On the determination of the degree of an equation obtained by elimination

Ferdinand Minding

ABSTRACT. In 1841, Ferdinand Minding published *Ueber die Bestimmung des Grades einer durch Elimination hervorgehenden Gleichung* in Crelle's Journal. His main theorem represents the first (implicit) appearance of the BKK bound (and even some of its extensions) in the mathematical literature. We present an English translation of this paper and a commentary which explains how Minding's formula relates to the mixed area of lattice polygons.

Translation and Commentary by D. Cox and J. M. Rojas

1. Translation of Minding's Paper [Min41]

From two given algebraic equations involving two unknowns, there are indeed different methods for deriving a new equation containing only one unknown. Frequently one wishes to know only the degree of this final equation, without writing down the equation itself. If I am not mistaken, there has until now been no rule given for finding this degree with the ease appropriate to so elementary an object. One knows, to be sure, that if the equations are respectively of degrees h and k, i.e., if the highest sum of the exponents of terms amounts to h in one and k in the other, then the desired degree can equal at most the product hk. This however is only a bound, from which the actual degree often deviates. In order to find the actual degree, it is therefore necessary to assess the true form of the final equation. We have the following equations:

$$(1) \qquad f(x,y) = A_0 y^m + A_1 y^{m-1} + A_2 y^{m-2} + \cdots + A_{m-1} y + A_m = 0,$$

$$(2) \qquad \theta(x,y) = B_0 y^n + B_1 y^{n-1} + B_2 y^{n-2} + \cdots + B_{n-1} y + B_n = 0,$$

in which the letters A and B with subscripts stand for arbitrary polynomials in x. We solve the equation (2) for y, denote its roots by $y_1, y_2, \ldots, y_n$, and form the product

$$(3) \qquad P = f(x, y_1) \cdot f(x, y_2) \cdots f(x, y_n).$$

Then $B_0^m \cdot P$ is a polynomial in x, and if one sets

(4) $$B_0^m \cdot P = \psi(x),$$

then

(5) $$\psi(x) = 0$$

is the desired final equation.

In order to show that $B_0^m \cdot P$ is a polynomial, first note that P is a rational function of x, in which, if the functions of $y_1, y_2, \ldots, y_n$ occurring symmetrically therein are expressed in terms of x using equation (2), the denominator can only contain a power of B_0. Designate any one of these symmetric functions by

$$S = y_1^{m_1} y_2^{m_2} y_3^{m_3} \cdots y_n^{m_n} + \cdots,$$

where the following terms indicated by the dots are obtained from the first through mixing $y_1, y_2, \ldots, y_n$, and none of the exponents $m_1, m_2, \ldots, m_n$ can be greater than m. Now set:

$$S_1 = \frac{1}{y_1^{m-m_1} \cdot y_2^{m-m_2} \cdots y_n^{m-m_n}} + \cdots,$$

therefore

$$S = (y_1 y_2 \cdots y_n)^m S_1 = (-1)^{mn} \frac{B_n^m}{B_0^m} S_1.$$

The symbol S_1 denotes a symmetric polynomial of the reciprocal roots of (2). Therefore the value of S_1 is a rational function, whose denominator can have only a power of B_n; so we set $S_1 = \frac{Z}{B_n^\lambda}$, where Z is a polynomial in x or more accurately a polynomial function of the polynomials $B_0, B_1, \ldots, B_n$, and λ is a positive integer. It follows that

$$S = (-1)^{mn} \frac{B_n^m \cdot Z}{B_0^m \cdot B_n^\lambda}.$$

Since S obviously can only have a power of B_0 in its denominator, B_n^λ must appear in the above numerator; therefore $B_0^m S$ and consequently $B_0^m P = \psi(x)$ are polynomials.

When equation (1) is solved for y, we denote the roots by $\eta_1, \eta_2, \ldots, \eta_m$. Set

$$Q = \theta(x, \eta_1) \cdot \theta(x, \eta_2) \cdots \theta(x, \eta_m)$$

and notice that

$$\theta(x, \eta_1) = B_0(\eta_1 - y_1)(\eta_1 - y_2) \cdots (\eta_1 - y_n), \text{ etc.},$$

so that

$$Q = B_0(\eta_1 - y_1) \cdots (\eta_1 - y_n) \times B_0(\eta_2 - y_1) \cdots (\eta_2 - y_n) \times \cdots$$
$$\cdots \times B_0(\eta_m - y_1) \cdots (\eta_m - y_n),$$

and because

$$A_0(y_1 - \eta_1)(y_1 - \eta_2) \cdots (y_1 - \eta_m) = f(x, y_1), \text{ etc.},$$

it follows that:

(6) $$A_0^n Q = (-1)^{mn} B_0^m P.$$

This proves that $\psi(x) = 0$ is the required final equation. Namely for every value assumed in this problem by x, $P = 0$ (as well as $Q = 0$) follows necessarily, thus $\psi(x) = 0$. Furthermore, should this equation contain a superfluous factor,

then neither $P = 0$ nor $Q = 0$ for such a $\psi(x) = 0$; but then, because of (4) and (6), A_0 and B_0 would vanish together, which in general[1] is not possible. If A_0 and B_0 have a common factor in a particular case, then the polynomial $\psi(x)$ is always divisible by this factor, because, as is easily seen, it always is of the form $\psi(x) = A_0 U + B_0 V$, in which U and V are polynomials. Because one can always remove such a case through an infinitely small change of the coefficients, and to be sure without changing the degree, so it follows, that in no case[2] does the equation $\psi(x) = 0$ have a root which doesn't come from a solution of our problem.

The degree of the polynomial $\psi(x)$ reveals itself in the following way. We have

$$\psi(x) = B_0^m \cdot f(x, y_1) \cdot f(x, y_2) \ldots f(x, y_n).$$

One develops the roots $y_1, y_2, \ldots, y_n$ of equation (2) according to decreasing powers of x and sets the series in place of each root in the above expression.[3] Then all fractional and negative powers of x will mutually cancel themselves, and we obtain the polynomial $\psi(x)$ unchanged. Since only the degree of $\psi(x)$ is desired, one instead replaces each series with only its first term, which for $y_1, y_2, \ldots, y_n$ are $c_1 x^{h_1}, c_2 x^{h_2}, \ldots, c_n x^{h_n}$. The procedure, through which the series and in particular the highest exponents $h_1, h_2, \ldots, h_n$ or the degree of the roots is found, is sufficiently known; one can compare for example *Lacroix Traite* page 223 of the first edition, where the development according to increasing powers is shown. One determines from this the highest exponent of x in each of the functions $f(x, c_1 x^{h_1})$, $f(x, c_2 x^{h_2}), \ldots$ or the *degree* of the functions $f(x, y_1)$, $f(x, y_2), \ldots$, which will be denoted by $k_1, k_2, \ldots, k_n$. These can be integers or fractions, but can never be negative since A_m has degree at least 0.[4] Finally, we denote the degree of B_0 by b, so that

(7) $$mb + k_1 + k_2 + k_3 + \cdots + k_n$$

is necessarily an integer, which gives the highest exponent of $\psi(x)$ or the sought-for degree of the final equation. In special cases, one can consider the values of $c_1, c_2, \ldots, c_n$ in order to see, whether the coefficient of the highest term in one of the factors $f(x, y_1), \ldots$ of $\psi(x)$, consequently of $\psi(x)$ itself, perhaps becomes identically zero, and in such a case, one will be compelled to include the subsequent terms of the series in the calculation; however it is not necessary to continue this suggestion any further. Rather it is clear, that in general the above-mentioned value (7) represents the actual degree of the polynomial $\psi(x)$.

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[1] **Translator's Note:** In Minding' paper, "in general" does *not* imply complete generality. In modern parlance, "in general" should be interpreted as "generically".

[2] **Translator's Note:** Here some care is required. One in general needs to perturb more coefficients than just those of A_0 and B_0. For example, taking $f = ay^4 + x^2 y^3 + x^3 y^2 + b$ and $\theta = cy^4 + x^2 y^3 + x^2 y^2 + d$, even with a, b, c, d generic, yields a system which always has one root at infinity, thus violating Minding's formula. See also Section 2.5.

[3] **Translator's Note:** By "decreasing powers", Minding means a series of the form

$$y(x) = cx^h + dx^{h-\alpha_1} + ex^{h-\alpha_2} + \cdots, \quad 0 < \alpha_1 < \alpha_2 < \cdots, \quad h, \alpha_1, \alpha_2, \ldots \in \mathbb{Q}.$$

Minding calls cx^h the *first term* and h the *highest exponent*.

[4] **Translator's Note:** One needs to forbid negative exponents in *all* A_i, not just A_m.

Let the following two equations be given, in which the symbol (x^μ) denotes a polynomial of degree μ in x:

$$f(x,y) = (x^2)y^4 + (x^2)y^3 + (x^4)y^2 + (x^5)y + (x^5) = 0,$$
$$\theta(x,y) = (x^8)y^5 + (x^6)y^4 + (x^9)y^3 + (x^4)y^2 + (x^3)y + (x^4) = 0.$$

These equations are of the 6th and 13th degree, so that the degree of the final equation is not higher than $6 \cdot 13 = 78$. To find it exactly, one calculates the degree of the roots y of $\theta(x,y) = 0$; one finds at once that $h_1 = h_2 = \frac{1}{2}, h_3 = h_4 = h_5 = -\frac{5}{3}$. It follows that the degrees of $f(x,y_1),\ldots$ are namely $k_1 = k_2 = \frac{11}{2}$ and $k_3 = k_4 = k_5 = 5$; furthermore $B_0 = (x^8)$, so that $b = 8$, and $m = 4$. Thus the degree of the final equation is $mb + k_1 + k_2 + k_3 + k_4 + k_5 = 4 \cdot 8 + 11 + 15 = 58$.

