METRIC ESTIMATES AND MEMBERSHIP COMPLEXITY FOR ARCHIMEDEAN AMOEBAE AND TROPICAL HYPERSURFACES

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Abstract. Given any complex Laurent polynomial f, Amoeba(f) is the image of its complex zero set under the coordinate-wise log absolute value map. We discuss an efficiently constructible polyhedral approximation, ArchTrop(f), of Amoeba(f), and derive explicit upper and lower bounds, solely as a function of the number of monomial terms of f, for the Hausdorff distance between these two sets. We also show that deciding whether a given point lies in ArchTrop(f) is doable in polynomial-time, for any fixed dimension, unlike the corresponding problem for Amoeba(f), which is **NP**-hard already in one variable. ArchTrop(f) can thus serve as a canonical low order approximation to start a higher order iterative polynomial system solving algorithm, e.g., homotopy continuation.

In memory of Mikael Passare.

1. Introduction

One of the happiest coincidences in algebraic geometry is that the norms of roots of polynomials can be estimated through polyhedral geometry. Perhaps the earliest incarnation of this fact was Isaac Newton's use of a polygon to determine initial exponents of series expansions for algebraic functions in one variable. This was detailed in a letter, dated October 24, 1676 [New76], that Newton wrote to Henry Oldenburg. In modern terminology, Newton counted, with multiplicity, the s-adic valuations of roots of univariate polynomials over the *Puiseux series field* $\mathbb{C}\langle\langle s\rangle\rangle$ (see, e.g., Theorem 5.7 from Section 5.2 below). Newton's result has since been extended to arbitrary non-Archimedean fields (see, e.g., [Dum06, Wei63]). Tropical geometry (see, e.g., [Ber71, Vir01, EKL06, LS09, IMS09, BR10, ABF13, MS15]) continues to deepen the links between algebraic, arithmetic, and polyhedral geometry.

We will use tropical methods to efficiently approximate complex amoebae in arbitrary dimension (see Theorem 3.4 and Corollary 5.1, respectively in Sections 3 and 5.1 below), and derive an Archimedean analogue of Newton's univariate result along the way (Theorem 1.5 in Section 1.1 below). While our approximations can be coarse, their computational cost is quite low (see Theorem 4.4 in Section 4 below), and initial experiments indicate that they are often good enough to yield high-quality start points for homotopy algorithms applied to sparse polynomial systems (see, e.g., [AGGR15, Sec. 3]). In what follows, Conv(S) denotes the convex hull of (i.e., smallest convex set containing) a subset $S\subseteq\mathbb{R}^n$.

Definition 1.1. *Let* $\mathbb{C}^*:=\mathbb{C}\setminus\{0\}$*, let* $c_1,\ldots,c_t\in\mathbb{C}^*$*, and call any* $f\in\mathbb{C}\left[x_1^{\pm 1},\ldots,x_n^{\pm 1}\right]$ *of the form* $f(x)=\sum_{j=1}^{t} c_j x^{a_j}$*, with* $\{a_1,\ldots,a_t\}$ *of cardinality* $t\geq 1$*, an* n-variate t-nomial. *(The notation* $x=(x_1,\ldots,x_n)$ *and* $x^{a_j}=x_1^{a_{1,j}}\cdots x_n^{a_{n,j}}$ *is understood.) We then define the* (ordinary) Newton polytope *of* f *to be* $\mathrm{Newt}(f):=\mathrm{Conv}\left(\{a_i\}_{i\in[t]}\right)$*, and the* Archimedean *Newton polytope of* f *to be* $\mathrm{ArchNewt}(f):=\mathrm{Conv}\left(\{(a_i,-\log|c_i|)\}_{i\in[t]}\right)$. $\diamond$

Hadamard defined the $n=1$ case of ArchNewt(f) around 1893, and observed a relationship between the absolute values of the complex roots of f and the slopes of certain edges of ArchNewt(f) [Had93, pp. 174–175 & 201] (see also [Ost40a, pp. 120–121] and [Val54, Ch.

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IX, pp. 193–202]). We'll see below that, for arbitrary n, approximating absolute values of complex roots can be reduced to maximizing certain linear forms over $\mathrm{ArchNewt}(f)$, and this ultimately leads us to a particular class of tropical varieties. In what follows, we let $\zeta = (\zeta_1, \ldots, \zeta_n) \in (\mathbb{C}^*)^n$ and define $\mathrm{Amoeba}(f) := \{(\log|\zeta_1|, \ldots, \log|\zeta_n|) \mid f(\zeta)=0 \text{ and } \zeta \in (\mathbb{C}^*)^n\}$. Clearly, $\mathrm{Amoeba}(f)$ is empty when f is a monomial.

Example 1.2. *For the trivariate binomial* $f(x) = 7 - x_1^2 x_2^3 x_3^5$, *it is easily checked that* $\mathrm{ArchNewt}(f)$ *is the line segment in* $\mathbb{R}^4$ *connecting* $(0,0,0,-\log 7)$ *and* $(2,3,5,0)$. *In particular,* $(\log|\zeta_1|, \log|\zeta_2|, \log|\zeta_3|) \in \mathrm{Amoeba}(f) \iff 7 = |\zeta_1^2 \zeta_2^3 \zeta_3^5|$, *and thus it is clear that* $\mathrm{Amoeba}(f)$ *is exactly the affine hyperplane in* $\mathbb{R}^3$ *defined by* $2w_1 + 3w_2 + 5w_3 = \log 7$. *Note also that any* $(w_1, w_2, w_3) \in \mathrm{Amoeba}(f)$ *makes the vector* $(w_1, w_2, w_3, -1)$ *perpendicular to* $\mathrm{ArchNewt}(f)$. $\diamond$

Example 1.3. *When* $f(x_1) := \frac{1}{89} - x_1^{16} + x_1^{49}$ *it turns out that* $\mathrm{Amoeba}(f)$ *consists of exactly* 26 *points. However, the points of* $\mathrm{Amoeba}(f)$ *cluster tightly about just* **2** *values: Exactly* 16 *complex roots of* f *have norm near* $\sqrt[16]{\frac{1}{89}} \approx 0.7553...$ *(to at least* 4 *decimal places) and exactly* 33 *complex roots of* f *have norm near* 1 *(to* 3 *decimal places). Here,* $\mathrm{ArchNewt}(f)$ *is the convex hull of* $\left\{\left(0, -\log\frac{1}{89}\right), (16,0), (49,0)\right\}$, *which is the triangle drawn below. Note also that the linear*

form $\frac{\log\frac{1}{89}}{16} v_1 - v_2$ *is maximized on the lower left edge of* $\mathrm{ArchNewt}(f)$, *while the linear form* $0 v_1 - v_2$ *is maximized on the lower right edge of* $\mathrm{ArchNewt}(f)$ *(if we restrict both linear forms to* $\mathrm{ArchNewt}(f)$*). More than coincidentally, every point of* $\mathrm{Amoeba}(f)$ *is within* 0.00034 *of some point of* $\left\{\frac{1}{16}\log\frac{1}{89}, 0\right\}$, *and the horizontal lengths (*16 *and* 33*) of the two lower edges count the number of roots with norm in the corresponding cluster. Note also that when* $\log|x_1| = \frac{1}{16}\log\frac{1}{89}$ *we have* $\frac{1}{89} = |-x_1^{16}| > |x_1^{49}|$, *and when* $\log|x_1| = 0$ *have* $\frac{1}{89} < |-x_1^{16}| = |x_1^{49}|$. *Furthermore, when* $\log|x_1| \notin \left\{\frac{1}{16}\log\frac{1}{89}, 0\right\}$, *the set* $\left\{\left|\frac{1}{89}\right|, |-x_1^{16}|, |x_1^{49}|\right\}$ *has cardinality* 3. $\diamond$

We refer the reader to the outstanding texts [Zie95, dLRS10] for further background on polytopes, faces, and inner and outer normals to faces.

Definition 1.4. *We define the* Archimedean *tropical variety of* f, $\mathrm{ArchTrop}(f)$, *to be*
$$\{v \in \mathbb{R}^n \mid (v,-1) \text{ is an outer normal of a positive-dimensional face of } \mathrm{ArchNewt}(f)\}$$
when $t \geq 2$ *and, when* $t=1$, *we set* $\mathrm{ArchTrop}(f) = \emptyset$. *We also call a face of* $\mathrm{ArchNewt}(f)$ *a* lower *face iff it has an outer normal of the form* $(v,-1)$ *for some* $v \in \mathbb{R}^n$. $\diamond$

For instance, in Examples 1.2 and 1.3, $\mathrm{ArchTrop}(f)$ was, respectively, a plane in $\mathbb{R}^3$ equal to $\mathrm{Amoeba}(f)$, and then a pair of points around which $\mathrm{Amoeba}(f)$ clustered. While $\mathrm{ArchTrop}(f)$ has appeared under different guises in earlier work (see, e.g., [Ost40a, Mik04, PR04, PRS11, TdW13]), explicit metric estimates for how well $\mathrm{ArchTrop}(f)$ approximates $\mathrm{Amoeba}(f)$ in *arbitrary* dimension have not yet appeared in the literature.[1]

1.1. Metric Estimates, With Multiplicity, in One Variable.

We now give explicit bounds on how $\mathrm{Amoeba}(f)$ clusters about the points of $\mathrm{ArchTrop}(f)$. In what follows, for any line segment $L \subset \mathbb{R}^2$ with vertices (a,b) and (c,d), its *horizontal length* is $\lambda(L) := |c-a|$.

[1] The only works in this direction that we are aware of are [AGS17, Rem. 4.2, Inequality 25] (for $n=1$, using new bounds for matrix polynomials) and [Mik05, Lemma 8.5, Pg. 360] (for the case $n=2$).

Theorem 1.5. *Given any univariate t-nomial f with $t \geq 2$, let $v_{\min} := \min \mathrm{ArchTrop}(f)$, $v_{\max} := \max \mathrm{ArchTrop}(f)$, let Γ be any connected component of the union of open intervals* $U_f := (v_{\min} - \log 2, v_{\max} + \log 2) \cap \bigcup\limits_{v \in \mathrm{ArchTrop}(f)} (v - \log 3, v + \log 3)$, *and let Λ_Γ be the sum of $\lambda(L)$ over all edges L of $\mathrm{ArchNewt}(f)$ with outer normal $(v, -1)$ satisfying $v \in \Gamma$. Then the number of roots $\zeta \in \mathbb{C}^*$ of f with $\log|\zeta| \in \Gamma$, counting multiplicity, is exactly Λ_Γ. In particular, $\mathrm{Amoeba}(f) \subset U_f$ and $\Lambda_\Gamma \geq 1$.*

We prove Theorem 1.5 below in Section 2.1, where a slight sharpening (Corollary 2.12) is also provided for $t = 3$.

Example 1.6. *If $f(x_1) := 1 + 19162399831 x_1^{16} + x_1^{49}$ then U_f is a disjoint union of two intervals, and Theorem 1.5 tells us that f has exactly 16 (resp. 33) roots with log-norm in the open interval $(-2.17292, -0.381151202193)$ (resp. $(-0.381151202190, 1.41061)$). In fact, for this example, the much smaller sub-intervals*

$$\tfrac{-\log 19162399831}{16} + 10^{-32}(-1,1) \quad \textit{and} \quad \tfrac{\log 19162399831}{33} + 10^{-16}(-1,1)$$

(respectively centered at the 2 points of $\mathrm{ArchTrop}(f)$) still respectively contain the same number of log-norms. $\diamond$

Remark 1.7. *The constants in the definition of U_f from Theorem 1.5 are optimal: Assertion (c) of Corollary 2.3 (resp. Lemma 2.5) in Section 2 below reveals that the $\log 2$ (resp. $\log 3$) in the definition of U_f can not be replaced by any smaller constant.* $\diamond$

We discuss in greater detail below how the neighborhood U_f improves or complements earlier root norm estimates from [Had93, Ost40a, AGS17]. Along the way, we will review some background on univariate roots estimates and prove our univariate results, before discussing our multivariate results.

The reader who wishes to see our higher-dimensional results now can skip to Section 3. Our results on membership complexity for $\mathrm{Amoeba}(f)$ and $\mathrm{ArchTrop}(f)$ are stated and proved in Section 4. Finally, we discuss connections to numerically solving polynomial systems, and compare our results to the older non-Archimedean case, in Section 5.

2. From Classical Approximations to Tropical Approximations

To prepare for the proof of Theorem 1.5 we will first review some classical root norm bounds in the univariate case. In particular, a key observation that will help us modernize some classical bounds is that $\mathrm{ArchTrop}(f)$ can be defined in at least 3 ways. In what follows, we assume $f(x_1) = \sum_{j=1}^{t} c_j x_1^{a_j} \in \mathbb{C}\left[x_1^{\pm 1}\right]$ is a univariate t-nomial (so the c_i are all nonzero).

