

Infinite Combinatorics and Applications to Banach space
Theory
Course Notes: Math 663-601

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September 10, 2023

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In this course we discuss several results on Infinite Combinatorics, and their applications to Banach space theory.

In the first chapter we present Ramsey's theorem for infinite subsets of the natural numbers. Assume you color all infinite subsequences of the natural numbers \mathbb{N} using finitely many colors. Then, under some mild topological condition on this coloring, Ramsey's theorem states that you can find a subsequence N of \mathbb{N} , so that all further subsequences M of N have the same color. In Chapter 2 we will apply Ramsey's theorem to obtain several results in Banach space theory, for example Haskell Rosenthal's celebrated ℓ_1 Theorem: Every semi normalized sequence in a Banach space either contains a weak Cauchy subsequence or it contains a subsequence which is equivalent to the ℓ_1 unit vector basis.

In view of Ramsey's theorem one might ask whether or not there exist versions of it for Banach spaces of the following kind. Assume you color the vectors of the sphere of a separable, infinite dimensional space X using finitely many colors. Is it possible, under certain conditions on the coloring, to conclude that X has an infinite dimensional Banach space whose sphere is monochromatic? The answer to this question is for most spaces negative, and we will present several examples which illustrate this in chapter 3. Nevertheless, parts of Ramsey's theorem still hold and we will present the following two examples: Gowers's dichotomy theorem, and its application to solve the homogenous Banach space problem, as well as a combinatorial result, recently obtained by the author in collaboration with E. Odell, and its application to several universality problems.

In chapter 4 we will give a short introduction to ordinal numbers, the arithmetic on them, and the principle of transfinite induction and recursion. We will use them to introduce in chapter 5 several Banach indices, which are isomorphic invariances of Banach spaces and therefore important tools to classify them.

Chapter 1

The Theorem of Ramsey

1.1 Ramsey's theorem for finite sequences

We begin with Ramsey's original theorem [Ra]. He was only 27 when he died in 1929 and this paper appeared the following year.

Notation. For two sets X and S we denote by X^S the set of all functions $f : S \rightarrow X$, or equivalently the set of all families $(x_i)_{i \in S} \subset X$. The *powerset of X* , i.e. the set of all subsets of X , is denoted by $\mathcal{P}(X)$.

Assume α is a cardinal number. At the moment we only worry about the cases $\alpha = n \in \mathbb{N}$, $\alpha = \omega$ (the countable infinite cardinal, which can be identified with the set of all natural numbers \mathbb{N}), $\alpha = \omega_1$ (the smallest uncountable cardinal) and $\alpha = \omega_c$ (the continuum). We denote by $[X]^\alpha$ the set of all subsets of X of cardinality α . $[X]^{<\alpha}$ and $[X]^{\leq\alpha}$ are the set of all subsets of X whose cardinality is less than, respectively at most α . In the case $X = \mathbb{N}$ or $X \subset \mathbb{N}$ or any other well ordered set we will identify $[X]^n$ and X^ω with the set of increasing sequences X .

Similarly $X^{<\alpha}$ and $X^{\leq\alpha}$ is the set of all families indexed over cardinalities smaller than α , respectively smaller then or equal to α .

In particular

- X^n , and X^ω , is the set of all sequences in X of cardinality n or all infinite sequences, and $X^{<\omega}$ the set of all finite sequences in X .
- $[X]^n$ and $[X]^\omega$, the set of all subsets of cardinality n , respectively ω or the set of all increasing sequences of cardinality n or ω , respectively.

Theorem 1.1.1. [Ra] *Let $k \in \mathbb{N}$ and $\mathcal{A} \subseteq [\mathbb{N}]^k$. Then there exists $M \in [\mathbb{N}]^\omega$ so that either $[M]^k \subseteq \mathcal{A}$ or $[M]^k \cap \mathcal{A} = \emptyset$.*

Remark. Note that the case $k = 1$ is simply the *Pigeon hole Principle*.

Proof of Theorem 1.1.1. We give first the proof for $k = 2$. Let $n_1 = 1$. Choose $M_1 \in [\mathbb{N}]^\omega$ with $1 < \min(M_1)$ so that

- a) either $(n_1, n) \in \mathcal{A}$ for all $n \in M_1$, or

b) or $(n_1, n) \notin \mathcal{A}$ for all $n \in M_1$.

Let $n_2 = \min M_1$ and choose $M_2 \in [M_1]^\omega$ so that $n_2 < M_2$ and either

a) $(n_2, n) \in \mathcal{A}$ for all $n \in M_2$, or

b) $(n_2, n) \notin \mathcal{A}$ for all $n \in M_2$.

