THE PUKÁNSZKY INVARIANT FOR MASAS IN GROUP VON NEUMANN FACTORS

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Abstract

The Pukánszky invariant associates to each maximal abelian self–adjoint subalgebra (masa) A in a type II_1 factor M a certain subset ot $\mathbb{N} \cup \{\infty\}$, denoted $\mathrm{Puk}(A)$. We study this invariant in the context of factors generated by infinite conjugacy class discrete countable groups G with masas arising from abelian subgroups H. Our main result is that we are able to describe $\mathrm{Puk}(VN(H))$ in terms of the algebraic structure of $H \subseteq G$, specifically by examining the double cosets of H in G. We illustrate our characterization by generating many new values for the invariant, mainly for masas in the hyperfinite type II_1 factor R.

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1 Introduction

Pukánszky introduced an invariant for maximal abelian self–adjoint subalgebras, (masas), A in separable type II_1 factors N in 1960, [13]. The invariant associates to each masa A a subset of $\mathbb{N}\cup\{\infty\}$. In his original paper, [13], Pukánszky showed that the values $\{n\}$, $n \in \mathbb{N}$, and $\{\infty\}$ are possible for singular masas in the hyperfinite type II_1 factor R. Subsequently, Popa obtained examples of masas in arbitrary type II_1 factors whose invariants contained 1, while he also showed that if 1 is not present, then the masa must be singular, [11]. This latter result was used by Popa, [11], Rădulescu, [14], Boca and Rădulescu [1], Niţică and Török, [8], and Robertson and Steeger, [16], to show that various masas in type II_1 factors arising from groups are singular by showing that their invariants are $\{\infty\}$. The known values of the invariant are $\{n\}$, $(n \in \mathbb{N})$, $\{\infty\}$, and any subset of $\mathbb{N}\cup\{\infty\}$ which contains 1. These latter subsets appear in recent work of Neshveyev and Størmer, [7], and the examples come from ergodic theory. Our objective in this paper is to analyze the case of a type II_1 factor $VN(G)$ arising from a countable discrete I.C.C. group G with a masa generated by an abelian subgroup H of G. Whenever we consider such a subgroup, it will be assumed that for each $g \in G\backslash H$, $\{hgh^{-1}\colon\ h \in H\}$ is infinite. This is equivalent to $VN(H)$ being a masa in $VN(G)$, [2]. Our work will generate many new examples of possible Pukánszky invariants for masas in R, which was also the setting for the results of [7], based on the crossed product of an abelian von Neumann algebra by an action of $\mathbb{Z}$. Theorem 2.1 allows us to move these values into nonhyperfinite factors, (see Remark 2.2).

Basing our approach on those of [13] and [16], we investigate the Pukánszky invariant of a group masa in a group factor by studying the space ${}_H\backslash G/_H$ of double cosets $HgH = \{hgk\colon\ h, k \in H\}$ and the associated subspaces of $\ell^2(G)$. We introduce an equivalence relation on the double cosets in terms of the stabilizer subgroups $K_g = \{(h, k) \in H \times H\colon\ hgk = g\}$. After some technical results in the third section, the main theorems are presented in the fourth section. Under an additional hypothesis, satisfied by our examples, the numbers appearing in the invariant will be the numbers of double cosets in the various equivalence classes (Theorem 4.1), giving a purely algebraic characterization. Theorem 4.3 describes the situation when this additional hypothesis is not assumed.

Notation and a discussion of standard background material are contained in the second section, where we define the terms used above. We also present a general result on tensor products, the only occasion in the paper when we do not assume that the type II_1 factors arise from groups. In the final section, we give examples of Pukánszky invariants based on the theorems of the preceding section; in particular, we obtain an uncountable family of distinct subsets of $\mathbb{N}\backslash\{1\}$, in contrast to the results of [7]. We are indebted to Christopher Smyth for suggesting the various finite index subgroups of the multiplicative group of nonzero rationals which we use to generate our examples.

This paper is in the long tradition of relating properties of discrete groups and their subgroups to properties of von Neumann algebras and their subalgebras.

2 Notation and preliminaries

Let N denote a separable type II_1 factor with normalized trace tr and let $\langle x, y\rangle = \mathrm{tr}(y^*x)$ be the usual inner product on N. Then $\|\cdot\|_2$ is defined by $\|x\|_2 = (\mathrm{tr}(x^*x))^{1/2}$, $x \in N$, and the completion of N in this norm is denoted $L^2(N)$. The map $x \mapsto x^*$ on N induces a conjugate linear isometry J on $L^2(N)$. When N is faithfully represented on $L^2(N)$ by left multiplication, the commutant N' is equal to JNJ. A maximal abelian self–adjoint subalgebra (masa) A of N induces an abelian subalgebra $\mathcal{A} = (A \cup JAJ)''$ of $B(L^2(N))$, since $JAJ \subseteq N' \subseteq A'$. Then $\mathcal{A}'$ is a type I von Neumann algebra whose center is $\mathcal{A}$, [6, Theorem 9.1.3].

The subspace $L^2(A) \subseteq L^2(N)$ is invariant for $\mathcal{A}$, since the operators in JAJ act by right multiplication and so the associated projection p lies in $\mathcal{A}'$. On the other hand, if $t \in \mathcal{A}'$ then, given $\varepsilon > 0$, choose $x \in N$ so that $\|t(1) - x\|_2 < \varepsilon$, pre– and post–multiply by u and u^* for any unitary $u \in A$ to obtain $\|t(1) - uxu^*\|_2 < \varepsilon$, and average over u to get $\|t(1) - \mathbb{E}_A(x)\|_2 \leq \varepsilon$ where $\mathbb{E}_A$ is the trace preserving conditional expectation of N onto A. Letting $\varepsilon \to 0$ gives $t(1) \in L^2(A)$ from which it follows that $p \in (\mathcal{A}')' = \mathcal{A}$. Thus p is central in $\mathcal{A}'$, and $(1-p)\mathcal{A}'$ is a type I von Neumann algebra, so is a direct sum of type I_n von Neumann algebras for some values of $n \in \mathbb{N} \cup \{\infty\}$. The set of n's which appear in this direct sum decomposition constitutes the Pukánszky invariant, which we denote by $\mathrm{Puk}(A)$. There seems to be no previously established notation for this set. If the projection p were not removed, then $1 \in \mathrm{Puk}(A)$ for all masas A. Even though removed, this projection has a major effect in tensor products (see below).

If ϕ is an automorphism of a finite factor N which is represented in standard form on $L^2(N)$, so that it has a cyclic and separating vector, then ϕ is spatially implemented and so extends to an automorphism of $B(L^2(N))$. This automorphism sends a masa A to another masa $B = \phi(A)$ in such a way that $\mathcal{A}$ and $\mathcal{B}$ are spatially isomorphic. Consequently $\mathrm{Puk}(A) = \mathrm{Puk}(B)$, so equality of the Pukánszky invariants is a necessary condition for two masas to be conjugated by an automorphism of the ambient factor. However, it is not a sufficient condition. In the case of a separable predual, $\mathcal{A}$ is a masa in $B(L^2(N))$ precisely when $\mathrm{Puk}(A) = \{1\}$, and this occurs for all Cartan masas, [4, 12], but can also happen for singular masas in the hyperfinite factor, as in Example 5.1 below.

