm{Trop}(g)$ there is a point $r \in \mathbf{Re}(Z(g))$ within distance $\frac{(\log 9)s - \log \frac{9}{2}}{\delta(g)}$. (The remaining inequalities follow from the fact that $s \leq t-1$ and an elementary calculation.) So let Γ be the unique connected component of U_g containing u and let $m := \#(\mathrm{Trop}(g) \cap \Gamma)$. We will prove that there is an $r \in \mathbf{Re}(Z(g))$ within the distance $\frac{(\log 9)m - \log \frac{9}{2}}{\delta(g)}$ of v, yielding an inequality at least as tight as needed. Toward this end, observe first that consecutive points of $\mathrm{Trop}(g) \cap \Gamma$ must be within distance strictly less than $\frac{\log 9}{\delta(g)}$. The maximal possible distance between v and r occurs when these two points lie at opposite extremes of the open interval Γ. Since v must be no closer than $\frac{\log 2}{\delta(g)}$ to an endpoint of Γ, the maximal possible distance must be $\mathrm{Length}(\Gamma) - \frac{\log 2}{\delta(g)} < \frac{\log 2}{\delta(g)} + \frac{\log 3}{\delta(g)} + \frac{\log 3}{\delta(g)} + (m-2)\frac{\log 9}{\delta(g)} - \frac{\log 2}{\delta(g)} = \frac{(\log 9)m - \log \frac{9}{2}}{\delta(g)}$, assuming (without loss of generality) that v is as far left as possible and r is as far right as possible. ■

3. Algorithmic Complexity: The Proofs of Theorems 1.1 and 1.9

The *BSS model over* $\mathbb{R}$ [BCSS98] naturally augments the classical *Turing machine* [Pap95, AB09, Sip12] by allowing field operations and comparisons over $\mathbb{R}$ in unit time. We are in fact forced to move beyond the Turing model since our exponential sums involve arbitrary real numbers, and the Turing model only allows finite bit strings as inputs. We refer the reader to [BCSS98] for further background.

We recall here some basic facts about the set of inputs on which a BSS machine over $\mathbb{R}$ terminates.

Theorem 3.1. [BCSS98, Thm. 1, Pg. 52] *The halting set of a BSS machine over* $\mathbb{R}$ *is a countable union of semi-algebraic sets.* ■

The converse of Theorem 3.1 fails in general: For instance, if S is any countably infinite subset of a transcendence basis for $\mathbb{R}$ over $\mathbb{Q}$, then S can not be the halting set of any BSS machine over $\mathbb{R}$. (One can even write such subsets in terms of infinite series, via an explicit basis found by von Neumann [vNeu28] around 1928.) This follows immediately from the following consequence of the development in [BCSS98, Sec. 2.3]:

Proposition BCS. *Any countable subset of* $\mathbb{R}$ *that is the halting set for a BSS machine over* $\mathbb{R}$ *must be a subset of the algebraic closure of a real extension of* $\mathbb{Q}$ *of finite transcendence degree.* ■

Let us also recall the following basic facts about *semi-algebraic* sets, i.e., the solution sets of finite collections of polynomial inequalities and polynomial equalities in $\mathbb{R}^n$: First, semi-algebraic sets are closed under all Boolean operations (intersection, union, and complement). Also, semi-algebraic sets admit a natural notion of dimension, via the largest d permitting a semi-algebraic embedding of a real d-ball (see, e.g., [BPR06, Ch. 5, Sec. 5.3, pp. 170–172]). Some additional qualitative facts we'll also need can be summarized as follows:

Semi-Algebraic Tameness Theorem. *Suppose* $S \subset \mathbb{R}^2$ *is semi-algebraic, and* $\bar{S}$ *and* S° *respectively denote the closure and interior of* S*. Then:*

1. $\bar{S}$, S°, *and* $\partial S := \bar{S} \setminus S^\circ$ *are semi-algebraic.*

2. S *has only finitely many connected components, each of which is semi-algebraic.*

3. If, in addition, S is a connected curve, then S has only finitely many singularities.
4. Let $\rho : \mathbb{R}^2 \longrightarrow \mathbb{R}$ denote the projection defined by $\rho(x,y)=x$. Then, continuing Assertion (3), there is an $n_S \in \mathbb{N}$ such that all fibers of ρ have cardinality at most n_S.