Let $n_3 = \min M_2$ and continue in this manner. The alternative a) or b) must occur infinitely many times. Suppose $M = \{n_i : \text{a) holds for } n_i\}$ is infinite. Then $[M]^2 \subseteq \mathcal{A}$. Otherwise $M = \{n_i : \text{b) holds for } n_i\}$ satisfies $[M]^2 \cap \mathcal{A} = \emptyset$.

Assume now the claim of our theorem to be true for $k - 1$, $k > 2$, and let $\mathcal{A} \subset [\mathbb{N}]^k$.

Using our induction hypothesis we can choose by induction on $\ell \in \mathbb{N}$ numbers $n_1 < n_2 < \dots$ in \mathbb{N} and sets $\mathbb{N} = M_0 \supset M_1 \supset M_2 \dots$, so that for all $\ell \in \mathbb{N}$

$$n_\ell = \min M_{\ell-1} \text{ and } n_\ell < \min M_\ell$$

$$\text{Either (Case 1) : } \forall (m_1, m_2 \dots m_k) \in [M_\ell]^{k-1} \quad (n_\ell, m_1, m_2, \dots, m_{k-1}) \in \mathcal{A}$$

$$\text{Or (Case 2) : } \forall (m_1, m_2 \dots m_k) \in [M_\ell]^{k-1} \quad (n_\ell, m_1, m_2, \dots, m_{k-1}) \notin \mathcal{A}$$

Then take infinite subsequence (n_{ℓ_i}) so that for all $i \in \mathbb{N}$ the same case happens and choose $M = \{n_{\ell_i} : i \in \mathbb{N}\}$ $i \in \mathbb{N}$. \square

Definition 1.1.2. By an r -coloring of a set A we mean a partition $(A_i)_{i=1}^r$ of A into r subsets. In that case we call $B \subset A$ *monochromatic* if for some $i \leq r$, $B \subset A_i$.

Corollary 1.1.3. For any $r \in \mathbb{N}$ and any r coloring of $[\mathbb{N}]^k$ there is an $M \in [\mathbb{N}]^\omega$ so that $[M]^k$ is monochromatic.

Using the *Compactness Principle* we can deduce the following strengthening of Theorem 1.1.1.

Corollary 1.1.4. Given k and m in \mathbb{N} there is an $n = n(k, m) \geq m$ so that the following holds. Assume that $\mathcal{A} \subset [\{1, 2, 3, \dots, n\}]^k$ then there exists $M \subset \{1, 2, 3, \dots, n\}$ with $\#M = m$ so that

$$\text{Either } [M]^k \subset \mathcal{A} \text{ or } [M]^k \cap \mathcal{A} = \emptyset.$$

Remark. For $k = 2$ Corollary 1.1.3 means the following: Given any number m there is a number $n \geq m$ so that: If ones invites any n people there at least m of them who either know each other or who do not know each other.

Proof of Corollary 1.1.4. Assume that for some k and m in \mathbb{N} our claim is wrong. Thus, assume that for any $n \in \mathbb{N}$, $n \geq m$ there is an $\mathcal{A}_n \subset [\{1, 2, \dots, n\}]^k$ so that

$$(*) \quad \forall M \subset [\{1, 2, \dots, n\}]^m \quad [M]^k \not\subset \mathcal{A}_n \text{ and } [M]^k \cap \mathcal{A}_n \neq \emptyset.$$

By induction choose $n_1 < n_2 < \dots$ and infinite sets $\mathbb{N} = N_0 \supset N_1 \supset N_2 \supset \dots$ so that for $\ell \in \mathbb{N}$

$$n_\ell = \min N_{\ell-1} < \min N_\ell$$

$$\forall n, n' \in \mathbb{N}_\ell \quad \mathcal{A}_n \cap [\{n_1, n_2, \dots, n_\ell\}]^k = \mathcal{A}_{n'} \cap [\{n_1, n_2, \dots, n_\ell\}]^k$$

In order to choose N_ℓ we can use the Pigeonhole Principle since for any $n \in N_{\ell-1}$ the set $\mathcal{A}_n \cap [\{n_1, n_2, \dots, n_\ell\}]^k$ is an element of the (finite) set $\mathcal{P}([\{n_1, n_2, \dots, n_\ell\}]^k)$.

Then put $N = \{n_i : i \in \mathbb{N}\}$ and let

$$\mathcal{A} = \bigcup_{\ell=1}^{\infty} \mathcal{A}_{n_\ell}.$$

Note that for $A \in [N]^k$

$$A \in \mathcal{A} \iff \exists \ell \in \mathbb{N} \quad A \in \mathcal{A}_{n_\ell} \iff \forall \ell \in \mathbb{N}, \max A \leq n_\ell \quad A \in \mathcal{A}_{n_\ell},$$

and, thus, for $\ell \in \mathbb{N}$ it follows that $\mathcal{A} \cap [n_1, \dots, n_\ell]^k = \mathcal{A}_{n_\ell}$.