Proposition 2.1. *For any univariate t-nomial f, the following three sets are identical:*

1. $\{v \in \mathbb{R} \mid (v, -1)$ is an outer normal of an edge of $\mathrm{ArchNewt}(f)\}$
2. *the set of slopes of the lower edges of* $\mathrm{ArchNewt}(f)$
3. $\{v \in \mathbb{R} \mid \max_j |c_j e^{a_j v}|$ *is attained for at least two distinct values of* $j\}$. ■

That our initial definition of $\mathrm{ArchTrop}(f)$ (in terms of outer normals to lower edges) can be replaced by the second set above is elementary. The equality of $\mathrm{ArchTrop}(f)$ with the third set follows immediately from the fact that an edge of a polygon P is a subset (containing at least two distinct points) where a non-trivial linear form is maximized on P (see, e.g., [Zie95, Ch. 7]). As we'll see in Section 5.2, these alternative characterizations are well-known in the tropical literature (see, e.g., [MS15]). In the Archimedean case, Hadamard and Ostrowski's original univariate root norm estimates were in fact stated in terms of edge slopes.

We now recall a pair of bounds dating back to 1893 and 1923.

Theorem 2.2.
(1) If $\zeta\in\mathbb{C}$ is a root of $\alpha_0+\cdots+\alpha_d x_1^d\in\mathbb{C}\left[x_1^{\pm 1}\right]$, and $\alpha_0\alpha_d\neq 0$, then

$$\frac{1}{2}\min_{\alpha_i\neq 0, i\neq 0}\left|\frac{\alpha_0}{\alpha_i}\right|^{1/i} < |\zeta| < 2\max_{i\in\{0,\ldots,d-1\}}\left|\frac{\alpha_i}{\alpha_d}\right|^{1/(d-i)}.$$

(2) If $g(x_1)=\beta_0+\cdots+\beta_p x_1^p+\gamma_1 x_1^{n_1}+\cdots+\gamma_q x_1^{n_q}\in\mathbb{C}[x_1]$ so that $\beta_0\beta_p\neq 0$ and $1\leq p<n_1<\cdots<n_q$, then g has a nonzero root with absolute value $\leq\left|\frac{\beta_0}{\beta_p}\right|^{1/p}\binom{p+q}{q}^{1/p}$. ■

Bound (1) is a paraphrase of a special case of [Had93, Pg. 201, Third Inequality], and is stated more explicitly in [Fuj16] (see also [RS02, pp. 243–249], particularly Bound 8.1.11 on Pg. 247). Bound (2) was proved by Montel [Mon23] (see also [RS02, Thm. 9.5.1, Pg. 304]).

Corollary 2.3. *Suppose $a_1<\cdots<a_t$ and $d:=a_t-a_1$. Label the roots of f in $\mathbb{C}^*$ by $\zeta_1,\ldots,\zeta_d$ (counting multiplicity) so that $|\zeta_1|\leq\cdots\leq|\zeta_d|$. Also, for each $i\in\{1,\ldots,d\}$, let v_i denote the slope of the (unique) lower edge of the polygon $\mathrm{ArchNewt}(f)\cap([a_1+i-1,a_1+i]\times\mathbb{R})$. Then:*

(a) $-\log 2<\log|\zeta_1|-\min\mathrm{ArchTrop}(f)\leq\log(t-1)$

(b) $-\log(t-1)\leq\log|\zeta_d|-\max\mathrm{ArchTrop}(f)<\log 2$

(c) The $\log 2$ (resp. $\log(t-1)$) terms above can not be replaced by any smaller constant (resp. function of t solely).

Remark 2.4. *Since any lower edge of $\mathrm{ArchNewt}(f)\cap([a_1+i-1,a_1+i]\times\mathbb{R})$ is in fact a line segment inside a lower edge of $\mathrm{ArchNewt}(f)$, Proposition 2.1 thus implies that $\{v_1,\ldots,v_d\}=\mathrm{ArchTrop}(f)$. Note also that $v_1\leq\cdots\leq v_d$ since $\mathrm{ArchNewt}(f)$ is convex and $a_1<\cdots<a_t$.* ⋄

Proof of Corollary 2.3: The lower bound from Part (a) and the upper bound from Part (b) follow immediately from Proposition 2.1, upon taking the log absolute value of both sides of Bound (1) from Theorem 2.2. In particular, we see that the lower and upper bounds from Bound (1) are exactly $\frac{1}{2}e^{\min\mathrm{ArchTrop}(f)}$ and $2e^{\max\mathrm{ArchTrop}(f)}$.

The upper bound from Part (a) follows similarly, but employing Bound (2) from Theorem 2.2 instead of Bound (1). In particular, one must apply Bound (2) in the following way: Take p so that the $(p,-\log|\beta_p|)$ is the right-hand vertex of the left-most lower edge of $\mathrm{ArchNewt}(f)$. By construction, this edge has slope $\frac{\log|\beta_0|-\log|\beta_p|}{p}$. Observing that $\binom{p+q}{q}^{1/p}=\left(\frac{(q+p)\cdots(q+1)}{p!}\right)^{1/p}=\left(\left(\frac{q}{p}+1\right)\cdots\left(\frac{q}{1}+1\right)\right)^{1/p}\leq((q+1)^p)^{1/p}=q+1$, and that the number of terms is $t=p+q+1$ with $p\geq 1$, we are done.

The lower bound from Part (b) follows by applying the preceding paragraph to the polynomial $x_1^{a_t+a_1}f(1/x_1)$: This has the effect of reflecting $\mathrm{ArchNewt}(f)$ across the vertical line $\frac{d}{2}\times\mathbb{R}$, and thus $\mathrm{ArchTrop}(f)$ is replaced by $-\mathrm{ArchTrop}(f)$. So we ultimately prove an upper bound of $\log(t-1)$ on $-\log|\zeta_d|-(-\max\mathrm{ArchTrop}(f))$ and we are done.

The optimality of the $\log 2$ terms is evinced by the polynomials

$$f_1(x_1):=x_1^{t-1}-x_1^{t-2}-\cdots-1 \quad\text{and}\quad f_2(x_1):=-1+x_1+\cdots+x_1^{t-1}:$$

One need only show that f_1 (resp. f_2) has a unique positive root increasing toward a limit of 2 (resp. decreasing toward a limit of $\frac{1}{2}$) as $t\longrightarrow\infty$. Uniqueness follows from Descartes' Rule, since each f_1 and f_2 have exactly one sign alternation in their ordered sequence of coefficients. The limiting behavior of their unique positive roots is easily obtained from Rolle's Theorem (since $f_1(0),f_2(0)<0$ and $f_1(x_1),f_2(x_1)\longrightarrow\infty$ as $x_1\longrightarrow\infty$), and the fact that the sum $\sum_{i=1}^r\frac{1}{2^i}$ is a strictly increasing function of $r\in\mathbb{N}$, and converges to 1 as $r\longrightarrow\infty$.

The optimality of the $\log(t-1)$ terms is easily seen via the polynomial $g(x_1):=(x_1+1)^{t-1}$: The left-most (resp. right-most) lower edge of $\mathrm{ArchNewt}(g)$ has slope $-\log(t-1)$ (resp. $\log(t-1)$), by the log-concavity of the binomial coefficients. So by Proposition 2.1, $\min \mathrm{ArchTrop}(g)=-\log(t-1)$ and $\max \mathrm{ArchTrop}(g)=\log(t-1)$. Since $\mathrm{Amoeba}(g)=\{0\}$, we are done. ■

By a slight variation of our last proof, we can easily obtain a family of examples showing that the $\log 3$ interval-width from Theorem 1.5 is also optimal, unless one refines further to incorporate t or other parameters.

Lemma 2.5. *For any $k\geq 1$ let $g_k(x_1):=1+x+\cdots+x^{k-1}-x^k+\frac{1}{9}x^{k+1}+\cdots+\frac{1}{9^k}x^{k+k}$. Then (1) $\mathrm{ArchTrop}(g_k)=\{0,\log 9\}$ for all k and (2) for any fixed $\varepsilon>0$ there is a k such that $\mathrm{Amoeba}(g_k)\cap(\log(3)-\varepsilon,\log(3)+\varepsilon)$ is non-empty.* ■

We now recall a seminal collection of bounds due to Ostrowski:

Theorem 2.6. [Ost40a, Cor. IX, Pg. 143][2] *Following the notation of Corollary 2.3, we have:*

(1) $-\log 2 < \log|\zeta_1| - v_1 \leq \log d,$

(2) $-\log d \leq \log|\zeta_d| - v_d < \log 2,$

(3) $\log\left(1-\frac{1}{2^{1/i}}\right) < \log|\zeta_i| - v_i < -\log\left(1-\frac{1}{2^{1/(d-i+1)}}\right)$ *for all* $i\in\{2,\dots,d-1\}$.

In particular,

$$-0.5348 \leq \log\left(1-\frac{1}{2^{1/i}}\right) - (-\log i) < -0.3665 \text{ and}$$

$$0.3665 < -\log\left(1-\frac{1}{2^{1/(d-i+1)}}\right) - \log(d-i+1) \leq 0.5348.$$ ■

Remark 2.7. *Since* $\mathrm{ArchTrop}(f)=\{v_1,\dots,v_d\}$ *(see Remark 2.4), Theorem 1.5 implies that any given* $\log|\zeta_i|$ *lies within distance* $\log 3$ *of some* v_j, possibly with $j\neq i$. *In this sense, the final assertion of Theorem 2.6 tells us that Theorem 1.5 isolates each* $\log|\zeta_i|$ *strictly better than Ostrowski's bounds, except possibly in the cases $i\in\{2,d-1\}$ or $t=d+1=3$. Corollary 2.12 in Section 2.1 below matches Ostrowski's bounds when $t=d+1=3$.* ⋄

Recently, Akian, Gaubert, and Sharify have derived metric bounds [AGS17] improving those of Hadamard and Ostrowski, but in a different direction from ours. Their focus was the *Matrix Polynomial Problem* (a.k.a. the *Polynomial Eigenvalue Problem*): Given matrices $A_0,\dots,A_d\in\mathbb{C}^{n\times n}$, find $\lambda\in\mathbb{C}$ such that $A_0+\lambda A_1+\cdots+\lambda^d A_d$ has determinant 0. The Matrix Polynomial Problem includes the classical eigenvalue problem for $d=1$, while the $n=1$ case is the problem of univariate polynomial solving. A special case of one of the main theorems of [AGS17] adds a new bound to the univariate t-nomial setting: The center of mass of the ℓ smallest points of $\mathrm{Amoeba}(f)$ is not too far to the left from the center of mass of the k smallest points of $\mathrm{ArchTrop}(f)$ (assuming repeated points are counted appropriately).

Theorem 2.8. *(Special case of* [AGS17, Thm. 4.1 & Rem. 4.2]*) Following the notation of Corollary 2.3, for any $\ell\in\{1,\dots,d\}$, we have $\sum_{i=1}^{\ell}\log|\zeta_i|\geq\left(\sum_{i=1}^{\ell}v_i\right)-\frac{1}{2}\log t$.* ■

Remark 2.9. *It should be noted that our notation differs significantly from that of* [AGS17]*. However, to keep the focus on proving Theorem 1.5, we postpone a more detailed discussion until Remark 5.4 of Section 5.2.* ⋄

[2]There was a typo in Ostrowski's original statement of the upper bound from Assertion (3), later corrected in an addendum by Ostrowski [Ost40b].

Note that while the $\ell=1$ case of Theorem 2.8 yields a weaker bound than Assertion (1) of our Theorem 2.6 when $t\geq 4$, a strength of Theorem 2.8 is its bound on a particular *amortized* error of approximating $\mathrm{Amoeba}(f)$ by $\mathrm{ArchTrop}(f)$: Applying Theorem 2.8 with $\ell=d$ to both $f(x_1)$ and $f(1/x_1)$ implies that $\left|\frac{\sum_{i=1}^d \log|\zeta_i|}{d} - \frac{\sum_{i=1}^d v_i}{d}\right| \leq \frac{1}{2d}\log t$. Such an estimate does not appear to be directly obtainable from our methods here. On the other hand, our methods usually imply tighter distance bounds for finding *some* v_j close to a given $\log|\zeta_i|$.

We have so far concentrated on showing that each $\log|\zeta_i|$ is close to some v_j, with optimal distance bounds. Showing that each v_j is close to some $\log|\zeta_i|$ requires more preparation, which we now detail.

