HOCHSCHILD COHOMOLOGY OF FACTORS WITH PROPERTY Γ

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Dedicated to the memory of Barry Johnson, 1937–2002

Abstract

The main result of this paper is that the k^{th} continuous Hochschild cohomology groups $H^k(\mathcal{M},\mathcal{M})$ and $H^k(\mathcal{M},B(H))$ of a von Neumann factor $\mathcal{M}\subseteq B(H)$ of type II_1 with property Γ are zero for all positive integers k. The method of proof involves the construction of hyperfinite subfactors with special properties and a new inequality of Grothendieck type for multilinear maps. We prove joint continuity in the $\|\cdot\|_2$-norm of separately ultraweakly continuous multilinear maps, and combine these results to reduce to the case of completely bounded cohomology which is already solved.

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1 Introduction

The continuous Hochschild cohomology of von Neumann algebras was initiated by Johnson, Kadison and Ringrose in a series of papers [21, 23, 24], where they developed the basic theorems and techniques of the subject. From their results, and from those of subsequent authors, it was natural to conjecture that the k^{th} continuous Hochschild cohomology group $H^k(\mathcal{M},\mathcal{M})$ of a von Neumann algebra over itself is zero for all positive integers k. This was verified by Johnson, Kadison and Ringrose, [21], for all hyperfinite von Neumann algebras and the cohomology was shown to split over the center. A technical version of their result has been used in all subsequent proofs and is applied below. Triviality of the cohomology groups has interesting structural implications for von Neumann algebras, [39, Chapter 7] (which surveys the original work in this area by Johnson, [20], and Raeburn and Taylor, [35]), and so it is important to determine when this occurs.

The representation theorem for completely bounded multilinear maps, [9], which expresses such a map as a product of $*$-homomorphisms and interlacing operators, was used by the first and third authors to show that the completely bounded cohomology $H^k_{cb}(\mathcal{M},\mathcal{M})$ is always zero [11, 12, 39]. Subsequently it was observed in [40, 41, 42] that to show that $H^k(\mathcal{M},\mathcal{M}) = 0$, it suffices to reduce a normal cocycle to a cohomologous one that is completely bounded in the first or last variable only, while holding fixed the others. The multilinear maps that are completely bounded in the first (or last) variable do not form a Hochschild complex; however it is easier to check complete boundedness in one variable only [40]. In joint work with Effros, [7], the first and third authors had shown that if the type II_1 central summand of a von Neumann algebra $\mathcal{M}$ is stable under tensoring with the hyperfinite type II_1 factor $\mathcal{R}$, then

$$H^k(\mathcal{M},\mathcal{M}) = H^k_{cb}(\mathcal{M},\mathcal{M}) = 0, \qquad k \geq 2. \tag{1.1}$$

This reduced the conjecture to type II_1 von Neumann algebras, and a further reduction to those von Neumann algebras with separable preduals was accomplished in [39, Section 6.5]. We note that we restrict to $k \geq 2$, since the case $k = 1$, in a different formulation, is the question of whether every derivation of a von Neumann algebra into itself is inner, and this was solved independently by Kadison and Sakai, [22, 38].

The non-commutative Grothendieck inequality for normal bilinear forms on a von Neumann algebra due to Haagerup, [19], (but building on earlier work of Pisier, [31]) and the existence of hyperfinite subfactors with trivial relative commutant due to Popa, [33], have been the main tools for showing that suitable cocycles are completely bounded in the first variable, [6, 40, 41, 42]. The importance of this inequality for derivation problems on von Neumann and C^*-algebras was initially observed in the work of Ringrose, [36], and of the first author, [4]. The current state of knowledge for the cohomology conjecture for type II_1 factors may be summarized as follows:

(i) $\mathcal{M}$ is stable under tensoring by the hyperfinite type II_1 factor $\mathcal{R}$, $k \geq 2$, [7];

(ii) $\mathcal{M}$ has property Γ and $k = 2$, [6, 11];

(iii) $\mathcal{M}$ has a Cartan subalgebra, [32, $k = 2$], [8, $k = 3$], [40, 41, $k \geq 2$];

(iv) $\mathcal{M}$ has various technical properties relating to its action on $L^2(\mathcal{M}, tr)$ for $k = 2$, [32], and conditions of this type were verified for various classes of factors by Ge and Popa, [18].

The two test questions for the type II_1 factor case are the following. Is $H^k(\mathcal{M}, \mathcal{M})$ equal to zero for factors with property Γ, and is $H^2(VN(\mathbb{F}_2), VN(\mathbb{F}_2))$ equal to zero for the von Neumann factor of the free group on two generators? The second is still open at this time; the purpose of this paper is to give a positive answer to the first (Theorems 6.4 and 7.2). If we change the coefficient module to be any containing $B(H)$, then the question arises of whether analogous results for $H^k(\mathcal{M}, B(H))$ are valid (see [7]). We will see below that our methods are also effective in this latter case.

The algebras of (i) above are called McDuff factors, since they were studied in [25, 26]. The hyperfinite factor $\mathcal{R}$ satisfies property Γ (defined in the next section), and it is an easy consequence of the definition that the tensor product of an arbitrary type II_1 factor with a

Γ–factor also has property Γ. Thus, as is well known, the McDuff factors all have property Γ, and so the results of this paper recapture the vanishing of cohomology for this class, [7]. However, as was shown by Connes, [13], the class of factors with property Γ is much wider. This was confirmed in recent work of Popa, [34], who constructed a family of Γ–factors with trivial fundamental group. This precludes the possibility that they are McDuff factors, all of which have fundamental group equal to $\mathbb{R}^+$.

The most general class of type II_1 factors for which vanishing of cohomology has been obtained is described in (iii). While there is some overlap between those factors with Cartan subalgebras and those with property Γ, the two classes do not appear to be directly related, since their definitions are quite different. It is not difficult to verify that the infinite tensor product of an arbitrary sequence of type II_1 factors has property Γ, using the $\|\cdot\|_2$–norm density of the span of elements of the form $x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes 1 \otimes 1 \cdots$. Voiculescu, [44], has exhibited a family of factors (which includes $VN(\mathbb{F}_2)$) having no Cartan subalgebras, but also failing to have property Γ. This suggests that the infinite tensor product of copies of this algebra might be an example of a factor with property Γ but without a Cartan subalgebra. This is unproved, and indeed the question of whether $VN(\mathbb{F}_2)\overline{\otimes}VN(\mathbb{F}_2)$ has a Cartan subalgebra appears to be open at this time. While we do not know of a factor with property Γ but with no Cartan subalgebra, these remarks indicate that such an example may well exist. Thus the results of this paper and the earlier results of [40] should be viewed as complementary to one another, but not necessarily linked.

We now give a brief description of our approach to this problem; definitions and a more extensive discussion of background material will follow in the next section. For a factor $\mathcal{M}$ with separable predual and property Γ, we construct a hyperfinite subfactor $\mathcal{R} \subseteq \mathcal{M}$ with trivial relative commutant which enjoys the additional property of containing an asymptotically commuting family of projections for the algebra $\mathcal{M}$ (fifth section). In the third section we prove a Grothendieck inequality for $\mathcal{R}$-multimodular normal multilinear maps, and in the succeeding section we show that separate normality leads to joint continuity in the $\|\cdot\|_2$-norm (or, equivalently, joint ultrastrong* continuity) on the closed unit ball of $\mathcal{M}$.

These three results are sufficient to obtain vanishing cohomology for the case of a separable predual (sixth section), and we give the extension to the general case at the end of the paper.

We refer the reader to our lecture notes on cohomology, [39], for many of the results used here and to [5, 13, 15, 25, 26, 27] for other material concerning property Γ. We also take the opportunity to thank Professors I. Namioka and Z. Piotrowski for their guidance on issues related to the fourth section of the paper.

2 Preliminaries

Throughout the paper $\mathcal{M}$ will denote a type II_1 factor with unique normalized normal trace tr. We write $\|x\|$ for the operator norm of an element $x \in \mathcal{M}$, and $\|x\|_2$ for the quantity $(tr(xx^*))^{1/2}$, which is the norm induced by the inner product $\langle x, y\rangle = tr(y^*x)$ on $\mathcal{M}$.

Property Γ for a type II_1 factor $\mathcal{M}$ was introduced by Murray and von Neumann, [27], and is defined by the following requirement: given $x_1, \ldots, x_m \in \mathcal{M}$ and $\varepsilon > 0$, there exists a unitary $u \in \mathcal{M}$, $tr(u) = 0$, such that

$$\|ux_j - x_j u\|_2 < \varepsilon, \qquad 1 \leq j \leq m. \tag{2.1}$$

Subsequently we will use both this definition and the following equivalent formulation due to Dixmier, [15]. Given $\varepsilon > 0$, elements $x_1, \ldots, x_m \in \mathcal{M}$, and a positive integer n, there exist orthogonal projections $\{p_i\}_{i=1}^n \in \mathcal{M}$, each of trace n^{-1} and summing to 1, such that

$$\|p_i x_j - x_j p_i\|_2 < \varepsilon, \qquad 1 \leq j \leq m, \quad 1 \leq i \leq n. \tag{2.2}$$

In [33], Popa showed that each type II_1 factor $\mathcal{M}$ with separable predual contains a hyperfinite subfactor $\mathcal{R}$ with trivial relative commutant ($\mathcal{R}' \cap \mathcal{M} = \mathbb{C}1$), answering positively an earlier question posed by Kadison. In the presence of property Γ, we will extend Popa's theorem by showing that $\mathcal{R}$ may be chosen to contain, within a maximal abelian subalgebra, projections which satisfy (2.2). This result is Theorem 5.3.