Note: *Neither Proposition BCS nor the preceding tameness theorem appeared in the published (Mathematische Annalen, Vol. 377, pp. 863–882 (2020)) version of this paper. In particular, we inserted the theorem above after Alexander Rashkovskii kindly pointed out in late July 2020 that our published proof of Theorem 1.1 had an error. We inserted Proposition BCS after a discussion with Lenore Blum, Felipe Cucker, and Mike Shub, on the simplest failures of the converse of Theorem 3.1. We apply the tameness theorem to give a corrected proof of Theorem 1.1 below, but we will have no further need for Proposition BCS.* $\diamond$

Proof of the Semi-Algebraic Tameness Theorem: The first two assertions of (1) are exactly the content of [BPR06, Ch. 3, Prop. 3.1, pg. 84]. The final assertion of (1) is then immediate since semi-algebraic sets are closed under Boolean operations by definition.

Assertion (2) is immediate from the notion of *cylindrical decomposition*. The latter is a refined decomposition of a semi-algebraic set into finitely many (semi-algebraic) connected components, and the existence of such a decomposition is a classical fact: See, e.g., [BPR06, Ch. 5, Thm. 5.6, pg. 163]. In particular, the tameness of fibers from Assertion (4) is also an immediate consequence of cylindrical decomposition.

Assertion (3) is a direct consequence of the notion of *semi-algebraic cell stratification of* $\mathbb{R}^2$ *adapted to* S. The latter is a partition S into finitely many semi-algebraic smooth manifolds (here, each diffeomorphic to an open interval or a point) called *strata*, such that the closure of any stratum is a union of strata. That such stratifications exist (and in much greater generality) is also a classical fact: See, e.g., [BPR06, Ch. 5, Thm. 5.38, pg. 177]. ■

3.1. **The Proof of Theorem 1.1.** Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and let
$$R := \mathbf{Re}(Z(1-e^{z_1}-e^{z_2})) \quad \text{and} \quad S := \{(\log|x|, \log|y|) \mid 1-x-y=0;\ x,y\in\mathbb{C}^*\}.$$
Via the equality $\log\left|e^{\alpha+\sqrt{-1}\beta}\right| = \alpha$ (valid for any $\alpha,\beta\in\mathbb{R}$) we see that
$(x,y)\in\mathbf{Re}(Z(1-e^{z_1}-e^{z_2})) \Longleftrightarrow (x,y)=(\log|e^{z_1}|, \log|e^{z_2}|)$ for some $(z_1,z_2)\in\mathbb{C}^2$ with $1-e^{z_1}-e^{z_2}=0$. Since the exponential function defines a surjection from $\mathbb{C}$ onto $\mathbb{C}^*$ we then clearly have $R=S$.

Now note that $J := \{(|w_1|,|w_2|) \mid 1-w_1-w_2=0;\ w_1,w_2\in\mathbb{C}^*\}$ is exactly the following semi-infinite strip with corners deleted: $I := \{(x,y)\in\mathbb{R}^2 \mid -1\leq y-x\leq 1,\ x+y\geq 1, \text{ and } xy\neq 0\}$. This is because $w_1+w_2=1 \Longrightarrow |w_1+w_2|=1$, $|w_1|=|1-w_2|=|w_2-1|$, and $|w_2|=|1-w_1|=|w_1-1|$. So by the Triangle Inequality we obtain $|w_1|+|w_2|\geq 1$, $|w_1|\geq ||w_2|-1|$, and $|w_2|\geq ||w_1|-1|$, and thus (setting $x=|w_1|$ and $y=|w_2|$) we obtain $J\subseteq I$. To see that $I\subseteq J$, assume $(x,y)\in I$ and consider $y_\theta := 1+xe^{\theta\sqrt{-1}}$ for $\theta\in[0,\pi]$. Clearly $x>0$. So then $|y_\theta|^2=(1+(\cos\theta)x)^2+(\sin\theta)^2x^2=1+2(\cos\theta)x+x^2$ is a decreasing differentiable function of θ, with $|y_0|=x+1$ and $|y_\pi|=|x-1|$. Since $|x-1|\leq y\leq x+1$ there must then be a $\theta\in[0,\pi]$ with $y=|y_\theta|$. Letting $w_1 := -xe^{\theta\sqrt{-1}}$ and $w_2 := y_\theta$, we then obtain $w_1+w_2=1$, $|w_1|=|x|=x$, and $|w_2|=y$. So we have obtained $I\subseteq J$ and thus $I=J$.