If one writes the given equations in terms of x instead of y, in order to find the degree in y of the final equation using the above-mentioned rule, one does not always find the same value for this as for the previous degree. To explain this circumstance, one must note, that the final equation in x only gives the finite values of x, which suffice to produce zero in both equations. If the final equation in y has a higher degree than that in x, then it is necessary that some root of the equation in y belong to an infinite value of x. It is always easy to make this value finite through an infinitely small change of the coefficients of one of the existing equations and to remove the inequality of the degrees of the final equations. Namely, let the equations (1) and (2) be written as follows:

$$f(x,y) = \alpha_0 x^\mu + \alpha_1 x^{\mu-1} + \cdots + \alpha_\mu = 0,$$
$$\theta(x,y) = \beta_0 x^\nu + \beta_1 x^{\nu-1} + \cdots + \beta_\nu = 0,$$

where $\alpha_0, \alpha_1, \ldots, \beta_0, \ldots, \beta_\nu$ are polynomials in y. If now neither A_0 has a common factor with B_0, nor α_0 with β_0, then neither can y have an infinite value for finite x, nor infinite x for finite y;[5] consequently there can then be no difference between the degrees of the final equations in x and in y. If a joint factor between A_0 and B_0 or α_0 and β_0 is present, one needs therefore only to change a coefficient in A_0 or one in α_0 in order for the final equations in x and y to have the same degree. If immediately afterwards one sets these changes to zero, then one can examine the coefficients of the highest term of the equation in order to decide how many values of x and how many of y become infinite, and finally how many finite solutions of the equations are present. This version of the calculation is however unnecessary if one uses the stated rule properly. Consider for example the following equations:

$$f(x,y) = (a + bx^2)y^4 + (c + ex)y^2 + gx^3 y + h + kx^2 + lx^3 = 0,$$
$$\theta(x,y) = \beta x^5 y^2 + (\gamma + \delta x^2)y + \lambda + \mu x^4 = 0,$$

or ordered with respect to x:

$$f(x,y) = (l + gy)x^3 + (k + by^4)x^2 + ey^2 x + h + cy^2 + ay^4 = 0,$$
$$\theta(x,y) = \beta y^2 x^5 + \mu x^4 + \delta y x^2 + \lambda + \gamma y = 0.$$

Here $A_0 = a + bx^2, B_0 = \beta x^5, \alpha_0 = l + gy, \beta_0 = \beta y^2$; consequently, unless $a = 0$ or $l = 0$, A_0 and B_0 as well as α_0 and β_0 have no common factor, so that the degrees of the final equation in x and y are both found to be $= 26$. However, if one sets

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[5]**Translator's Note:** From a modern point of view, the roots lie on a toric surface such that roots will not have finite x or y coordinates when they lie on curves corresponding to edges that are neither vertical nor horizontal in the Newton polygon. It is amazing that Minding knew this.

$a = 0$ and $l = 0$ at the same time, and calculates the degree of the final equations, then one finds 25 for the equation in x, and 24 for that in y. Through the vanishing of a and l, two roots of the previous final equation in y and one of the previous final equation in x become therefore infinite; at the same time however also the new final equation in x is divisible by x^2, the common factor of A_0 and B_0, so likewise the new final equation in y is divisible by y, the common factor of α_0 and β_0. Of the 26 finite solutions, which come from the initial equations, 23 in general remain finite when a and l vanish; the three remaining become $x_{24} = 0$, $y_{24} = \infty$; $x_{25} = 0$, $y_{25} = \infty$; $x_{26} = \infty$, $y_{26} = 0$. One sees, how here the desired number of finite solutions is found through repeated use of the stated rule and comparison of the results.