We now briefly recall the definition of continuous Hochschild cohomology for von Neumann algebras. Let $\mathcal{X}$ be a Banach $\mathcal{M}$-bimodule and let $\mathcal{L}^k(\mathcal{M}, \mathcal{X})$ be the Banach space of k-linear bounded maps from the k-fold Cartesian product $\mathcal{M}^k$ into $\mathcal{X}$, $k \geq 1$. For $k = 0$, we define $\mathcal{L}^0(\mathcal{M}, \mathcal{X})$ to be $\mathcal{X}$. The coboundary operator $\partial^k \colon \mathcal{L}^k(\mathcal{M}, \mathcal{X}) \to \mathcal{L}^{k+1}(\mathcal{M}, \mathcal{X})$ (usually abbreviated to just ∂) is defined, for $k \geq 1$, by

$$\begin{aligned}\partial\phi(x_1, \ldots, x_{k+1}) = x_1\phi(x_2, \ldots, x_{k+1}) &\\ + \sum_{i=1}^{k}(-1)^i\phi(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_{k+1}) &\\ + (-1)^{k+1}\phi(x_1, \ldots, x_k)x_{k+1}, &\end{aligned} \tag{2.3}$$

for $x_1, \ldots, x_{k+1} \in \mathcal{M}$. When $k = 0$, we define $\partial\xi$, for $\xi \in \mathcal{X}$, by

$$\partial\xi(x) = x\xi - \xi x, \qquad x \in \mathcal{M}. \tag{2.4}$$

It is routine to check that $\partial^{k+1}\partial^k = 0$, and so Im ∂^k (the space of coboundaries) is contained in Ker ∂^{k+1} (the space of cocycles). The continuous Hochschild cohomology groups $H^k(\mathcal{M}, \mathcal{X})$ are then defined to be the quotient vector spaces Ker ∂^k/Im ∂^{k-1}, $k \geq 1$. When $\mathcal{X}$ is taken to be $\mathcal{M}$, an element $\phi \in \mathcal{L}^k(\mathcal{M}, \mathcal{M})$ is normal if ϕ is separately continuous in each of its variables when both range and domain are endowed with the ultraweak topology induced by the predual $\mathcal{M}_*$.

Let $\mathcal{N} \subseteq \mathcal{M}$ be a von Neumann subalgebra, and assume that $\mathcal{M}$ is represented on a Hilbert space H. Then $\phi\colon\ \mathcal{M}^k \to B(H)$ is $\mathcal{N}$-multimodular if the following conditions are satisfied by all $a \in \mathcal{N}$, $x_1, \ldots, x_k \in \mathcal{M}$, and $1 \leq i \leq k-1$:

$$a\phi(x_1, \ldots, x_k) = \phi(ax_1, x_2, \ldots, x_k), \tag{2.5}$$

$$\phi(x_1, \ldots, x_k)a = \phi(x_1, \ldots, x_{k-1}, x_k a), \tag{2.6}$$

$$\phi(x_1, \ldots, x_i a, x_{i+1}, \ldots, x_k) = \phi(x_1, \ldots, x_i, ax_{i+1}, \ldots, x_k). \tag{2.7}$$

A fundamental result of Johnson, Kadison and Ringrose, [21], states that each cocycle ϕ on $\mathcal{M}$ is cohomologous to a normal cocycle $\phi - \partial\psi$, which can also be chosen to be $\mathcal{N}$-multimodular for any given hyperfinite subalgebra $\mathcal{N} \subseteq \mathcal{M}$. This has been the starting point for all subsequent theorems in von Neumann algebra cohomology, since it permits the substantial simplification of considering only $\mathcal{N}$-multimodular normal cocycles for a suitably chosen hyperfinite subalgebra $\mathcal{N}$, [39, Chapter 3]. The present paper will provide another instance of this.

The matrix algebras $\mathbb{M}_n(\mathcal{M})$ over a von Neumann algebra (or C^*-algebra) $\mathcal{M}$ carry natural C^*-norms inherited from $\mathbb{M}_n(B(H)) = B(H^n)$, when $\mathcal{M}$ is represented on H. Each bounded map $\phi\colon\ \mathcal{M} \to B(H)$ induces a sequence of maps $\phi^{(n)}\colon\ \mathbb{M}_n(\mathcal{M}) \to \mathbb{M}_n(B(H))$ by applying ϕ to each matrix entry (it is usual to denote these by ϕ_n but we have adopted $\phi^{(n)}$ to avoid notational difficulties in the sixth section). Then ϕ is said to be completely bounded if

$\sup_{n\geq 1} \|\phi^{(n)}\| < \infty$, and this supremum defines the completely bounded norm $\|\phi\|_{cb}$ (see [17, 29] for the extensive theory of such maps). A parallel theory for multilinear maps was developed in [9, 10], using $\phi\colon\ \mathcal{M}^k \to \mathcal{M}$ to replace the product in matrix multiplication. We illustrate this with $k = 2$. The n-fold amplification $\phi^{(n)}\colon\ \mathbb{M}_n(\mathcal{M}) \times \mathbb{M}_n(\mathcal{M}) \to \mathbb{M}_n(\mathcal{M})$ of a bounded bilinear map $\phi\colon\ \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is defined as follows. For matrices (x_{ij}), $(y_{ij}) \in \mathbb{M}_n(\mathcal{M})$, the (i, j) entry of $\phi^{(n)}((x_{ij}), (y_{ij}))$ is $\sum_{k=1}^{n} \phi(x_{ik}, y_{kj})$. We note that if ϕ is $\mathcal{N}$-multimodular, then it is easy to verify from the definition of $\phi^{(n)}$ that this map is $\mathbb{M}_n(\mathcal{N})$-multimodular for each $n \geq 1$, and this will be used in the next section.

As before, ϕ is said to be completely bounded if $\sup_{n\geq 1} \|\phi^{(n)}\| < \infty$. By requiring all cocycles and coboundaries to be completely bounded, we may define the completely bounded Hochschild cohomology groups $H^k_{cb}(\mathcal{M}, \mathcal{M})$ and $H^k_{cb}(\mathcal{M}, B(H))$ analogously to the continuous case. It was shown in [11, 12] (see also [39, Chapter 4]) that $H^k_{cb}(\mathcal{M}, \mathcal{M}) = 0$ for $k \geq 1$ and all von Neumann algebras $\mathcal{M}$, exploiting the representation theorem for completely bounded multilinear maps, [9], which is lacking in the bounded case. This built on earlier work, [7], on completely bounded cohomology when the module is $B(H)$. Subsequent investigations have focused on proving that cocycles are cohomologous to completely bounded ones, [8, 32], or to ones which exhibit complete boundedness in one of the variables [6, 40, 41, 42]. We will also employ this strategy here.

3 A multilinear Grothendieck inequality

The non-commutative Grothendieck inequality for bilinear forms, [31], and its normal counterpart, [19], have played a fundamental role in Hochschild cohomology theory [39, Chapter 5]. The main use has been to show that suitable normal cocycles are completely bounded in at least one variable [8, 40, 41, 42]. In this section we prove a multilinear version of this inequality which will allow us to connect continuous and completely bounded cohomology in the sixth section.

If $\mathcal{M}$ is a type II_1 factor and n is a positive integer, we denote by tr_n the normalized trace on $\mathbb{M}_n(\mathcal{M})$, and we introduce the quantity $\rho_n(X) = (\|X\|^2 + n\ tr_n(X^*X))^{1/2}$, for $X \in \mathbb{M}_n(\mathcal{M})$. We let $\{E_{ij}\}_{i,j=1}^n$ be the standard matrix units for $\mathbb{M}_n$ ($\{e_{ij}\}_{i,j=1}^n$ is the more usual way of writing these matrix units, but we have chosen upper case letters to conform to our conventions on matrices). If $\phi^{(n)}$ is the n-fold amplification of the k-linear map ϕ on $\mathcal{M}$ to $\mathbb{M}_n(\mathcal{M})$, then

$$\phi^{(n)}(E_{11}X_1E_{11}, \ldots, E_{11}X_kE_{11}), \qquad X_i \in \mathbb{M}_n(\mathcal{M}),$$

is simply ϕ evaluated at the (1,1) entries of these matrices, leading to the inequality

$$\|\phi^{(n)}(E_{11}X_1E_{11}, \ldots, E_{11}X_kE_{11})\| \leq \|\phi\|\|X_1\| \ldots \|X_k\|. \tag{3.1}$$

Our objective in Theorem 3.3 is to successively remove the matrix units from (3.1), moving from left to right, at the expense of increasing the right hand side of this inequality. The following two variable inequality will allow us to achieve this for certain multilinear maps.