2.1. **Proving Theorem 1.5.** We will need three technical results on bounding the norms of summands of sparse polynomials, and counting roots of polynomials in annuli, before proving Theorem 1.5. As before, we let $f(x_1)=\sum_{j=1}^t c_j x_1^{a_j}$ be a univariate t-nomial, with exponents in strictly increasing order: $a_1<\cdots<a_t$.

Proposition 2.10. *Suppose $t\geq 3$, $v\in\mathrm{ArchTrop}(f)$, and ℓ is the unique index such that $(a_\ell,-\log|c_\ell|)$ is the right-hand vertex of the lower edge of $\mathrm{ArchNewt}(f)$ of slope v (so $2\leq\ell$). Then for any $N\in\mathbb{N}$ and x_1 with $|x_1|\geq(N+1)e^v$ we have $\sum\limits_{j=1}^{\ell-1}\left|c_j x_1^{a_j}\right| < \frac{1}{N}\left|c_\ell x_1^{a_\ell}\right|$.*

Proof of Proposition 2.10: First note that $2\leq\ell\leq t$ by construction. Letting $r:=\log|x_1|$ and $\beta_j:=\log|c_j|$ we obtain $\sum_{j=1}^{\ell-1}\left|c_j x_1^{a_j}\right| = \sum_{j=1}^{\ell-1} e^{a_j r+\beta_j} = \sum_{j=1}^{\ell-1} e^{a_j(r-v)+a_j v+\beta_j}$. Clearly, $a_j\leq a_\ell-(\ell-j)$, so for $r\geq v$ we have $\sum\limits_{j=1}^{\ell-1} e^{(a_\ell-(\ell-j))(r-v)+a_j v+\beta_j} \leq \sum\limits_{j=1}^{\ell-1} e^{(a_\ell-(\ell-j))(r-v)+a_\ell v+\beta_\ell}$, thanks to Proposition 2.1 and the definition of $\mathrm{ArchTrop}(f)$. So then

$$\begin{aligned}
\textstyle\sum_{j=1}^{\ell-1}\left|c_j x_1^{a_j}\right| &\leq e^{(a_\ell-(\ell-1))(r-v)+a_\ell v+\beta_\ell}\textstyle\sum_{j=1}^{\ell-1} e^{(j-1)(r-v)}\\
&= e^{(a_\ell-(\ell-1))(r-v)+a_\ell v+\beta_\ell}\left(\frac{e^{(\ell-1)(r-v)}-1}{e^{(r-v)}-1}\right)\\
&< e^{(a_\ell-(\ell-1))(r-v)+a_\ell v+\beta_\ell}\left(\frac{e^{(\ell-1)(r-v)}}{e^{r-v}-1}\right) = \frac{e^{a_\ell r+\beta_\ell}}{e^{r-v}-1}
\end{aligned}$$

So to prove our desired inequality it clearly suffices to enforce $e^{r-v}-1\geq N$. The last inequality clearly holds for all $r\geq v+\log(N+1)$, so we are done. ■

Pellet's Theorem. [Pel81][3] *If the Laurent polynomial $|c_\ell|x_1^{a_\ell} - \sum\limits_{i\in\{1,\ldots,t\}\setminus\{\ell\}}|c_i|x_1^{a_i}$ has exactly 2 positive roots $\zeta_1<\zeta_2$ then, counting multiplicities, f has exactly $a_\ell-a_1$ (resp. a_t-a_ℓ) roots with norm in $(0,\zeta_1]$ (resp. $[\zeta_2,\infty)$). In particular, f has no roots with norm in (ζ_1,ζ_2).* ■

Lemma 2.11. *Set $v_{\min}:=\min\mathrm{ArchTrop}(f)$ and $v_{\max}:=\max\mathrm{ArchTrop}(f)$. Also let u_1 and u_2 be* consecutive *points of $\mathrm{ArchTrop}(f)$ satisfying $u_2\geq u_1+\log 9$, and let ℓ be the unique index such that $(a_\ell,-\log|c_\ell|)$ is the unique vertex of $\mathrm{ArchNewt}(f)$ incident to lower edges of slopes u_1 and u_2 (so $2\leq\ell\leq t-1$). Then, counting multiplicities, f has exactly $a_\ell-a_1$ (resp. a_t-a_ℓ) roots $\zeta\in\mathbb{C}$ satisfying $\frac{1}{2}e^{v_{\min}}<|\zeta|<3e^{u_1}$ (resp. $\frac{1}{3}e^{u_2}<|\zeta|<2e^{v_{\max}}$).*

[3] See also [RS02, Thm. 9.2.2, Pg. 285]. Our paraphrase here follows immediately from Pellet's original theorem simply by multiplying f by $x_1^{-a_1}$.

Proof of Lemma 2.11: By symmetry (with respect to replacing x_1 by $\frac{1}{x_1}$) it clearly suffices to prove the first root count. Setting $x_1 := 3e^{u_1}$, Proposition 2.10 tells us that $\frac{1}{2}|c_\ell|x_1^{a_\ell} > \sum_{j=1}^{\ell-1}|c_j|x_1^{a_j}$. Observing that $\frac{1}{x_1} = \frac{1}{3e^{u_1}} \geq 3e^{-u_2}$ (since $u_2 - u_1 \geq \log 9$), another application of Proposition 2.10 to $f(1/x_1)$ then implies that $\frac{1}{2}|c_\ell|x_1^{a_\ell} > \sum_{j=\ell+1}^{t}|c_j|x_1^{a_j}$. So $g(x_1) := |c_\ell|x_1^{a_\ell} - \sum_{j\neq\ell}^{t}|c_j|x_1^{a_j}$ is positive at $x_1 = 3e^{u_1}$. Note also that both $g(\varepsilon)$ and $g(1/\varepsilon)$ are negative for all sufficiently small $\varepsilon > 0$. So by Rolle's Theorem, g has at least 2 positive roots. Moreover, by Descartes' Rule (since g has exactly 2 sign alternations in its ordered sequence of coefficients), g has at most 2 positive roots. So we may apply Pellet's Theorem and, applying Assertion (1) of Theorem 2.2 as well, we are done. ■

Proof of Theorem 1.5: We first prove that Corollary 2.3 and Lemma 2.11 imply that the roots of f lie in a particular union of annuli: More precisely, we will first prove that $\mathrm{Amoeba}(f) \subset U_f$.

Let $v_{\min} := \min \mathrm{ArchTrop}(f)$ and $v_{\max} := \max \mathrm{ArchTrop}(f)$. Then, using Corollary 2.3, the left-hand inequality of Assertion (a), together with the right-hand inequality of Assertion (b), imply that $\mathrm{Amoeba}(f)$ lies within the interval $(v_{\min} - \log 2, v_{\max} + \log 2)$. In particular, if $w \in \mathrm{Amoeba}(f)$ satisfies $w \leq v_{\min}$ (resp. $w \geq v_{\max}$) then w must be within distance $\log 2$ of the left-most (resp. right-most) point of $\mathrm{ArchTrop}(f)$.

Now assume $w \in \mathrm{Amoeba}(f)$ satisfies $v_{\min} < w < v_{\max}$. Let u_1 and u_2 be the unique consecutive points of $\mathrm{ArchTrop}(f)$ satisfying $u_1 < w < u_2$. Then $u_2 - u_1 \leq \log 9$ immediately implies that w must be within distance $\log 3$ of at least one of u_1 or u_2. Otherwise, Lemma 2.11 implies that w can *not* lie in the interval $[u_1 + \log 3, u_2 - \log 3]$, and thus w is within distance $\log 3$ of either u_1 or u_2.

In other words, w must always be within $\log 3$ of some point of $\mathrm{ArchTrop}(f)$. Combined with the containment $\mathrm{Amoeba}(f) \subset (v_{\min} - \log 2, v_{\max} + \log 2)$, we thus obtain that $\mathrm{Amoeba}(f) \subset U_f$.

We now study Λ_Γ: Having ordered the monomial terms of f so that the exponents a_i are in increasing order, the Fundamental Theorem of Algebra then tells us that the number of roots of f in $\mathbb{C}^*$ is exactly $a_t - a_1$. Also, we clearly have

$$\sum_{\substack{\Gamma \text{ a connected} \\ \text{component of } U_f}} \Lambda_\Gamma \quad = \quad a_t - a_1. \tag{1}$$

In particular, when $t \geq 2$, U_f is non-empty and any Γ must contain at least 1 point of $\mathrm{ArchTrop}(f)$. So any Λ_Γ is a positive integer when $t \geq 2$.

To conclude, we merely need to prove that f has exactly Λ_Γ roots $\zeta \in \mathbb{C}^*$ with $\log|\zeta| \in \Gamma$. Toward this end, assume U_f has exactly C connected components, and we label them $\Gamma_1, \ldots, \Gamma_C$ so that $i < j \Longrightarrow$ every point of Γ_i is strictly less than any point of Γ_j.

Now, if $C = 1$, then $U_f = \Gamma$ is connected and $\Lambda_\Gamma = a_t - a_1$. Since we already know that $\mathrm{Amoeba}(f) \subset U_f$, the stated root count is thus true.

If $C \geq 2$ then Lemma 2.11 immediately implies that f has exactly Λ_{Γ_1} roots $\zeta \in \mathbb{C}^*$ with $\log|\zeta| \in \Gamma_1$: Simply apply Lemma 2.11 with u_1 the right-most point of $\Gamma_1 \cap \mathrm{ArchTrop}(f)$ and u_2 the left-most point of $\Gamma_2 \cap \mathrm{ArchTrop}(f)$. More generally, it is easy to prove that, for any $i \geq 2$, f has exactly $\Lambda_{\Gamma_1} + \cdots + \Lambda_{\Gamma_i}$ roots $\zeta \in \mathbb{C}^*$ with $\log|\zeta| \in \Gamma_1 \cup \cdots \cup \Gamma_i$: Simply apply Lemma 2.11 with u_1 the right-most point of $(\Gamma_1 \cup \cdots \cup \Gamma_i) \cap \mathrm{ArchTrop}(f)$ and u_2 the left-most point of $\Gamma_{i+1} \cap \mathrm{ArchTrop}(f)$. Note also that $(a_2 - a_1) + (a_3 - a_2) + \cdots + (a_\ell - a_{\ell-1}) = a_\ell - a_1$, and thus $\Lambda_{\Gamma_1} + \cdots + \Lambda_{\Gamma_i}$ is always of the form $a_\ell - a_1$ for ℓ an increasing function of i.

Since the Γ_i are disjoint, we then obtain that the number of roots $\zeta \in \mathbb{C}^*$ of f with $\log|\zeta| \in \Gamma_i$ is exactly $(\Lambda_{\Gamma_1} + \cdots + \Lambda_{\Gamma_i}) - (\Lambda_{\Gamma_1} + \cdots + \Lambda_{\Gamma_{i-1}}) = \Lambda_{\Gamma_i}$. So we are done. ■

We can tighten the union of intervals U_f further when $t=3$: Combining Theorem 1.5 with Assertion (1a) of Theorem 3.4 (stated and proved in Section 3 below) immediately yields the following refinement.

Corollary 2.12. *Let* $v_{\min} := \min \mathrm{ArchTrop}(f)$, $v_{\max} := \max \mathrm{ArchTrop}(f)$, *and assume that* $t=3$ *and* $v_{\max} - v_{\min} > \log 4$. *Then there are exactly* $a_2 - a_1$ *(resp.* $a_3 - a_2$*) roots* $\zeta \in \mathbb{C}$ *of* f *with* $v_{\min} - \log 2 < \log|\zeta| \leq v_{\min} + \log 2$ *(resp.* $v_{\max} - \log 2 \leq \log|\zeta| < v_{\max} + \log 2$*).* ■

3. Approximating Amoebae in Arbitrary Dimension

Moving on to the multivariate case, let us first review some basic facts on the structure of $\mathrm{ArchTrop}(f)$.

Proposition 3.1. *If* f *is an* n*-variate binomial then* $\mathrm{Amoeba}(f)$ *and* $\mathrm{ArchTrop}(f)$ *are identical affine hyperplanes in* $\mathbb{R}^n$. ■

Lemma 3.2. *Suppose* f *is an* n*-variate* t*-nomial with* $\mathrm{Newt}(f)$ *of dimension* k. *Then:*

(0) $k \leq \min\{n, t-1\}$.

(1) $k=1 \Longrightarrow \mathrm{ArchTrop}(f)$ *is a non-empty disjoint union of at most* $t-1$ *parallel affine hyperplanes in* $\mathbb{R}^n$.