Postscript. After completing this project, a new work came to my attention: *System der Algebra von Dr. P.J.E. Finck, Professor zu Straßburg; Leipzig bei Barth, 1841*, which on page 405 indicates a much more concise rule for the degree of the final equation than the one mentioned in this project. The rule (whose proof is given below) is the following: if all coefficients $A_0, A_1, \ldots, A_m$ of equation (1) have degree m' and all coefficients $B_0, B_1, \ldots, B_n$ of equation (2) have degree n', then the degree of the polynomial $\psi(x)$, which gives the final equation, is the following: $mn' + nm'$. The proof of this theorem, which fills two pages of the text, follows very easily from the above, for in this case all roots y of equation (2) have degree 0, that is, $h_1 = h_2 = \cdots = h_n = 0$, so that $k_1 = k_2 = \cdots = k_n = m'$; at the same time, $b = n'$ because B_0 has degree n'. Consequently the degree of the final equation is $mb + k_1 + k_2 + \cdots + k_n = mn' + nm'$, qed. If this rule is used in cases in which the coefficients are of unequal degree, then the result is no longer dependable because it assumes equality of every degree. This can result in extraneous roots. On the other hand, in the rule stated in the present treatise, in order to find the degree of $\psi(x)$, the degrees of the coefficients are used correctly in the calculation no matter how they are given.

2. Commentary on Minding's Paper

The main result of Minding's paper [**Min41**] is the formula

$$(7) \qquad mb + k_1 + k_2 + k_3 + \cdots + k_n$$

for the degree of the *final equation (Endgleichung)* obtained by eliminating y from equations (1) and (2). Minding notes that this degree is the number of solutions of the equations, at least when the coefficients are generic. On the other hand, the BKK bound states that if P_1 and P_2 are the Newton polygons of the polynomials in (1) and (2), then for generic coefficients, the number of solutions with all coordinates nonzero is the mixed area $\mathcal{M}(P_1, P_2)$ [**Ber75, BZ91, Roj03**]. The goal of this commentary to explain why these formulas give the same answer for the types of polynomials considered by Minding, and how Minding's result also foreshadowed later extensions of the BKK bound to counting all complex solutions. We will also clarify the implicit genericity assumptions.

Although (7) seems far removed from mixed areas, recall that Minding expresses the roots y of $\theta(x, y)$ as certain fractional power series, now called *Puiseux series*. The technique for computing these series is due to Newton and is where he introduced the Newton polygon. Once we understand how this works, we will see that (7) reduces to standard formulas for the mixed area of lattice polygons.

An introduction to Puiseux series appears in [**Wal50**, Ch. IV] and the properties we need about mixed areas can be found in [**CLO98**, Ch. 7] and [**Roj03**, Sec. 7–8]. For consistency, we will use the notation of [**Roj03**] for mixed areas.

2.1. Puiseux Series and Newton Polygons. If $y_1 \ldots, y_n$ are the roots of $\theta(x, y) = 0$, then standard resultant formulas [**CLO98**, (1.4) of Ch. 3] show that the right-hand side of

$$\psi(x) = B_0^m f(x, y_1) f(x, y_2) \cdots f(x, y_n)$$

is the univariate resultant $\mathrm{Res}(\theta, f, y)$. This gives another proof that ψ is the polynomial obtained by eliminating y from (1) and (2).

Minding's innovation was to express the y_i in terms of x using Puiseux series

$$(8) \qquad y(x) = cx^h + dx^{h-\alpha_1} + ex^{h-\alpha_2} + \cdots, \quad 0 < \alpha_1 < \alpha_2 < \cdots,$$

where $h, \alpha_1, \alpha_2, \ldots \in \mathbb{Q}$. To see how this relates to Newton polygons, consider Minding's bivariate polynomial

$$(9) \qquad \theta(x, y) = (x^8)y^5 + (x^6)y^4 + (x^9)y^3 + (x^4)y^2 + (x^3)y + (x^4) = 0,$$

where, following Minding, (x^μ) indicates a polynomial in x of degree μ. If all the coefficient polynomials (x^μ) are generic, then the Newton polygon P_2 of $\theta(x, y)$ is

(10)
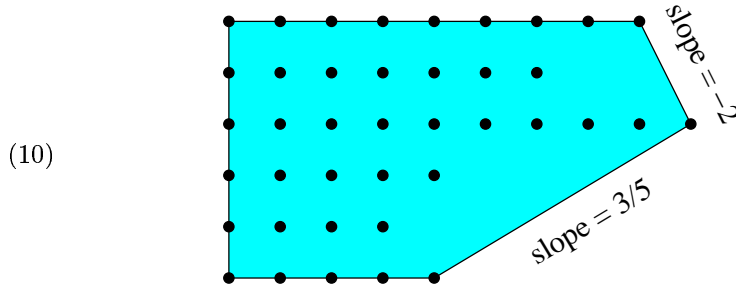
To see how (8) relates to P_2, we write

$$\theta(x, y) = \cdots + B_{8,5}x^8y^5 + \cdots + B_{9,3}x^9y^3 + \cdots + B_{4,0}x^4.$$

If we substitute (8) into this and set the resulting expression to zero, we obtain

$$0 = \cdots + B_{8,5}x^8(cx^h + \cdots)^5 + \cdots + B_{9,3}x^9(cx^h + \cdots)^3 + \cdots + B_{4,0}x^4.$$