Lemma 3.1. *Let $\mathcal{M} \subseteq B(H)$ be a type II_1 factor with a hyperfinite subfactor $\mathcal{N}$ of trivial relative commutant, let $C > 0$ and let n be a positive integer. If $\psi\colon\ \mathbb{M}_n(\mathcal{M}) \times \mathbb{M}_n(\mathcal{M}) \to B(H)$ is a normal bilinear map satisfying*

$$\psi(XA, Y) = \psi(X, AY), \qquad A \in \mathbb{M}_n(\mathcal{N}), \qquad X, Y \in \mathbb{M}_n(\mathcal{M}), \tag{3.2}$$

and

$$\|\psi(XE_{11}, E_{11}Y)\| \leq C\|X\|\|Y\|, \qquad X, Y \in \mathbb{M}_n(\mathcal{M}), \tag{3.3}$$

then

$$\|\psi(X,Y)\| \le C\rho_n(X)\rho_n(Y), \qquad X,Y \in \mathbb{M}_n(\mathcal{M}). \tag{3.4}$$

Proof. Let η and ν be arbitrary unit vectors in H^n and define a normal bilinear form on $\mathbb{M}_n(\mathcal{M}) \times \mathbb{M}_n(\mathcal{M})$ by

$$\theta(X,Y) = \langle \psi(XE_{11}, E_{11}Y)\eta, \nu\rangle \tag{3.5}$$

for $X,Y \in \mathbb{M}_n(\mathcal{M})$. Then $\|\theta\| \le C$ by (3.3). By the non-commutative Grothendieck inequality for normal bilinear forms on a von Neumann algebra, [19], there exist normal states f, F, g and G on $\mathbb{M}_n(\mathcal{M})$ such that

$$|\theta(X,Y)| \le C(f(XX^*) + F(X^*X))^{1/2}(g(YY^*) + G(Y^*Y))^{1/2} \tag{3.6}$$

for all $X,Y \in \mathbb{M}_n(\mathcal{M})$. From (3.2), (3.5) and (3.6),

$$\begin{aligned} |\langle \psi(X,Y)\eta,\nu\rangle| &= \left|\sum_{j=1}^{n} \langle \psi(XE_{j1}E_{11}, E_{11}E_{1j}Y)\eta,\nu\rangle\right| \\ &\le \sum_{j=1}^{n} |\theta(XE_{j1}, E_{1j}Y)|, \end{aligned} \tag{3.7}$$

which we can then estimate by

$$C\sum_{j=1}^{n}(f(XE_{j1}E_{1j}X^*) + F(E_{1j}X^*XE_{j1}))^{1/2}(g(E_{1j}YY^*E_{j1}) + G(Y^*E_{j1}E_{1j}Y))^{1/2}, \tag{3.8}$$

and this is at most

$$C\left(f(XX^*) + \sum_{j=1}^{n} F(E_{1j}X^*XE_{j1})\right)^{1/2}\left(\sum_{j=1}^{n} g(E_{1j}YY^*E_{j1}) + G(Y^*Y)\right)^{1/2}, \tag{3.9}$$

by the Cauchy-Schwarz inequality.

Let $\{\mathcal{N}_\lambda\}_{\lambda\in\Lambda}$ be an increasing net of matrix subalgebras of $\mathcal{N}$ whose union is ultraweakly dense in $\mathcal{N}$. Let $\mathcal{U}_\lambda$ denote the unitary group of $\mathbb{M}_n(\mathcal{N}_\lambda)$ with normalized Haar measure dU. Since $\mathbb{M}_n(\mathcal{N})' \cap \mathbb{M}_n(\mathcal{M}) = \mathbb{C}1$, a standard argument (see [39, 5.4.4]) gives

$$tr_n(X)1 = \lim_{\lambda} \int_{\mathcal{U}_\lambda} U^*XU\ dU \tag{3.10}$$

in the ultraweak topology. Substituting XU and U^*Y respectively for X and Y in (3.7)–(3.9), integrating over $\mathcal{U}_\lambda$ and using the Cauchy-Schwarz inequality give

$$\begin{aligned}|\langle\psi(X,Y)\eta,\nu\rangle| &= |\langle\psi(XU,U^*Y)\eta,\nu\rangle| \\ &\leq C\left(f(XX^*)+\sum_{j=1}^{n} F\left(E_{1j}\int_{\mathcal{U}_\lambda} U^*X^*XU\ dU\ E_{j1}\right)\right)^{1/2}\times \\ &\quad \left(\sum_{j=1}^{n} g\left(E_{1j}\int_{\mathcal{U}_\lambda} U^*YY^*U\ dU\ E_{j1}\right)+G(Y^*Y)\right)^{1/2}. \end{aligned} \tag{3.11}$$

Now take the ultraweak limit over $\lambda\in\Lambda$ in (3.11) to obtain

$$\begin{aligned}&|\langle\psi(X,Y)\eta,\nu\rangle| \leq \\ &C\left(f(XX^*)+\sum_{j=1}^{n} F(E_{1j}tr_n(X^*X)E_{j1})\right)^{1/2}\left(\sum_{j=1}^{n} g(E_{1j}\ tr_n(YY^*)E_{j1})+G(Y^*Y)\right)^{1/2}, \end{aligned} \tag{3.12}$$

using normality of F and g. Since η and ν were arbitrary, (3.12) immediately implies that

$$\begin{aligned}\|\psi(X,Y)\| &\leq C(\|XX^*\|+ntr_n(X^*X))^{1/2}(ntr_n(YY^*)+\|Y^*Y\|)^{1/2} \\ &= C\rho_n(X)\rho_n(Y), \end{aligned} \tag{3.13}$$

completing the proof. □

Remark 3.2. The inequality (3.12) implies that

$$|\langle\psi(X,Y)\eta,\nu\rangle| \leq C(f(XX^*)+ntr_n(X^*X))^{1/2}(G(Y^*Y)+ntr_n(YY^*))^{1/2} \tag{3.14}$$

for $X,Y\in\mathbb{M}_n(\mathcal{M})$, which is exactly of Grothendieck type. The normal states F and g have both been replaced by ntr_n. The type of averaging argument employed above may be found in [16]. □

We now come to the main result of this section, a multilinear inequality which builds on the bilinear case of Lemma 3.1. We will use three versions $\{\psi_i\}_{i=1}^3$ of the map ψ in the previous lemma, with various values of the constant C. The multilinearity of ϕ below will guarantee that each map satisfies the first hypothesis of Lemma 3.1.

Theorem 3.3. *Let $\mathcal{M} \subseteq B(H)$ be a type II_1 factor and let $\mathcal{N}$ be a hyperfinite subfactor with trivial relative commutant. If $\phi\colon\ \mathcal{M}^k \to B(H)$ is a k-linear $\mathcal{N}$-multimodular normal map, then*

$$\|\phi^{(n)}(X_1, \ldots, X_k)\| \leq 2^{k/2}\|\phi\|\rho_n(X_1)\ldots\rho_n(X_k) \tag{3.15}$$

for all $X_1, \ldots, X_k \in \mathbb{M}_n(\mathcal{M})$ and $n \in \mathbb{N}$.

Proof. We may assume, without loss of generality, that $\|\phi\| = 1$. We take (3.1) as our starting point, and we will deal with the outer and inner variables separately. Define, for $X, Y \in \mathbb{M}_n(\mathcal{M})$,

$$\psi_1(X,Y) = \phi^{(n)}(X^*E_{11}, E_{11}X_2E_{11}, \ldots, E_{11}X_kE_{11})^*\phi^{(n)}(YE_{11}, E_{11}X_2E_{11}, \ldots, E_{11}X_kE_{11}), \tag{3.16}$$

where we regard $X_2, \ldots, X_k \in \mathbb{M}_n(\mathcal{M})$ as fixed. Then (3.1) implies that

$$\|\psi_1(XE_{11}, E_{11}Y)\| \leq \|X_2\|^2\ldots\|X_k\|^2\|X\|\|Y\|, \tag{3.17}$$

and (3.2) is satisfied. Taking C to be $\|X_2\|^2\ldots\|X_k\|^2$ in Lemma 3.1 gives

$$\begin{aligned}\|\psi_1(X,Y)\| &= \|\psi_1(E_{11}X, YE_{11})\| \\ &\leq \|X_2\|^2\ldots\|X_k\|^2\rho_n(E_{11}X)\rho_n(YE_{11}).\end{aligned} \tag{3.18}$$

Now

$$\begin{aligned}\rho_n(E_{11}X) &= (\|E_{11}XX^*E_{11}\| + n\,tr_n(E_{11}XX^*E_{11}))^{1/2} \\ &\leq 2^{1/2}\|X\|,\end{aligned} \tag{3.19}$$

since $tr_n(E_{11}) = n^{-1}$, and a similar estimate holds for $\rho_n(YE_{11})$. If we replace X by X_1^* and Y by X_1 in (3.18), then (3.16) and (3.19) combine to give

$$\|\phi^{(n)}(X_1E_{11}, E_{11}X_2E_{11}, \ldots, E_{11}X_kE_{11})\| \leq 2^{1/2}\|X_1\|\|X_2\|\ldots\|X_k\|. \tag{3.20}$$

Now consider the bilinear map

$$\psi_2(X,Y) = \phi^{(n)}(X, YE_{11}, E_{11}X_3E_{11}, \ldots, E_{11}X_kE_{11}) \tag{3.21}$$

where $X_3, \dots X_k$ are fixed. By (3.20), this map satisfies (3.3) with $C = 2^{1/2}\|X_3\| \dots \|X_k\|$, and multimodularity of ϕ ensures that (3.2) holds. By Lemma 3.1,

$$\begin{aligned}\|\psi_2(X,Y)\| &= \|\psi_2(X, YE_{11})\| \\ &\le 2^{1/2}\|X_3\| \dots \|X_k\|\rho_n(X)\rho_n(YE_{11}) \\ &\le 2\|X_3\| \dots \|X_k\|\rho_n(X)\|Y\|. \end{aligned} \tag{3.22}$$

Replace X by X_1 and Y by X_2 to obtain

$$\|\phi^{(n)}(X_1, X_2E_{11}, E_{11}X_3E_{11}, \dots, E_{11}X_kE_{11})\| \le 2\rho_n(X_1)\|X_2\| \dots \|X_k\|. \tag{3.23}$$

We repeat this step $k-2$ times across each succeeding consecutive pair of variables, gaining a factor of $2^{1/2}$ each time and replacing each $\|X_i\|$ by $\rho_n(X_i)$, until we reach the inequality

$$\|\phi^{(n)}(X_1, X_2, \dots, X_{k-1}, X_kE_{11})\| \le 2^{k/2}\rho_n(X_1) \dots \rho_n(X_{k-1})\|X_k\|. \tag{3.24}$$

To complete the proof, we now define

$$\psi_3(X,Y) = \phi^{(n)}(X_1, \dots, X_{k-1}, X)\phi^{(n)}(X_1, \dots, X_{k-1}, Y^*)^*, \tag{3.25}$$

where $X_1, \dots, X_{k-1}$ are fixed. We may apply Lemma 3.1 with $C = 2^k\rho_n(X_1)^2 \dots \rho_n(X_{k-1})^2$ to obtain

$$\|\psi_3(X,Y)\| \le C\rho_n(X)\rho_n(Y). \tag{3.26}$$

Put $X = X_k$ and $Y = X_k^*$. Then (3.25) and (3.26) give the estimate

$$\|\phi^{(n)}(X_1, \dots, X_k)\| \le 2^{k/2}\rho_n(X_1) \dots \rho_n(X_k), \tag{3.27}$$

as required, since $\rho_n(X_k^*) = \rho_n(X_k)$. $\square$

We will use Theorem 3.3 subsequently in a modified form which we now state.

