New Complexity Bounds for Certain Real Fewnomial Zero Sets (Extended Abstract)

Joel Gomez* Andrew Niles† J. Maurice Rojas‡

September 18, 2007

Rojas dedicates this paper to his friend, Professor Tien-Yien Li.

Abstract

Consider real bivariate polynomials f and g, respectively having 3 and m monomial terms. We prove that for all $m \geq 3$, there are systems of the form (f, g) having exactly $2m-1$ roots in the positive quadrant. Even examples with $m=4$ having 7 positive roots were unknown before this paper, so we detail an explicit example of this form. We also present an $O(n^{11})$ upper bound for the number of diffeotopy types of the real zero set of an n-variate polynomial with $n+4$ monomial terms.

1 Introduction

Finding the correct combinatorics governing the real zero sets of sparse polynomials is a major open problem within real algebraic geometry (see, e.g., [BS07, DRRS07]). In particular, while the maximal number of real roots of a sparse polynomial in one variable has been well-understood for centuries (dating back to $17^{\underline{\text{th}}}$ century work of Descartes [SL54]), only loose upper bounds are known in higher-dimensions. Nevertheless, the bounds currently known have already proved of great use in arithmetic geometry [CZ02] and Hilbert's $16^{\underline{\text{th}}}$ Problem [Kal03] (to name just a few areas), and it is known that **optimal** bounds would have significant applications in many areas of engineering. Here, with an eye toward tightening known upper bounds, we exhibit sparse polynomial systems with more roots than previously known (Theorem 1 below), and a new **polynomial** upper bound on the number of diffeotopy types of real zero sets of n-variate polynomials with $n+4$ monomial terms (Theorem 2 below).

1.1 New Lower Bounds

Consider the following system of analytic equations:

$$(\star) \quad \begin{cases} \alpha_1 + \alpha_2 x^{a_2} y^{b_2} + \alpha_3 x^{a_3} y^{b_3} \\ \beta_1 + \beta_2 x^{c_2} y^{d_2} + \cdots + \beta_m x^{c_m} y^{d_m}, \end{cases}$$

for nonzero real α_i and β_i and distinct nonzero real vectors (a_2, b_2), (a_3, b_3), $(c_2, d_2), \ldots, (c_m, d_m)$. The number of isolated roots (x, y) in the positive quadrant $\mathbb{R}_+^2$ is of course bounded above by some function of the coefficients and exponents, but it wasn't until Askold Khovanski's invention of

*Department of Mathematics, Box 8205, NC State University, Raleigh, NC 27695-8205. Partially supported by an undergraduate fellowship from NSF REU grant DMS-0552610.

†CPU Box 274436, University of Rochester, Rochester, NY 14627. Partially supported by an undergraduate fellowship from NSF REU grant DMS-0552610.

‡Department of Mathematics, Texas A&M University TAMU 3368, College Station, Texas 77843-3368, USA, rojas@math.tamu.edu , www.math.tamu.edu/~rojas . Partially supported by NSF CAREER grant DMS-0349309 and NSF REU grant DMS-0552610.

Fewnomial Theory around the early 1980s [Kho80, Kho91] that an upper bound depending solely on m was known: $3^{m+2}2^{(m+1)(m+2)/2}$ (just for the number of non-degenerate roots),[1] invoking a very special case of his famous **Theorem on Real Fewnomials**.

Later, Li, Rojas, and Wang proved an upper bound of 2^m-2 [LRW03], while more recently Avendaño has proved that for the special case where $(a_2,b_2,a_3,b_3)=(1,0,0,1)$ — and the remaining (c_i,d_i) lie in $\mathbb{Z}^n$ — the *polynomial* system $(\star)$ never has more than $2m-2$ isolated positive roots [Ave07]. It has been conjectured that the correct general upper bound for the number of isolated positive roots of $(\star)$ should be polynomial in m, but this remains an open problem. So we provide the following new lower bound.