Finally, recall that a projection q in a von Neumann algebra N with center Z is said to be abelian if qNq is abelian, and then $qNq = qZ$, [6, pp. 419–422]. All of the preceding discussion is standard except for the introduction of the symbol $\mathrm{Puk}(A)$. Before moving on to the main results about group von Neumann algebras, we first discuss a general theorem on tensor products.

Theorem 2.1. *Let N_1 and N_2 be type II_1 factors with masas respectively A_1 and A_2. Then $A = A_1 \overline{\otimes} A_2$ is a masa in $N_1 \overline{\otimes} N_2$, and*

$$\mathrm{Puk}(A_1 \overline{\otimes} A_2) = \mathrm{Puk}(A_1) \cup \mathrm{Puk}(A_2) \cup \{n_1 n_2 \colon\ n_i \in \mathrm{Puk}(A_i)\}. \tag{2.1}$$

Proof. As a special case of Tomita's commutation theorem, [19], $(A_1 \overline{\otimes} A_2)' \cap (N_1 \overline{\otimes} N_2) = A_1 \overline{\otimes} A_2$, and so $A_1 \overline{\otimes} A_2$ is a masa. The underlying Hilbert space for the standard form of $N_1 \overline{\otimes} N_2$ is $L^2(N_1) \otimes_2 L^2(N_2)$, and the adjoint operator J is the tensor product $J_1 \otimes J_2$ of the respective adjoint operators on $L^2(N_i)$, $i = 1, 2$. If p_1 and p_2 are respectively the projections in $\mathcal{A}_1$ and $\mathcal{A}_2$ onto $L^2(A_1)$ and $L^2(A_2)$, then $p = p_1 \otimes p_2$ is the projection onto $L^2(A)$. Thus $1 - p = 1 - (p_1 \otimes p_2) = (1 - p_1) \otimes (1 - p_2) + (1 - p_1) \otimes p_2 + p_1 \otimes (1 - p_2)$.

If

$$(1 - p_i) = \sum_{j=1}^{\infty} z_{i,j}, \qquad i = 1, 2, \tag{2.2}$$

are the central splittings for which $\mathcal{A}_i' z_{i,j}$ is $n_{i,j}$–homogeneous, then we obtain a central splitting in $\mathcal{A}'$ of $1 - p$ by

$$1 - p = \sum_{j=1}^{\infty} z_{1,j} \otimes p_2 + \sum_{k=1}^{\infty} p_1 \otimes z_{2,k} + \sum_{j,k=1}^{\infty} z_{1,j} \otimes z_{2,k}, \tag{2.3}$$

and the three sums contribute respectively the three terms in (2.1) to $\mathrm{Puk}(A)$. Note that p_1 and p_2 disappear from $\mathrm{Puk}(A_1)$ and $\mathrm{Puk}(A_2)$, but not from $\mathrm{Puk}(A_1 \overline{\otimes} A_2)$. □

Remark 2.2. Let S be any subset of $\mathbb{N} \cup \{\infty\}$ containing 1 and let $A \subseteq R$ be a masa for which $\mathrm{Puk}(A) = S$, [7]. Then let $B \subseteq N$ be a Cartan masa in a nonhyperfinite factor N; such examples may be found in [5]. Then $\mathcal{B}$ is a masa in $B(L^2(N))$, [4], and so $\text{Puk}(B) = \{1\}$. Then $A\overline{\otimes}B$ is a masa in the nonhyperfinite factor $R\overline{\otimes}N$, and Theorem 2.1 gives $\text{Puk}(A\overline{\otimes}B) = S$. Thus the values of $\text{Puk}(\cdot)$ found in [7] also occur in the nonhyperfinite setting. $\square$

In general, the behavior of $\text{Puk}(\cdot)$ for the standard constructions in von Neumann algebras is unclear. If A and B are masas in the hyperfinite type II_1 factor R, then $\mathbb{M}_2(R)$ is isomorphic to R, so we may view $A \oplus B$ as another masa in R. We do not know how to relate $\text{Puk}(A \oplus B)$ to $\text{Puk}(A)$ and $\text{Puk}(B)$ except in trivial cases, for example when B is a unitary conjugate of A.

3 Technical results

This section is concerned with some technical results on equivalent projections in $\mathcal{A}'$. We continue to assume throughout that G is a countable discrete I.C.C. group containing an abelian subgroup H such that $\{hgh^{-1}\colon\ h \in H\}$ is infinite for each $g \in G\backslash H$. As already noted, $A = VN(H)$ is then a masa in the type II_1 factor $VN(G)$. We let HgH denote the double coset $\{hgk\colon\ h, k \in H\}$ and write ${}_H\backslash G/_H$ for the set of double cosets. We will wish to exclude the trivial double coset $H = HeH$, and when we do this we will refer to the remainder of ${}_H\backslash G/_H$ as the *nontrivial double cosets*. For a subset $S \subseteq G$, $[S]$ is the closed span of S in $\ell^2(G)$ while $p_{[S]}$ is the projection onto $[S]$. We adopt this symbol over $\ell^2(S)$ for ease of notation. Since $[HgH]$ is invariant under left and right multiplications by elements of H, we see that $p_{[HgH]} \in \mathcal{A}'$ for all $g \in G$. In the theory of subfactors, $p_{[H]}$ is, in different notation, the Jones projection e_A. We denote by K_g the subgroup of $H \times H$ given by $\{(h, k) \in H \times H\colon\ hgk = g\}$. This is the stabilizer subgroup for g. Recall that two groups F_1, F_2 are *commensurable* if they possess isomorphic finite index subgroups $G_i \subseteq F_i$, $i = 1, 2$. In this paper we will assign a stronger meaning to this term by requiring the isomorphic subgroups to be equal. Thus, for two subgroups F_1, F_2 of $H \times H$, commensurability will mean that $F_1 \cap F_2$ has finite index in F_1F_2, equivalent to the requirements that $F_1 \cap F_2$ be of finite index in F_1 and in F_2, by elementary group theory. We define an equivalence relation on the nontrivial double cosets in ${}_H\backslash G/_H$ by $HcH \sim HdH$ if K_c and K_d are commensurable. This is well defined because if $Hc_1H = Hc_2H$ then $K_{c_1} = K_{c_2}$. It is also transitive, as the following simple lemma shows. Although it is undoubtedly known, we include it for the reader's convenience.

Lemma 3.1. *Let F_1, F_2, F_3 be subgroups of an abelian group L and suppose that $F_1F_2/F_1 \cap F_2$ and $F_2F_3/F_2 \cap F_3$ are finite groups. Then $F_1F_3/F_1 \cap F_3$ is a finite group.*

Proof. The hypotheses imply that the orders of $F_1/F_1 \cap F_2$, $F_2/F_1 \cap F_2$, $F_2/F_2 \cap F_3$, and $F_3/F_2 \cap F_3$ are all finite. Let $\pi\colon\ L \to L/F_2 \cap F_3$ be the quotient homomorphism and let ρ be its restriction to $F_1 \cap F_2$. Then ρ maps $F_1 \cap F_2$ into $F_2/F_2 \cap F_3$ with kernel $F_1 \cap F_2 \cap F_3$.