Corollary 3.4. *Let $\mathcal{M} \subseteq B(H)$ be a type II_1 factor and let $\mathcal{N}$ be a hyperfinite subfactor with trivial relative commutant. Let $n \in \mathbb{N}$, let $P \in \mathbb{M}_n(\mathcal{M})$ be a projection of trace n^{-1}, and let $\phi\colon\ \mathcal{M}^k \to B(H)$ be a k-linear $\mathcal{N}$-multilinear normal map. Then, for $X_1, \dots, X_k \in \mathbb{M}_n(\mathcal{M})$,*

$$\|\phi^{(n)}(X_1P, \dots, X_kP)\| \le 2^k\|\phi\|\|X_1\| \dots \|X_k\|. \tag{3.28}$$

Proof. For $1 \leq i \leq k$,

$$\begin{aligned}\rho_n(X_iP) &= (\|PX_i^*X_iP\| + ntr_n(PX_i^*X_iP))^{1/2} \\ &\leq (\|X_i^*X_i\|(1 + ntr_n(P)))^{1/2} \\ &= 2^{1/2}\|X_i\|. \end{aligned} \tag{3.29}$$

The result follows immediately from (3.15) with each X_i replaced by X_iP. □

4 Joint continuity in the $\|\cdot\|_2$-norm

There is an extensive literature on the topic of joint and separate continuity of functions of two variables (see [3, 28] and the references therein) with generalizations to the multivariable case. In this section we consider an n-linear map $\phi\colon\ \mathcal{M}\times\ldots\times\mathcal{M}\to\mathcal{M}$ on a type II_1 factor $\mathcal{M}$ which is ultraweakly continuous (or normal) separately in each variable. The restriction of ϕ to the closed unit ball will be shown to be separately continuous when both range and domain have the $\|\cdot\|_2$-norm, and from this we will deduce joint continuity in the same metric topology. Many such joint continuity results hinge on the Baire category theorem, and this is true of the following lemma, which we quote as a special case of a result from [3], and which also can be found in [37, p.163]. Such theorems stem from [2].

Lemma 4.1. *Let $\mathcal{X},\mathcal{Y}$ and $\mathcal{Z}$ be complete metric spaces, and let $f\colon\ \mathcal{X}\times\mathcal{Y}\to\mathcal{Z}$ be continuous in each variable separately. For each $y_0\in\mathcal{Y}$, there exists an $x_0\in\mathcal{X}$ such that $f(x,y)$ is jointly continuous at (x_0,y_0).*

We now use this lemma to obtain a joint continuity result which is the first step in an induction argument. Let B denote the closed unit ball of a type II_1 factor $\mathcal{M}$, to which we give the metric induced by the $\|\cdot\|_2$-norm. Then B is a complete metric space. We assume that multilinear maps ϕ below satisfy $\|\phi\|\le 1$, so that ϕ maps $B\times\ldots\times B$ into B. The k^{th} copy of B in such a Cartesian product will be written as B_k.

Lemma 4.2. *Let $\phi\colon\ \mathcal{M}\times\mathcal{M}\to\mathcal{M}$, $\|\phi\|\le 1$, be a bilinear map which is separately continuous in the $\|\cdot\|_2$-norm on $B_1\times B_2$. Then $\phi\colon\ B_1\times B_2\to B$ is jointly continuous in the $\|\cdot\|_2$-norm.*

Proof. If we apply Lemma 4.1 with y_0 taken to be 0, then there exists $a\in B$ such that the restriction of ϕ to $B_1\times B_2$ (which we also write as ϕ) is jointly continuous at $(a,0)$. We now prove joint continuity at (0,0), first under the assumption that $a\ge 0$, and then deducing the general case from this. Suppose, then, that $a\ge 0$.

Consider sequences $\{h_n\}_{n=1}^{\infty} \in B_1$ and $\{k_n\}_{n=1}^{\infty} \in B_2$, both having limit 0 in the $\|\cdot\|_2$-norm. If $h_n \geq 0$, then $a - h_n \in B_1$ since for positive elements

$$\|a - h_n\| \leq \max\{\|a\|, \|h_n\|\} \leq 1. \tag{4.1}$$

Thus $\{(a - h_n, k_n)\}_{n=1}^{\infty}$ converges to $(a, 0)$ in $B_1 \times B_2$. Since

$$\begin{aligned}\|\phi(h_n, k_n)\|_2 &= \|\phi((h_n - a) + a, k_n)\|_2 \\ &\leq \|\phi(a - h_n, k_n)\|_2 + \|\phi(a, k_n)\|_2,\end{aligned} \tag{4.2}$$

we see that

$$\lim_{n\to\infty} \|\phi(h_n, k_n)\|_2 = 0 \tag{4.3}$$

from joint continuity at $(a, 0)$.

Now suppose that each h_n is self-adjoint, and write $h_n = h_n^+ - h_n^-$ with $h_n^+ h_n^- = 0$, and $h_n^{\pm} \geq 0$. Then $\|h_n\|_2^2 = \|h_n^+\|_2^2 + \|h_n^-\|_2^2$, so $h_h^{\pm} \in B_1$ and $\lim_{n\to\infty} \|h_n^{\pm}\|_2 = 0$. This shows that

$$\lim_{n\to\infty} \phi(h_n, k_n) = \lim_{n\to\infty} \phi(h_n^+, k_n) - \lim_{n\to\infty} \phi(h_n^-, k_n) = 0 \tag{4.4}$$

for a self-adjoint sequence in the first variable. This easily extends to a general sequence from B_1 by taking real and imaginary parts. Thus ϕ is jointly continuous at (0,0) when $a \geq 0$.

For the general case, take the polar decomposition $a = bu$ with $b \geq 0$ and u unitary, which is possible because $\mathcal{M}$ is type II_1. Then the map $\psi(x, y) = \phi(xu, y)$ is jointly continuous at $(b, 0)$, and thus at (0,0) from above. Since $\phi(x, y) = \psi(xu^*, y)$, joint continuity of ϕ at (0,0) follows immediately.

We now show joint continuity at a general point $(a, b) \in B_1 \times B_2$. If $\lim_{n\to\infty} (a_n, b_n) = (a, b)$ for a sequence in $B_1 \times B_2$, then the equations

$$a_n = a + 2h_n, \quad b_n = b + 2k_n \tag{4.5}$$

define a sequence $\{(h_n, k_n)\}_{n=1}^{\infty}$ in $B_1 \times B_2$ convergent to (0,0). Then

$$\phi(a_n, b_n) - \phi(a, b) = 2\phi(h_n, b) + 2\phi(a, k_n) + 4\phi(h_n, k_n), \tag{4.6}$$

and the right hand side converges to 0 by joint continuity at (0,0) and separate continuity in each variable. This shows joint continuity at (a, b). □

Proposition 4.3. *Let $\phi\colon\ \mathcal{M} \times \ldots \times \mathcal{M} \to \mathcal{M}$, $\|\phi\| \le 1$, be a bounded n-linear map which is separately continuous in the $\|\cdot\|_2$-norm on $B_1 \times \ldots \times B_n$. Then $\phi\colon\ B_1 \times \ldots \times B_n \to B$ is jointly continuous in the $\|\cdot\|_2$-norm.*

Proof. The case $n = 2$ is Lemma 4.2, so we proceed inductively and assume that the result is true for all $k \le n-1$. Then consider a separately continuous $\phi\colon\ B_1 \times \ldots \times B_n \to B$. If we fix the first variable then the resulting $(n-1)$-linear map is jointly continuous on $B_2 \times \ldots \times B_n$ by the induction hypothesis. If we view this Cartesian product as $B_1 \times (B_2 \times \ldots \times B_n)$, then we have separate continuity, so Lemma 4.1 ensures that there exists $a \in B_1$ so that ϕ is jointly continuous at $(a, 0, \ldots, 0)$. We may then follow the proof of Lemma 4.2 to show firstly that ϕ is jointly continuous at $(0, \ldots, 0)$, and subsequently that ϕ is jointly continuous at a general point $(a_1, \ldots, a_n)$, using the induction hypothesis. □

Theorem 4.4. *Let $\phi\colon\ \mathcal{M} \times \ldots \times \mathcal{M} \to \mathcal{M}$, $\|\phi\| \le 1$, be separately normal in each variable. Then the restriction of ϕ to $B_1 \times \ldots \times B_n$ is jointly continuous in the $\|\cdot\|_2$-norm.*