Theorem 1 *For all $m\geq 3$, there exist polynomial systems of the form $(\star)$ above with at least $2m-1$ roots in $\mathbb{R}^2_+$. In particular, the polynomial system*

$$\begin{gathered} x^6+\tfrac{44}{31}y^3-y \\ y^{14}+\tfrac{44}{31}x^3y^8-xy^8+\alpha x^{133} \end{gathered}$$

has exactly 7 roots in $\mathbb{R}^2_+$ for $1936254\leq\alpha\leq 1936838$.

While it easy to construct systems of the form $(\star)$ with exactly $2m-2$ positive roots, examples with $m=3$ having 5 positive roots weren't known until 2000 [Haa02, DRRS07]. Moreover, even examples with $m=4$ having 7 positive roots appear to have been unknown before this paper.

1.2 New Topological Upper Bounds

Recall that while a smooth, real, degree d projective plane curve has at most $1+\binom{d-1}{2}$ connected components [Har76], determining the possible nestings of these ovals — a piece of the first part of Hilbert's famous $16^{\underline{\text{th}}}$ Problem [Kal03] — is quite complicated. In more general language, this is the determination of possible **diffeotopy types** of such curves.

Definition 1 *Recall that a* **diffeotopy** *between two sets $X,Y\subseteq\mathbb{R}^n$ is a differentiable function $H:[0,1]\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$ such that $H(t,\cdot)$ is a diffeomorphism for all $t\in[0,1]$, $H(0,\cdot)$ is the identity on X, and $H(1,X)=Y$. Equivalently, we simply say that X and Y are* **diffeotopic**. $\diamond$

Note that diffeotopy is a more refined equivalence than diffeomorphism, since diffeotopy implies an entire continuous family of "infinitesimal" diffeomorphisms that deform X to Y and back again. Returning to nestings of ovals of real degree d projective plane curves, an asymptotic formula of e^{d^2} is now known [OK00], and the **exact** number is currently known (as of early 2007) only for $d\leq 8$.

Via our techniques here, we can count diffeotopy types in a dramatically different setting. Recall that $\mathbb{R}^*:=\mathbb{R}\setminus\{0\}$.

Definition 2 *Given any polynomial f, its* **support** *(or* **spectrum***) — written $\mathrm{Supp}(f)$ — is the set of exponent vectors in its monomial terms. Also, we let $Z_+(f)$ (resp. $Z^*_{\mathbb{R}}(f)$) denote the set of roots of f in $\mathbb{R}^n_+$ (resp. $(\mathbb{R}^*)^n$). Finally, given any $\mathcal{A}\subset\mathbb{R}^n$, we let $\mathrm{Conv}\mathcal{A}$ denote its* **convex hull**. $\diamond$

Theorem 2 *For any fixed $\mathcal{A}\subset\mathbb{Z}^n$ with $\#\mathcal{A}=n+4$ and $\mathrm{Conv}\mathcal{A}$ of positive volume, there are no more than $O(n^{11})$ diffeotopy types for any smooth $Z^*_{\mathbb{R}}(f)$ with $\mathrm{Supp}(f)=\mathcal{A}$. In particular, our bound is completely independent of the coordinates of $\mathcal{A}$.*

[1] Via now standard tricks, Khovanski's bound easily implies an upper bound of $2^{O(m^2)}$ for the number of isolated roots.

The positive volume assumption is natural, for otherwise one would in fact be studying (up to an invertible monomial change of variables) an instance where $\#\mathcal{A} \geq n' + 5$ with $n' \leq n$ and Conv$\mathcal{A}$ has positive n'-dimensional volume (see, e.g., [BRS07, Cor. 1, Sec. 2]).

That there exists any sort of upper bound depending solely on n is already a non-trivial fact, first observed by Lou van den Dries around the 1990s via o-minimality (see, e.g., [vdD98, Prop. 3.2, Pg. 150]). Bounds exponential in n were then (implicitly) discovered in [GVZ04], and appear explicitly in [BV06]. Our polynomial bound above is thus a great improvement. In particular, **polynomial** bounds for the number of diffeotopy types were previously known only for $\#\mathcal{A} \leq n+3$ [BRS07, DRRS07].