The terms indicated by $\cdots$ in the parentheses have smaller degree in x, so that the only way the highest degree x terms can cancel is for

$$B_{8,5}c^5x^{8+5h} + B_{9,3}c^3x^{9+3h} = 0 \quad \text{or} \quad B_{9,3}c^3x^{9+3h} + B_{4,0}x^4 = 0.$$

The first equality implies that $h = \frac{1}{2}$, and the (nonzero) values of c giving $B_{8,5}c^5 + B_{9,3}c^3 = 0$ yield two solutions y_1, y_2 of $\theta(x, y) = 0$. The second equality above implies that $h = -\frac{5}{3}$, and the (nonzero) values of c giving $B_{9,3}c^3 + B_{4,0} = 0$ yield the remaining three solutions y_3, y_4, y_5. The exponents $\frac{1}{2}$ and $-\frac{5}{3}$ are the ones mentioned by Minding in his paper. Note that they are the negative reciprocals of the slopes indicated in (10).

In general, assume that the Newton polygon of $\theta(x, y)$ has the following form:
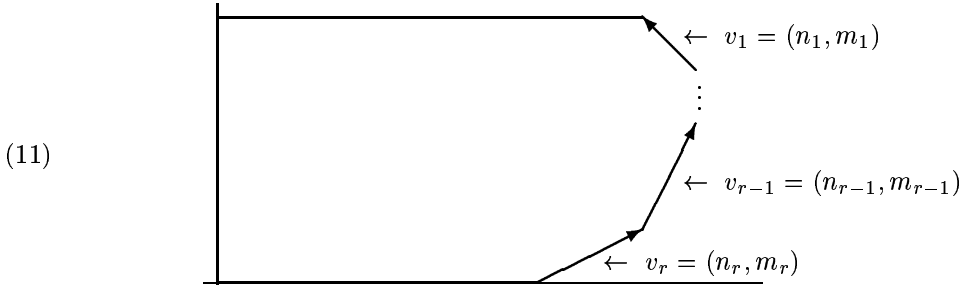
Note that the edges on the right are considered as vectors. Then one can show that $\theta(x, y) = 0$ has n solutions, where

$$m_1 \text{ solutions have the form } y(x) = c_{1,j}x^{-\frac{n_1}{m_1}} + \cdots, \ j = 1, \ldots, m_1,$$

$$m_2 \text{ solutions have the form } y(x) = c_{2,j}x^{-\frac{n_2}{m_2}} + \cdots, \ j = 1, \ldots, m_2,$$

(12)

$$\vdots$$

$$m_r \text{ solutions have the form } y(x) = c_{r,j}x^{-\frac{n_r}{m_r}} + \cdots, \ j = 1, \ldots, m_r.$$

and the dots $\cdots$ indicate lower order terms. The solutions involve the edges on the *right-hand* side of the Newton polygon because Minding writes his Puiseux series with *decreasing* exponents. Note also that this $19^{\underline{\text{th}}}$ century construction is a foreshadowing of general results for systems of multivariate polynomial equations that form the core of what is now called non-Archimedean amoeba theory or tropical algebraic geometry.

2.2. Mixed Area of Lattice Polygons. Consider the standard lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. The area of a lattice polygon P will be denoted $\text{Area}(P)$, normalized so that the unit square has area 1. Given two lattice polygons P_1, P_2, there are four methods to compute the mixed area $\mathcal{M}(P_1, P_2)$:

- (Definition) $\mathcal{M}(P_1, P_2)$ is the coefficient of $\lambda\mu$ in the expansion of the quantity $\text{Area}(\lambda P_1 + \mu P_2)$ as a polynomial in λ, μ [**Roj03**, Def. 7.0.25].
- (Inclusion-Exclusion) $\mathcal{M}(P_1, P_2) = \text{Area}(P_1 + P_2) - \text{Area}(P_1) - \text{Area}(P_2)$ [**Roj03**, Lem. 7.0.29].
- (Mixed Subdivision) $\mathcal{M}(P_1, P_2)$ is the sum of the areas of the mixed cells in a mixed subdivision of $P_1 + P_2$ [**Roj03**, Lem. 7.0.29].
- (Recursion) $\mathcal{M}(P_1, P_2) = -\sum_{\mathcal{F}} \min_{u \in P_1}(u \cdot \nu_{\mathcal{F}})\text{Length}(\mathcal{F})$. The sum is over all edges $\mathcal{F}$ of P_2 where $\nu_{\mathcal{F}}$ is the primitive[6] inward-pointing normal vector to $\mathcal{F}$ and $\text{Length}(\mathcal{F})$ is the normalized length of $\mathcal{F}$ [**CLO98**, Thm. 4.12 of Ch. 7].