Thus $F_1 \cap F_2 / F_1 \cap F_2 \cap F_3$ is a finite group. Then each of the inclusions

$$F_1 \cap F_2 \cap F_3 \subseteq F_1 \cap F_2 \subseteq F_1 \tag{3.1}$$

is of finite index, so $F_1 / F_1 \cap F_2 \cap F_3$ is a finite group, as is $F_1 \cap F_3 / F_1 \cap F_2 \cap F_3$. Finiteness of $F_1 / F_1 \cap F_3$ now follows from the inclusions

$$F_1 \cap F_2 \cap F_3 \subseteq F_1 \cap F_3 \subseteq F_1, \tag{3.2}$$

and similarly $F_3 / F_1 \cap F_3$ is a finite group. Since $F_1 F_3 / F_1$ is isomorphic to $F_3 / F_1 \cap F_3$, finiteness of $F_1 F_3 / F_1 \cap F_3$ is a consequence of the finite index inclusions

$$F_1 \cap F_3 \subseteq F_1 \subseteq F_1 F_3. \tag{3.3}$$

$\square$

Theorem 3.2. *(i) Let c and d be elements of $G \backslash H$. Then there exists an operator $t \in \mathcal{A}'$ such that $p_{[HdH]}\, t\, p_{[HcH]} \neq 0$ if and only if $HcH \sim HdH$.*

(ii) Let q be the projection onto the closed subspace spanned by all the group elements in an equivalence class of nontrivial double cosets. Then $q \in \mathcal{A}$.

(iii) The projections $p_{[HcH]}$ and $p_{[HdH]}$ are equivalent in $\mathcal{A}'$ if and only if $K_c = K_d$.

Proof. (i) Let $t \in \mathcal{A}'$ be such that $p_{[HdH]}\, t\, p_{[HcH]} \neq 0$. To obtain a contradiction, suppose that $K_d / K_c \cap K_d$ is infinite, and let $\{(h_n, k_n) K_c \cap K_d\}_{n=1}^{\infty}$ be a listing of the $K_c \cap K_d$–cosets in K_d. Then the group elements $\{h_n c k_n\}_{n=1}^{\infty}$ are distinct, since equality of $h_n c k_n$ and $h_m c k_m$ would imply that $(h_n h_m^{-1}, k_n k_m^{-1}) \in K_c$ and (h_n, k_n) and (h_m, k_m) would define the same $K_c \cap K_d$ – coset. Thus $\{h_n c k_n\}_{n=1}^{\infty}$ are distinct as orthonormal vectors in $\ell^2(G)$. Each vector in $[HdH]$ is left invariant by all elements of K_d, and in particular $t(h_n c k_n) = t(c)$ for $n \geq 1$ since $t \in \mathcal{A}'$. For each $m \geq 1$, the vector $\sum_{n=1}^{m} n^{-1} h_n c k_n$ (whose norm is bounded by $\left(\sum_{n=1}^{\infty} n^{-2}\right)^{1/2} = \pi/\sqrt{6}$) is thus mapped by t to $\sum_{n=1}^{m} n^{-1} h_n t(c) k_n = \left(\sum_{n=1}^{m} n^{-1}\right) t(c)$, which forces $t(c) = 0$, otherwise t is an unbounded operator. But then $t(hck) = ht(c)k = 0$ for $h, k \in H$ and so $p_{[HdH]}\, t\, p_{[HcH]} = 0$, a contradiction. Thus $K_d / K_c \cap K_d$ is finite, and the same conclusion holds for $K_c/K_c \cap K_d$ by considering t^*. This proves that K_c and K_d are commensurable and $HcH \sim HdH$.

Conversely, suppose that K_c and K_d are commensurable. Then there is an action of the finite group $K_cK_d/K_c \cap K_d$ on both HcH and HdH. Let $\{(h_i, k_i)K_c \cap K_d\}_{i=1}^n$ be a listing of the cosets of $K_c \cap K_d$ in K_c and define $t\colon\ [HcH] \to [HdH]$ on the vectors arising from group elements by

$$t(xcy) = \sum_{i=1}^{n} h_i x d y k_i, \qquad x, y \in H. \tag{3.4}$$

This is well defined because if $xcy = wcz$ for $w, x, y, z \in H$ then $(w^{-1}x, yz^{-1}) \in K_c$ and $(w^{-1}x, yz^{-1}) = (h_j r, k_j s)$ for some $j \in \{1, \ldots, n\}$ and $(r, s) \in K_c \cap K_d$, leading to

$$t(xcy) = \sum_{i=1}^{n} h_i x d y k_i = \sum_{i=1}^{n} h_i h_j r w d s k_i k_j z = t(wcz) \tag{3.5}$$

since $rds = d$ and $\{(h_i h_j, k_i k_j)\}_{i=1}^n$ gives another listing of the $K_c \cap K_d$ – cosets.

If $\{(x_j, y_j)\}_{i=1}^\infty$ are representatives of the cosets of K_c in $H \times H$ then $\{x_j c y_j\}_{j=1}^m$ are distinct in G and $\left\| \sum\limits_{j=1}^{m} \alpha_j x_j c y_j \right\|^2 = \sum\limits_{j=1}^{m} |\alpha_j|^2$. Then

$$t\left(\sum_{j=1}^{m} \alpha_j x_j c y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_j h_i x_j d y_j k_i \tag{3.6}$$

and the right hand side of (3.6) has norm at most $n \left\| \sum\limits_{j=1}^{m} \alpha_j x_j d y_j \right\|$, by the triangle inequality. The vectors of the form $x_j d y_j$ are either equal or orthogonal and so

$$n \left\| \sum_{j=1}^{m} \alpha_j x_j d y_j \right\| \le n \left\| \sum_{j=1}^{m} |\alpha_j| x_j d y_j \right\|. \tag{3.7}$$

To estimate the latter sum, define an equivalence relation on $\{1, \ldots, m\}$ by $r \sim s$ if $x_r d y_r = x_s d y_s$ and note that each equivalence class has at most $|K_d/K_c \cap K_d|$ elements. To see this, let $\{(a_i, b_i)K_c \cap K_d\}_{i=1}^\ell$ be a listing of the cosets of $K_c \cap K_d$ in K_d and fix s. Then $r \sim s$ if and only if $(x_r x_s^{-1}, y_r y_s^{-1}) \in K_d$ and so (x_r, y_r) has the form $(x_s, y_s)(a_i, b_i)(h, k)$ for some $i \in \{1, \ldots, \ell\}$ and $(h, k) \in K_c \cap K_d$ so can lie in at most ℓ cosets of K_c in $H \times H$. If we replace each $|\alpha_j|$ in (3.7) by $\max\{|\alpha_p|:\ p \sim j\}$, then we obtain the estimate

$$\left\| t\left(\sum_{j=1}^{m} \alpha_j x_j c y_j\right)\right\| \le n \left\| \sum_{j=1}^{m} |\alpha_j| x_j d y_j \right\| \le n\ell \left\| \sum_{j=1}^{m} |\alpha_j|^2 \right\|^{1/2} \tag{3.8}$$

and $\|t\| \le |K_c/K_c \cap K_d| \cdot |K_d/K_c \cap K_d|$ by letting m and the α_j's vary. Thus t is a bounded operator and it extends with the same norm to the whole space by setting it equal to 0 on the orthogonal complement of $[HcH]$. It is clear from the definition that this extension, also denoted by t, commutes with left and right multiplications by elements of H and so $t \in \mathcal{A}'$. It is also clear from (3.4) that $t(c) \neq 0$ so $p_{[HdH]}\, t\, p_{[HcH]} \neq 0$.