2 Outlines of the Proofs

The complete proofs of our two main theorems will appear in the full version of this paper. In this extended abstract, we will simply sketch the main ideas.

2.1 Proving Theorem 1 by Induction

The special case $m=3$ follows immediately from earlier work of Haas [Haa02], where an example consisting of two degree 106 polynomials was detailed. The more recent paper [DRRS07] gives a far simpler example, consisting of a pair of degree 6 polynomials, and gives an explanation of the paucity of such extremal examples via $\mathcal{A}$-discriminants.

The special case $m=4$ can be checked by computationally verifying our stated example, e.g., via rational univariate reduction and an application of Sturm-Habicht sequences to count real roots. This approach is dates back to the 19$^{\underline{\text{th}}}$ century and has undergone recent algorithmic revivals [Kro82, Can88, GLS99], so the verification of our example is a simple exercise in `Maple` (see [DRRS07] for an extended illustration of these and other techniques when $m=3$). The general construction of such examples is more subtle, however, so let us now assume $m \geq 5$.

By rescaling the variables, dividing by suitable monomial terms, and employing a monomial change of variables [LRW03], we can reduce to the following univariate problem: Find real a_i, b_j, c_k such that

$$1 + c_2 x^{a_2}(1-x)^{b_2} + \cdots + c_m x^{a_m}(1-x)^{b_m}$$

has at least $2m-1$ roots in the open interval $(0,1)$.

We now proceed by induction on m: Let us assume that we have constructed our desired example for some fixed $m \geq 4$, and that it in fact has the following form:

$$f_m(x) := 1 - \left(\frac{31}{44}\right)^{35/12} x^{-1/6}(1-x)^{35/12} - \left(\frac{44}{31}\right)^{5/6} x^{1/3}(1-x)^{1/6} - c_4 x^{a_4}(1-x)^{b_4} - \cdots - c_m x^{a_m}(1-x)^{b_m},$$

for some positive integers $a_4, b_4, \ldots, a_m, b_m$. We will then show that we can find a positive real c and a positive integer a such that

$$g(x) := f_m(x) - c x^a (1-x)^7$$

has at least $2m+1$ roots in $(0,1)$.

Toward this end, first note that (a) for any m, the function f_m satisfies $\lim_{x \to 0^+} f_m(x) = -\infty$ and $f_m(1) = 1$, and (b) the graph of $x^a(1-x)^7$ is an arc in the first quadrant, **nearly flat aside from a single peak**, connecting $(0,0)$ and $(1,0)$. In particular, the graph of $x^a(1-x)^7$ becomes "flatter" as a increases. The main trick will then be to pick a sufficiently large (and $c > 0$ a bit more carefully) so that the graphs of f_m and $x^{a_{m+1}}(1-x)^7$ intersect in at least $2m+1$ points in the first quadrant.

Labelling the roots of f_m in $(0,1)$ as $z_1 < \cdots < z_{2m-1}$, we then need only make some elementary calculations to find conditions on c and a so that guarantee that our graph condition holds. Toward this end, let y_1 (resp. y_{2m} and y_{2m+1}) be the unique point in $(0,z_1)$ (resp. $(z_{2m-1},1)$ and $(y_{2m},1)$) such that $f_m(y_1)$ is $-1/4$ (resp. $1/4$ and $1/2$). Also, by Rolle's Theorem, for all $i \in \{2,\ldots,2m-1\}$, we can find a point $y_i \in (z_{i-1},z_i)$ such that $f'_m(y_i)=0$.

In order to enforce our graph condition, it is easily checked (employing Rolle's Theorem once more) that it suffices to find conditions on c and a that imply $cy_{2m}^a(1-y_{2m})^7 < \min\limits_{1<i<2m} |f_m(y_i)|$ and $cy_{2m+1}^a(1-y_{2m+1})^7 > f_m(y_{2m+1}) = 1/2$. Letting $\alpha := \min\limits_{1<i<2m} |f_m(y_i)|$, an elementary calculation then yields the sufficients conditions $cy_{2m}^a(1-y_{2m})^7 < \alpha$ and $cy_{2m+1}^a(1-y_{2m+1})^7 > 1/2$.