(2) $k \geq 2 \Longrightarrow \mathrm{ArchTrop}(f)$ *is a path-connected polyhedral complex, of pure dimension* $n-1$, *with at most* $t(t-1)/2$ *faces of dimension* $n-1$.

(3) $t=k+1 \Longrightarrow \mathrm{ArchTrop}(f) \subseteq \mathrm{Amoeba}(f)$ *and both* $\mathrm{Amoeba}(f)$ *and* $\mathrm{ArchTrop}(f)$ *are contractible.* ■

Proposition 3.1 is elementary. Assertions (0)–(2) of Lemma 3.2 follows easily from the definition of $\mathrm{ArchTrop}(f)$, thanks to polyhedral duality [Zie95]. (See also [MS15, Ch. 3, Sec. 3] for a much more detailed discussion in the non-Archimedean setting.) Assertion (3) of Lemma 3.2 was one of the first basic topological results on amoebae and can be found, for instance, in [For98, Prop. 3.1.8], [Rul03, Thms. 8 & 12], and [TdW13, Lemma 3.4 (a)].

Our main multivariate result is that every point of $\mathrm{Amoeba}(f)$ is within an explicit distance of some point of $\mathrm{ArchTrop}(f)$, *and vice-versa*, independent of the degree or number of variables of f. We use $|\cdot|$ for the standard ℓ_2-norm on $\mathbb{C}^n$.

Definition 3.3. *For any* $\varepsilon > 0$ *and* $X \subseteq \mathbb{R}^n$ *we define the* open ε-neighborhood of X *to be* $X_\varepsilon := \{x \in \mathbb{R}^n \mid |x - x'| < \varepsilon$ *for some* $x' \in X\}$, *and let* $\overline{X}_\varepsilon$ *denote its Euclidean closure.* ◇

Theorem 3.4. *For any* $f \in \mathbb{C}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$ *with exactly* $t \geq 2$ *monomial terms and* $\mathrm{Newt}(f)$ *of dimension* k *we have:*

(1) *(a)* $\mathrm{Amoeba}(f) \subseteq \overline{\mathrm{ArchTrop}(f)}_{\log(t-1)}$ *and, for* $k=1$, $\mathrm{Amoeba}(f) \subsetneq \mathrm{ArchTrop}(f)_{\log 3}$.

(b) $\mathrm{ArchTrop}(f) \subseteq \mathrm{Amoeba}(f)_{\varepsilon_{k,t}}$ *where* $\varepsilon_{1,t} := (\log 9)t - \log\frac{81}{2} < 2.2t - 3.7$, $\varepsilon_{2,t} := \sqrt{2}(t-2)\left((\log 9)t - \log\frac{81}{2}\right) < (t-2)(3.11t - 5.23)$, *and* $\varepsilon_{k,t} := \sqrt{k}\left\lceil \frac{1}{4}t(t-1)\right\rceil \left((\log 9)t - \log\frac{81}{2}\right)$ *for* $k \geq 3$.
In particular, $\varepsilon_{k,t} < \frac{3}{5}t^{3/2}(t-1)^2$ *for all* $k \geq 1$ *and* $t \geq 2$.

(2) *Let* $\varphi(x) := 1 + x_1 + \cdots + x_{t-1}$ *and* $\psi(x) := (x_1+1)^{t-k} + x_2 + \cdots + x_k$. *Then*

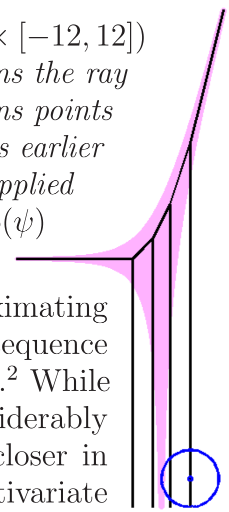
(a) $\mathrm{Amoeba}(\varphi)$ *contains a point at distance* $\log(t-1)$ *from* $\mathrm{ArchTrop}(\varphi)$ *and*

(b) $\mathrm{ArchTrop}(\psi)$ *contains points approaching distance* $\log(t-k)$ *from* $\mathrm{Amoeba}(\psi)$.

We prove Theorem 3.4 below. For multivariate polynomials, our bounds appear to be the first allowing dependence on just the number of terms t. In particular, Assertion (1a) sharpens, and extends to arbitrary dimension, an earlier bound of Mikhalkin for the case $n=2$: Letting L denote the number of lattice points in the Newton polygon of f, [Mik05, Lemma 8.5, Pg. 360] asserts that $\mathrm{Amoeba}(f)$ is contained in the possibly larger neighborhood $\overline{\mathrm{ArchTrop}(f)}_{\log(L-1)}$. Assertion (2a) of Theorem 3.4 shows that the size of the neighborhood from Assertion (1a) is in fact optimal for the infinite family of cases $t=k+1\geq 3$.

Finding the tightest neighborhood of $\mathrm{Amoeba}(f)$ containing $\mathrm{ArchTrop}(f)$ appears to be an open problem: We are unaware of any earlier multivariate version of Assertion (1b). The only other earlier distance bound between an amoeba (of positive dimension) and a polyhedral approximation we know of is a result of Viro [Vir01, Sec. 1.5] on the distance between the graph of a univariate polynomial (drawn on log paper) and a piecewise linear curve that is ultimately a piece of the $n=2$ case of $\mathrm{ArchTrop}(f)$ here.

Example 3.5. *Setting* $\psi(x)=(x_1+1)^4+x_2$ *we see* $\mathrm{Amoeba}(\psi)\cap([-7,7]\times[-12,12])$ *and* $\mathrm{ArchTrop}(\psi)\cap([-7,7]\times[-12,12])$ *on the right.* $\mathrm{ArchTrop}(\psi)$ *contains the ray* $(\log 4, 4\log 4)+\mathbb{R}_+(0,-1)$ *and this rightmost downward-pointing ray contains points with distance from* $\mathrm{Amoeba}(\psi)$ *approaching* $\log 4$. *We also observe that Viro's earlier polygonal approximation of graphs of univariate polynomials on log paper, applied here, would result in the polygonal curve that is the subcomplex of* $\mathrm{ArchTrop}(\psi)$ *obtained by deleting all* 4 *downward-pointing rays.* $\diamond$



It is worth comparing Theorem 3.4 to two other methods for approximating complex amoebae: Purbhoo, in [Pur08], describes a uniformly convergent sequence of outer polyhedral approximations to any amoeba, using cyclic resultants.[2] While $\mathrm{ArchTrop}(f)$ lacks this refinability, the computation of $\mathrm{ArchTrop}(f)$ is considerably simpler: See Section 4 below and [AGGR15]. $\mathrm{ArchTrop}(f)$ is actually closer in spirit to the *spine* of $\mathrm{Amoeba}(f)$. The latter construction, based on a multivariate version of Jensen's Formula from complex analysis, is due to Passare and Rullgård [PR04, Sec. 3] and results in a polyhedral complex that is always contained in, and is homotopy equivalent to, $\mathrm{Amoeba}(f)$. While $\mathrm{ArchTrop}(f)$ is not always homotopy equivalent to $\mathrm{Amoeba}(f)$, $\mathrm{ArchTrop}(f)$ at least has polynomial membership complexity in fixed dimension (see Section 4 below). Further background on the computational complexity of amoebae can be found in [The02, SdW13, TdW15].

Remark 3.6. *The* Matrix Polynomial Problem *(see our discussion from Section 2) can be naturally phrased as a polynomial system with solutions in* $\mathbb{C}\times\mathbb{P}^{n-1}_{\mathbb{C}}$ *(i.e., an* intersection of several *hypersurfaces) by considering the vector equality* $(A_0+\lambda A_1+\cdots+\lambda^d A_d)x=\mathbf{O}$. *In particular,* [AGS17, Thm. 4.1] *describes a tropical approximation to the solutions of the Matrix Polynomial Problem. However, the metric bounds of* [AGS17] *for the case* $n\geq 2$ *(which have exponential dependence on the dimension* n*) are not directly comparable to the bounds from Theorem 3.4, which apply to a* single *hypersurface (and have dependence sub-quartic in the number of terms).* $\diamond$

3.1. **Proof of Assertion (1a) of Theorem 3.4.** When $k=1$ it is clear that f is of the form $g(x^a)$ for some $g\in\mathbb{C}\left[x_1^{\pm 1}\right]$ and $a\in\mathbb{Z}^n\setminus\{\mathbf{O}\}$. Assertion (1) of Lemma 3.2, and Theorem 1.5 applied to g — noting in particular that both $\mathrm{ArchTrop}(f)$ and $\mathrm{Amoeba}(f)$ consist of

[4]See also [FMMdW17] for recent computational improvements to Purbhoo's outer approximation.

affine hyperplanes perpendicular to the vector a — then imply the second bound from Part (a). So let us now assume $k\geq 2$.

Let $w:=(\log|\zeta_1|,\ldots,\log|\zeta_n|)\in\mathrm{Amoeba}(f)$, $\zeta=(\zeta_1,\ldots,\zeta_n)$, and assume without loss of generality that $|c_1\zeta^{a_1}|\geq|c_2\zeta^{a_2}|\geq\cdots\geq|c_t\zeta^{a_t}|$. Since $f(\zeta)=0$ implies that $|c_1\zeta^{a_1}|=|c_2\zeta^{a_2}+\cdots+c_t\zeta^{a_t}|$, the Triangle Inequality immediately implies that $|c_1\zeta^{a_1}|\leq(t-1)|c_2\zeta^{a_2}|$. Taking logarithms, we then obtain

$$a_1\cdot w+\log|c_1|\geq\cdots\geq a_t\cdot w+\log|c_t| \tag{2}$$

and

$$a_1\cdot w+\log|c_1|\leq\log(t-1)+a_2\cdot w+\log|c_2| \tag{3}$$

For each $i\in\{2,\ldots,t\}$ let us then define δ_i to be the shortest vector such that

$$a_1\cdot(w+\delta_i)+\log|c_1|=a_i\cdot(w+\delta_i)+\log|c_i|.$$

Note that $\delta_i=\lambda_i(a_i-a_1)$ for some nonnegative λ_i since we are trying to affect the dot-product $\delta_i\cdot(a_1-a_i)$. In particular, $\lambda_i=\frac{(a_1-a_i)\cdot w+\log|c_1/c_i|}{|a_1-a_i|^2}$ so that $|\delta_i|=\frac{(a_1-a_i)\cdot w+\log|c_1/c_i|}{|a_1-a_i|}$. (Indeed, Inequality (2) implies that $(a_1-a_i)\cdot w+\log|c_1/c_i|\geq 0$.)

Inequality (3) implies that $(a_1-a_2)\cdot w+\log|c_1/c_2|\leq\log(t-1)$. We thus obtain $|\delta_2|\leq\frac{\log(t-1)}{|a_1-a_2|}\leq\log(t-1)$. So let $i_0\in\{2,\ldots,t\}$ be any i minimizing $|\delta_i|$. We of course have $|\delta_{i_0}|\leq\log(t-1)$, and by the definition of δ_{i_0} we have

$$a_1\cdot(w+\delta_{i_0})+\log|c_1|=a_{i_0}\cdot(w+\delta_{i_0})+\log|c_{i_0}|.$$

Moreover, the fact that δ_{i_0} is the shortest among the δ_i implies that

$$a_1\cdot(w+\delta_{i_0})+\log|c_1|\geq a_i\cdot(w+\delta_{i_0})+\log|c_i|$$

for all i. Otherwise, we would have $a_1\cdot(w+\delta_{i_0})+\log|c_1|<a_i\cdot(w+\delta_{i_0})+\log|c_i|$ and $a_1\cdot w+\log|c_1|\geq a_i\cdot w+\log|c_i|$ (the latter following from Inequality (2)). Taking a convex linear combination of the last two inequalities, it is then clear that there must be a $\mu\in[0,1)$ such that $a_1\cdot(w+\mu\delta_{i_0})+\log|c_1|=a_i\cdot(w+\mu\delta_{i_0})+\log|c_i|$. Thus, by the definition of δ_i, we would obtain $|\delta_i|\leq\mu|\delta_{i_0}|<|\delta_{i_0}|$ — a contradiction.

We thus have the following:

$$a_1\cdot(w+\delta_{i_0})-(-\log|c_1|)=a_{i_0}\cdot(w+\delta_{i_0})-(-\log|c_{i_0}|),$$
$$a_1\cdot(w+\delta_{i_0})-(-\log|c_1|)\geq a_i\cdot(w+\delta_{i_0})-(-\log|c_i|)$$

for all i, and $|\delta_{i_0}|\leq\log(t-1)$. This implies that $w+\delta_{i_0}\in\mathrm{ArchTrop}(f)$. In other words, we've found a point in $\mathrm{ArchTrop}(f)$ sufficiently near $(\log|\zeta_1|,\ldots,\log|\zeta_n|)$ to prove our desired upper bound. ■

3.2. **Proving Assertion (1b) of Theorem 3.4.** We begin with a refinement of the special case $n=1$. Let $\#S$ denote the cardinality of a set S.