(ii) The projection q commutes with left and right multiplications by group elements from H, and so $q \in \mathcal{A}'$. If HcH is in the equivalence class but HdH is not, then (i) shows that any $t \in \mathcal{A}'$ mapping $[HcH]$ to $[HdH]$ must be 0. Thus the range of q is invariant for $\mathcal{A}'$ and $q \in \mathcal{A}'' = \mathcal{A}$.

(iii) If $v \in \mathcal{A}'$ is a partial isometry mapping $[HcH]$ onto $[HdH]$ then the relation

$$v(c) = v(hck) = hv(c)k, \quad (h,k) \in K_c, \tag{3.9}$$

shows that the range of v is contained in the set of K_c–invariant vectors in $[HdH]$, so surjectivity of v implies that $K_c \subseteq K_d$. The reverse containment follows from consideration of v^*, and so $K_c = K_d$.

Conversely suppose that $K_c = K_d$. Using (3.4) and the work in (i), there is a well defined operator $v \in \mathcal{A}'$ which is 0 on $[HcH]^\perp$, and on $[HcH]$ is given by

$$v(xcy) = xdy, \qquad x, y \in H. \tag{3.10}$$

Moreover $\|v\| \le |K_c/K_c \cap K_d| \cdot |K_d/K_c \cap K_d| = 1$. Its adjoint maps xdy to xcy and also has norm at most 1. Then $v^*v = p_{[HcH]}$ and $vv^* = p_{[HdH]}$, showing the equivalence of these projections. □

4 The main theorems

In this section we present the main results on computing the Pukánszky invariant for a masa $VN(H) \subseteq VN(G)$ arising from an abelian subgroup $H \subseteq G$ satisfying the properties already discussed. In the previous section we introduced an equivalence relation on the nontrivial double cosets ${}_H\backslash G/{}_H$ in terms of the commensurability of the stabilizer subgroups K_c for elements $c \in G\backslash H$. The first result determines the Pukánszky invariant in terms of the algebraic structure of H and G under an extra technical hypothesis (which will be satisfied by all our examples in the next section).

Theorem 4.1. *Let G be a countable I.C.C. group with an abelian subgroup H such that $\{hgh^{-1}\colon\ h \in H\}$ is infinite for each $g \in G\backslash H$. Moreover suppose that, for each pair of elements $c, d \in G\backslash H$, the stabilizer subgroups K_c and K_d are either equal or noncommensurable. Then $n \in \mathbb{N} \cup \{\infty\}$ lies in Puk($VN(H)$) if and only if there is an equivalence class of nontrivial double cosets in ${}_H\backslash G/{}_H$ with n elements.*

Proof. Let $W = \{Hg_iH\colon\ 1 \leq i \leq n,\ g_i \notin H\}$, $n \in \mathbb{N} \cup \{\infty\}$, be an equivalence class of nontrivial double cosets and let q be the projection onto the closed span of the subspaces $[Hg_iH]$. By Theorem 3.2 (ii), $q \in \mathcal{A}$ and so is in the center of $\mathcal{A}'$. We will show that $\mathcal{A}'q$ is n–homogeneous, and thus that its contribution to Puk(A) is precisely $\{n\}$. Since $(1-p)\ell^2(G)$ is the closed span of vectors from the nontrivial double cosets, this will prove the theorem.

By hypothesis, $K_{g_i} = K_{g_j}$ for $1 \leq i, j \leq n$, and so Theorem 3.2 (iii) implies that the projections onto the subspaces $[Hg_iH]$ are pairwise equivalent in $\mathcal{A}'$. Each $[Hg_iH]$ is separable, is invariant for $\mathcal{A}$, and possesses a cyclic vector g_i for $\mathcal{A}$. By [10, pp. 35], $\mathcal{A}p_{[Hg_iH]}$ is maximal abelian in $B([Hg_iH])$, and since the compression of $\mathcal{A}'$ by $p_{[Hg_iH]}$ commutes with this algebra, we see that each $p_{[Hg_iH]}$ is an abelian projection in $\mathcal{A}'$. This proves that $\mathcal{A}'q$ is n–homogeneous and that this part of $\mathcal{A}'$ contributes exactly $\{n\}$ to Puk(A). □

We now discuss the more complicated general case where two stabilizer subgroups could be commensurable but unequal, and for this we need to establish some notation. Since different equivalence classes contribute to the Pukánszky invariant independently, by Theorem 3.2 (ii), we will make the simplifying assumption that there is only one equivalence class of double cosets in the nontrivial part of ${}_H\backslash G/_H$ and that the elements are labeled $\{Hc_iH\}_{i=1}^{\infty}$. The analysis for a finite set of double cosets is no different. We write K_i for the stabilizer subgroup of c_i, and for each finite subset μ of $\mathbb{N}$ we let K_μ denote the product of the groups $\{K_i\colon\ i \in \mu\}$. For a particular K_μ, let $\sigma = \{i\colon\ K_i \subseteq K_\mu\}$, which clearly contains μ, but could be larger, even infinite. We then relabel K_μ as K_σ. Each K_σ is a finite product of K_j's, so we note that if $i \in \sigma$ then K_σ/K_i is a finite group. It is then clear from the construction that $K_\sigma \subseteq K_{\sigma'}$ when $\sigma \subseteq \sigma'$, and $K_\sigma = K_{\sigma'}$ precisely when $\sigma = \sigma'$. We thus have a set S of subsets of $\mathbb{N}$ consisting of those σ's appearing above to label the groups. Under the hypothesis of Theorem 4.1, all K_i's are equal and so there is only one label, $\mathbb{N}$, in this case, and $K_i = K_{\mathbb{N}}$ for all i. For each $\sigma \in S$, we let q_σ denote the projection onto the vectors in $(1-p)L^2(VN(G))$ which are K_i–invariant for every $i \in \sigma$. The range of q_σ is invariant for $\mathcal{A}'$ and thus each such projection lies in the center $Z(\mathcal{A}') = \mathcal{A}$. We then define $z_\sigma = q_\sigma - \bigvee\{q_{\sigma'}\colon\ \sigma \subsetneqq \sigma'\} \in \mathcal{A}$, while setting $z_\sigma = q_\sigma$ when $\{\sigma'\colon\ \sigma \subsetneqq \sigma'\}$ is empty. Finally let $p_i \in \mathcal{A}'$ denote the projection onto $[Hc_iH]$.

Our assumption that there is only one equivalence class of double cosets, and the above notation, will be in force for the remainder of the section.

Lemma 4.2. *If $\sigma \in S$ and $z_\sigma \neq 0$, then $\mathcal{A}'z_\sigma$ is $|\sigma|$–homogeneous and $|\sigma| \in \mathrm{Puk}(A)$. Moreover, if $\sigma \neq \sigma'$ then $z_\sigma z_{\sigma'} = 0$.*

Proof. Consider $i \notin \sigma$. Then any vector $\xi \in [Hc_iH]$ is K_i–invariant and so $z_\sigma\xi$ is $K_\sigma K_i$–invariant. This group is $K_{\sigma'}$ for some σ' strictly containing σ and so $z_\sigma\xi = q_{\sigma'}z_\sigma\xi$. Since $z_\sigma q_{\sigma'} = 0$ by construction, this implies that z_σ annihilates $[Hc_iH]$ for $i \notin \sigma$.

