Research Program

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Group actions are ubiquitous in mathematics. To understand a mathematical object, it is often helpful to understand its symmetries as expressed by a group. For example, a group acts on a ring by automorphisms, preserving its structure. Analogously, a Lie algebra acts on a ring by derivations. Unifying these two types of actions are Hopf algebras acting on rings. A Hopf algebra is not only an algebra, but also a coalgebra, and the notion of an action preserving the structure of a ring uses this structure. The category of representations of a Hopf algebra is correspondingly rich—it is a tensor category. Mathematicians studying tensor categories and their many applications are led to study Hopf algebras, and vice versa.

Hopf algebras can behave quite differently from groups and Lie algebras. Their representations can have a noncommutative flavor (due to asymmetry in the tensor product), making them more challenging to study. I take a homological and geometric approach to the representation theory of Hopf algebras, expanding what is known for finite groups and Lie algebras to accommodate noncommutativity. My research program involves collaborations with many mathematicians including postdocs and graduate students. Part of it targets understanding of Hopf algebras and their categories of representations, algebras on which they act, and related structures such as smash products (i.e. rings built out of Hopf algebras and rings on which they act) and their deformations. Most of my techniques are homological, involving Hopf algebra cohomology and Hochschild cohomology. Another part of my research program is directly aimed at some fundamental questions about cohomology for algebras or tensor categories more generally. My recent and current projects can be grouped into three areas as described below: support variety theory, cohomology of Hopf algebras, and algebraic deformations and structure of Hochschild cohomology.

Background

Let k be a field and $\otimes = \otimes_k$. A *Hopf algebra* is a k-algebra H with k-linear maps $\Delta : H \to H \otimes H$ (*coproduct*), $\varepsilon : H \to k$ (*counit* or *augmentation*), and $S : H \to H$ (*coinverse* or *antipode*) satisfying some properties, for example Δ and ε are algebra homomorphisms and S is an anti-algebra homomorphism. (See [17].) Hopf algebras first arose in algebraic topology, and have turned up in many other fields, such as representation theory, mathematical physics, and combinatorics.

Categories of (left) H-modules are *tensor categories* (see [5]): Given two H-modules M, N, their tensor product $M \otimes N$ (over the field k) is again an H-module via Δ.[1] We say that H is *cocommutative* if the coproduct Δ is symmetric, that is, if transposing tensor factors is the identity map. In this case, the tensor product of modules is commutative (up to isomorphism). This includes the cases that H is the group algebra kG of a group G (where $\Delta(g) = g \otimes g$ for all $g \in G$) or the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ (where $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$). Many Hopf algebras are not cocommutative yet have module categories with commutative (up to isomorphism) tensor product. This commutativity is typically functorial as in the case of quantum universal

[1] That is, $h \cdot (m \otimes n) = \sum (h_1 \cdot m) \otimes (h_2 \cdot n)$ for all $h \in H$, $m \in M$, $n \in N$, where $\Delta(h) = \sum h_1 \otimes h_2$ symbolically (Sweedler's notation).

enveloping algebras $U_q(\mathfrak{g})$, where there are operators on highest weight modules giving commutativity of tensor product up to isomorphism. Yet many other Hopf algebras do not have commutative tensor categories.

Hopf algebras act on other algebras, preserving their structure: An algebra A over k is a (left) *H-module algebra* if A is a (left) H-module and

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b) \quad \text{and} \quad h \cdot 1 = \varepsilon(h)1$$

for all $h \in H$ and $a, b \in A$. This generalizes both the notions of a group acting by automorphisms and of a Lie algebra acting by derivations. The *smash product* $A\#H$ is $A \otimes H$ as a vector space, with multiplication $(a \otimes h)(b \otimes \ell) = \sum a(h_1 \cdot b) \otimes h_2\ell$ for all $a, b \in A$ and $h, \ell \in H$. This smash product algebra encodes both the structure of A and of H as well as the action of H on A. In case H is a group algebra kG of a group G, the smash product is also called a *semidirect product* and denoted $A \rtimes G$. In that case the multiplication is simply given by $(a \otimes g)(b \otimes \ell) = a(g \cdot b) \otimes g\ell$ for all $a, b \in A$ and $g, \ell \in G$.