Theorem 3.7. *Suppose f is any univariate t-nomial with $t\geq 3$ and $s:=\#\mathrm{ArchTrop}(f)$. (So $1\leq s\leq t-1$.) Then for any $v\in\mathrm{ArchTrop}(f)$ there is a root $\zeta\in\mathbb{C}^*$ of f with $|v-\log|\zeta||<\log 2$, $|v-\log|\zeta||\leq\log\min\{18,t-1\}$, or $|v-\log|\zeta||<(\log 9)s-\log\frac{9}{2}<2.2s-1.5$, according as s is 1, 2, or ≥ 2. In particular, $|v-\log|\zeta||<(\log 9)t-\log\frac{81}{2}<2.2t-3.7$ for all $t\geq 3$.*

Proof of Theorem 3.7: Following the notation of Theorem 1.5, let Γ be the unique connected component of U_f containing $v\in\mathrm{ArchTrop}(f)$ and let $m:=\#(\Gamma\cap\mathrm{ArchTrop}(f))$. (So $1\leq m\leq s$.) The quantity $|v-\log|\zeta||$ is thus clearly maximized when v is as far to the left as possible and $\log|\zeta|\in\Gamma$ is as far to the right as possible, or vice-versa. Without loss of generality, we may assume the first possibility:

$$|v-\log|\zeta||<\log(3)+(\log 9)(m-2)+\log(3)+\delta,$$

where δ is $\log 3$ or $\log 2$, according as $m<s$ or $m=s$. We thus obtain the largest possible upper bound of $(\log 9)s-\log\frac{9}{2}$ when $m=s$. Note also that $s\leq t-1$. So now we merely need to refine the cases with $s\in\{1,2\}$.

The case $s=1$ follows from Corollary 2.3 since $\min \mathrm{ArchTrop}(f)=\max \mathrm{ArchTrop}(f)$ here.

The case $s=2$ proceeds as follows: If $m=1$ then Γ is an open interval of width $\log(2)+\log(3)=\log 6$, with v at distance $\log 2$ from the left limit, so we must have $|v-\log|\zeta||<\log 3$. If $m=2$ then Γ is an open interval of width at most $\log(2)+2\log(3)+\log(2)=\log 36$, but we have $v_{\min}-\log 2<\log|\zeta|<v_{\max}+\log 2$ and $\mathrm{ArchTrop}(f)=\{v_{\min},v_{\max}\}$. So we have $|v-\log|\zeta||<\log(3)+\log(3)+\log(2)=\log 18$. In addition, we can apply Corollary 2.3 to observe that there is always a root $\zeta\in\mathbb{C}^*$ of f with $|v_{\min}-\log|\zeta||\leq\log(t-1)$, and the same bound can be attained for $|v_{\max}-\log|\zeta||$, possibly with a different root ζ. So we obtain $|v-\log|\zeta||\leq\log\min\{18,t-1\}$. ■

We will handle the case $n\geq 2$ of Assertion (1b) of Theorem 3.4 by showing that any point $v\in\mathrm{ArchTrop}(f)$ lies close to the intersection of $\mathrm{Amoeba}(f)$ with a specially chosen line also containing v. With some care, this enables us to reduce to the case $n=1$. In particular, intersecting a line with $\mathrm{Amoeba}(f)$ is the same as evaluating f along a monomial curve, and we'll need a technical lemma to pick exponents that permit an easy reduction to $n=1$.

Theorem 3.8. *Given any subset $\{a_1,\ldots,a_t\}\subset\mathbb{Z}^n$ of cardinality $t\geq n+1$, there exists an $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n\setminus\{\mathbf{O}\}$ such that the dot-products $\alpha\cdot a_1,\ldots,\alpha\cdot a_t$ are pair-wise distinct and, for all $i\in[n]$, $|\alpha_i|\leq\left\lceil\frac{1}{4}t(t-1)\right\rceil$ or $|\alpha_i|\leq t-2$, according as $n\geq 3$ or $n=2$.*

Proof of Theorem 3.8: Observe that for the $\alpha\cdot a_i$ to remain distinct we must have α avoid a set of $\leq t(t-1)/2$ hyperplanes, depending on $\{a_1,\ldots,a_t\}$. This is equivalent to α avoiding the zero set of an n-variate polynomial of degree $t(t-1)/2$. Schwartz's Lemma (see, e.g., [Sch80]) then tells us that for *any* $S\subset\mathbb{Z}$ with $\#S>t(t-1)/2$ there is an $\alpha\in S^n$ avoiding our aforementioned set of hyperplanes. Picking $S=\left\{-\left\lceil\frac{1}{4}t(t-1)\right\rceil,\ldots,\left\lceil\frac{1}{4}t(t-1)\right\rceil\right\}$ then gives us the case $n\geq 3$.

For the case $n=2$, it is enough to prove that the set of lattice points

$$X:=\{-(t-2),\ldots,t-2\}\times\{1,\ldots,t-2\}$$

contains at least $1+t(t-1)/2$ distinct directions (and thus we can always find a suitable $\alpha\in X$). In other words, we need to prove that X has at least $1+t(t-1)/2$ points with relatively prime coordinates. Throwing out the directions $(1,0)$ and $(0,1)$, it is then enough to show that $Y:=\{1,\ldots,t-2\}^2$ contains at least $\frac{t(t-1)}{4}-\frac{1}{2}$ points with relatively prime coordinates. The number of such points, for arbitrary t, forms the sequence A018805 in Sloane's Online Encyclopedia of Integer Sequences [Slo10]. A routine, but tedious calculation then yields the $t\in\{3,\ldots,45\}$ portion of the $n=2$ case.

The remaining cases can be settled as follows: By a standard Möbius inversion argument, the number of points with relatively prime coordinates in Y is exactly $\sum\limits_{d=1}^{t-2}\mu(d)\left\lfloor(t-2)/d\right\rfloor^2$ where μ is the classical *Möbius function* (see, e.g., [HWW08]). A simple expansion then yields our desired number of points to be bounded from below by

$$A(t):=\frac{(t-2)^2}{\zeta(2)}-4(t-2)-2(t-2)\log(t-2)-2\zeta(2)(t-1),$$

where $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ is the classical Riemann ζ-function.[5] A simple derivative calculation

[5]Please note that this paragraph of the proof of Theorem 3.8 is the only place where ζ indicates something other than a complex root of a polynomial.

then yields that $A(t) - \frac{t(t-1)}{4} + \frac{1}{2}$ is increasing for all $t \geq 25$. So it's enough to prove that $A(46) > 517$. One can check via `Maple` that $A(46) > 519.9$, so we are done. ■

Proof of Assertion (1b) of Theorem 3.4: The very last assertion follows easily from the fact that $\left\lceil \frac{1}{4}t(t-1) \right\rceil \leq \frac{1}{4}t(t-1) + \frac{1}{2}$ and an elementary computation.

Now let $v = (v_1, \ldots, v_n)$ be any point of ArchTrop(f). If $v \in$ Amoeba(f) then there is nothing to prove. So let us assume $v \notin$ Amoeba(f). Since the case $n=1$ is immediate from Proposition 3.1 and Theorem 3.7, we will assume henceforth that $n \geq 2$.

To reduce to the case $k=n$, let us temporarily assume that $k<n$. Without loss of generality, we can order the variables $x_1, \ldots, x_n$ so that the image of Newt(f) under the coordinate projection sending $\mathbb{R}^n$ onto $\mathbb{R}^k \times \{0\}^{n-k}$ has dimension k, and the restriction of the projection to Newt(f) is a bijection. Define $g(x_1, \ldots, x_k) := f(x_1, \ldots, x_k, e^{v_{k+1}}, \ldots, e^{v_n})$. By the definition of ArchTrop(f), $\max_{i \in [t]} |c_i e^{a_i \cdot v}|$ is attained for at least two distinct values of i. By our construction of g, this monomial norm condition implies that $(v_1, \ldots, v_k) \in$ ArchTrop(g). Clearly then, if we can find a root $(\zeta_1, \ldots, \zeta_k)$ of g with $|(v_1, \ldots, v_k) - (\log|\zeta_1|, \ldots, \log|\zeta_k|)| < \varepsilon_{k,t}$, then $\zeta := (\zeta_1, \ldots, \zeta_k, e^{v_{k+1}}, \ldots, e^{v_n})$ will be a root of f yielding a point of Amoeba(f) within distance $\varepsilon_{k,t}$ of v. But finding such a $(\zeta_1, \ldots, \zeta_k)$ for g is nothing more than a lower-dimensional instance of the case where the dimension of the underlying Newton polytope is the same as the underlying number of variables.

We may thus assume $k = n \geq 2$ henceforth. Now consider a monomial curve $C(t) := (\gamma_1 t^{\alpha_1}, \ldots, \gamma_n t^{\alpha_n})$ with $\alpha = (\alpha_1, \ldots, \alpha_n) \neq \mathbf{O}$. (Note that the image of $C(\mathbb{C}^*)$ under the coordinate-wise log absolute map is a line in $\mathbb{R}^n$.) Setting $\gamma_i := e^{v_i}$ for all i we obtain $|C(1)| = (e^{v_1}, \ldots, e^{v_n})$, independent of α. So let us pick α satisfying the conclusion of Theorem 3.8 and set $h(t) := f(C(t))$. Then by the definition of ArchTrop(f), and especially because the $\alpha \cdot a_i$ are pair-wise distinct, we can conclude that $h \in \mathbb{C}[t]$ has exactly t monomial terms *and* $0 \in$ ArchTrop(h). So to find a root $\zeta \in (\mathbb{C}^*)^n$ with $(\log|\zeta_1|, \ldots, \log|\zeta_n|)$ close to v, it's enough to prove that h has a root ρ close to 1. Thanks to Theorem 3.7, we can do the latter, so now we simply have to account for metric distortion from specializing f along $C(t)$.

Taking logarithms, Amoeba(h) containing a point at distance ε from 0 implies that Amoeba(f) contains a point at distance $\leq |\alpha|\varepsilon$ from v. So by the coordinate bounds of Theorem 3.8, we are done. ■

3.3. **Proving Assertion (2) of Theorem 3.4.** We first note an alternative characterization of ArchTrop(f), valid in all dimensions.

Proposition 3.9. *For any n-variate t-nomial $f(x) = \sum_{j=1}^{t} c_j x^{a_j} \in \mathbb{C}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$, we have* ArchTrop$(f) = \{v \in \mathbb{R}^n \mid \max_j |c_i e^{a_j \cdot v}|$ *is attained for at least two distinct values of* $j\}$. ■

Like Proposition 2.1, Proposition 3.9 follows easily from the fact that a positive-dimensional face of a polytope P is a subset where a particular linear form attains its maximum (over P) at least twice (see, e.g., [Zie95, Ch. 7]).

Proof of Assertion (2) of Theorem 3.4: To prove Part (a), note that $(1, \ldots, 1)/(1-t)$ is a root of φ and thus $p := -\log(t-1)(1, \ldots, 1) \in$ Amoeba(φ). Note that Newt(φ) is the standard n-simplex $\Delta_n \subset \mathbb{R}^n$. So, by polyhedral duality [Zie95], and the definition of ArchTrop(φ), we have that ArchTrop(φ) is the positive codimension locus of the outer normal fan of Δ_n. In particular, ArchTrop$(\varphi) \cap \overline{\mathbb{R}^{t-1}_{-}}$ is the boundary of the negative orthant. So the distance from p to ArchTrop(φ) is $\log(t-1)$.

To prove Part (b), note that $(x_1+1)^{t-k}$ has unique root -1, and this root has multiplicity $t-k$. Recall that the roots of a monic univariate polynomial are continuous functions of the (non-leading) coefficients, e.g., [RS02, Thm. 1.3.1, Pg. 10].[6] So then, for any $\varepsilon>0$, we can find a $\delta_\varepsilon>0$ so that for all $\delta\in\mathbb{C}$ with $|\delta|\in[0,\delta_\varepsilon)$, *all* the roots ζ_1 of $(x_1+1)^{t-k}-\delta$ satisfy $|\zeta_1+1|<\varepsilon$. Clearly then, for any $\varepsilon'>0$, taking $|\rho_2|,\ldots,|\rho_n|$ sufficiently small (or $u_2:=\log|\rho_2|,\ldots,u_n:=\log|\rho_n|$ sufficiently negative) implies that the distance from any point $u\in\text{Amoeba}(f)$ of the form $(u_1,u_2,\ldots,u_n)$ to the hyperplane $\{0\}\times\mathbb{R}^{n-1}$ is at most ε': Simply take ε so that $\varepsilon'=\log(1+\varepsilon)$ and $|x_2|+\cdots+|x_n|<\delta_\varepsilon$.