Since $1 - p = \sum\limits_{i=1}^{\infty} p_i$, we obtain

$$z_\sigma = \sum_{i\in\sigma} z_\sigma p_i = \sum_{i\in\sigma} z_\sigma q_\sigma p_i. \tag{4.1}$$

As in the proof of Theorem 3.2 (i), we now construct, for each pair $i, j \in \sigma$, a partial isometry v which exhibits $q_\sigma p_i$ and $q_\sigma p_j$ as equivalent projections in $\mathcal{A}'$. The groups $F_i = K_\sigma/K_i$

and $F_j = K_\sigma/K_j$ are finite groups, so let $\{(x_\ell, y_\ell)\}_{\ell=1}^r$ and $\{(u_m, w_m)\}_{m=1}^s$ be respectively representatives of the cosets in these groups. Then let $\{(h_n, k_n)\}_{n=1}^\infty$ be representatives of the cosets of K_σ in $H \times H$. Then $\{(x_\ell h_n c_i y_\ell k_n)\}$ and $\{(u_m h_n c_j w_m k_n)\}$ are orthonormal bases for $[Hc_iH]$ and $[Hc_jH]$ respectively. The vectors in the ranges of $q_\sigma p_i$ and $q_\sigma p_j$ have the respective forms

$$\sum_{\ell=1}^{r}\sum_{n=1}^{\infty} \lambda_n x_\ell h_n c_i y_\ell k_n \quad \text{and} \quad \sum_{m=1}^{s}\sum_{n=1}^{\infty} \lambda_n u_m h_n c_j w_m k_n \tag{4.2}$$

for $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$. These vectors have respective norms

$$r^{1/2}\left(\sum_{n=1}^{\infty}|\lambda_n|^2\right)^{1/2} \quad \text{and} \quad s^{1/2}\left(\sum_{n=1}^{\infty}|\lambda_n|^2\right)^{1/2}.$$

Thus there is a well defined operator t from the range of $q_\sigma p_i$ to that of $q_\sigma p_j$ which takes the first vector in (4.2) to the second. Division of t by $\sqrt{s/r}$ gives an isometry which becomes a partial isometry v by setting it to be 0 on $(\text{Ran } q_\sigma p_i)^\perp$. Since any element of $H \times H$ is a product of an (x_ℓ, y_ℓ), an (h_n, k_n) and an element of K_i, it is easy to check that $v \in \mathcal{A}'$. We have thus proved that the set of projections $\{q_\sigma p_i\}_{i\in\sigma}$ are pairwise equivalent, as is also the case for $\{z_\sigma q_\sigma p_i\}_{i\in\sigma}$. Each is an abelian projection, and this shows, from (4.1), that $\mathcal{A}' z_\sigma$ is $|\sigma|$–homogeneous, and that $|\sigma| \in \text{Puk}(A)$.

Suppose now that there exist two distinct sets $\sigma, \sigma' \in S$ such that $z_\sigma z_{\sigma'} \neq 0$, and fix a nonzero vector ξ for which $z_\sigma \xi = z_{\sigma'}\xi = \xi$. There exists $\sigma'' \in S$ containing both σ and σ' such that $K_\sigma K_{\sigma'} = K_{\sigma''}$. Then σ'' strictly contains at least one of σ and σ', say σ, and ξ is an invariant vector for $K_{\sigma''}$. Thus $q_{\sigma''}\xi = \xi$, and consequently $z_\sigma \xi = 0$. This contradiction completes the proof. □

We now come to the general result on $\text{Puk}(A)$.

Theorem 4.3. *Let G be a countable discrete I.C.C. group with an abelian subgroup H such that $A = VN(H)$ is a masa in $VN(G)$. Assume that there is one equivalence class of double cosets, and let $z_\infty = (1-p) - \sum_{\sigma\in S} z_\sigma$. If $z_\infty = 0$, then* $\text{Puk}(A) = \{|\sigma|:\ \sigma \in S,\ z_\sigma \neq 0\}$. *If $z_\infty \neq 0$, then* $\text{Puk}(A) = \{|\sigma|:\ \sigma \in S,\ z_\sigma \neq 0\} \cup \{\infty\}$.

Proof. Lemma 4.2 shows that $\{|\sigma|:\ \sigma \in S,\ z_\sigma \neq 0\}$ is the contribution to Puk(A) of $\sum_{\sigma\in S} z_\sigma$, so we need only consider $z_\infty \neq 0$. We will show that this contributes precisely $\{\infty\}$. Let z be any nonzero central projection under z_∞. We will show that $\mathcal{A}'z$ is not k–homogeneous for any finite integer k, and this will force $\mathcal{A}'z_\infty$ to be ∞–homogeneous.

Since $z \neq 0$, there is an integer i such that $zp_i \neq 0$, and so there is a vector $\xi \neq 0$ in both $[Hc_iH]$ and the range of z. Consider such a vector ξ, and write it as $\xi = \sum_{n=1}^{\infty} \alpha_n h_n c_i k_n$, where $\{(h_n, k_n)\}_{n=1}^{\infty}$ is a set of representatives of the cosets of K_i in $H \times H$. Renumbering if necessary, we may assume that $\alpha_1 = \alpha_2 = \cdots = \alpha_m \neq 0$, and $\alpha_r \neq \alpha_1$ for $r \geq m+1$. If F is any subgroup of $H \times H$ containing K_i, then F/K_i permutes the basis vectors and fixes none of them. If F/K_i fixes ξ then this group permutes $\{h_n c_i h_n\}_{n=1}^{m}$, and so has order bounded by $m!$. This shows that there is a maximal set $\sigma(\xi) \in S$ containing i such that $K_{\sigma(\xi)}$ fixes ξ. We consider two cases.

Case 1: $\{|\sigma(\xi)|:\ \xi = zp_i\xi \neq 0\}$ is bounded as ξ and i vary.

In this case choose a vector ξ and associated set σ at which $|\sigma(\xi)|$ is a maximum. If no $\sigma' \in S$ is larger, then $z_\sigma = q_\sigma$ and so $\xi = zp_i\xi = zz_\sigma p_i\xi = 0$ while $\xi = z_\sigma\xi \neq 0$, a contradiction. Thus there is a $\sigma' \in S$ which is strictly larger than σ. For any such σ', $q_{\sigma'}(\xi)$ is $K_{\sigma'}$–invariant and in the range of zp_i, so by maximality $q_{\sigma'}(\xi) = 0$. Thus $z_\sigma\xi = \xi$, so $\xi = zz_\sigma\xi = 0$, a contradiction. Thus case 1 cannot occur. This forces us into

Case 2: Given $k \in \mathbb{N}$ there exist $i \in \mathbb{N}$ and a nonzero vector ξ in $[Hc_iH]$, which is also in the range of z, such that $|\sigma(\xi)| \geq k+1$.