Clearly then, R is simply the image of I under the (differentiable) coordinate-wise logarithm map. In particular, we see that the curve Y defined by $y=\log(1+e^x)$, as x ranges over all of $\mathbb{R}$, is a connected component of the boundary ∂R.

By Theorem 3.1, if membership in R is decidable, then R must be a countable union $\bigcup_{i\in\mathbb{N}} S_i$ of semi-algebraic sets S_i. Let $W := Y\cap([0,1]\times\mathbb{R})$, abusing notation slightly by identifying $\mathbb{C}$ with $\mathbb{R}^2$. Then W is compact and infinite, and thus some S_i must have $W\cap S_i$ infinite. Note in particular that $W\cap S_i^\circ=\emptyset$ (since S_i° is in the interior of R) and thus (by the Semi-Algebraic Tameness Theorem) $S_i\setminus S_i^\circ$ must be a finite union of isolated points and smooth connected semi-algebraic curves. In particular, S_i must contain a smooth connected semi-algebraic curve C such that $W\cap C$ is infinite. Recalling that $\rho:\mathbb{R}^2\longrightarrow\mathbb{R}$ is the projection defined by $\rho(x,y)=x$, we may assume further that C is the graph of a smooth algebraic function f on a non-empty open sub-interval of $(0,1)$, via the Implicit Function Theorem and Assertion (4) of the Semi-Algebraic Tameness Theorem. (In particular, this might entail replacing C with a non-empty, connected (and semi-algebraic), open subset of C.)

Now observe that $W\cap C$ (resp. $\rho(W\cap C)$) must have at least one accumulation point since W (resp. $[0,1]$) is compact, and thus the graphs of the smooth functions $\log(1+e^x)$ and f agree on an infinite sequence of points with a limit point. But this is impossible, since $\log(1+e^x)$ is a transcendental function. In particular, since $\log(1+e^x)$ is analytic on the domain $\mathbb{R}\times(-\pi,\pi)$, the function f must have an analytic continuation to an algebraic function with an essential singularity at ∞ [Ahl79, Pg. 127]. Since algebraic functions can only have zeroes or poles of finite fractional order at ∞, we obtain a contradiction. ■

Note: *The key obstruction to membership in* $\mathbf{Re}(Z(g))$ *being decidable — demonstrated above — is that the* boundary *of* $\mathbf{Re}(Z(1-e^{z_1}-e^{z_2}))$ *is not expressible as a countable union of semi-algebraic sets. Indeed, it is easy to express the interior* R° *of* $R:=\mathbf{Re}(Z(1-e^{z_1}-e^{z_2}))$ *as a countable union of disks: simply consider the union of all open disks of maximal radius that lie in* R° *and are centered at some rational point in* R°. ⋄

3.2. **Proving Theorem 1.9.** We will need some supporting results on linear programming before starting our proof of Theorem 1.9. The results we'll need are covered with great clarity in well-known monographs such as [Sch86, GLS93, Gri13].

Definition 3.2. *Given any matrix* $M\in\mathbb{R}^{N\times n}$ *with* $i^{\underline{th}}$ *row* m_i, *and* $c:=(c_1,\ldots,c_N)\in\mathbb{R}^N$, *the notation* $Mx\leq c$ *means that* $m_1\cdot x\leq c_1,\ldots,m_N\cdot x\leq c_N$ *all hold. These inequalities are called* constraints, *and the set of all* $x\in\mathbb{R}^N$ *satisfying* $Mx\leq c$ *is called the* feasible region *of* $Mx\leq c$. *We also call a constraint* active *if and only if it holds with equality. Finally, we call a constraint* redundant *if and only if the corresponding row of* M *and corresponding entry of* c *can be deleted without affecting the feasible region of* $Mx\leq c$. ⋄