The field k is a (left) H-module via the augmentation ε. The *cohomology of H* is

$$\mathrm{H}^*(H, k) := \mathrm{Ext}^*_H(k, k).$$

This is a graded ring under cup product (equivalently, Yoneda composition) and in fact this cup product is graded commutative. When it is also finitely generated (as it is known to be for many classes of Hopf algebras discussed in Section 2 below) one may use it to study representations of H geometrically through their support varieties (as discussed in Section 1 below).

The *Hochschild cohomology* of an algebra A over the field k is

$$\mathrm{HH}^*(A) := \mathrm{Ext}^*_{A \otimes A^{op}}(A, A),$$

where A^{op} is the algebra A with opposite multiplication, and $A \otimes A^{op}$ acts on A (on the left) by left and right multiplication. Equivalently, $\mathrm{HH}^*(A)$ is the cohomology of the A-bimodule A. This Hochschild cohomology $\mathrm{HH}^*(A)$ is also graded commutative, and when A is a Hopf algebra, it contains a copy of the cohomology $\mathrm{H}^*(A, k)$ of A.

1. Representations and Support

One wants to understand representations—of groups, rings, and algebras—and their numerous connections to other parts of mathematics. This can be a difficult task with all possible outcomes seemingly incomplete: Categories of representations tend to be wild, with no hope of classifying indecomposable objects as one might first set out to do. A useful approach is to understand modules via some invariants.

The theory of "varieties for modules" introduced by Quillen [28] has been very successful, with many applications for finite groups (where it started) and myriad other settings. The setting of cocommutative Hopf algebras was developed in many papers by Friedlander, Pevtsova, and others. My research focuses on noncocommutative Hopf algebras and tensor categories where there remain many open questions. In work with Dave Benson [2] and with former postdoc Julia Plavnik [27], we constructed classes of examples where representations behave quite differently from how one expects when looking at more traditional categories of representations of finite group schemes or other symmetric categories. (See Theorem 2 below.) We are left wondering exactly what *is* true in general. Can we expect a satisfactory support variety theory with applications to understanding tensor categories of representations? Part of my research program aims at an answer to this question, beginning with [2, 27] as catalysts.

I will need some assumptions on the tensor categories, all of which will arise from representations of Hopf algebras: Let H be a finite dimensional Hopf algebra whose cohomology $\mathrm{H}^*(H,k)$ is finitely generated and for which $\mathrm{Ext}^*_H(M,M)$ is finitely generated as a module over $\mathrm{H}^*(H,k)$ for each finite dimensional (left) H-module M (under the left action given by cup product, equivalently, by $-\otimes M$ followed by Yoneda composition). There are many classes of examples of Hopf algebras H satisfying these hypotheses (see Section 2 below). Given a finite dimensional (left) H-module M, its *support variety* is

$$V(M) := \mathrm{Max}(\mathrm{H}^*(H,k)/I_M),$$

that is the maximal ideal spectrum of $\mathrm{H}^*(H,k)/I_M$, where I_M is the ideal of $\mathrm{H}^*(H,k)$ consisting of all elements that annihilate $\mathrm{Ext}^*_H(M,M)$. (This is simply the set of maximal ideals of $\mathrm{H}^*(H,k)/I_M$ considered as a topological space under the Zariski topology.) Depending on context, one may wish instead to consider prime ideals. Many times one also prefers first to restrict to the subalgebra of $\mathrm{H}^*(H,k)$ generated by all elements of even degree, which is strictly commutative rather than just graded commutative. These support varieties enjoy many useful properties, for example they behave well with respect to extensions of modules. If H is cocommutative, Friedlander and Pevtsova [7] proved that the variety of a tensor product is the intersection of varieties, that is

$$V(M\otimes N) = V(M)\cap V(N) \tag{1}$$

for all finite dimensional H-modules M, N. Yet this is not always true when H is noncocommutative. Not even containment holds in general, as my result with Benson states:

Theorem 2. [2] There are Hopf algebras H, and H-modules M, N, for which

$$V(M\otimes N) \not\subseteq V(M)\cap V(N).$$

There are nonprojective H-modules M for which $M^{\otimes n}$ is projective for some n, and H-modules M, N for which $M\otimes N$ is projective while $N\otimes M$ is not.