By renumbering, we may assume that $i = 1$ and $\sigma(\xi)$ contains $\{1, 2, \ldots, k+1\}$. In Lemma 4.2 the projections $q_{\sigma(\xi)}p_i$, $1 \leq i \leq k+1$, were shown to be equivalent in $\mathcal{A}'$ and $zq_{\sigma(\xi)}p_1\xi = \xi \neq 0$, so $\{zq_{\sigma(\xi)}p_i\}_{i=1}^{k+1}$ is a set of $k+1$ equivalent nonzero orthogonal projections under z. Thus $\mathcal{A}'z$ is not k–homogeneous for any $k \in \mathbb{N}$. $\square$

5 Examples

In [7] it was shown, using ergodic theory, that any subset of $\mathbb{N} \cup \{\infty\}$ which contains 1 can be the Pukánszky invariant of a masa in the hyperfinite type II_1 factor R. Since R is equal to $VN(G)$ for any amenable countable discrete I.C.C. group, we will use Theorem 4.1 to present examples of other sets which exclude the value 1. These arise from matrix groups, as did the examples constructed by Pukánszky [13], and his have motivated ours. In each case, the calculations are similar and so we will only give full details in the first example. In [18], we introduced the notion of strong singularity of a masa A in a type II_1 factor M. These are the masas for which the inequality

$$\|\mathbb{E}_A - \mathbb{E}_{uAu^*}\|_{\infty,2} \geq \|u - \mathbb{E}_A(u)\|_2, \tag{5.1}$$

for all unitaries $u \in M$. The inequality implies that u must lie in A whenever $uAu^* = A$, so singularity of A follows immediately from (5.1), and it is a useful criterion for determining singularity. In the case of a masa generated by an abelian subgroup H of an I.C.C. group G, we found a sufficient condition for strong singularity in terms of the algebraic structure: given $g_1, \ldots, g_n \in G \backslash H$, there exists $h \in H$ such that

$$g_i h g_j \notin H, \qquad 1 \leq i, j \leq n. \tag{5.2}$$

This is [15, Lemma 2.1] adapted to the case of a group von Neumann factor.

Let $\mathbb{Q}$ denote the additive group of rationals and let $\mathbb{Q}^\times$ be the multiplicative group of nonzero rationals. For each n, we define a subgroup $F_n \subseteq \mathbb{Q}^\times$ of index n by

$$F_n = \left\{ \frac{p}{q} 2^{kn} \colon \ k \in \mathbb{Z}, \quad p, q \in \mathbb{Z}_{\text{odd}} \right\}. \tag{5.3}$$

We let F_∞ denote the subgroup $\{p/q \colon \ p, q \in \mathbb{Z}_{\text{odd}}\}$ of $\mathbb{Q}^\times$ of infinite index.

Example 5.1. Let $n \in \mathbb{N} \cup \{\infty\}$ and let

$$G = \left\{ \begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix} : \ f \in F_n, \quad x \in \mathbb{Q} \right\}, \quad H = \left\{ \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} : \ f \in F_n \right\}. \tag{5.4}$$

Then $VN(G)$ is the hyperfinite type II_1 factor, $VN(H)$ is a strongly singular masa and $\mathrm{Puk}(VN(H)) = \{n\}$.

Proof. For fixed $f \in F_n$, $g \in F_n \backslash \{1\}$, and $y \in \mathbb{Q}\backslash\{0\}$, the relations

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g & x(1-g) \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{Q}, \tag{5.5}$$

and

$$\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} f & hy \\ 0 & 1 \end{pmatrix}, \qquad h \in F_n, \tag{5.6}$$

show that G is I.C.C. by varying x and h to get infinitely many distinct conjugates in each case. The group G has an abelian normal subgroup

$$K = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Q} \right\} \tag{5.7}$$

and the quotient G/K is isomorphic to the abelian subgroup H. Then G is an extension of an abelian group by an abelian group and hence is amenable, [9, p. 31]. It follows that $VN(G)$ is the hyperfinite type II_1 factor, [17]. For elements $\left(\begin{smallmatrix} f_i & x_i \\ 0 & 1 \end{smallmatrix}\right) \in G\backslash H$, $1 \le i \le k$, choose $h \in F_n$ such that $h \neq -\frac{x_i}{f_i x_j}$, $1 \le i, j \le n$, possible because $x_j \neq 0$. Then the identity

$$\begin{pmatrix} f_i & x_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_j & x_j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} f_i h f_j & f_i h x_j - x_i \\ 0 & 0 \end{pmatrix} \notin H \tag{5.8}$$

shows that (5.2) is satisfied, and so $VN(H)$ is a strongly singular masa in $VN(G)$. If $\left(\begin{smallmatrix} f & x \\ 0 & 1 \end{smallmatrix}\right) \in G\backslash H$, so that $x \neq 0$, then the identity

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} gfh & gx \\ 0 & 1 \end{pmatrix} \tag{5.9}$$

shows that $(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)) \in H \times H$ is the only stabilizing element for $\left(\begin{smallmatrix} f & x \\ 0 & 1 \end{smallmatrix}\right)$, and the hypothesis of Theorem 4.1 is met. Let $\{x_i F_n\}_{i=1}^n$ be a listing of the cosets of F_n in $\mathbb{Q}^\times$. Then the equation

$$\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} gf^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g & x_i f \\ 0 & 1 \end{pmatrix} \tag{5.10}$$

shows that the elements $\left(\begin{smallmatrix}1 & x_i\\ 0 & 1\end{smallmatrix}\right) \in G$, $1 \le i \le n$, generate n distinct equivalent double cosets, and $\mathrm{Puk}(VN(H)) = \{n\}$. $\square$

Our remaining examples, with one exception, are all subgroups of the invertible $k \times k$ upper triangular matrices T_k over $\mathbb{Q}$ which naturally form a tower by embedding $X \in T_k$ as $\left(\begin{smallmatrix}1 & 0\\ 0 & X\end{smallmatrix}\right) \in T_{k+1}$. Each T_k has an abelian normal subgroup K_k consisting of those matrices in T_k with 1's on the diagonal and 0's in all other positions except the first row. The quotient T_k/K_k is isomorphic to

$$\left\{\begin{pmatrix} f & 0\\ 0 & X\end{pmatrix} :\ f \in \mathbb{Q}^{\times},\ X \in T_{k-1}\right\} \tag{5.11}$$

and thus each T_k is amenable, by induction. The one exception, mentioned above, is a subgroup of $\bigcup_{k=1}^{\infty} T_k$, and so this is also amenable. We remark that Pukánszky's examples in [13] are 2×2 upper triangular matrices where the $(1, 2)$–entries are taken from fields which are unions of finite fields. His examples yield the hyperfinite factor directly since his groups are exhibited as unions of finite subgroups.