Lemma 3.3. *Suppose* n *is fixed. Then, given any* $c\in\mathbb{R}^N$ *and* $M\in\mathbb{R}^{N\times n}$, *we can, in time polynomial in* N, *find a submatrix* M' *of* M, *and a subvector* c' *of* c, *such that the feasible regions of* $Mx\leq c$ *and* $M'x\leq c'$ *are equal, and* $M'x\leq c'$ *has no redundant constraints. Furthermore, in time polynomial in* N, *we can also enumerate all maximal sets of active constraints defining vertices of the feasible region of* $Mx\leq c$. ■

Note that we are using the BSS model over $\mathbb{R}$ in the preceding lemma. In particular, we are counting just field operations and comparisons over $\mathbb{R}$, and these are the only operations needed above. There are many possible choices for the underlying algorithm: For instance, the classical Simplex Algorithm (using, say, Bland's Anticycling Rule) very easily yields Lemma 3.3. Note that the assumption that n be fixed is critical: As of October 2018, it is still an open problem whether Linear Programming can be done in time polynomial in *both* n and N in the BSS model over $\mathbb{R}$ (a.k.a. *strongly polynomial-time* in older terminology).

Proof of Theorem 1.9: Let $r\in\mathbb{R}^n$ be our input query point. Let $b_j:=\mathbf{Re}(\beta_j)$ for all j. Using $O(t\log t)$ comparisons and $O(n)$ arithmetic operations, we can first isolate all indices such that $\max_j\{a_j\cdot r+b_j\}$ is attained, so let j_0 be any such index. (Note that these are the same indices we would obtain if we were to maximize $\left|e^{a_j\cdot z+\beta_j}\right|$.) We then obtain, say, J equations of the form $a_j\cdot r+b_j=a_{j_0}\cdot r+b_{j_0}$ and K inequalities of the form $a_j\cdot r+b_j<a_{j_0}\cdot r+b_{j_0}$.

Thanks to Lemma 3.3, we can determine the exact cell of $\mathrm{Trop}(f)$ containing r if $J\geq 2$. Otherwise, we obtain the unique cell of $\mathbb{R}^n\setminus\mathrm{Trop}(f)$ with relative interior containing r. Note also that an $(n-1)$-dimensional face of either kind of cell must be contained in a hyperplane of the form $\{u\in\mathbb{R}^n \mid (a_{j_1}-a_{j_2})\cdot u+(b_{j_1}-b_{j_2})=0\}$ for some distinct indices j_1 and j_2. So there are at most $t(t-1)/2$ such $(n-1)$-dimensional faces, and thus σ_r is the intersection of at most $t(t-1)/2$ half-spaces. So we are done. ■

4. The Proof of Our Main Multivariate Bound: Theorem 1.10

Let us first observe that $d\leq\min\{n,t-1\}$ follows immediately from the basic fact that any d-polytope in $\mathbb{R}^n$ has dimension at most n and at least $d+1$ vertices.

In what follows, for any real $n\times n$ matrix M and $z\in\mathbb{R}^n$, we assume that z is a column vector when we write Mz. Also, for any subset $S\subseteq\mathbb{R}^n$, the notation $MS:=\{Mz \mid z\in S\}$ is understood. The following simple functoriality properties of $\mathrm{Trop}(g)$ and $\mathbf{Re}(Z(g))$ will prove useful.

Proposition 4.1. *Suppose g_1 and g_2 are n-variate exponential t-sums, $\alpha\in\mathbb{C}^*$, $a\in\mathbb{R}^n$, $\beta:=(\beta_1,\ldots,\beta_n)\in\mathbb{C}^n$, and g_2 satisfies the identity $g_2(z)=\alpha e^{a\cdot z}g_1(z_1+\beta_1,\ldots,z_n+\beta_n)$. Then $\mathbf{Re}(Z(g_2))=\mathbf{Re}(Z(g_1))-\mathbf{Re}(\beta)$ and $\mathrm{Trop}(g_2)=\mathrm{Trop}(g_1)-\mathbf{Re}(\beta)$. Also, if $M\in\mathbb{R}^{n\times n}$ and we instead have the identity $g_2(z)=g_1(Mz)$, then $M\mathbf{Re}(Z(g_2))=\mathbf{Re}(Z(g_1))$ and $M\mathrm{Trop}(g_2)=\mathrm{Trop}(g_1)$.* ■