Proof. If we can show that the restriction of ϕ is separately continuous in the $\|\cdot\|_2$-norm, then the result will follow from Proposition 4.3. By fixing all but one of the variables, we reduce to the case of a normal map $\psi\colon\ \mathcal{M} \to \mathcal{M}$, $\|\psi\| \le 1$. By [39, 5.4.3], there exist normal states $f, g \in \mathcal{M}_*$ such that

$$\|\psi(x)\|_2 \le f(x^*x)^{1/2} + g(xx^*)^{1/2}, \qquad x \in \mathcal{M}. \tag{4.7}$$

We may suppose that $\mathcal{M}$ is represented in standard form, so that every normal state is a vector state. Thus choose $\xi, \eta \in L^2(\mathcal{M}, tr)$ such that

$$f(x^*x) = \langle x^*x\xi, \xi\rangle = \|x\xi\|_2^2 \tag{4.8}$$

and

$$g(xx^*) = \langle xx^*\eta, \eta\rangle = \|x^*\eta\|_2^2. \tag{4.9}$$

Consider now a sequence $\{x_n\}_{n=1}^{\infty} \in B$ which converges to $x \in B$ in the $\|\cdot\|_2$-norm. Given $\varepsilon > 0$, choose $y, z \in \mathcal{M}$ such that

$$\|\xi - y\|_2, \quad \|\eta - z\|_2 < \varepsilon. \tag{4.10}$$

Then (4.7)–(4.10) combine to give

$$\begin{aligned} \|\psi(x - x_n)\|_2 &\leq \|(x - x_n)\xi\|_2 + \|(x - x_n)^*\eta\|_2 \\ &\leq \|(x - x_n)y\|_2 + \|(x - x_n)^*z\|_2 + 4\varepsilon \\ &\leq \|y\|\|x - x_n\|_2 + \|z\|\|x - x_n\|_2 + 4\varepsilon. \end{aligned} \tag{4.11}$$

Thus, from (4.11),

$$\limsup_{n\geq 1} \|\psi(x - x_n)\|_2 \leq 4\varepsilon, \tag{4.12}$$

and since $\varepsilon > 0$ was arbitrary, we conclude that $\lim\limits_{n\to\infty} \|\psi(x - x_n)\|_2 = 0$. This proves the result. □

Corollary 4.5. *Let $\mathcal{M} \subseteq B(H)$ be a type II_1 factor and let $\phi\colon \mathcal{M} \times \ldots \times \mathcal{M} \to B(H)$, $\|\phi\| \leq 1$, be a bounded n–linear map which is separately normal in each variable. For an arbitrary pair of unit vectors ξ, $\eta \in H$, the n–linear form*

$$\psi(x_1, \ldots, x_n) = \langle \phi(x_1, \ldots, x_n)\xi, \eta\rangle, \qquad x_i \in \mathcal{M}, \tag{4.13}$$

is jointly continuous in the $\|\cdot\|_2$–norm when restricted to $B_1 \times \ldots \times B_n$.

Proof. View ψ as having range in $\mathbb{C}1 \subseteq \mathcal{M}$, and apply Theorem 4.4. □

Remark 4.6. Restriction to the unit ball is necessary in the previous results. The bilinear map $\phi(x, y) = xy$, $x, y \in \mathcal{M}$, is separately normal, but if we take a sequence of projections $p_n \in \mathcal{M}$ of trace n^{-4}, then $\lim\limits_{n\to\infty} \|np_n\|_2 = 0$, but $\|\phi(np_n, np_n)\|_2 = n^2(tr(p_n))^{1/2} = 1$. This shows that ϕ is not jointly continuous in the $\|\cdot\|_2$-norm for the whole of $\mathcal{M}$. However, a simple scaling argument shows that ϕ may have arbitrary norm and that restriction to the closed ball of any finite radius allows the same conclusion concerning joint continuity.

If $\mathcal{M}$ is faithfully represented on a Hilbert space H, then the ultrastrong* topology is defined by the family of seminorms

$$x \mapsto \left(\sum_{n=1}^{\infty} \|x\xi_n\|^2 + \|x^*\xi_n\|^2\right)^{1/2}, \quad \xi_n \in H, \quad \sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty, \qquad x \in \mathcal{M}. \tag{4.14}$$

Thus convergence of a net $\{x_\lambda\}_{\lambda\in\Lambda}$ to x in the ultrastrong* topology is equivalent to ultraweak convergence of the nets

$$\{(x - x_\lambda)(x - x_\lambda)^*\}_{\lambda\in\Lambda} \quad \text{and} \quad \{(x - x_\lambda)^*(x - x_\lambda)\}_{\lambda\in\Lambda}$$

to 0, showing that the ultrastrong* topology is independent of the particular representation. By [43, III.5.3], this topology, when restricted to the unit ball of $\mathcal{M}$, equals the topology arising from the $\|\cdot\|_2$-norm. Thus the conclusion of Theorem 4.4 could have been stated as the joint ultrastrong* continuity of ϕ when restricted to closed balls of finite radius. In [1], Akemann proved the equivalence of continuity in the ultraweak and ultrastrong* topologies for bounded maps restricted to balls, so these results give another proof of Theorem 4.4. We have preferred to argue directly from Grothendieck's inequality. □

5 Hyperfinite subfactors

In [33], Popa showed the existence of a hyperfinite subfactor $\mathcal{N}$ of a separable factor $\mathcal{M}$ with trivial relative commutant ($\mathcal{N}' \cap \mathcal{M} = \mathbb{C}1$). In this section we use Popa's result to construct such a subfactor with some additional properties in the case that $\mathcal{M}$ has property Γ. The second lemma below is part of the inductive step in the main theorem. We begin with a technical result which is a special case of a more general result in [30, Prop. 1.11]. In our situation the proof is short and so we include it for completeness.

Lemma 5.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be type II_1 factors and suppose that there exists a matrix algebra $\mathbb{M}_r$ such that $\mathcal{M}$ is isomorphic to $\mathbb{M}_r \otimes \mathcal{N}$. If $\mathcal{M}$ has property Γ, then so too does $\mathcal{N}$.*

Proof. Fix a free ultrafilter ω on $\mathbb{N}$, and let $\mathcal{M}^\omega$ denote the resulting ultraproduct factor, which contains a naturally embedded copy of $\mathcal{M}$ with relative commutant denoted $\mathcal{M}_\omega$. Then $\mathcal{M}$ has property Γ if and only if $\mathcal{M}_\omega \neq \mathbb{C}1$, [13]. Since $\mathcal{M}^\omega$ is isomorphic to $\mathbb{M}_r \otimes \mathcal{N}^\omega$, and $\mathcal{M}_\omega$ is then isomorphic to $I_r \otimes \mathcal{N}_\omega$, the result follows. $\square$

Lemma 5.2. *Let $\mathcal{M}$ be type II_1 factor with property Γ and let $\mathcal{M} = \mathbb{M}_r \otimes \mathcal{N}$ be a tensor product decomposition of $\mathcal{M}$. Given $x_1, \ldots, x_k \in \mathcal{M}$, $n \in \mathbb{N}$, and $\varepsilon > 0$, there exists a set of orthogonal projections $\{p_i\}_{i=1}^n \in \mathcal{N}$, each of trace n^{-1}, such that*

$$\|[1 \otimes p_i, x_j]\|_2 < \varepsilon, \qquad 1 \leq i \leq n, \quad 1 \leq j \leq k. \tag{5.1}$$

Proof. Write each x_j as an $r \times r$ matrix over $\mathcal{N}$, and let $\{y_i\}_{i=1}^{kr^2}$ be a listing of all the resulting matrix entries. By Lemma 5.1, $\mathcal{N}$ has property Γ, so given $\delta > 0$ we can find a set $\{p_i\}_{i=1}^n \in \mathcal{N}$ of orthogonal projections of trace n^{-1} satisfying

$$\|[p_i, y_j]\|_2 < \delta, \qquad 1 \leq i \leq n, \quad 1 \leq j \leq kr^2, \tag{5.2}$$

from [15]. It is clear that (5.1) will hold for $\delta < r^{-2}\varepsilon$. $\square$

Since the projections found above have equal trace, they may be viewed as the minimal projections on the diagonal of an $n \times n$ matrix subalgebra of $\mathcal{N}$; we will use this subsequently.

Theorem 5.3. *Let $\mathcal{M}$ be a type II_1 factor with separable predual and with property Γ. Then there exists a hyperfinite subfactor $\mathcal{R}$ with trivial relative commutant satisfying the following condition. Given $x_1, \ldots, x_k \in \mathcal{M}$, $n \in \mathbb{N}$, and $\varepsilon > 0$, there exist orthogonal projections $\{p_i\}_{i=1}^n \in \mathcal{R}$, each of trace n^{-1}, such that*

$$\|[p_i, x_j]\|_2 < \varepsilon, \qquad 1 \le i \le n, \quad 1 \le j \le k. \tag{5.3}$$

Proof. We will construct $\mathcal{R}$ as the ultraweak closure of an ascending union of matrix subfactors $\mathcal{A}_n$ which we define inductively. We first fix a sequence $\{\theta_i\}_{i=1}^\infty$ of normal states (with θ_1 the trace) which is norm dense in the set of all normal states in $\mathcal{M}_*$. We then choose a sequence $\{m_i\}_{i=1}^\infty$ from the unit ball of $\mathcal{M}$ which is $\|\cdot\|_2$-norm dense in the unit ball. For these choices, the induction hypothesis is

(i) for each $k \le n$ there exist orthogonal projections $p_1, \ldots, p_k \in \mathcal{A}_n$, $tr(p_i) = k^{-1}$, satisfying

$$\|[p_i, m_j]\|_2 < n^{-1}, \qquad 1 \le i \le k, \quad 1 \le j \le n; \tag{5.4}$$

(ii) if $\mathcal{U}_n$ is the unitary group of $\mathcal{A}_n$ with normalized Haar measure du, then

$$\left|\theta_i\left(\int_{\mathcal{U}_n} u m_j u^*\, du\right) - tr(m_j)\right| < n^{-1} \tag{5.5}$$

for $1 \le i \le n,\ 1 \le j \le n$.