In the cocommutative setting, none of this can happen: Not only are $M\otimes N$ and $N\otimes M$ always isomorphic, but since the support variety measures the complexity of a module (that is, how far it is from being projective), one sees from the tensor product property (1) that $M^{\otimes n}$ is projective if, and only if, M is projective.

To prove Theorem 2, we explicitly constructed Hopf algebras from finite groups and their actions on other groups. The curious behavior of their modules stated Theorem 2 results from group actions. In [27] with Plavnik, we put these results in a more general context and provided further classes of examples, showing that *any* Hopf algebra for which the tensor product property (1) holds can be embedded as a subalgebra of a Hopf algebra for which it does not! This statement and related representation-theoretic consequences may seem surprising to those familiar with standard support variety theory.

There do exist noncommutative, noncocommutative Hopf algebras for which support varieties behave in a more traditional fashion: In work with Julia Pevtsova [26], we investigated the quantum elementary abelian groups. These are simply tensor products of several copies of Taft algebras: $H = T_\ell^{\otimes n}$ where T_ℓ is generated by g, x subject to relations $x^\ell = 0$, $g^\ell = 1$, $gx = qxg$, where q is a primitive ℓth root of unity in $\mathbb{C}$, $\Delta(g) = g \otimes g$, and $\Delta(x) = x \otimes g + 1 \otimes x$. The Taft algebra T_ℓ may be considered to be a Borel subalgebra $u_q(\mathfrak{sl}_2)^{\geq 0}$ of the small quantum group $u_q(\mathfrak{sl}_2)$, a finite dimensional quotient of $U_q(\mathfrak{sl}_2)$. We showed that the tensor product property (1) holds for these quantum elementary abelian groups H, and we classified thick tensor ideals in the stable category of finite dimensional H-modules. For proofs, we required another version of variety for modules, the "rank varieties," defined representation-theoretically in our joint paper [25]. These were the first examples of noncommutative, noncocommutative Hopf algebras for which the tensor product property (1) is known to hold. Again one is left wondering what is true in general. We pose the following:

Questions 3. What are conditions on a Hopf algebra H (or more generally on a tensor category) that will guarantee the tensor product property (1) holds? Does it always hold when the tensor product is commutative (up to isomorphism)? When it does not hold, what are some consequences for the structure of the tensor categories?

It is always true that $V(M \otimes N) \subseteq V(M)$: One may define the action of $\mathrm{H}^*(H, k)$ on $\mathrm{Ext}^*_H(M \otimes N, M \otimes N)$ by first tensoring with M, then with N. However if the tensor product of modules is noncommutative, it is not necessarily true that $V(M \otimes N) \subseteq V(N)$, as Theorem 2 shows. When the tensor product is commutative up to isomorphism, we do at least have containment, $V(M \otimes N) \subseteq V(M) \cap V(N)$.

In [3] with Petter Bergh and Julia Plavnik, we began a more comprehensive study of support varieties for finite tensor categories in what is planned to be the first of several papers exploring Questions 3 and further developing the general theory. For this and related work, we wish to understand also *when* there is a well-behaved support variety theory, and with this goal in mind, we are led to those classes of Hopf algebras for which the cohomology is known to be finitely generated. Such finite generation questions are the subject of the next section.

2. Cohomology of Hopf Algebras

More than 50 years ago, Golod, Venkov, and Evens proved that the cohomology of a finite group is finitely generated, laying groundwork for Quillen's geometric approach to group representation theory. This approach was later expanded and applied by many mathematicians with great success to group representations and beyond. One wants to do the same for Hopf algebras and tensor categories (see Section 1), but there is an obstacle: Mathematicians do not understand the structure of Hopf algebras or of tensor categories well enough in general. Many have asked if the following conjecture is true:

Conjecture 4. If H is a finite dimensional Hopf algebra, then $\mathrm{H}^*(H, k)$ is finitely generated.