With a bit more work, we then obtain that any c satisfying $\frac{\alpha}{y_{2m}^a(1-y_{2m})^7} > c > \frac{1}{2y_{2m+1}^a(1-y_{2m+1})^7}$ will work, provided that $a > \log\left(\frac{1}{2\alpha}\left(\frac{1-y_{2m}}{1-y_{2m+1}}\right)^7\right) \Big/ \log\left(\frac{y_{2m+1}}{y_{2m}}\right)$. Since $y_{2m} < y_{2m+1}$, the natural logarithm of their quotient is nonzero, so we are not dividing by zero. As $y_{2m}, y_{2m+1} \in (0,1)$, and as $\alpha > 0$, both of the natural logarithms involve only positive numbers, so they are well defined. It is then easily checked that if we first pick b sufficiently large, then we can always find our required c, and thus we can construct the g we desire. So we are done. ■

2.2 Proving Theorem 2 via the Horn-Kapranov Uniformization

Before outlining our proof, we will have to rapidly review $\mathcal{A}$-discriminants and some recent bounds for the number of real roots of certain generalizations of binomial systems. We point out that the necessary background is detailed further in [DRRS07], so our key contribution here is that we extend the results of [DRRS07] to higher-dimensional (reduced) discriminant varieties.

2.3 Background on $\mathcal{A}$-Discriminants

First, recall that to any set $\mathcal{A} \subset \mathbb{Z}^n$, one can consider the family of polynomials f supported on $\mathcal{A}$. The closure of the set of coefficient values $(c_a)_{a\in\mathcal{A}}$ yielding f with a degenerate zero set in $(\mathbb{C}^*)^n$ is defined to be the $\mathcal{A}$-discriminant variety $\nabla_{\mathcal{A}}$ [Loe91, GKZ94]. The $\mathcal{A}$-discriminant variety indeed turns out to be an algebraic variety, and when $\operatorname{codim}\nabla_{\mathcal{A}} = 1$, it possesses a defining polynomial irreducible over $\mathbb{Z}[c_a \mid a \in \mathcal{A}]$ which we will call $\Delta_{\mathcal{A}}$.

$\mathcal{A}$-discriminant varieties turn out also to be fibered over a variety of even lower dimension, known as the **reduced** $\mathcal{A}$-discriminant variety, $\overline{\nabla}_{\mathcal{A}}$. More to the point, the topology of $Z^*_{\mathbb{R}}(f)$ is constant on the **real** complement of $\overline{\nabla}_{\mathcal{A}}$. We now detail the necessary definitions and results.

Theorem 3 *[DFS05, Prop. 4.1] Given $\mathcal{A} := \{a_1,\ldots,a_m\} \in \mathbb{Z}^n$, the discriminant locus $\nabla_{\mathcal{A}}$ is exactly the closure of*

$$\{[u_1 t^{a_1} : \cdots : u_m t^{a_m}] \mid u := (u_1,\ldots,u_m) \in \mathbb{C}^m,\ \mathcal{A}u = \mathbf{O},\ \textstyle\sum_{i=1}^m u_i = 0,\ t = (t_1,\ldots,t_n) \in (\mathbb{C}^*)^n\}. \ ■$$

Lemma 1 *Suppose $\mathcal{A} = \{a_1,\ldots,a_m\} \subset \mathbb{Z}^n$ affinely generates $\mathbb{Z}^n$ and $a_1 = \mathbf{O}$. Then there are $i_1,\ldots,i_n \in \{2,\ldots,m\}$ such that $\det[a_{i_1},\ldots,a_{i_n}]$ is odd.* ■