To begin the induction, let $\mathcal{A}_1 = \mathbb{C}1$ and let $p_1 = 1$, which commutes with m_1, so (i) holds. The second part of the hypothesis is also satisfied because θ_1 is the trace. Now suppose that $\mathcal{A}_{n-1}$ has been constructed. We apply Lemma 5.2 n times to the set $\{m_1, \ldots, m_n\}$, taking ε to be n^{-1} and k to be successively $1, 2, \ldots, n$. At the k^{th} step we acquire a copy of $\mathbb{M}_k$, leading to a matrix algebra

$$\mathcal{B}_n = \mathcal{A}_{n-1} \otimes \mathbb{M}_1 \otimes \mathbb{M}_2 \otimes \cdots \otimes \mathbb{M}_n = \mathcal{A}_{n-1} \otimes \mathbb{M}_{n!} \tag{5.6}$$

containing sets of projections which satisfy (i).

Now decompose $\mathcal{M}$ as $\mathcal{B}_n \otimes \mathcal{N}$ for some type II_1 factor $\mathcal{N}$, and choose a hyperfinite subfactor $\mathcal{S} \subseteq \mathcal{N}$ with trivial relative commutant, [33]. There exists an ascending sequence

$\{\mathcal{F}_r\}_{r=1}^{\infty}$ of matrix subalgebras of $\mathcal{S}$ whose union is ultraweakly dense in $\mathcal{S}$, as is $\bigcup_{r\geq 1} \mathcal{B}_n \otimes \mathcal{F}_r$ in $\mathcal{B}_n \otimes \mathcal{S}$. Let $\mathcal{V}_r$ denote the unitary group of $\mathcal{B}_n \otimes \mathcal{F}_r$ with normalized Haar measure dv. Since $\mathcal{B}_n \otimes \mathcal{S}$ has trivial relative commutant in $\mathcal{M}$, a standard computation (see, for example, [39, 5.4.4]) shows that

$$\lim_{r\to\infty} \int_{\mathcal{V}_r} vxv^*\ dv = tr(x)1 \tag{5.7}$$

ultraweakly for all $x \in \mathcal{M}$. Since each θ_i is normal, we may select r so large that

$$\left|\theta_i\left(\int_{\mathcal{V}_r} vm_jv^*\ dv\right) - tr(m_j)\right| < n^{-1} \tag{5.8}$$

for $1 \leq i \leq n$ and $1 \leq j \leq n$. For this choice of r, define $\mathcal{A}_n$ to be $\mathcal{B}_n \otimes \mathcal{F}_r$. Now both (i) and (ii) are satisfied.

Let $\mathcal{R} \subseteq \mathcal{M}$ be the ultraweak closure of the union of the $\mathcal{A}_n$'s. We now verify that (5.3) holds for a given set $\{x_1, \ldots, x_k\} \in \mathcal{M}$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Let $\delta = \varepsilon/3$ and, without loss of generality, assume that $\|x_j\| \leq 1$ for $1 \leq j \leq k$. Then choose elements m_{n_j} from the sequence so that

$$\|x_j - m_{n_j}\|_2 < \delta, \qquad 1 \leq j \leq k. \tag{5.9}$$

Now select $r \in \mathbb{N}$ to be so large that

$$r > \delta^{-1},\ n,\ \max\{n_j : 1 \leq j \leq k\}. \tag{5.10}$$

By (i), (with n and r replacing respectively k and n) there exist orthogonal projections $p_i \in \mathcal{A}_r \subseteq \mathcal{R}$, $1 \leq i \leq n$, each of trace n^{-1}, such that

$$\|[p_i, m_{n_j}]\|_2 < r^{-1} < \delta, \qquad 1 \leq i \leq n, \quad 1 \leq j \leq k. \tag{5.11}$$

Then (5.9), (5.11) and the triangle inequality give

$$\|[p_i, x_j]\|_2 < 3\delta = \varepsilon, \qquad 1 \leq i \leq n, \quad 1 \leq j \leq k, \tag{5.12}$$

as required. It remains to show that $\mathcal{R}' \cap \mathcal{M} = \mathbb{C}1$, which will also show that $\mathcal{R}$ is a factor.

Consider $x \in \mathcal{R}' \cap \mathcal{M}$, which we may assume to be of unit norm. Then choose a subsequence $\{m_{n_j}\}_{j=1}^{\infty}$ converging to x in the $\|\cdot\|_2$–norm. We note that $\|x - m_{n_j}\| \leq 2$, so this sequence converges to x ultraweakly, and $\lim_j tr(m_{n_j}) = tr(x)$, by normality of the trace. The inequality

$$\left\| \left(\int_{\mathcal{U}_{n_j}} u m_{n_j} u^* \, du \right) - x \right\|_2 = \left\| \int_{\mathcal{U}_{n_j}} u(m_{n_j} - x) u^* \, du \right\|_2 \leq \|m_{n_j} - x\|_2, \tag{5.13}$$

which is valid because $x \in \mathcal{R}'$, shows that these integrals also converge ultraweakly to x. For any fixed value of i, the sequence $\left\{\theta_i \left(\int_{\mathcal{U}_{n_j}} u m_{n_j} u^* \, du \right)\right\}_{j=1}^{\infty}$ converges to $\theta_i(x)$, since each θ_i is ultraweakly continuous, and also to $tr(x)$, by (5.8). This shows that $\theta_i(x) = tr(x)$ for each $i \geq 1$. By norm density of $\{\theta_i\}_{i=1}^{\infty}$ in the set of normal states, we conclude that $x = tr(x)1$, and so $\mathcal{R}$ has trivial relative commutant in $\mathcal{M}$. □

Remark 5.4. We note, from the construction of the $\mathcal{A}_n$'s, that the projections in the previous theorem are contained in a Cartan masa in $\mathcal{R}$. It is not clear whether this is a masa in $\mathcal{M}$ in general (and we would not expect it to be Cartan in $\mathcal{M}$). We do not pursue this point as it will not be needed subsequently. □

6 The separable predual case

In this section we show that the cohomology groups $H^k(\mathcal{M},\mathcal{M})$ and $H^k(\mathcal{M},B(H))$, $k \geq 2$, are 0 for any type II_1 factor $\mathcal{M} \subseteq B(H)$ with property Γ and separable predual (but note that we place no restriction on H). The general case is postponed to the next section. We will need an algebraic lemma, for which we now establish some notation.

Let S_k, $k \geq 2$, be the set of non-empty subsets of $\{1, 2, \ldots, k\}$, and let T_k be the collection of subsets containing k. The cardinalities are respectively $2^k - 1$ and 2^{k-1}. If $\sigma \in S_k$ then we also regard it as an element of S_r for all $r > k$, and we denote its cardinality by $|\sigma|$. We note that S_{k+1} is then the disjoint union of S_k and T_{k+1}. If $\phi\colon\ \mathcal{M}^k \to B(H)$ is a k-linear map, $p \in \mathcal{M}$ is a projection and $\sigma \in S_k$, then we define $\phi_{\sigma,p}\colon\ \mathcal{M}^k \to B(H)$ by

$$\phi_{\sigma,p}(x_1, \ldots, x_k) = \phi(y_1, \ldots, y_k), \tag{6.1}$$

where $y_i = px_i - x_ip$ for $i \in \sigma$, and $y_i = x_i$ otherwise. For convenience of notation we denote the commutator $[p, x_i]$ by $\hat{x}_i$ since we will only be concerned with one projection at this time. For example, if $k = 3$ and $\sigma = \{2, 3\}$, then

$$\begin{aligned}\phi_{\sigma,p}(x_1, x_2, x_3) &= \phi(x_1, px_2 - x_2p, px_3 - x_3p)\\ &= \phi(x_1, \hat{x}_2, \hat{x}_3), \qquad x_i \in \mathcal{M}.\end{aligned} \tag{6.2}$$

If $\sigma \in S_k$, denote by $\ell(\sigma)$ the least integer in σ. Then define $\phi_{\sigma,p,i}(x_1, \ldots, x_k)$ by changing the i^{th} variable in $\phi_{\sigma,p}$ from x_i to $\hat{x}_i$, $1 \leq i < \ell(\sigma)$, and replacing $\hat{x}_i$ by $p\hat{x}_i$ when $i = \ell(\sigma)$. In the above example $\ell(\sigma) = 2$, and

$$\phi_{\sigma,p,1}(x_1, x_2, x_3) = \phi(\hat{x}_1, \hat{x}_2, \hat{x}_3), \qquad \phi_{\sigma,p,2}(x_1, x_2, x_3) = \phi(x_1, p\hat{x}_2, \hat{x}_3). \tag{6.3}$$