We will show that Minding's formula is equivalent to the last two of these methods. To make the relation to (7) easier to see, we will write the recursion formula as

$$(13) \qquad \mathcal{M}(P_1, P_2) = \sum_{\mathcal{F}} \max_{u \in P_1}(u \cdot \tilde{\nu}_{\mathcal{F}}).$$

where as above, $\mathcal{F}$ is an edge of P_2 and $\tilde{\nu}_{\mathcal{F}} = -\text{Length}(\mathcal{F})\,\nu_{\mathcal{F}}$.

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[6]Thus $\nu_{\mathcal{F}}$ is the first lattice point on the open ray generated by any inner normal of $\mathcal{F}$.

In practice, the vector $\tilde{\nu}_{\mathcal{F}}$ is easy to read off from the polygon P_2. Write $\mathcal{F}$ as a vector $v = (n, m)$ with the polygon to the left of v. This is the convention used in (2.1). Then one can easily show that

$$(14) \qquad\qquad\qquad \tilde{\nu}_{\mathcal{F}} = (m, -n).$$

So $\tilde{\nu}_{\mathcal{F}}$ is the outward-pointing normal vector of $\mathcal{F}$ whose length equals that of $\mathcal{F}$.

2.3. Minding's Formula and Recursion. We can now prove that Minding's formula (7) computes the mixed area of $P_1 =$"Newton polygon of $f(x, y)$" and $P_2 =$"Newton polygon of $\theta(x, y)$", assuming that both polygons have the form (2.1). (See Section 2.4, just after Example 1, for a rigorous statement of the required conditions for P_1 and P_2.)

In P_2, (14) shows the edges on the right in (2.1) give the following outward-pointing normal vectors: $\tilde{\nu}_1 = (m_1, -n_1), \ldots, \tilde{\nu}_r = (m_r, -n_r)$. We also need to consider the top, left and bottom edges of P_2, which have the following outward-pointing normal vectors:

$$\tilde{\nu}_{top} = (0, b), \ \tilde{\nu}_{left} = (-n, 0), \ \tilde{\nu}_{bottom} = (0, -c),$$

where n is the degree of $\theta(x, y)$ in y, b is the degree in x of the coefficient of y^n, and c is the degree of x of the coefficient of y^0. Note that n and b are the same as in Minding's paper.

Since P_1 lies in the first quadrant and contains the origin, and $f(x, y)$ has degree m in y, we easily obtain

$$\max_{u \in P_1} (u \cdot \tilde{\nu}_{top}) = mb, \ \max_{u \in P_1} (u \cdot \tilde{\nu}_{left}) = \max_{u \in P_1} (u \cdot \tilde{\nu}_{bottom}) = 0.$$

Using (13), it follows that the mixed area is given by

$$(15) \qquad\qquad \mathcal{M}(P_1, P_2) = mb + \sum_{i=1}^{r} \max_{u \in P_1} \big(u \cdot (m_i, -n_i)\big).$$

We can interpret the sum in (15) as follows. Recall that Minding took the n solutions $y = c_{i,j} x^{-\frac{n_i}{m_i}} + \cdots, \ i = 1, \ldots, r, \ j = 1, \ldots, m_i$ from (12) and substituted them into $f(x, y)$. If $f(x, y) = \sum_{(k,l) \in P_1 \cap \mathbb{Z}^2} A_{k,l} x^k y^l$, then:

$$\deg \big(f(x, c_{i,j} x^{-\frac{n_i}{m_i}} + \cdots) \big) = \deg \Big(\sum_{(k,l) \in P_1 \cap \mathbb{Z}^2} A_{k,l} x^k (c_{i,j} x^{-\frac{n_i}{m_i}} + \cdots)^l \Big)$$

$$= \deg \Big(\sum_{(k,l) \in P_1 \cap \mathbb{Z}^2} A_{k,l} c_{i,j} x^{k+l(-\frac{n_i}{m_i})} + \cdots \Big)$$

$$= \max_{(k,l) \in P_1 \cap \mathbb{Z}^2} \big(k + l(-\tfrac{n_i}{m_i}) \big),$$

where the final equality holds for generic coefficients. Doing this for each of the n solutions gives the n numbers that Minding denoted $k_1, \ldots, k_n$. Then the sum of the first m_1 of these numbers is

$$m_1 \max_{(k,l) \in P_1 \cap \mathbb{Z}^2} \big(k + l(-\tfrac{n_1}{m_1}) \big) = \max_{(k,l) \in P_1 \cap \mathbb{Z}^2} \big(m_1 k + l(-n_1) \big)$$

$$= \max_{(k,l) \in P_1 \cap \mathbb{Z}^2} \big((k, l) \cdot (m_1, -n_1) \big).$$

This is clearly the first term of the sum in (15). The other terms match up in exactly the same way, which gives the formula

$$\mathcal{M}(P_1, P_2) = mb + k_1 + \cdots + k_n.$$

This proves that Minding's formula is the mixed area of the Newton polygons.