More generally, Etingof and Ostrik [6] stated this conjecture for finite tensor categories; this more general form of the conjecture will not be considered here. Conjecture 4 is known to be true for cocommutative Hopf algebras (Friedlander and Suslin [8]), for small quantum groups (Ginzburg and Kumar [11]), for finite quantum function algebras

(Gordon [12]), some pointed Hopf algebras generalizing the small quantum groups (my paper with Mastnak, Pevtsova, and Schauenburg [16], further generalized in my paper with Andruskiewitsch, Angiono, and Pevtsova [1]), and many other classes of Hopf algebras. Snashall and Solberg [40] made a related conjecture about Hochschild cohomology of finite dimensional algebras to which Xu [46] found a counterexample. By contrast, there is no known counterexample nor proof of Conjecture 4.

As explained in Section 1, those Hopf algebras for which Conjecture 4 is true are thereby endowed with a potentially powerful tool, namely a well behaved support variety theory, for understanding their representations. Of course for such applications we want not only to know that cohomology is finitely generated, but also to understand its structure. With collaborators, I have worked to prove Conjecture 4 for some classes of Hopf algebras while at the same time unveiling enough of the structure of cohomology as to apply it effectively. In addition to [1, 16], some other steps I have taken in this direction are my papers [23] with former PhD student Van Nguyen and [21, 22] with Nguyen and Xingting Wang. Our results provide further evidence for Conjecture 4. However there is clearly much more to do, and I am continuing to work with collaborators on the conjecture as well as its representation-theoretic implications.

3. Algebraic Deformations and Hochschild Cohomology

A deformation of an algebra is another algebra that is very similar to it. Many algebras are obtained as deformations of simpler algebras. For example, if A is a graded algebra with an action of a Hopf algebra H that preserves the grading, one might be interested in a *Poincaré-Birkhoff-Witt (PBW) deformation* of the smash product $A\#H$, that is, a filtered algebra whose associated graded algebra is $A\#H$ (we give elements of H the degree 0). In case $A = S(V)$ is a symmetric algebra (i.e. polynomial ring) and $H = kG$ is a finite group algebra, PBW deformations of $A\#kG$ were considered by Drinfeld, and later by other mathematicians, and termed Drinfeld (graded) Hecke algebras, symplectic reflection algebras, or rational Cherednik algebras, depending on the context. Other classes of algebras A and Hopf algebras H have also appeared in a number of places. In [36] with Anne Shepler, we surveyed techniques for understanding the structure of such deformations, beginning with the classical Poincaré-Birkhoff-Witt Theorem (in which $H = k$ and $A = S(V)$). For some specific types of algebras A and Hopf algebras H, we propose:

Problem 5. Understand PBW deformations of $A\#H$.

In work with then MIT postdoc Chelsea Walton [43, 44], we considered a general context containing many known classes of examples: Let $A = T(V)/(R)$ be a Koszul algebra, where V is an H-module and $R \subset V \otimes V$ is an H-submodule, that is, A is a homogeneous quadratic algebra that is an H-module algebra, and the A-module k has a linear free resolution (a Koszul resolution). Let κ be a k-linear map from R to $H \oplus (V \otimes H)$ and

$$\mathcal{H}_\kappa := T(V)\#H/\left(r - \kappa(r) \mid r \in R\right). \tag{6}$$

If $\kappa \equiv 0$, then $\mathcal{H}_\kappa \cong A\#H$. In general, we gave in [43] necessary and sufficient conditions for $\mathcal{H}_\kappa$ to be a PBW deformation of $A\#H$. In our general setting, we use the Koszul resolution of A to construct a larger resolution for $A\#H$, accounting for the possible nonsemisimplicity of H. (The case that H is a nonsemisimple group algebra kG is my joint work with Shepler [34].) We gave many examples, some old and some new, and in [44] we considered actions on pairs of module algebras to capture and generalize some double constructions in the literature. There is still much to do to understand various PBW deformations and their representations.

I have further work on Problem 5 in the special case $H = kG$, a group algebra. When $A = S_{\mathbf{q}}(V)$, a skew polynomial ring (or quantum symmetric algebra), in work with former graduate student Piyush Shroff [39], we examined PBW deformations termed quantum Drinfeld orbifold algebras, building on earlier work of Shroff [38] and giving it a homological interpretation. I co-led, with Anne Shepler, a research team at the Women in Noncommutative Algebra and Representation Theory Workshop at Banff in 2016, and we worked on connections to color Lie algebras, resulting in the paper [9] with Sian Fryer, Tina Kanstrup, and Ellen Kirkman.