On the other hand, by the log-concavity of the binomial coefficients, $\text{ArchNewt}\big((x_1+1)^{t-k}\big)$ must have an edge of slope $t-k$. This will enable us to prove that $\text{ArchTrop}(\psi)$ contains a ray of the form $\{(\log(t-k),N,\ldots,N)\}_{N\rightarrow+\infty}$. and thus conclude: The points along this ray have distance to $\text{Amoeba}(\psi)$ approaching $\log(t-k)$, by the preceding paragraph.

To see why such a ray lies in $\text{ArchTrop}\big((x_1+1)^{t-k}\big)$ simply note that as $N\longrightarrow-\infty$, the linear form $\log(t-k)u_1+Nu_2+\cdots+Nu_n-u_{n+1}$ is maximized exactly at the vertices
$$(t-k-1,0,\ldots,0,-\log(t-k))\quad\text{and}\quad(t-k,0,\ldots,0,0)$$
of $\text{ArchNewt}\big((x_1+1)^{t-k}\big)$. (Indeed, the only other possible vertices of $\text{ArchNewt}\big((x_1+1)^{t-k}\big)$ are the basis vectors $e_2,\ldots,e_k$ of $\mathbb{R}^{n+1}$.) So, by Proposition 3.9, we are done. ■

4. On the Computational Complexity of ArchTrop(f) and Amoeba(f)

The complexity classes **P**, **NP**, **PSPACE**, and **EXPTIME** — from the classical Turing model of computation — can be identified with families of *decision* problems, i.e., problems with a yes or no answer. Larger complexity classes correspond to problems with larger worst-case complexity. We refer the reader to [Sip92, Pap95, AB09, Sip12] for further background. Aside from the basic definitions of input size and **NP**-hardness, it will suffice here to simply recall that $\mathbf{P}\subseteq\mathbf{NP}\subseteq\mathbf{PSPACE}\subseteq\mathbf{EXPTIME}$, and that the properness of each inclusion (aside from $\mathbf{P}\subsetneqq\mathbf{EXPTIME}$, which has been known for some time [HS65, Rob83]) is a famous open problem. All algorithmic complexity results below count bit operations, and do so as a function of some underlying notion of input size.

Deciding membership in an amoeba can easily be rephrased as a problem within the *Existential Theory of the Reals*. The latter setting has been studied extensively in the $20^{\underline{\text{th}}}$ century (see, e.g., [Tar51, Coh69, BKR86, Can88]) and the current state of the art implies that amoeba membership is in **PSPACE**, i.e., it can be solved in polynomial-time by a parallel algorithm[7], provided one allows exponentially many processors. More precisely, we define the *input size* of a polynomial $f\in\mathbb{Z}[x_1,\ldots,x_n]$, written $f(x)=\sum_{j=1}^t c_jx^{a_j}$, to be $\text{size}(f):=\sum_{j=1}^t\log_2\left((2+|c_j|)\prod_{i=1}^n(2+|a_{i,j}|)\right)$, where $a_{i,j}$ is the $i^{\underline{\text{th}}}$ entry of the column vector a_j. (Put another way, up to a bounded additive error, $\text{size}(f)$ is just the sum of the bit-sizes of all the coefficients and exponents.) Similarly, we define $\text{size}(v)$, for any $v=(v_1,\ldots,v_n)\in\mathbb{Q}^n$, to be the sum of the sizes of the numerators and denominators of the v_i (written in lowest terms). We similarly extend the notion of input size to polynomials in $\mathbb{Q}[x_1,\ldots,x_n]$. Considering real and imaginary parts, we can extend further still to polynomials in $\mathbb{Q}\big[\sqrt{-1}\big][x_1,\ldots,x_n]$.

[6] The statement there excludes roots of multiplicity equal to the degree of the polynomial, but the proof in fact works in this case as well.

[7] i.e., an algorithm distributed across several processors running simultaneously on some shared memory.

Remark 4.1. *For our notion of input size, sufficiently sparse polynomials have size polynomial in the* logarithm *of the degree of the polynomial (among other parameters). For instance, our definition implies that* $c+x_1+x_2^d$ *has size* $O(\log(c)+\log(d))$. *This is in contrast to other definitions of input size in older papers (see, e.g.,* [The02]*) where degree is counted in such a way that* $1+x_1+x_2^d$ *has size* $\geq d$. $\diamond$

Theorem 4.2. *There is a* **PSPACE** *algorithm to decide, for any input pair* $(z,f) \in \bigcup_{n\in\mathbb{N}} \left(\mathbb{Q}^n \times \mathbb{Q}\left[\sqrt{-1}\right][x_1,\ldots,x_n]\right)$, *whether* $(\log|z_1|,\ldots,\log|z_n|) \in \mathrm{Amoeba}(f)$. *Furthermore, the special case where* $z=1$ *and* $f\in\mathbb{Z}[x_1]$ *in the preceding membership problem is already* **NP**-*hard.*

Theorem 4.2 is implicit in the papers [Pla84, BKR86, Can88] so, for the convenience of the reader, we provide an outline of the proof in Section 4.1 below.

Remark 4.3. *While* [The02, Cor. 2.7] *mentions "polynomial-time" amoeba membership detection in fixed dimension, the definition of input size implicitly used in* [The02] *differs from ours and yields complexity polynomial in the degree, among other parameters. So the method underlying* [The02, Cor. 2.7] *in fact has* exponential *worst-case complexity relative to the input size we use here. Indeed, the* **NP**-*hardness lower bound from Theorem 4.2 tells us that the existence of a polynomial-time amoeba membership algorithm for* $n=1$ *(relative to our notion of input size here) would imply* $\mathbf{P}=\mathbf{NP}$. $\diamond$

Since we now know that $\mathrm{ArchTrop}(f)$ is provably close to $\mathrm{Amoeba}(f)$, $\mathrm{ArchTrop}(f)$ would be of great practical value if $\mathrm{ArchTrop}(f)$ were easier to work with than $\mathrm{Amoeba}(f)$. This indeed appears to be the case: When the dimension n is fixed and all the coefficient absolute values of f have rational logarithms, standard high-dimensional convex hull algorithms (see, e.g., [Ede87]) enable us to describe every face of $\mathrm{ArchTrop}(f)$, as an explicit intersection of half-spaces, in polynomial-time.

The case of rational coefficients presents some subtleties because the underlying computations, done naively, involve arithmetic on rational numbers with exponentially large bit-size. Nevertheless, point membership for $\mathrm{ArchTrop}(f)$ has polynomial bit complexity when n is fixed.

Theorem 4.4. *Fix any* $\varepsilon>0$. *Then there is an* $O(nt(\log d)^{1+\varepsilon}(20.8\sigma(\log\sigma)^{1+\varepsilon})^{2n+2})$ *algorithm to decide, for any input* $(z,f)\in\bigcup_{n\in\mathbb{N}}\left(\mathbb{Q}^n\times\mathbb{Q}\left[\sqrt{-1}\right][x_1,\ldots,x_n]\right)$ *(with* $f(x)$ *of the form* $\sum_{j=1}^t c_j x^{a_j}$, *with degree at most* d *with respect to any variable,* $z=(z_1,\ldots,z_n)$, *and the bit-sizes of the* z_i *and* c_i *at most* σ*), whether* $(\log|z_1|,\ldots,\log|z_n|)\in\mathrm{ArchTrop}(f)$.

Furthermore, if we instead assume that both $\log|z_i|,\log|c_i|\in\mathbb{Q}$ *have bit size* $\leq\sigma$ *for all* i, *then there is an* $O\big(nt(\sigma+\log d)\log^2(\sigma d)\big)$ *algorithm to decide whether* $(\log|z_1|,\ldots,\log|z_n|)\in\mathrm{ArchTrop}(f)$.

We prove Theorem 4.4 in Section 4.2 below. An important relaxation of the point membership problem is the problem of finding the *distance* to $\mathrm{ArchTrop}(f)$ from a given query point v. The complexity of the latter problem, and its relevance to polynomial system solving, is explored further in [AGGR15].

4.1. **From Classical Computational Algebra to the Proofs of Theorems 4.2 and 4.4.** In what follows, all O-constants are effective and absolute. Let us first recall the following results of Plaisted and Ben-Or, Kozen, and Reif.

Theorem 4.5. [Pla84] *The problem* "Decide whether an arbitrary input $f\in\mathbb{Z}[x_1]$ has a complex root of norm 1." *is* **NP**-*hard.* ■

Theorem 4.6. [BKR86, Can88] *There is an algorithm that, given any collection of polynomials $f_1,\ldots,f_p,g_1,\ldots,g_q,h_1,\ldots,h_r\in\mathbb{Q}[x_1,\ldots,x_n]$, decides whether there is a $\zeta=(\zeta_1,\ldots,\zeta_n)\in\mathbb{R}^n$ with $f_1(\zeta)=\cdots=f_p(\zeta)=0$, $g_1(\zeta),\ldots,g_q(\zeta)>0$, and $h_1(\zeta),\ldots,h_r(\zeta)\geq 0$, in time*

$$\left[\sum_{i=1}^p \mathrm{size}(f_i)) + (\sum_{i=1}^q \mathrm{size}(g_i)) + (\sum_{i=1}^r \mathrm{size}(h_i))\right]^{O(1)},$$

using $\left[\sum_{i=1}^p \mathrm{size}(f_i)) + (\sum_{i=1}^q \mathrm{size}(g_i)) + (\sum_{i=1}^r \mathrm{size}(h_i))\right]^{O(1)}$ *processors.* ■

Theorem 4.2 will then follow easily from two elementary propositions. The first is a well-known trick from computational algebra for re-expressing polynomial systems in a simpler form. The second efficiently reduces complex root detection to real root detection.

Proposition 4.7. *Given any $f_1,\ldots,f_m\in\mathbb{Q}\left[\sqrt{-1}\right]\left[x_1^{\pm1},\ldots,x_n^{\pm1}\right]$, we can find $g_1,\ldots,g_M\in\mathbb{Q}\left[\sqrt{-1}\right]\left[x_1^{\pm1},\ldots,x_n^{\pm1},y_1^{\pm1},\ldots,y_N^{\pm1}\right]$ satisfying the following properties:*

1. *$f_1=\cdots=f_m=0$ has a root in $\mathbb{C}^n \iff g_1=\cdots=g_M=0$ has a root in $\mathbb{C}^N$.*
2. *Each g_i is either a quadratic binomial or a linear trinomial.*
3. *$\sum_{i=1}^M \mathrm{size}(g_i)=O(\sum_{i=1}^m \mathrm{size}(f_i))$.*

Moreover, $g_1,\ldots,g_M$ can be found in time $O(\sum_{i=1}^m \mathrm{size}(f_i))$. ■

Proposition 4.8. *Given any $f_1,\ldots,f_m\in\mathbb{Q}\left[\sqrt{-1}\right]\left[x_1^{\pm1},\ldots,x_n^{\pm1}\right]$ with each f_i of degree at most 2, we can find $g_1,\ldots,g_M\in\mathbb{Q}\left[x_1^{\pm1},\ldots,x_n^{\pm1},y_1^{\pm1},\ldots,y_N^{\pm1}\right]$ satisfying the following properties:*

1. *$f_1=\cdots=f_m=0$ has a root in $\mathbb{C}^n \iff g_1=\cdots=g_M=0$ has a root in $\mathbb{R}^N$.*
2. *$\sum_{i=1}^M \mathrm{size}(g_i)=O(\sum_{i=1}^m \mathrm{size}(f_i))$.*

Moreover, $g_1,\ldots,g_M$ can be found in time $O(mn)$. ■

A simple example of Proposition 4.7 is the replacement of $f(x_1):=1-2x_1+x_1^5$ by the system $G:=(y_1-x_1^2, y_2-y_1^2, y_3-y_2x_1, y_4-1+2x_1, y_5-y_4-y_3)$: It is easy to see that at a root of G, we must have $y_5=1-2x_1+x_1^5=0$. The proof of Proposition 4.7 is not much harder: One simply substitutes new variables to break down sums with more than 2 terms and (employing the binary expansions of the underlying exponents) monomials of degree more than 2. Proposition 4.8 follows easily upon expanding every complex multiplication (resp. complex addition) into 4 real multiplications (resp. 2 real additions), by introducing new variables for the real and imaginary parts of the x_i.