Lemma 6.1. *Let $p \in \mathcal{M} \subseteq B(H)$ be a fixed but arbitrary projection, and let $\mathcal{C}_k$, $k \geq 2$, be the class of k-linear maps $\phi\colon\ \mathcal{M}^k \to B(H)$ which satisfy*

$$p\phi(x_1, \ldots, x_k) = \phi(px_1, x_2, \ldots, x_k), \tag{6.4}$$

$$\phi(x_1, \ldots, x_ip, x_{i+1}, \ldots, x_k) = \phi(x_1, \ldots, x_i, px_{i+1}, \ldots, x_k), \tag{6.5}$$

for $x_j \in \mathcal{M}$ and $1 \le i \le k-1$. If $\phi \in \mathcal{C}_k$ then

$$p\phi(x_1,\ldots,x_k) - p\phi(x_1p,\ldots,x_kp) = \sum_{\sigma\in S_k}(-1)^{|\sigma|+1}p\phi_{\sigma,p}(x_1,\ldots,x_k). \tag{6.6}$$

Moreover, for each $\sigma \in S_k$,

$$p\phi_{\sigma,p}(x_1,\ldots,x_k) = \sum_{i=1}^{\ell(\sigma)}\phi_{\sigma,p,i}(x_1,\ldots,x_k). \tag{6.7}$$

Proof. We will show (6.6) by induction, so consider first the case $k=2$ and take $\phi \in \mathcal{C}_2$. Then, using (6.4) and (6.5) repeatedly,

$$\begin{aligned} p\phi(x_1,x_2) &= p\phi(px_1,x_2) \\ &= p\phi(\hat{x}_1,x_2) + p\phi(x_1p,x_2) \\ &= p\phi(\hat{x}_1,x_2) + p\phi(x_1p,px_2) \\ &= p\phi(\hat{x}_1,x_2) + p\phi(x_1p,\hat{x}_2) + p\phi(x_1p,x_2p) \\ &= p\phi(\hat{x}_1,x_2) - p\phi(\hat{x}_1,\hat{x}_2) + p\phi(px_1,\hat{x}_2) + p\phi(x_1p,x_2p) \\ &= p\phi(\hat{x}_1,x_2) - p\phi(\hat{x}_1,\hat{x}_2) + p\phi(x_1,\hat{x}_2) + p\phi(x_1p,x_2p) \end{aligned} \tag{6.8}$$

and the result follows by moving $p\phi(x_1p,x_2p)$ to the left hand side.

Suppose now that (6.6) is true for maps in $\mathcal{C}_r$ with $r<k$, and consider $\phi \in \mathcal{C}_k$. Note that if we fix x_k, the resulting map is an element of $\mathcal{C}_{k-1}$, so the induction hypothesis gives

$$p\phi(x_1,\ldots,x_k) - p\phi(x_1p,\ldots,x_{k-1}p,x_k) = \sum_{\sigma\in S_k\setminus T_k}(-1)^{|\sigma|+1}p\phi_{\sigma,p}(x_1,\ldots,x_k). \tag{6.9}$$

Since this is an algebraic identity, we may replace x_k by $\hat{x}_k$ to obtain

$$p\phi(x_1,\ldots,x_{k-1},\hat{x}_k) - p\phi(x_1p,\ldots,x_{k-1}p,\hat{x}_k) = \sum_{\sigma\in S_k\setminus T_k}(-1)^{|\sigma|+1}p\phi_{\sigma,p}(x_1,\ldots,x_{k-1},\hat{x}_k). \tag{6.10}$$

By (6.5),

$$\begin{aligned} p\phi(x_1p,\ldots,x_{k-1}p,x_k) &= p\phi(x_1p,\ldots,x_{k-1}p,px_k) \\ &= p\phi(x_1p,\ldots,x_{k-1}p,\hat{x}_k) + p\phi(x_1p,\ldots,x_kp). \end{aligned} \tag{6.11}$$

Now use (6.10) to replace $p\phi(x_1p,\ldots,x_{k-1}p,\hat{x}_k)$ in (6.11), and add the resulting equation to (6.9). After rearranging, we obtain

$$\begin{aligned} p\phi(x_1,\ldots,x_k)-p\phi(x_1p,\ldots,x_kp) &= \sum_{\sigma\in S_k\setminus T_k}(-1)^{|\sigma|+1}p\phi_{\sigma,p}(x_1,\ldots,x_k)\\ &\quad-\sum_{\sigma\in S_k\setminus T_k}(-1)^{|\sigma|+1}p\phi_{\sigma,p}(x_1,\ldots,x_{k-1},\hat{x}_k)\\ &\quad+p\phi(x_1,\ldots,x_{k-1},\hat{x}_k), \end{aligned} \tag{6.12}$$

and the right hand side of (6.12) is equal to $\sum\limits_{\sigma\in S_k}(-1)^{|\sigma|+1}p\phi_{\sigma,p}(x_1,\ldots,x_k)$. This completes the inductive step.

We now prove the second assertion. The idea is to bring the projection in on the left, then past each variable (introducing a commutator each time) until the first existing commutator is reached. To avoid technicalities we illustrate this in the particular case of $k=3$ and $\sigma=\{2,3\}$. We use (6.4) and (6.5) to move p to the right, and the general procedure should then be clear. Thus

$$\begin{aligned} p\phi(x_1,\hat{x}_2,\hat{x}_3) &= \phi(px_1,\hat{x}_2,\hat{x}_3)\\ &= \phi(\hat{x}_1,\hat{x}_2,\hat{x}_3)+\phi(x_1p,\hat{x}_2,\hat{x}_3)\\ &= \phi(\hat{x}_1,\hat{x}_2,\hat{x}_3)+\phi(x_1,p\hat{x}_2,\hat{x}_3), \end{aligned} \tag{6.13}$$

as required. □

Lemma 6.2. *Let $\mathcal{M}\subseteq B(H)$ be a type II_1 factor and let $\phi\colon\mathcal{M}^k\to B(H)$ be a bounded k–linear separately normal map. Let $\{p_r\}_{r=1}^\infty$ be a sequence of projections in $\mathcal{M}$ which satisfy (6.4), (6.5) and*

$$\lim_{r\to\infty}\|[p_r,x]\|_2=0,\qquad x\in\mathcal{M}. \tag{6.14}$$

Then for each $\sigma\in S_k$, each integer $i\le\ell(\sigma)$ and each pair of unit vectors ξ, $\eta\in H$,

$$\lim_{r\to\infty}\langle\phi_{\sigma,p_r,i}(x_1,\ldots,x_k)\xi,\eta\rangle=0. \tag{6.15}$$

Proof. By Lemma 6.1, each variable in $\phi_{\sigma,p_r,i}$ is one of three types, and at least one of the latter two must occur: x_j, $p_rx_j-x_jp_r$ and $p_r(p_rx_j-x_jp_r)$. Thus, as $r\to\infty$, the variables either remain the same (first type) or tend to 0 in the $\|\cdot\|_2$–norm (second and third types), by hypothesis. The result follows from the joint continuity of Corollary 4.5. $\square$

We now come to the main result of this section, the vanishing of cohomology for property Γ factors with separable predual. The heart of the proof is to show complete boundedness of certain multilinear maps and we state this as a separate theorem.

Theorem 6.3. *Let $\mathcal{M} \subseteq B(H)$ be a type II_1 factor with property Γ and a separable predual. Let $\mathcal{R} \subseteq \mathcal{M}$ be a hyperfinite subfactor with trivial relative commutant and satisfying the conclusion of Theorem 5.3. Then a bounded k–linear $\mathcal{R}$–multimodular separately normal map $\phi\colon \mathcal{M}^k \to B(H)$ is completely bounded and $\|\phi\|_{cb} \le 2^k\,\|\phi\|$.*

Proof. Fix an integer n, and a set $X_1, \ldots, X_k \in \mathbb{M}_n(\mathcal{M})$. By Theorem 5.3, we may find sets of orthogonal projections $\{p_{i,r}\}_{i=1}^n$, $r \ge 1$, in $\mathcal{R}$ with trace n^{-1} such that for each $x \in \mathcal{M}$

$$\lim_{r\to\infty} \|[p_{i,r}, x]\|_2 = 0, \qquad 1 \le i \le n. \tag{6.16}$$

Let $P_{i,r} \in \mathbb{M}_n(\mathcal{M})$ be the diagonal projection $I_n \otimes p_{i,r}$. These projections satisfy the analog of (6.16) for elements of $\mathbb{M}_n(\mathcal{M})$.

The n-fold amplification $\phi^{(n)}$ of ϕ to $\mathbb{M}_n(\mathcal{M})$ is an $\mathbb{M}_n(\mathcal{R})$-multimodular map, so (6.4) and (6.5) are satisfied. Thus, for each $r \ge 1$, it follows from Lemma 6.1 that

$$\begin{aligned}\sum_{i=1}^n P_{i,r}\phi^{(n)}(X_1,\ldots,X_k) &- \sum_{i=1}^n \sum_{\sigma\in S_k} (-1)^{|\sigma|+1} P_{i,r}\phi^{(n)}_{\sigma,P_{i,r}}(X_1,\ldots,X_k)\\ &= \sum_{i=1}^n P_{i,r}\phi^{(n)}(X_1P_{i,r},\ldots,X_kP_{i,r}) = \sum_{i=1}^n P_{i,r}\phi^{(n)}(X_1P_{i,r},\ldots,X_kP_{i,r})P_{i,r},\end{aligned} \tag{6.17}$$

where the last equality results from multimodularity of $\phi^{(n)}$. Since $\{P_{i,r}\}_{i=1}^n$ is a set of orthogonal projections for each $r \ge 1$, the right hand side of (6.17) has norm at most

$$\max_{1\le i\le n}\{\|\phi^{(n)}(X_1P_{i,r},\ldots,X_kP_{i,r})\|\} \le 2^k\,\|\phi\|\,\|X_1\|\ldots\|X_k\|, \tag{6.18}$$

using (3.28) in Corollary 3.4.