4.1. The Proof of Assertion (1) of Theorem 1.10.

First note that, thanks to Proposition 4.1, an invertible linear change of variables allows us to reduce to the special case $\{a_1,\ldots,a_{n+1}\}=\{\mathbf{O},e_1,\ldots,e_n\}$, where $\mathbf{O}$ and $\{e_1,\ldots,e_n\}$ are respectively the origin and standard basis vectors of $\mathbb{R}^n$. But this special case is well-known: One can either prove it directly, or avail to earlier work on the spines of amoebae, e.g., [PR04, Thms. 1 & 2]. (See also [For98, Prop. 3.1.8] or the remark following Theorem 8 on Page 33, and Theorem 12 on Page 36, of [Rul03] for precursors.) ■

4.2. The Proof of Assertion (2a) of Theorem 1.10.

By Assertion (2b) (proved independently in the next section) $Z(g)$ is non-empty. So pick any $z\in Z(g)$, let $r:=\mathbf{Re}(z)$, and assume without loss of generality that $\left|e^{a_1\cdot z+\beta_1}\right|\geq\left|e^{a_2\cdot z+\beta_2}\right|\geq\cdots\geq\left|e^{a_t\cdot z+\beta_t}\right|$. Since $g(z)=0$ implies $\left|e^{a_1\cdot z+\beta_1}\right|=\left|e^{a_2\cdot z+\beta_2}+\cdots+e^{a_t\cdot z+\beta_t}\right|$, the Triangle Inequality immediately implies that $\left|e^{a_1\cdot z+\beta_1}\right|\leq(t-1)\left|e^{a_2\cdot z+\beta_2}\right|$. Letting $b_j:=\mathbf{Re}(\beta_j)$ for all j and then taking logarithms we obtain

$$a_1\cdot r+b_1\geq\cdots\geq a_t\cdot r+b_t \quad\text{and} \tag{3}$$

$$a_1\cdot r+b_1\leq\log(t-1)+a_2\cdot r+b_2 \tag{4}$$

For each $j\in\{2,\ldots,t\}$ let us then define η_j to be the shortest vector such that

$$a_1\cdot(r+\eta_j)+b_1=a_j\cdot(r+\eta_j)+b_j.$$

Note that $\eta_j=\lambda_j(a_j-a_1)$ for some nonnegative λ_j since we are trying to affect the dot-product $\eta_j\cdot(a_1-a_j)$. In particular, $\lambda_j=\frac{(a_1-a_j)\cdot r+b_1-b_j}{|a_1-a_j|^2}$ so that $|\eta_j|=\frac{|(a_1-a_j)\cdot r+b_1-b_j|}{|a_1-a_j|}$. (Indeed, Inequality (3) implies that $(a_1-a_j)\cdot r+b_1-b_j\geq 0$.)

Inequality (4) implies that $(a_1 - a_2) \cdot r + b_1 - b_2 \leq \log(t-1)$. We thus obtain $|\eta_2| \leq \frac{\log(t-1)}{|a_1-a_2|} \leq \frac{\log(t-1)}{\delta(g)}$. So let $j_0 \in \{2,\ldots,t\}$ be any j minimizing $|\eta_j|$. We of course have $|\eta_{j_0}| \leq (\log(t-1))/\delta(g)$ and, by the definition of η_{j_0}, we have

$$a_1 \cdot (r+\eta_{j_0}) + b_1 = a_{j_0} \cdot (r+\eta_{j_0}) + b_{j_0}.$$

Moreover, the fact that η_{j_0} is the shortest among the η_j implies that

$$a_1 \cdot (r+\eta_{j_0}) + b_1 \geq a_j \cdot (r+\eta_{j_0}) + b_j$$

for all j: Otherwise, for some j', we would have $a_1 \cdot (r+\eta_{j_0}) + b_1 < a_{j'} \cdot (r+\eta_{j_0}) + b_{j'}$ and $a_1 \cdot r + b_1 \geq a_{j'} \cdot r + b_{j'}$ (the latter following from Inequality (3)). Taking a convex linear combination of the last two inequalities, we would then obtain a $\mu \in [0,1)$ such that