Definition 3 *Suppose $\mathcal{A} = \{a_1,\ldots,a_m\} \subset \mathbb{Z}^n$ affinely generates $\mathbb{Z}^n$, has cardinality $m \geq n+2$, and $a_1 = \mathbf{O}$. We call any set $C = \{i_1,\ldots,i_n\}$ with $\det[a_{i_1},\ldots,a_{i_n}]$ odd as in Lemma 1 above, an* **odd cell** *of $\mathcal{A}$. For any $n\times m$ matrix B, we then let B_C (resp. $B_{C'}$) denote the submatrix of B defined by columns of B with index in C (resp. $\{2,\ldots,m\}\setminus C$). For any vectors $v,w \in (\mathbb{C}^*)^m$, let us denote* *their coordinate-wise product by* $v\cdot w := (v_1w_1,\ldots,v_mw_m)$*. Also let* Γ *be the multivalued*[2] *function from* $(\mathbb{C}^*)^m$ *to* $(\mathbb{C}^*)^{m-n-1}$ *defined by* $\Gamma(y) := \frac{y_{C'}}{y_1}\cdot\left(\frac{y_C}{y_1}\right)^{-A_C^{-1}A_{C'}}$*. Finally, we define the* **reduced** $\mathcal{A}$**-discriminant variety**, $\overline{\nabla}_{\mathcal{A}}\subset\mathbb{C}^{m-n-1}$*, to be the closure of*

$$\{\Gamma(u)\ \mid u := (u_1,\ldots,u_m)\in(\mathbb{C}^*)^m,\ \mathcal{A}u=\mathbf{O},\ \textstyle\sum_{i=1}^m u_i=0\},$$

and call any connected component of $(\mathbb{R}^*)^{m-n-1}\setminus\overline{\nabla}_{\mathcal{A}}$ *a* **reduced (**$\mathcal{A}$**-)discriminant chamber**. $\diamond$

Remark 1 *Since we always implicitly assume that an odd cell has been fixed a priori for our reduced* $\mathcal{A}$*-discriminant varieties,* Γ *in fact restricts to a single-valued function from* $(\mathbb{R}^*)^m$ *to* $(\mathbb{R}^*)^{m-n-1}$. $\diamond$

Proposition 1 *Suppose* $\mathcal{A}=\{a_1,\ldots,a_m\}\subset\mathbb{Z}^n$ *affinely generates* $\mathbb{Z}^n$*, has cardinality* $m\geq n+2$*, and* $a_1=\mathbf{O}$*. Also let* C *be any odd cell of* $\mathcal{A}$*, let* $f(x) := \sum_{i=1}^m \delta_i x^{a_i}$ *with* $\delta := (\delta_1,\ldots,\delta_m)\in(\mathbb{R}^*)^m$*, and let* $\bar{\delta}\in(\mathbb{R}^*)^m$ *be the unique vector with* $\bar{\delta}_1=1$*,* $\bar{\delta}_C=(1,\ldots,1)$ *and* $\bar{\delta}_{C'}=\Gamma(\delta)$*. Finally, let* $\bar{f} := \sum_{i=1}^m \bar{\delta}_i x^{a_i}$ *and let* $\mathrm{Conv}\mathcal{A}$ *denote the convex hull of* $\mathcal{A}$*. Then:*

1. Γ *induces a surjection from the set of connected components of*
$$\mathbb{P}^{m-1}_{\mathbb{R}}\setminus\left(\nabla_{\mathcal{A}}\cup\left\{[y_1:\cdots:y_m]\in\mathbb{P}^{m-1}_{\mathbb{R}}\ \middle|\ y_1\cdots y_m=0\right\}\right)$$
to the set of reduced $\mathcal{A}$*-discriminant chambers.*

2. *If, for all facets* Q' *of* $\mathrm{Conv}\mathcal{A}$*, we have that* $\#(\mathcal{A}\cap Q')=n$*, then* $Z^*_{\mathbb{R}}(f)$ *and* $Z^*_{\mathbb{R}}(\bar{f})$ *are diffeotopic. Furthermore, for any* f_1 *and* f_2 *with* $\bar{f}_1$ *and* $\bar{f}_2$ *lying in the same reduced* $\mathcal{A}$*-discriminant chamber,* $Z^*_{\mathbb{R}}(\bar{f}_1)$ *and* $Z^*_{\mathbb{R}}(\bar{f}_2)$ *are diffeotopic.* $\blacksquare$

Proposition 1 follows easily from a routine application of the Smith normal form and the implicit function theorem. In particular, the crucial trick is to observe that exponentiation by A_C, when C is an odd cell, induces an automorphism of orthants of $(\mathbb{R}^*)^n$. Our assumption on the intersection of $\mathcal{A}$ with the facets of $\mathrm{Conv}\mathcal{A}$ ensures that any topological change in the zero sets of f and $\bar{f}$ (in the underlying real toric variety corresponding to $\mathrm{Conv}\mathcal{A}$ [Ful93]) occurs within $(\mathbb{R}^*)^n$.