Sketch of Proof of Theorem 4.2: First observe that $(\log|z_1|,\ldots,\log|z_n|)\in\mathrm{Amoeba}(f)\iff f$ has a complex root ζ with $|\zeta_i|=|z_i|$ for all i. Letting A and B denote the real and imaginary parts of f, and letting α_i and β_i denote the real and imaginary parts of ζ_i, we thus obtain that $(\log|z_1|,\ldots,\log|z_n|)\in\mathrm{Amoeba}(f)$ if and only if the polynomial system

$$A(\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n)=B(\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n)=0,\ \alpha_1^2+\beta_1^2=|z_1|^2,\ldots,\alpha_n^2+\beta_n^2=|z_n|^2$$

has a root $(\alpha,\beta)=(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n)\in\mathbb{R}^{2n}$. Now, while the preceding system of equations has size significantly larger than $\mathrm{size}(z)+\mathrm{size}(f)$ (due to the underlying expansions of powers of $\zeta_i=\alpha_i+\beta_i\sqrt{-1}$), we can introduce new variables and equations (via Propositions 4.7 and 4.8) to obtain *another* polynomial system, also with a real solution if and only if $(\log|z_1|,\ldots,\log|z_n|)\in\mathrm{Amoeba}(f)$, with size linear in $\mathrm{size}(z)+\mathrm{size}(f)$ instead. Applying Theorem 4.6, we obtain our **PSPACE** upper bound.

Our **NP**-hardness complexity lower bound for the special case $n=1$ follows immediately from Theorem 4.5, since $|\zeta_1|=1\iff\log|\zeta_1|=0$. ■

Remark 4.9. *A reduction of amoeba membership to the Existential Theory of the Reals, with an* **EXPTIME** *complexity upper bound instead, was observed in* [The02, Sec. 2.2]. ◇

4.2. **Proving Theorem 4.4.** Let us first recall the following result on comparing monomials in rational numbers.

Theorem 4.10. [BRS09, Sec. 2.4] *Suppose $\alpha_1,\ldots,\alpha_N \in \mathbb{Q}$ are positive and $\beta_1,\ldots,\beta_N \in \mathbb{Z}$. Also let A be the maximum of the numerators and denominators of the α_i (when written in lowest terms) and $B := \max_i\{|\beta_i|\}$. Then, within*

$$O\big(N30^N \log(B)(\log\log B)^2 \log\log\log(B)(\log(A)(\log\log A)^2 \log\log\log A)^N\big)$$

bit operations, we can determine the sign of $\alpha_1^{\beta_1}\cdots\alpha_N^{\beta_N}-1$. ■

While the underlying algorithm is a simple application of Arithmetic-Geometric Mean Iteration (see, e.g., [Ber03]), its complexity bound hinges on a deep estimate of Nesterenko [Nes03], which in turn refines seminal work of Matveev [Mat00] and Alan Baker [Bak77] on linear forms in logarithms.

Proof of Theorem 4.4: From Proposition 3.9, it is clear that we merely need an efficient method to compare quantities of the form $|c_i z^{a_i}|$, and there are exactly $t-1$ such comparisons to be done. So our first complexity bound follows immediately from the special case of Theorem 4.10 where $A = 2^\sigma$, $B = d$, and $N = 2n+2$. In particular, $30\log 2 < 20.8$.

The second assertion follows almost trivially: Thanks to the exponential form of the coefficients and the query point, one can take logarithms to reduce to comparing integer linear combinations of rational numbers of bit size linear in $\max\{\sigma,\log d\}$. So the underlying monomial norm comparisons can be reduced to standard techniques for fast integer multiplication (see, e.g., [BS96, Pg. 43]). ■

5. Connections to Numerical Solutions and Non-Archimedean Tropical Geometry

5.1. **Coarse, but Fast, Isolation of Roots of Polynomial Systems.** An immediate consequence of Assertion (1a) of Theorem 3.4 is an estimate for isolating the possible absolute value vectors of complex roots of arbitrary systems of multivariate polynomial equations.

Corollary 5.1. *Suppose $f_1,\ldots,f_m \in \mathbb{C}\big[x_1^{\pm 1},\ldots,x_n^{\pm 1}\big]$ where f_i has exactly t_i monomial terms for all i. Then any root $\zeta \in (\mathbb{C}^*)^n$ of $F = (f_1,\ldots,f_m)$ satisfies*

$$(\log|\zeta_1|,\ldots,\log|\zeta_n|) \in \overline{\mathrm{ArchTrop}(f_1)}_{\varepsilon_1} \cap \cdots \cap \overline{\mathrm{ArchTrop}(f_m)}_{\varepsilon_m},$$

where $\varepsilon_i := \log(t_i - 1)$ for each i. ■

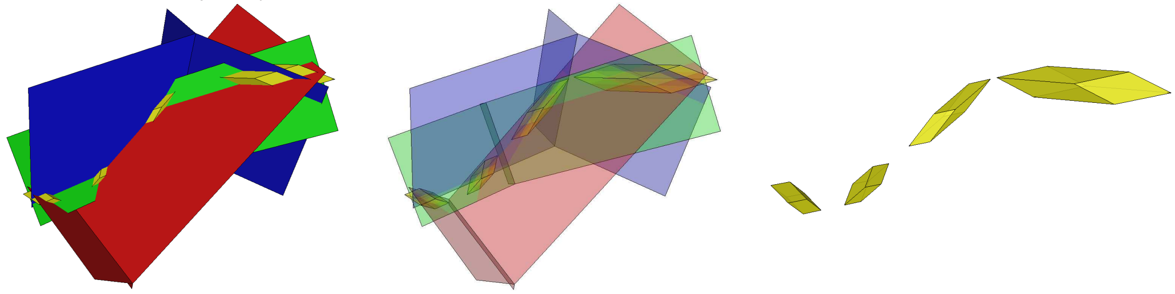
Example 5.2. *We can isolate the log absolute value vectors of the complex roots of the 3×3 system*

$$F := (f_1, f_2, f_3) := (x_1^{12}x_2^{11} - x_1^{22} - 1/16^6, x_2^{11}x_3^{12} - 1 - x_1^{22}/16^6, x_3^{11} - 1 - x_1^{22}/16^{18})$$

via Corollary 5.1 as follows: Find the points of $X := \mathrm{ArchTrop}(f_1) \cap \mathrm{ArchTrop}(f_2) \cap \mathrm{ArchTrop}(f_3)$ by searching through suitable triplets of edges of the $\mathrm{ArchNewt}(f_i)$, and then create isolating parallelepipeds about the points of X. More precisely, observe that $\mathrm{Conv}(\{(12,11,0,0),(22,0,0,0)\})$, $\mathrm{Conv}(\{(0,12,11,0),(0,0,0,0)\})$, $\mathrm{Conv}(\{(0,0,11,0),(0,0,0,0)\})$ are respective edges of $\mathrm{ArchNewt}(f_1)$, $\mathrm{ArchNewt}(f_2)$, and $\mathrm{ArchNewt}(f_3)$, and the vector $(0,0,0,-1)$ is an outer normal to each of these edges. So $(0,0,0)$ is a point of X. Running through the remaining triplets of edges we then obtain that X in fact consists of exactly 4 points:

$$(-1,0,0)\log 4 \quad , \quad (0,0,0) \quad , \quad \left(1,\tfrac{10}{11},0\right)\log 4 \quad , \quad \text{and} \quad \left(\tfrac{25}{11},\tfrac{250}{121},\tfrac{14}{11}\right)\log 4.$$

So Corollary 5.1 tells us that the points of $Y := \mathrm{Amoeba}(f_1) \cap \mathrm{Amoeba}(f_2) \cap \mathrm{Amoeba}(f_3)$ lie in the union of the 4 parallelepipeds drawn below to the right: Truncations of $\mathrm{ArchTrop}(f_1)$, ArchTrop(f_2), *and* ArchTrop(f_3) *are drawn below on the left, and the middle illustration uses transparency to further detail the intersection.*



(The parallelepipeds are simply intersections of slabs of thickness $\log 4$*, which are portions of the* $(\log 2)$*-neighborhoods arising from applying Corollary 5.1.) Suitably ordered, each point of* X *is actually within distance* $\sqrt{3}\times 10^{-6}$ *(*$< 0.693... = \log 2$*) of some point of* Y *(and vice-versa), well in accordance with Corollary 5.1.*

Finding Y *took about* 2500 *seconds via* `Bertini`[8] *(set to adaptive precision*[9]*), on a Dell XPS13 laptop running Ubuntu Linux 14.4. Finding* X *took a fraction of a second via a short* `Matlab` *program running on the same system.* $\diamond$

In our preceding 3×3 example, each parallelepiped corresponded naturally to a 3×3 *binomial* system, easily obtainable from each triplet of edges mentioned above. For instance, if one considers $(0,0,0)\in X$, then the binomial summand $x_1^{12}x_2^{11}-x_1^{22}$ of f_1 corresponds naturally to the lower edge E_1 of ArchNewt(f_1) with inner normal $(0,0,0,1)$: Each term of the binomial corresponds to some vertex of E_1. Repeating this construction with f_2 and f_3, one can then associate the *lower binomial system* $G := (x_1^{12}x_2^{11}-x_1^2, x_2^{11}x_3^{12}-1, x_3^{11}-1)$ to the point $(0,0,0)\in X$.

The intersections of the ArchTrop(f_i) in fact lead to approximations of the *roots* of F in $(\mathbb{C}^*)^3$ — not just their absolute value vectors: This is accomplished via the roots of the lower binomial system corresponding to an isolated point of the tropical intersection X. For example, for our preceding F, there is a set S of 1210 complex roots of F with coordinate-wise log absolute value clustering within distance $\sqrt{3}\times 10^{-6}$ of $(0,0,0)$. The corresponding lower binomial system G has exactly 1210 complex roots: They form a set T consisting of points of the form

$$\left(e^{\left(\frac{24p}{110}-\frac{2q}{10}+\frac{2r}{10}\right)\pi\sqrt{-1}}, e^{\left(\frac{2q}{11}-\frac{24p}{121}\right)\pi\sqrt{-1}}, e^{\frac{2p}{11}\pi\sqrt{-1}}\right),$$

as (p,q) ranges over $\{0,\ldots,10\}$ and r ranges over $\{0,\ldots,9\}$. The Hausdorff distance between S and T (see Section 5.2 below) is < 0.2846, so S is a rather coarse approximation of T. However, 110 *of the true roots of* F (in the cluster S) are in fact within distance 1.26×10^{-8} of some root of the lower binomial system G.

The remaining lower binomial systems of F are

$\left(x_1x_2-\frac{1}{16^6}, x_2x_3-1, x_3-1\right)$, $\left(x_1x_2-x_1^2, x_2x_3-\frac{x_1^2}{16^6}, x_3-1\right)$, and $\left(x_1x_2-x_1^2, x_2x_3-\frac{x_1^2}{16^6}, x_3-\frac{x_1^2}{16^{18}}\right)$. The total number of complex roots for our 4 lower binomial systems is 5566 — exactly the number of complex roots of F, thanks to a classical result of Bernstein [Ber75]. Using lower binomial systems to obtain canonical start points for homotopy continuation is pursued further in [AGGR15, Sec. 3].

[8]Bertini is a state of the art homotopy solver freely downloadable from `https://bertini.nd.edu` .

[9]For our example, fixed precision of 200 digits was insufficient for `Bertini` to correctly find all complex roots. However, default adaptive precision gave a correct count.

Remark 5.3. *One should also observe that the total number of* real *roots for all the lower binomial systems of F is 8, which also happens to be the exact number of real roots of F. One isn't always this lucky, but the probability of successfully counting real roots this way can be quantified: See* [BHPR11] *for initial results in this direction, and connections to $\mathcal{A}$-discriminants.* $\diamond$

5.2. **Non-Archimedean Precursors and Simplified Maslov Dequantization.** Recall that $\mathbb{C}\langle\langle s\rangle\rangle$ is the union $\bigcup_{d\in\mathbb{N}}\mathbb{C}\left(\left(s^{1/d}\right)\right)$ of formal Laurent series fields. While $\mathbb{C}$ is perhaps a more popular field in applications than $\mathbb{C}\langle\langle s\rangle\rangle$, $\mathbb{C}$ is more exceptional algebraically: $\mathbb{C}$ is the unique (up to isomorphism) algebraically closed field that is complete with respect to an absolute value that is unbounded on $\mathbb{Z}$ (see, e.g., [EP05, Thm. 1.2.3]). Such an absolute value is called *Archimedean*, so let us now review what a *non*-Archimedean *valuation* is.