2.4. Minding's Formula and Mixed Subdivisions. Here we present an alternative approach to seeing how Minding's formula relates to mixed area. The key idea is a combinatorial construction due to Huber and Sturmfels [**HS95**].

For polygons $P_1, P_2 \subset \mathbb{R}^2$, a *mixed subdivision*[7] is a collection of polygons $\mathcal{C} = \{C_i\}_{i=1}^N$, called *cells*, satisfying the following conditions:
(a) $\bigcup_{C_i \in \mathcal{C}} C_i = P_1 + P_2$.
(b) For any two distinct C_i and C_j in $\mathcal{C}$, $C_i \cap C_j$ is a (possibly empty) face of both C_i and C_j.
(c) There are two distinguished cells C_i and C_j, called *unmixed cells*, such that $C_i = P_1 + p$ and $C_j = P_2 + q$ for some $p, q \in \mathbb{R}^2$.
(d) All remaining cells, called the *mixed cells*, are parallelograms.

EXAMPLE 1. Taking P_1 and P_2 to respectively be the Newton polygons of Minding's f and θ, one possible mixed subdivision is the following:
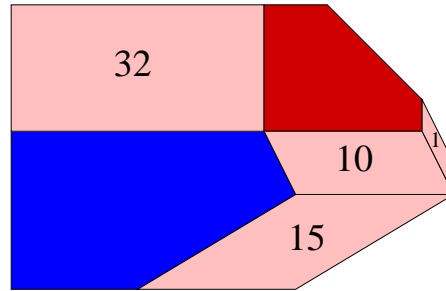
Note in particular that a mixed subdivision of the Minkowski sum $P_1 + P_2$ allows us to express the Inclusion-Exclusion formula for mixed area as a sum of parallelogram areas. For instance, the mixed area from Minding's example is the sum of the parallelogram areas above: $32 + 10 + 1 + 15 = 58$.

We can identify the individual terms of Minding's formula as polygon areas as follows. Let P_1 and P_2 be lattice polygons of the form (2.1) from Section 2.1. The precise meaning of this is that P_1 and P_2 lie in the first quadrant, contain the origin, and satisfy

$$\max_{(s,t) \in P_i} (t) = \text{Length}(P_i \cap \{x = 0\}), \quad i = 1, 2.$$

Define

$$m = \text{Length}(P_1 \cap \{x = 0\})$$
$$n = \text{Length}(P_2 \cap \{x = 0\})$$
$$b = \text{Length}(P_2 \cap \{y = n\})$$

---

[7]Mixed subdivisions are usually defined in terms of n-tuples of polytopes in $\mathbb{R}^n$. For simplicity, our definition for $n = 2$ is more restrictive than usual. A complete exposition on mixed subdivisions can be found in [**Roj03**] in this volume or [**HS95**].

and let F_i, $i = 1, \ldots, r$ be the edges of P_2 corresponding to the vectors $v_i = (n_i, m_i)$ in (2.1). Using [**Roj03**, Sec. 7], one can show that there is a mixed subdivision $\mathcal{C}$ of P_1 and P_2 such that:

(a) $P_1 + (b, n)$ and $P_2 + (0, 0)$ are the unmixed cells.

(b) $[0, b] \times [n, n + m]$ is a mixed cell of area mb.

(c) All other mixed cells are of the form $E + F_i$ for certain edges E of P_1 and $i \in \{1, \ldots, r\}$.

(d) For any $i \in \{1, \ldots, r\}$, let $\widetilde{C}_i$ be the union of all mixed cells of $\mathcal{C}$ of the form $E + F_i$ where E is an edge of P_1. Then there is an rational number ℓ_i such that the parallelogram $[0, \ell_i] + F_i$ has the same area as $\widetilde{C}_i$. Furthermore,

$$\text{Area}(\widetilde{C}_i) = \text{Area}([0, \ell_i] + F_i) = m_i \ell_i = \max_{u \in P_1} \big(u \cdot (m_i, -n_i) \big),$$

where $\max_{u \in P_1} \big(u \cdot (m_i, -n_i) \big)$ is the ith term of the sum in (15).