2.4 Background on Sheared Binomial Systems

Definition 4 *Suppose* $\ell_1,\ldots,\ell_j\in\mathbb{R}[\lambda_1,\ldots,\lambda_k]$ *are polynomials of degree* ≤ 1*. We then call any system of equations of the form* $S := \left(1-\prod_{i=1}^j \ell_i^{b_{1,i}}(\lambda_1,\ldots,\lambda_k),\ldots,1-\prod_{i=1}^j \ell_i^{b_{k,i}}(\lambda_1,\ldots,\lambda_k)\right)$*, with* $b_{i,i'}\in\mathbb{R}$ *for all* i,i'*, and the vectors* $(b_{1,1},\ldots,b_{1,j}),\ldots,(b_{k,1},\ldots,b_{k,j})$ *linearly independent, a* $k\times k$ **sheared binomial system with** j **factors***. We also call each* ℓ_i *a* **factor** *of the system. A sheared binomial system is referred to as a* **Gale Dual System** *in [BS07].*$\diamond$

Note that our definition implies that $j\geq k$. For $j=k$, it is easy to reduce any $k\times k$ sheared binomial system with j factors to a $k\times k$ linear system, simply by multiplying and dividing equations (mimicking Gaussian elimination). For $j>k$, sheared polynomial systems become much more complicated.

Theorem 4 *[BS07] The number of non-degenerate roots* $\lambda\in\mathbb{R}^k$ *of any* $k\times k$ *sheared binomial system with* $n+k$ *factors, and all factors positive, is bounded above by* $(e^2+3)2^{(k-4)(k+1)/2}n^k$*, for all* $k\geq 1$*. In particular,* $e^2+3\approx 10.38905610$. $\blacksquare$

[2] The multiple values arise from the presence of rational exponents, and the number of images of a point is always bounded above by a constant depending only on $\mathcal{A}$.

2.5 The Proof of Theorem 2

Let $\mathcal{T}_{\mathcal{A}}$ denote the toric variety corresponding to the convex hull of $\mathcal{A}$ [Ful93]. We will separate into two cases: (1) those $\mathcal{A}$ that are generic in the sense that every subset of cardinality $n+1$ has convex hull of positive volume, and (2) any remaining $\mathcal{A}$. The most difficult case is Case (1), so we will restrict to this case, leaving Case (2) for the full version of this paper.

By our genericity assumption, every facet of $\mathcal{A}$ corresponds to the vertices of a simplex, and thus the complex zero set of any f with $\mathrm{Supp}(f)=\mathcal{A}$ is always nonsingular at infinity, relative to $\mathcal{T}_{\mathcal{A}}$ (see, e.g., [BRS07, Sec. 3.2]). By Proposition 1, it then suffices to show that our desired bound applies to the number of **reduced** $\mathcal{A}$-discriminant chambers. Note also that by Proposition 1 and [DRRS07, Lemma 3.3], the real part of the reduced $\mathcal{A}$-discriminant variety — $\mathbb{R}^3 \cap \overline{\nabla}_{\mathcal{A}}$ — must be the union of a finite set of curves and the closure of $\{\Psi(\lambda) \mid \lambda \in \mathbb{R}^2,\ \ell_1(\lambda)\cdots\ell_{n+4}(\lambda) \neq 0\}$, where $\Psi(\lambda):=(\psi_1(\lambda),\psi_2(\lambda),\psi_3(\lambda)):=\left(\prod_{i=1}^{n+4}\ell_i^{b_{1,i}}(\lambda),\prod_{i=1}^{n+4}\ell_i^{b_{2,i}}(\lambda),\prod_{i=1}^{n+4}\ell_i^{b_{3,i}}(\lambda)\right)$, and $\ell_1,\ldots,\ell_{n+4}$ are bivariate polynomials in λ of degree ≤ 1 Let $\Omega\subset\mathbb{R}^3$ denote the aforementioned closure. Since curves do not disconnect connected components of the complement of a (locally closed) real algebraic surface, it thus suffices to focus on Ω. In particular, the connected components of $(\mathbb{R}^*)^3\setminus\Omega$ are (up to the deletion of finitely many curves) exactly the reduced $\mathcal{A}$-discriminant chambers. Note also, by observing the poles of the ψ_i, that Ω is the closure of the union of no more than $(n+4)(n+5)/2$ topological disks.