$$a_1 \cdot (r+\mu\eta_{j_0}) + b_1 = a_{j'} \cdot (r+\mu\eta_{j_0}) + b_{j'}.$$

Thus, by the definition of $\eta_{j'}$, we would obtain $|\eta_{j'}| \leq \mu|\eta_{j_0}| < |\eta_{j_0}|$ — a contradiction.

We thus have (i) $a_1 \cdot (r+\eta_{j_0}) + b_1 = a_{j_0} \cdot (r+\eta_{j_0}) + b_{j_0}$, (ii) $a_1 \cdot (r+\eta_{j_0}) + b_1 \geq a_j \cdot (r+\eta_{j_0}) + b_j$ for all j, and (iii) $|\eta_{j_0}| \leq (\log(t-1))/\delta(g)$. Together, these inequalities imply that $u := r+\eta_{j_0} \in \mathrm{Trop}(g)$ and $|r-u| \leq (\log(t-1))/\delta(g)$. ■

4.3. **The Proof of Assertion (2b) of Theorem 1.10.** Thanks to Proposition 4.1, we can apply a suitable orthogonal linear change of variables to assume that $d = n$. By the $k=1$ case of Proposition 1.14 we then deduce that there exists a unit vector $\theta \in \mathbb{R}^n$ such that

$$\min_{i \neq j} |a_i \cdot \theta - a_j \cdot \theta| \geq \frac{\delta(g)}{\sqrt{e}nt^2} \tag{5}$$

Now let $u \in \mathrm{Trop}(g)$ and write $u = u_\theta \theta + u_\theta^\perp$ for some $u_\theta \in \mathbb{R}$ and $u_\theta^\perp \in \mathbb{R}^n$ perpendicular to θ. Our goal is to find $z \in \mathbb{C}^n$ with $g(z)=0$ and $|\mathbf{Re}(z) - u| \leq \frac{\sqrt{e}nt^2\left((\log 9)t - \log\frac{81}{2}\right)}{\delta(g)}$.

For $z_1 \in \mathbb{C}$ we then define the univariate exponential t-sum $\tilde{g}(z_1) := \sum_{j=1}^t e^{a_j \cdot z_1\theta + a_j \cdot u_\theta^\perp + \beta_j}$. $\tilde{g}$ is in fact the restriction of g to the complex line parametrized by $l(z_1) = z_1\theta + u_\theta^\perp$. In particular, $\tilde{g}$ has the same number of terms as g thanks to our choice of θ, and the definition of $\mathrm{Trop}(\tilde{g})$ implies that $u_\theta \in \mathrm{Trop}(\tilde{g})$. By Theorem 1.5 there is an $\omega \in \mathbb{C}$ such that $0 = \tilde{g}(\omega) = g(l(\omega))$ and $|\mathbf{Re}(\omega) - u_\theta| \leq \frac{\left((\log 9)t - \log\frac{81}{2}\right)}{\delta(\tilde{g})}$. Since $|\mathbf{Re}(l(\omega)) - u| = |(\mathbf{Re}(\omega) - u_\theta)\theta|$, and $\delta(\tilde{g}) \geq \frac{\delta(g)}{\sqrt{e}nt^2}$ by Inequality (5), we can conclude by taking $z := l(\omega)$. ■

4.4. **The Proof of Assertion (3) of Theorem 1.10.** Since $\varphi(r) = 0$ it is clear that $\mathbf{Re}(Z(g)) \ni r$. It is easily checked that $\mathrm{Trop}(g)$ is the codimension 1 part of the outer normal fan of the standard n-simplex in $\mathbb{R}^n$. So r is in fact at distance $(\log(t-1))/\delta$ from $\mathrm{Trop}(g)$ because r lies in the negative orthant and is at distance $(\log(t-1))/\delta$ from each of the coordinate hyperplanes of $\mathbb{R}^n$. ■

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