The idea is that the parallelogram $[0, \ell_i] + F_i$ "straightens out" $\widetilde{C}_i$. For instance, in the subdivision of Example 1, let F_1 be the edge of P_2 corresponding to $v_1 = (-1, 2)$. Since F_1 has height 2, the two mixed cells in Example 1 involving this edge give the following polygons of the same area:

$$\ell_1 = 5 + \tfrac{1}{2} = \tfrac{11}{2}$$
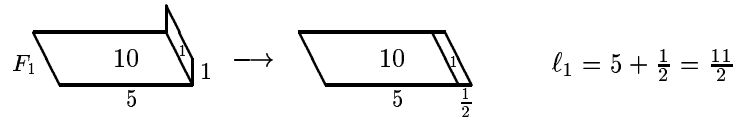
In general, comparing part (d) to the formulas at the end of Section 2.3, we see that ℓ_i is the degree of $f(x, c_{i,j} x^{-\frac{n_i}{m_i}} + \cdots)$ for $j = 1, \ldots, m_i$. It follows that
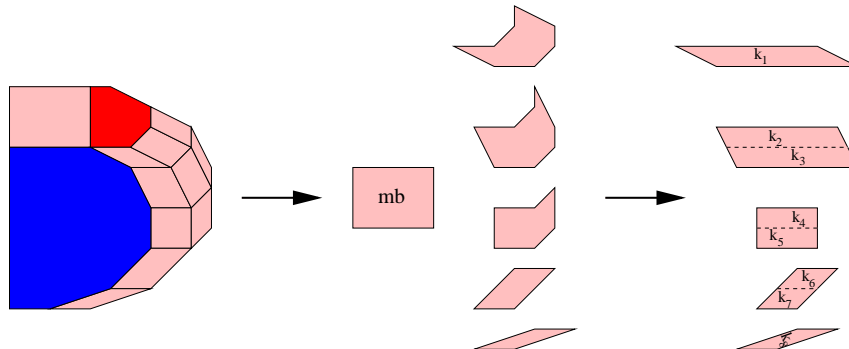
$$\underbrace{\ell_1, \ldots, \ell_1}_{m_1 \text{ times}}, \quad \cdots \quad, \underbrace{\ell_r, \ldots, \ell_r}_{m_r \text{ times}}$$

are Minding's numbers $k_1, \ldots, k_n$. Thus (a)–(d) imply that

$$\mathcal{M}(P_1, P_2) = \text{sum of areas of mixed cells in the mixed subdivision}$$

$$= mb + \sum_{i=1}^{r} m_i \ell_i = mb + k_1 + \cdots + k_n.$$

so that Minding's formula indeed gives the mixed area.

EXAMPLE 2. Here is an example more complicated than Minding's:

Note that $k_2 = k_3$, $k_4 = k_5$, and $k_5 = k_6$. As noted above, the right-hand side of our illustration above shows how we can calculate the area of an entire "strip" of mixed cells as the area of a single parallelogram. ∎

In general, the connections between algebraic geometry and combinatorics are very deep. In our context, this can be made into a "mantra" of sorts: any algorithm for computing mixed area yields a proof of Bernstein's Theorem in the plane, and vice-versa.

2.5. Affine Roots and Genericity. Minding's formula reduces to the BKK bound only when we make certain assumptions on the coefficients. On the other hand, Minding's formula is without qualification a special case of the following result due to Li and Wang:

THEOREM 2.1. [**LW96**] *The number of isolated roots in $\mathbb{C}^n$ of a system of polynomial equations $f_1(x_1, \ldots, x_n) = \cdots = f_n(x_1, \ldots, x_n) = 0$ is no more than $\mathcal{M}(P_1', \ldots, P_n')$ where, for all i, P_i' is the convex hull of the union of the origin and the Newton polytope of f_i.* ∎

Note in particular that one need not assume that the polynomials have a nonzero constant term or that their exponents satisfy any restriction. That Li and Wang's result holds for generic coefficients (assuming that all the polynomials have nonzero constant term) follows easily from the BKK bound, so the theorem is interesting because it states an upper bound which is true regardless of degeneracies or the presence of constant terms.

However, this brings to mind an important question: How much genericity is necessary for the BKK bound to hold? From Bernstein's seminal paper [**Ber75**], it is easy to show that it suffices to have all coefficients corresponding to Newton polytope vertices be generic.[8] That one could get away with less genericity appears to have been first observed in [**CR91**], and a complete characterization of which sets of coefficients need to be generic (for the BKK bound, the Li-Wang bound, and even more general bounds) appears in [**Roj99**]. Interestingly, although he refrains from explicitly stating a clear characterization of which coefficients should be chosen generically, Minding did appear to understand that not all coefficients need to be generic for his formula to be true.

2.6. Final Comments. For a nice discussion of how Minding's argument relates to the BKK bound, we refer the reader to Khovanskii's essay in [**BZ91**, Ch. 4, Sec. 27, Addendum 3]. Also, [**Roj03**, Sec. 6] explains how mixed volume arisely naturally when considering the solutions of polynomial equations.

Acknowledgements

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[8]It is certainly not enough to have all the vertex coefficients simply be nonzero.

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