When $A = S(V)$ and $H = kG$, a paper with Anne Shepler [35] highlights differences in the modular setting, that is when the characteristic of k divides the order of G. We see a completely new phenomenon: When $A = S(V)$ and $H = \mathbb{C}G$, Ram and Shepler [29] showed that algebras of the form $\mathcal{H}_\kappa$ (see (6)) defined by Drinfeld (deforming the symmetric algebra relations) and those of a slightly different form defined by Lusztig for Weyl groups (deforming the group action) are isomorphic. In the modular setting, this does not always happen. Shepler and I found PBW deformations of $S(V)\#kG$ for which both the symmetric algebra relations and the group action are deformed. This happens already in the smallest such example, where a cyclic group G acts nonsemisimply on a vector space V of dimension 2. In fact we see the possibility of such deformations in our explicit resolution for realizing Hochschild cohomology of $S(V)\#kG$, and the resolution also provides a proof that the deformations are all distinct. This resolution is essentially a tensor product of the Koszul resolution of $S(V)$ with the bar (or other) resolution of kG, suitably modified, inspiring a general theory of resolutions for twisted tensor products that I developed with Shepler in [37]. (In characteristic 0, in retrospect, it is clear from the much smaller resolution valid in that case that the deformations of Drinfeld and Lusztig had to be isomorphic.) We developed a general theory of PBW deformations of $S(V)\#kG$ in the modular setting, tying them in to the structure of Hochschild cohomology as we understood it from our resolution. We continue to seek a better understanding of the cohomology that informs the deformation theory:

Problem 7. In the modular setting, understand Hochschild cohomology $\mathrm{HH}^*(S(V)\#kG)$ as an algebra and as a graded Lie algebra. Apply this knowledge to understand better the deformations of $S(V)\#kG$.

The graded Lie structure of Hochschild cohomology, defined by Gerstenhaber [10], governs the possible deformations of an algebra. In particular, the "infinitesimal" of a deformation is a Hochschild 2-cocycle with square bracket zero. Shepler and I wrote several papers on the case of characteristic 0 ([30, 31, 32, 33]). A general theory of Hochschild cohomology of algebras and the graded Lie structure can be found in my book [45].

More generally for questions in algebraic deformation theory and more, one wants to solve the following problem for various types of algebras A:

Problem 8. Understand the graded Lie structure of Hochschild cohomology $\mathrm{HH}^*(A)$.

This graded Lie structure is defined on the bar complex of A. In contrast, cup products may be defined on any projective resolution, and one chooses an efficient resolution in order to compute them. The bar complex is useful theoretically, but it is not suitable for computation, and can obscure important information about an algebra since it is so large.

In order to compute and understand the graded Lie structure on Hochschild cohomology, historically, one was forced to translate it laboriously from the bar complex to a favorite efficient complex by means of chain maps, and it still seems necessary to do this in many settings. In work with collaborators, I have been developing better techniques. In [19] with then MIT postdoc Cris Negron, we gave a method for computing the graded Lie structure directly on a choice of resolution satisfying some properties, of which Koszul resolutions are examples. In [20] with Negron, we used these techniques to prove a theoretical result: Gerstenhaber brackets on the Hochschild cohomology of a skew group algebra formed from a polynomial ring and a finite group in characteristic 0 can always be expressed in terms of Schouten brackets on polyvector fields. In [13] with my former graduate students Lauren Grimley and Van Nguyen, we used these techniques from [19] to give some computational results for some classes of examples. Yury Volkov [41] generalized these techniques, removing all restrictions on the resolutions. In [18] with Negron and Volkov, we put these techniques into a larger context by showing there is an A_∞-structure behind them, and in [42] with Volkov, we generalized some of the theory to some exact monoidal categories. Further work of mine on improving techniques for understanding Gerstenhaber brackets, as well as using them to obtain further theoretical and computational results, are in [4, 14, 15, 24], written with University of Utah postdoc Benjamin Briggs and with my recently graduated PhD students Dustin McPhate, Tekin Karadag, Pablo Ocal, and Tolulope Oke.

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