Example 5.2. Fix $k \in \mathbb{N}$ and let $S = \{n_1, \ldots, n_k\}$ be a set of elements of $\mathbb{N} \cup \{\infty\}$ with possible repeats. Let G be the group of $(k + 1) \times (k + 1)$ matrices of the form

$$\begin{pmatrix} 1 & x_1 & \ldots & x_k\\ & f_1 & & \\ & & \ddots & \\ & & & f_k\end{pmatrix}, \qquad x_i \in \mathbb{Q}, \quad f_i \in F_{n_i}, \tag{5.12}$$

and let H be the diagonal subgroup. Then $VN(H)$ is a strongly singular masa in $VN(G)$ with

$$\mathrm{Puk}(VN(H)) = \left\{\prod_{i\in\sigma} n_i :\ \sigma \subseteq \{1, \ldots, k\},\ \sigma \neq \emptyset\right\}. \tag{5.13}$$

Proof. An argument similar to that of the previous example establishes that $VN(G)$ is the hyperfinite factor and that $VN(H)$ is a strongly singular masa. The numbers in the Pukánszky invariant are determined by the numbers of nonzero entries in the first row of a particular group element. To avoid excessive complications we will discuss only the case

$k=2$ and $n_1, n_2 \in \mathbb{N}$; this contains all the ingredients of the general situation. Thus we fix integers m, n and let

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & f & 0 \\ 0 & 0 & g \end{pmatrix} : \ x, y \in \mathbb{Q}, \ f \in F_m, \ g \in F_n \right\} \tag{5.14}$$

with H the diagonal subgroup. If $\{x_i F_m\}_{i=1}^m$ and $\{y_j F_n\}_{j=1}^n$ are respectively the cosets of F_m and F_n in $\mathbb{Q}^\times$, then the double cosets of H in G are generated by three types of elements,

$$\begin{pmatrix} 1 & x_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & y_j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & x_i & y_j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.15}$$

with respectively m, n and mn of each type. In the same order, the stabilizer subgroups are

$$\left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g^{-1} \end{pmatrix} \right) : \ g \in F_n \right\}, \left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & f^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) : \ f \in F_m \right\},$$
$$\left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \right\} \tag{5.16}$$

and each pair is either equal or noncommensurable. Theorem 4.1 then gives that $\mathrm{Puk}(VN(H)) = \{m, n, mn\}$, and the general proof of (5.13) proceeds in the same way. To illustrate this result, take $m=2$, $n=3$ to get $\{2,3,6\}$ or take $m=2$, $n=2$ to get $\{2,4\}$.

We note that the results of this example could also be obtained by combining Example 5.1 with Theorem 2.1 and performing an induction argument. □

Example 5.3. This is a modification of the previous example where we now allow $k=\infty$. In the definition of G in (5.12) we allow $x_i \neq 0$ for only finitely many i's and $f_j \neq 1$ for only finitely many j's. Then G is still a countable amenable group. The result of (5.13) still holds, where σ is an arbitrary finite nonempty subset of $\mathbb{N}$. By letting S vary over the uncountably many infinite subsets of the primes, we then obtain an uncountable family of distinct Pukánszky invariants. □

Example 5.4. Fix three integers a, b, c and let

$$G = \left\{ \begin{pmatrix} f_1 & x & y \\ 0 & f_2 2^{am} & 0 \\ 0 & 0 & f_3 2^{bm+bcn} \end{pmatrix} : \ f_i \in F_\infty, \quad x, y \in \mathbb{Q}, \quad m, n \in \mathbb{Z} \right\} \tag{5.17}$$

with abelian diagonal subgroup H. There are three types of double cosets generated respectively by

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x, y \neq 0, \tag{5.18}$$

giving rise to three noncommensurable stabilizing subgroups of $H \times H$. For the first two types it is straightforward to see that there are respectively a and b distinct double cosets; representatives are

$$\begin{pmatrix} 1 & 2^i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad 1 \leq i \leq a, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 2^j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad 1 \leq j \leq b. \tag{5.19}$$

In the third case, it is clear that the elements

$$\begin{pmatrix} 1 & 2^i & 2^j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad i, j \in \mathbb{Z}, \tag{5.20}$$

account for all the possible double cosets HgH, but there is duplication. Two elements $\left(\begin{smallmatrix} 1 & 2^i & 2^j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 2^k & 2^\ell \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ are in the same double coset if and only if (i, j) and (k, ℓ) define the same element in $K = \mathbb{Z}^2/\{(am, bm + bcn) \colon \ m, n \in \mathbb{Z}\}$, so the number of distinct double cosets is the order of this quotient group. Let $\pi \colon \ \mathbb{Z}^2 \to K$ be the quotient homomorphism and note that K is generated by $\pi((1,0))$ and $\pi((0,1))$. By taking $m = 0$, $n = 1$, we see that $\pi((0,1))$ has order bc. By taking $m = 1$, $n = 0$, we see that $p\pi((1,0))$ is in the group generated by $\pi((0,1))$ for the first time when $p = a$. Thus the order of K is abc and $\mathrm{Puk}(VN(H)) = \{a, b, abc\}$.

Note that sets of this type do not appear from tensoring as in Theorem 2.1. $\square$

The above examples all satisfy the hypotheses of Theorem 4.1, so it is natural to ask whether they are automatically satisfied. We show that this is not so by exhibiting two stabilizing subgroups which are unequal but commensurable since both are finite groups.

Example 5.5. Let $F_n^+ = \{f \in F_n \colon\ f > 0\}$ and let $J = \{\pm 1\}$. Define

$$G = \left\{ \begin{pmatrix} f & x & y \\ 0 & j & 0 \\ 0 & 0 & k \end{pmatrix} :\ f \in F_n^+, \quad j, k \in J, \quad x, y \in \mathbb{Q} \right\} \tag{5.21}$$

with diagonal subgroup H. Then it is easy to see that stabilizing subgroups of $H \times H$ for $\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ are respectively

$$\left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k^{-1} \end{pmatrix} \right) :\ k \in J \right\}, \quad \left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) :\ j \in J \right\}, \tag{5.22}$$

and these are finite and unequal. $\square$

Our final example examines the situation of $H \subseteq G \subseteq K$, where $VN(H)$ is a masa in both $VN(G)$ and $VN(K)$. The two containing factors give rise to two Pukánszky invariants, and the question is whether they are related to one another; we answer this negatively. We let $\mathbb{Q}(\sqrt{2})$ denote the finite extension $\{r + s\sqrt{2} \colon\ r, s \in \mathbb{Q}\}$ of $\mathbb{Q}$.

Example 5.6. Let H and G be as in Example 5.1, where we showed that $\mathrm{Puk}(VN(H))$ (which we now denote by $\mathrm{Puk}(VN(H), VN(G))$ to indicate the ambient factor) is $\{n\}$. Let

$$K = \left\{ \begin{pmatrix} f & y \\ 0 & 1 \end{pmatrix} :\ f \in F_n,\ y \in \mathbb{Q}(\sqrt{2}) \right\}. \tag{5.23}$$

Arguing as in Example 5.1, we see that $VN(H)$ is a masa in $VN(K)$, and since

$$\left\{ \begin{pmatrix} 1 & m+\sqrt{2} \\ 0 & 1 \end{pmatrix} :\ m \in \mathbb{Z} \right\} \tag{5.24}$$

generate distinct double cosets, we easily obtain $\mathrm{Puk}(VN(H), VN(K)) = \{\infty\}$. Note that $VN(G)$ has trivial relative commutant in $VN(K)$, otherwise $VN(H)$ would not be a masa in $VN(K)$.

In this example, $VN(K)$ is hyperfinite because K is amenable. We now present one where $VN(K)$ fails to be hyperfinite, based on wreath products of groups. Let G and H be as above, except that we require the $(1,1)$ entries of the matrices to lie in F_n^+ rather than F_n. Let J be any countable nonamenable discrete I.C.C. group ($\mathbb{F}_2$ for example), and let

$$L = \{\phi\colon G \to J:\ \phi(g) = e_J \text{ except on a finite set}\}. \tag{5.25}$$

Then L is countable and we can define an action $\alpha\colon G \to \mathrm{Aut}(L)$ by

$$(\alpha_g\phi)(h) = \phi(gh), \quad g, h \in G. \tag{5.26}$$

Now let $K = L \rtimes_\alpha G$.