To count the number of connected components of $(\mathbb{R}^*)^3\setminus\Omega$, we will use the classical **critical points method** [CG84], combined with our more recent tools. In particular, let us first bound the number of critical values of the map from Ω to $\mathbb{R}$ defined by x_1.

A simple derivative calculation then reveals that some critical values are given by $\psi_3(\lambda)$, where λ satisfies $\frac{\partial\psi_1}{\partial\lambda_1}\frac{\partial\psi_2}{\partial\lambda_2}=\frac{\partial\psi_2}{\partial\lambda_1}\frac{\partial\psi_1}{\partial\lambda_2}$ and $\frac{\partial\psi_1}{\partial\lambda_1}\frac{\partial\psi_3}{\partial\lambda_2}=\frac{\partial\psi_3}{\partial\lambda_1}\frac{\partial\psi_1}{\partial\lambda_2}$. An elementary calcluation then yields that the preceding 2×2 system is equivalent to a 2×2 polynomial system consisting of two polynomials of degree $\leq n+3$. So by Bézout's Theorem, the number of critical values is bounded above by $(n+3)^2$.

Next, there are contributions from more complicated nodal singularities. It is then easily checked that these reduce to counting the number of roots $(\lambda,\lambda')\in(\mathbb{R}^*)^4$ of a sheared binomial system of the following form: $(\psi_1(\lambda),\psi_2(\lambda))=(\psi_1(\lambda),\psi_2(\lambda))$ — a 4×4 sheared binomial system, with $\leq 2n+8$ factors. By Theorem 4, and by counting sign conditions, we then obtain a contribution of $\frac{(2n+8)(2n+9)}{2}\cdot(e^2+3)n^4=O(n^6)$.

To count the number of connected components of $(\mathbb{R}^*)^3\setminus\Omega$, let us now introduce planes $H_1,\ldots,H_N$ exactly at the locations of our preceding critical values. Clearly, any connected component of

$$T:=(\mathbb{R}^*)^3\setminus(\Omega\cup H_1\cup\cdots\cup H_N)$$

is contained in a unique connected component of $(\mathbb{R}^*)^3\setminus\Omega$. So it suffices to count the connected components of T. To do the latter, observe that $N=O(n^6)$ and our planes thus divide $(\mathbb{R}^*)^3$ into $O(n^6)$ vertical slabs.

Now note that within the interior of each slab, Ω does **not** intersect the x_2 or x_3 coordinate planes, and has no vertical tangents. So to count components of T within any particular vertical strip, we need only bound from above the number of connected components of the complement of $\Omega\cup\{x_1=0\}$ within a vertical plane distinct from $H_1,\ldots,H_N$. This clearly reduces to the critical points method once more, in one dimension lower. In particular, with a bit of work, one is reduced asymptotically to counting the number of real roots of a 2×2 sheared binomial system with $n+4$ factors. So by Theorem 4 once again, and a sign condition count, the desired upper bound is $O(n^5)$. Thus, each of our vertical slabs contains no more than $O(n^5)$ connected components of T. Taking into account the number of vertical strips, we thus finally arrive at our stated upper bound of $O(n^{11})$ for the number of connected components of T, so we are done. ■

3 Acknowledgements

We thank Martin Avendaño for useful discussions.

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