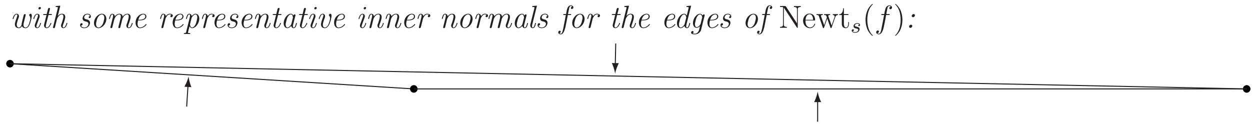
A *(non-Archimedean) valuation* on a field K is a function $\nu : K \longrightarrow \mathbb{R}\cup\{\infty\}$ such that $\nu(0)=\infty$ and, for all $a,b\in K$, (1) $\nu(ab)=\nu(a)+\nu(b)$ and (2) $\nu(a+b)\geq\min\{\nu(a),\nu(b)\}$ with equality if $\nu(a)\neq\nu(b)$. Inequality (2) is sometimes called the *Ultrametric Inequality*. Note in particular that $-\log(a+b)\geq\min\{-\log a,-\log b\}-\log 2$ for any $a,b\in\mathbb{R}_+$, so $-\log|\cdot|$ violates the Ultrametric Inequality when $a=b=1$. $-\log|\cdot|$ is thus sometimes called the *Archimedean* valuation on $\mathbb{C}$, and the minus sign on the log is one of the reasons behind various sign discrepancies when comparing Archimedean and non-Archimedean tropical varieties.

Remark 5.4. *Let us note some additional notational divergences in the tropical literature: In the notation of* [AGS17], *our* $\mathrm{ArchTrop}(f)$ *would be the* log *of the set of "tropical roots" of the "tropical polynomial"* $\max_{i\in\{1,\ldots,t\}}|c_i|x^{a_i}$ *defined over the nonnegative reals. However, most other authors (e.g.,* [Pin98, MS15]*) would instead call* $\max_{i\in\{1,\ldots,t\}}\{a_iw+\log|c_i|\}$ *or* $\min_{i\in\{1,\ldots,t\}}\{a_iw-\log|c_i|\}$ *a tropical polynomial, depending on what semi-ring they prefer. This in turn introduces a sign flip in the corresponding definitions of tropical root or tropical variety. (The oldest definition of tropical polynomial is in fact via the minimum of a collection of linear forms* [Pin98]*.) Because of this sign discrepancy, some authors (*[AGS17] *included) use a variant of* $\mathrm{ArchNewt}(f)$ *which implies examining* upper *hulls, instead of the lower hulls used here, to define tropical varieties.* $\diamond$

More to the point, let us recall a particular classical non-Archimedean valuation used often in the current tropical geometry literature.

Definition 5.5. *We define the* s-adic valuation *of any element* $c=\sum_{j=k}^{\infty}\gamma_j s^{j/d}\in\mathbb{C}\langle\langle s\rangle\rangle\setminus\{0\}$ *to be* $\nu_s(c):=\min_{\gamma_j\neq 0} j/d$, *and set* $\nu_s(0):=\infty$. *We then define the* s-adic Newton polytope *of any* $f\in\mathbb{C}\langle\langle s\rangle\rangle\left[x_1^{\pm 1},\ldots,x_n^{\pm 1}\right]$ *to be* $\mathrm{Newt}_s(f):=\mathrm{Conv}\left(\{(a_i,\nu_s(c_i))\}_{i\in[t]}\right)$. *We also define the* s-adic tropical variety of f *to be*
$\mathrm{Trop}_s(f):=\{v\in\mathbb{R}^n\mid(v,1)$ *is an inner normal of a face of* $\mathrm{Newt}_s(f)$ *of positive dimension*$\}$. $\diamond$

Example 5.6. *We have drawn* $\mathrm{Newt}_s(f)$ *for the trinomial* $f(x_1):=s-x_1^{16}+x_1^{49}$ *below, along with some representative inner normals for the edges of* $\mathrm{Newt}_s(f)$*:*



There are just two upward-pointing inner normals, and thus just two inner normals of the form $(v,1)$*:* $\left(\frac{1}{16},1\right)$ *and* $(0,1)$. *So* $\mathrm{Trop}_s(f)=\left\{\frac{1}{16},0\right\}$ *here.* $\diamond$

Letting $Z_s^*(f)$ denote the roots of f in $(\mathbb{C}\langle\langle s\rangle\rangle\setminus\{0\})^n$, Newton's $17^{\underline{\text{th}}}$-century result on Puiseux series expansions [New76], in modern language, can then be paraphrased as follows:

Theorem 5.7. [New76] *If $f \in \mathbb{C}\langle\langle s\rangle\rangle\left[x_1^{\pm 1}\right]$, $v \in \mathbb{Q}$, and L is the face of* $\mathrm{Newt}_s(f)$ *with inner normal $(v,-1)$, then f has exactly $\lambda(L)$ roots, counting multiplicity, with s-adic valuation v. In particular, $\nu_s(Z_s^*(f)) = \mathrm{Trop}_s(f)$.* ■

Example 5.8. *The trinomial $f(x_1) := s - x_1^{16} + x_1^{49}$ from our last example has exactly* 49 *roots in $\mathbb{C}\langle\langle s\rangle\rangle$:* 16 *of the form $e^{2\pi\sqrt{-1}j/16}s^{1/16} + \sum_{i=2}^{\infty}\alpha_{i,j}s^{i/16}$ (for $j \in [16]$) and* 33 *of the form $e^{2\pi\sqrt{-1}j/33} + \sum_{i=1}^{\infty}\beta_{i,j}s^{i}$ (for $j \in [33]$), where $\alpha_{i,j} \in \mathbb{Q}\left[e^{2\pi\sqrt{-1}/16}\right]$ and $\beta_{i,j} \in \mathbb{Q}\left[e^{2\pi\sqrt{-1}/33}\right]$. So the horizontal lengths (*16 *and* 33*) of the two lower edges of* $\mathrm{Newt}_s(f)$ *indeed count exactly the number of roots with corresponding valuation.* ◇

Note how the valuations $\nu_s(Z_s^*(f))$ are *exactly* determined by the lower edges of $\mathrm{Newt}_s(f)$, unlike the Archimedean setting where approximation is unavoidable (witness Example 1.3). Our Theorem 1.5 is thus an Archimedean analogue of Newton's result, including counting norms up to some variant of multiplicity. Dumas, around 1906, extended Theorem 5.7 to the p-adic complex numbers $\mathbb{C}_p$ [Dum06]. In fact, one can replace $\mathbb{C}\langle\langle s\rangle\rangle$ by any algebraically closed field with non-Archimedean valuation [Wei63].

There are two important additional characterizations of $\mathrm{Trop}_s(f)$, completely parallel to our earlier Propositions 2.1 and 3.9.

Proposition 5.9. *If $f \in \mathbb{C}\langle\langle s\rangle\rangle\left[x_1^{\pm 1}\right]$ then $-\mathrm{Trop}_s(f)$ is the set of slopes of the lower edges of* $\mathrm{Newt}_s(f)$*. More generally, if $f \in \mathbb{C}\langle\langle s\rangle\rangle\left[x_1^{\pm 1},\ldots,x_n^{\pm 1}\right]$, then*

$$\mathrm{Trop}_s(f) = \left\{ v \in \mathbb{R}^n \;\middle|\; \min_i\{a_i\cdot v + \nu_s(c_i)\} \text{ is attained for at least two distinct values of } i \right\}. \blacksquare$$

The first assertion is elementary, while the second follows easily from the definition of an inner face normal (see, e.g., [Zie95, Ch. 7]).

For any $\zeta_1,\ldots,\zeta_n \in \mathbb{C}\langle\langle s\rangle\rangle \setminus \{0\}$, let $\nu_s(\zeta) := (\nu_s(\zeta_1),\ldots,\nu_s(\zeta_n))$. That the last assertion of Theorem 5.7 can be extended to multivariate polynomials was first observed by Kapranov.

Kapranov's Non-Archimedean Amoeba Theorem. *(Special Case)*[10] [EKL06] *For any $f \in \mathbb{C}\langle\langle s\rangle\rangle\left[x_1^{\pm 1},\ldots,x_n^{\pm 1}\right]$, we have $\nu_s(Z_s^*(f)) = \mathrm{Trop}_s(f) \cap \nu_s(\mathbb{C}\langle\langle s\rangle\rangle)^n$.* ■

Our Theorem 3.4 is thus an Archimedean analogue of Kapranov's Theorem.

We close with some topological observations. First observe that $\mathrm{ArchTrop}(f)$ need *not* be contained in $\mathrm{Amoeba}(f)$, nor even have the same homotopy type as $\mathrm{Amoeba}(f)$, already for $n = 1$: The example $f(x_1) = (x_1+1)^2$ yields $\mathrm{ArchTrop}(f) = \{\pm\log 2\}$ but $\mathrm{Amoeba}(f) = \{0\}$. However, one can always recover $\mathrm{ArchTrop}(f)$ as the Hausdorff limit of a *sequence of suitably scaled* amoebae. To clarify this, first recall that the *Hausdorff distance* between any two subsets $X, Y \subseteq \mathbb{R}^n$ is

$$\Delta(X,Y) := \max\left\{ \sup_{x\in X}\inf_{y\in Y}|x-y|, \sup_{y\in Y}\inf_{x\in X}|x-y| \right\}.$$

Also, the *support* of a Laurent polynomial $f(x) = \sum_{j=1}^{t} c_j x^{a_j}$ is $\mathrm{Supp}(f) := \{a_j \mid c_j \neq 0\}$.

Corollary 5.10. *Let f be any n-variate t-nomial with $t \geq 2$ and $k := \dim\mathrm{Newt}(f)$. Then:*

1. $\Delta(\mathrm{ArchTrop}(f), \mathrm{Amoeba}(f)) \leq \sqrt{k}\left\lceil \frac{1}{4}t(t-1)\right\rceil\left((\log 9)t - \log\frac{81}{2}\right) = O\left(t^{7/2}\right)$.
2. *There exists a family of Laurent polynomials $(f_\mu)_{\mu\geq 1}$ with $\mathrm{Supp}(f_\mu) = \mathrm{Supp}(f)$ for all $\mu \geq 1$ and $\Delta\left(\frac{1}{\mu}\mathrm{Amoeba}(f_\mu), \mathrm{ArchTrop}(f)\right) \longrightarrow 0$ as $\mu \longrightarrow \infty$.*

[10] Kapranov's Theorem was originally stated for any algebraically closed field with a non-Archimedean valuation. Note in particular that $\nu_s(\mathbb{C}\langle\langle s\rangle\rangle) = \mathbb{Q}$ here.

We will prove Corollary 5.10 momentarily, but let us first recall one of the consequences of *Maslov dequantization* (see, e.g., [Mas86, LMS01, Vir01] and [Mik04, Cor. 6.4]): a method to obtain any *non*-Archimedean tropical variety as a limit of a family of scaled Archimedean amoebae. Assertion (2) thus shows how $\mathrm{ArchTrop}(f)$ provides a fully *Archimedean* version of this limit. Another precursor to Assertion (2), involving the piecewise linear structure approached by the intersection of $\mathrm{Amoeba}(f)$ with a large sphere, appears in [Ber71] and [GKZ94, Prop. 1.9, Pg. 197]. Thanks to Assertion (1), we can prove Assertion (2) in just three lines.

Proof of Corollary 5.10: Assertion (1) of Corollary 5.10 follows immediately from Assertion (1b) of Theorem 3.4, and the fact that $k \leq t-1$. Let us write $f(x) = \sum_{j=1}^{t} c_j x^{a_j}$, define $f_\mu(x) := \sum_{j=1}^{t} c_j^\mu x^{a_j}$, and observe that $f_1 = f$.

Since $|c_i e^{a_i \cdot v}| \geq |c_j e^{a_j \cdot v}| \iff |c_i e^{a_i \cdot v}|^\mu \geq |c_j e^{a_j \cdot v}|^\mu$, we immediately obtain that $\mathrm{ArchTrop}(f_\mu) = \mu \mathrm{ArchTrop}(f)$. So then $\Delta(\mathrm{Amoeba}(f_\mu), \mathrm{ArchTrop}(f_\mu)) = \mu\Delta\left(\frac{1}{\mu}\mathrm{Amoeba}(f_\mu), \mathrm{ArchTrop}(f)\right)$ and Assertion (1) thus implies $\Delta\left(\frac{1}{\mu}\mathrm{Amoeba}(f_\mu), \mathrm{ArchTrop}(f)\right) = \frac{O\left(t^{7/2}\right)}{\mu}$ for all $\mu \geq 1$. ■

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