The elements of K are formal products ϕg with multiplication given by $g\phi = \alpha_g(\phi)g$. Since K contains a copy of L which in turn contains a copy of J, we see that K is nonamenable and we have $H \subseteq G \subseteq K$. If $\phi \in L$, $g \in G$ and $h \in H$, then $h\phi g h^{-1} = \alpha_h(\phi) h g h^{-1}$. If $g \notin H$, then $\{h\phi g h^{-1}:\ h \in H\}$ is infinite. On the other hand, if $g \in H$ and $\phi g \notin H$, then $\phi \neq e_L$. In this case there exists $g_0 \in G$ such that $\phi(g_0) \neq e_J$, and so $h\phi h^{-1} = \alpha_h(\phi)$, when evaluated at g_0, gives $\alpha_h(\phi)(g_0) = \phi(hg_0)$. Thus $\{\alpha_h(\phi):\ h \in H\}$ is infinite, otherwise there will be an infinite set of h's where $\phi(hg_0) \neq e_J$, contrary to the definition of L. Thus H meets the criterion for $VN(H)$ to be a masa in $VN(K)$. If, for $h_1, h_2 \in H$, we have $h_1\phi g h_2 = \phi g$, then $\alpha_{h_1}(\phi) h_1 g h_2 = \phi g$. This forces h_2 to be h_1^{-1}, and also that $\alpha_{h_1}(\phi) = \phi$, leading to

$$\phi(h_1^r g) = \phi(g), \quad g \in G,\ r \in \mathbb{Z}. \tag{5.27}$$

Since F_n^+ has no nontrivial finite subgroups, this relation shows that if $\phi \neq e_L$ then $h_1 = e_H$, and in all cases the stabilizer subgroup of ϕg is (e_H, e_H) when $\phi g \notin H$. It is then clear that

$\mathrm{Puk}(VN(H), VN(K)) = \{\infty\}$, while $\mathrm{Puk}(VN(H), VN(G)) = \{2n\}$, following the methods of Example 5.1. □

Remark 5.7. Although we do not know that an arbitrary subset of $\mathbb{N} \cup \{\infty\}$ can be the Pukánszky invariant of a masa in the hyperfinite factor, we suspect that this is so. However, in other group factors there are obstructions to achieving this. If a masa has a Pukánszky invariant consisting of a finite set of integers then $\mathcal{A}' = \sum_{i=1}^{k} C(\Omega_i) \otimes \mathbb{M}_{n_i}$ and $\mathcal{A} = \sum_{i=1}^{k} C(\Omega_i) \otimes I_{n_i}$. In this case $\mathcal{A}$ has a finite cyclic set of vectors. For the free group factors, this is ruled out by a result of Dykema [3], so in $VN(\mathbb{F}_n)$, $n \geq 2$, no finite set of integers appears as the Pukánszky invariant of a masa. □

Remark 5.8. Let M be a type II_1 factor, let $p \in M$ be a projection of trace $1/n$ for a fixed but arbitrary integer $n \geq 2$, and let $N = pMp$. Then $M = \mathbb{M}_n \otimes N$. Pick a masa $A \subseteq N$ and let $\mathbb{D}_n$ be the diagonal masa in $\mathbb{M}_n$, giving rise to a masa $B = \mathbb{D}_n \otimes A$ in M. We will show below that the von Neumann algebra $N(B)''$ generated by the normalizer of B is equal to $\mathbb{M}_n \otimes N(A)''$. This gives $N(B)'' = \mathbb{M}_n \otimes A$ when A is singular in N, so $\mathbb{D}_n \otimes A$ is never singular in M and $1 \in \mathrm{Puk}(\mathbb{D}_n \otimes A)$. We thank Ken Dykema for pointing out to us the following concrete examples of this phenomenon.

Let $\mathbb{F}_k$, $k \geq 2$, be the free group on k generators. Then $VN(\mathbb{F}_k)$ is isomorphic to $\mathbb{M}_n \otimes VN(\mathbb{F}_{n^2(k-1)+1})$ for $n \geq 1$, [20]. In particular, $VN(\mathbb{F}_2)$ is isomorphic to $\mathbb{M}_n \otimes VN(\mathbb{F}_{n^2+1})$, for each $n \geq 1$. There is a singular masa A_n in each $VN(\mathbb{F}_{n^2+1})$, corresponding to a choice of generator in $\mathbb{F}_{n^2+1}$, leading to a sequence $\{B_n = \mathbb{D}_n \otimes A_n\}_{n \geq 1}$ of masas whose normalizers generate pairwise nonisomorphic von Neumann algebras $\{\mathbb{M}_n \otimes A_n\}_{n \geq 1}$. The formula $\mathrm{Puk}(B_n) = \{1, \infty\}$, for $n \geq 1$, can be established as in Theorem 2.1.

We now justify the assertion, made above, that $N(\mathbb{D}_n \otimes A)''$ is $\mathbb{M}_n \otimes A$. The second algebra is clearly contained in the first, by considering normalizing unitaries of the form $u \otimes v$, so we must show that a unitary normalizer of $\mathbb{D}_n \otimes A$ lies in the second algebra. We give full details for $n = 2$, and then indicate how to obtain the general case.

Let $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in N(\mathbb{D}_2 \otimes A)$. Then there is a projection $\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right) \in \mathbb{D}_2 \otimes A$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}. \tag{5.28}$$

This leads to

$$x = xp, \quad zp = 0, \quad \text{and} \quad z(1-p) = z. \tag{5.29}$$

Since $x^*x + z^*z = 1$, we have $px^*xp + (1-p)z^*z(1-p) = 1$, showing that $px^*xp = p$ and that $x = xp$ is a partial isometry. For any $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \in \mathbb{D}_2 \otimes A$, there exists $\left(\begin{smallmatrix} c & 0 \\ 0 & d \end{smallmatrix}\right) \in \mathbb{D}_2 \otimes A$ such that

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \tag{5.30}$$

so $ax = xc$ and x is then in the normalizing groupoid of A, which generates $N(A)''$, (see [5, Lemma 2.1]). The same argument places y, z and w in $N(A)''$ by moving these elements to the $(1,1)$ position in the matrix. For example,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} z & w \\ x & y \end{pmatrix}, \tag{5.31}$$

and so $\left(\begin{smallmatrix} z & w \\ x & y \end{smallmatrix}\right)$ is also a normalizing unitary. The general case is then easily established from the following two observations:

(i) if the result is true for $\mathbb{D}_n \otimes A$ for a particular integer n then it is also true for all $k \leq n$;

(ii) the relation $\mathbb{D}_2 \otimes (\mathbb{D}_{2^n} \otimes A) \cong \mathbb{D}_{2^{n+1}} \otimes A$ allows us to prove the result by induction for the integers 2^n, $n \geq 1$. □

We end with some open problems arising from our work. Is every subset of $\mathbb{N} \cup \infty$ the Pukánszky invariant of some masa in the hyperfinite factor R? Does every singular masa A in $VN(\mathbb{F}_n)$, $n \geq 2$, have $\text{Puk}(A) = \{\infty\}$? In a property T factor, is $\text{Puk}(A)$ always a finite set, and is there only a countable set of possible invariants for all masas in such a factor?

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