Now fix an arbitrary pair of unit vectors $\xi, \eta \in H^n$, and apply the vector functional $\langle \cdot \xi, \eta \rangle$ to (6.17). When we let $r \to \infty$ in the resulting equation, the terms in the double sum tend to 0 by Lemmas 6.1 and 6.2, leaving the inequality

$$|\langle \phi^{(n)}(X_1, \dots, X_k)\xi, \eta \rangle| \leq 2^k \, \|\phi\| \, \|X_1\| \dots \|X_k\|, \tag{6.19}$$

since the projections in the first term of (6.17) sum to 1. Now n, ξ and η were arbitrary, so complete boundedness of ϕ follows from (6.19), as does the inequality $\|\phi\|_{cb} \leq 2^k \, \|\phi\|$. □

In the following theorem we restrict to $k \geq 2$ since the two cases of $k = 1$ are in [22, 38] and [5] respectively.

Theorem 6.4. *Let $\mathcal{M} \subseteq B(H)$ be a type II_1 factor with property Γ and a separable predual. Then*

$$H^k(\mathcal{M}, \mathcal{M}) = H^k(\mathcal{M}, B(H)) = 0, \qquad k \geq 2. \tag{6.20}$$

Proof. Let $\mathcal{R}$ be a hyperfinite subfactor of $\mathcal{M}$ with trivial relative commutant and satisfying the additional property of Theorem 5.3. We consider first the cohomology groups $H^k(\mathcal{M}, \mathcal{M})$. By [39, Chapter 3], it suffices to consider an $\mathcal{R}$-multimodular separately normal k-cocycle ϕ, which is then completely bounded by Theorem 6.3. It now follows from [11, 12] (see also [39, 4.3.1]) that ϕ is a coboundary. When $B(H)$ is the module, we appeal instead to [7] to show that each completely bounded cocycle is a coboundary, completing the proof. □

Remark 6.5. By [39, Chapter 3], cohomology can be reduced to the consideration of normal $\mathcal{R}$–multimodular maps which, in the case of property Γ factors, are all completely bounded from Theorem 6.3. Thus we reach the perhaps surprising conclusion that

$$H^k(\mathcal{M}, \mathcal{X}) = H^k_{cb}(\mathcal{M}, \mathcal{X}), \qquad k \geq 1, \tag{6.21}$$

for any property Γ factor $\mathcal{M}$ and any ultraweakly closed $\mathcal{M}$–bimodule $\mathcal{X}$ lying between $\mathcal{M}$ and $B(H)$. □

Remark 6.6. Theorem 6.4 shows that each normal k-cocycle ϕ may be expressed as $\partial\psi$ where $\psi\colon \mathcal{M}^{k-1} \to \mathcal{M}$ (or into $B(H)$). Lemma 3.2.4 of [39] and the proof of Theorem 5.1 of [40] make it clear that ψ can be chosen to satisfy

$$\|\psi\| \leq K_k \|\phi\| \tag{6.22}$$

for some absolute constant K_k. □

7 The general case

We now consider the general case of a type II_1 factor $\mathcal{M}$ which has property Γ, but is no longer required to have a separable predual. We will, however, make use of the separable predual case of the previous section. The connection is established by our first result.

Proposition 7.1. *Let $\mathcal{M}$ be a type II_1 factor with property Γ, let F be a finite subset of $\mathcal{M}$, and let $\phi\colon\ \mathcal{M}^k \to \mathcal{M}$ be a bounded k-linear separately normal map. Then F is contained in a subfactor $\mathcal{M}_F$ which has property Γ and a separable predual. Moreover, $\mathcal{M}_F$ may be chosen so that ϕ maps $(\mathcal{M}_F)^k$ into $\mathcal{M}_F$.*

Proof. We will construct inductively an ascending sequence of separable unital C^*-subalgebras $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of $\mathcal{M}$, each containing F, with the following properties:

(i) ϕ maps $(\mathcal{A}_n)^k$ into $\mathcal{A}_{n+1}$;

(ii) given $x_1, \ldots, x_r \in \mathcal{A}_n$ and $\varepsilon > 0$, there exists a unitary $u \in \mathcal{A}_{n+1}$ of trace 0 such that

$$\|[x_i, u]\|_2 < \varepsilon, \qquad 1 \leq i \leq r; \tag{7.1}$$

(iii) there exists a sequence of unitaries $\{v_i\}_{i=1}^{\infty}$ in $\mathcal{A}_{n+1}$ such that

$$tr(x)1 \in \overline{\text{conv}}^{\|\cdot\|}\{v_i x v_i^*\colon\ i \geq 1\} \tag{7.2}$$

for each $x \in \mathcal{A}_n$.

Define $\mathcal{A}_1$ to be the separable C^*-algebra generated by the elements of F and the identity element. We will only show the construction of $\mathcal{A}_2$, since the inductive step from $\mathcal{A}_n$ to $\mathcal{A}_{n+1}$ is identical.

The restriction of ϕ to $(\mathcal{A}_1)^k$ has separable range which, together with $\mathcal{A}_1$, generates a separable C^*-algebra $\mathcal{B}$. Then ϕ maps $(\mathcal{A}_1)^k$ into $\mathcal{B}$. Now fix a countable sequence $\{a_n\}_{n=1}^{\infty}$ which is norm dense in the unit ball of $\mathcal{A}_1$. For each finite subset σ of this sequence and each integer j we may choose a trace 0 unitary $u_{\sigma,j}$ such that

$$\|[a, u_{\sigma,j}]\|_2 < j^{-1}, \qquad a \in \sigma. \tag{7.3}$$

There are a countable number of such unitaries, so together with $\mathcal{B}$ they generate a larger separable C^*-algebra $\mathcal{C}$. By the Dixmier approximation theorem, [14], we may choose a countable set of unitaries $\{v_i\}_{i=1}^{\infty} \in \mathcal{M}$ so that (7.2) holds when x is any element of $\{a_n\}_{n=1}^{\infty}$. Then these unitaries, combined with $\mathcal{C}$, generate a separable C^*-algebra $\mathcal{A}_2$. By construction of $\mathcal{B}$, ϕ maps $(\mathcal{A}_1)^k$ into $\mathcal{A}_2$, while the second and third properties follow from a simple approximation argument using the norm density of $\{a_n\}_{n=1}^{\infty}$.

Let $\mathcal{A}_F$ be the norm closure of $\bigcup\limits_{n\geq 1} \mathcal{A}_n$, and denote the ultraweak closure by $\mathcal{M}_F$. Then $\mathcal{M}_F$ has separable predual and property Γ, from (7.1) and the $\|\cdot\|_2$-norm density of $\mathcal{A}_F$ in $\mathcal{M}_F$. It remains to show that $\mathcal{M}_F$ is a factor. If τ is a normalized normal trace on $\mathcal{M}_F$ then (7.2) shows that τ and tr agree on $\mathcal{A}_F$. By normality they agree on $\mathcal{M}_F$, so this von Neumann algebra has a unique normalized normal trace and is thus a factor. This completes the proof. □

Theorem 7.2. *Let $\mathcal{M} \subseteq B(H)$ be a type II_1 factor with property Γ. Then*

$$H^k(\mathcal{M}, \mathcal{M}) = H^k(\mathcal{M}, B(H)) = 0, \qquad k \geq 2. \tag{7.4}$$

Proof. We first consider $H^k(\mathcal{M}, \mathcal{M})$. By [39, Chapter 3], we may restrict attention to a separately normal k-cocycle ϕ. For each finite subset F of $\mathcal{M}$, let ϕ_F be the restriction of ϕ to the subfactor $\mathcal{M}_F$ of Proposition 7.1. By Theorem 6.4, there exists a $(k-1)$-linear map $\psi_F\colon\ (\mathcal{M}_F)^{k-1} \to \mathcal{M}_F$ such that $\phi_F = \partial\psi_F$ and there is a uniform bound on $\|\psi_F\|$ (Remark 6.6). Let $\mathbb{E}_F$ be the normal conditional expectation of $\mathcal{M}$ onto $\mathcal{M}_F$, and define $\theta_F\colon\ \mathcal{M}^{k-1} \to \mathcal{M}$ by the composition $\psi_F \circ (\mathbb{E}_F)^{k-1}$. Any F which contains a given set $\{x_1, \ldots, x_k\}$ of elements of $\mathcal{M}$ satisfies

$$\phi(x_1, \ldots, x_k) = \phi_F(x_1, \ldots, x_k) = \partial\theta_F(x_1, \ldots, x_k). \tag{7.5}$$

Now order the finite subsets of $\mathcal{M}$ by inclusion and take a point ultraweakly convergent subnet of $\{\theta_F\}$ with limit $\theta\colon\ \mathcal{M}^{k-1} \to \mathcal{M}$. It is then a simple matter to check that $\phi = \partial\theta$, and thus $H^k(\mathcal{M}, \mathcal{M}) = 0$.

The case of $H^k(\mathcal{M}, B(H))$ is essentially the same. The only difference is that ψ_F and θ_F now map into $B(H)$ in place of $\mathcal{M}_F$. □

Remark 7.3. A more complicated construction of $\mathcal{M}_F$ in the preceding two results would have given the additional property that $\mathcal{M}_F \subseteq \mathcal{M}_G$ whenever $F \subseteq G$ is an inclusion of finite subsets of $\mathcal{M}$. However, this was not needed for Theorem 7.2. □

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