By Theorem 1.1.1 we can now choose an $L \in [N]^\omega$ so that $[L]^k \subset \mathcal{A}$ or $[L]^k \cap \mathcal{A}$. Write L as

$$L = \{n_{\ell_1}, n_{\ell_2}, \dots\},$$

then note that either

$$[\{n_{\ell_1}, n_{\ell_2}, \dots, n_{\ell_m}\}]^k \subset \mathcal{A} \cap [\{1, 2, \dots, \dots, n_{\ell_m}\}]^k = \mathcal{A}_{n_{\ell_m}},$$

or

$$[\{n_{\ell_1}, n_{\ell_2}, \dots, n_{\ell_m}\}]^k \cap \mathcal{A} = [\{n_{\ell_1}, n_{\ell_2}, \dots, n_{\ell_m}\}]^k \cap \mathcal{A}_{n_{\ell_m}} = \emptyset$$

which is a contradiction to our assumption. \square

Exercise 1.1.5. For $m \in \mathbb{N}$ let $R(m)$ (sometimes also denoted by $R(2, m)$) be the minimum of all $n \in \mathbb{N}$ so that the conclusion of Corollary 1.1.3 holds, i.e so that for all $\mathcal{A} \subset [\{1, 2, 3, \dots, n\}]^k$ there exists $M \subset \{1, 2, 3, \dots, n\}$ with $\#M = m$ so that either $[M]^k \subset \mathcal{A}$ or $[M]^k \cap \mathcal{A} = \emptyset$. Show that $R(3) = 6$.

Remark. $R(4) = 18$ (hard, possible only by using computer), $R(5)$ is unknown. It is only known that $R(5) \in [43, 49]$.

Erdős: "If Aliens land on earth and threaten the to eliminate all human beings if they do not provide $R(5)$, then, probably combining all the avialable computing power, we probably would be able to compute $R(5)$. If they want to know $R(6)$ we can as well start preying" (This was about 30 years ago, maybe nowadays we have enough computing power to find $R(6)$)

1.2 Infinite Games

Let X be a non empty set and let $\mathcal{A} \subset X^\omega$ we consider the following game $G(\mathcal{A}, X)$.

Player I: chooses $x_1 \in X$

Player II: chooses $x_2 \in X$

Player I: chooses $x_3 \in X$

\vdots

Player I has won if the resulting sequence (x_n) is in \mathcal{A} . Otherwise Player II is declared winner.

Remark. We will use above setup in the case where Player II chooses in his/her moves certain subsets, say S_1, S_2 etc. of S (like infinite subsets of \mathbb{N} , or closed infinite dimensional subspaces of a Banach space X) and Player II chooses elements s_i out of the set S_i (for example \mathbb{N} , or a Banach space). Player I has won if he can assure that the resulting sequence (s_i) lies in some set $\mathcal{A} \subset S^\omega$.

In order to reduce that game to the above described game we simply put

$X = S \times \mathcal{P}$, where $\mathcal{P} \subset \mathcal{P}(X)$ are the allowed choices for Player I

$$\tilde{\mathcal{A}} = \{(s_i, S_i) \in X^\omega : (s_{2i}) \in \mathcal{A} \text{ and } \forall i \in \mathbb{N} \quad s_{2i} \in S_{2i-1}\}.$$

Exercise 1.2.1. Assume S is a set and $\mathcal{A} \subset S^\omega$ is closed with respect to the product of the discrete topology. Let $\mathcal{P} \subset \mathcal{P}(X)$

Then show that $\tilde{\mathcal{A}}$, as defined above is closed in the product topology of $(S \times \mathcal{P})^\omega$.

Let $\mathcal{A} \subset X^\omega$, X some non empty set. For Player I having a winning strategy for $G(\mathcal{A}, X)$ means informally that

$$\exists x_1 \in X \quad \forall x_2 \in X \quad \exists x_3 \in X \quad \forall x_4 \in X \dots \quad (x_i) \in \mathcal{A}.$$

Given that this is an *infinite phrase* one might want to be careful.

This is the formal definition:

Definition 1.2.2. Let $\mathcal{A} \subset [X]^\omega$, $X \neq \emptyset$. We say that *Player I has a winning strategy for $G(\mathcal{A}, X)$* if

($W_I(\mathcal{A}, X)$) There is a sequence of functions $(f_n)_{n=0}^\infty$ with $f_n : X^n \rightarrow X$ ($f_0 \in X$) so that for any sequence $(z_n)_{n=1}^\infty$ the sequence (x_n) defined by

$$x_{2n-1} = f_{n-1}(z_1, z_2, \dots, z_{n-1}) \text{ and } x_{2n} = z_n \text{ whenever } n \in \mathbb{N}$$

is in \mathcal{A} .

We say that *Player II has a winning strategy for $G(\mathcal{A}, X)$* if

($W_{II}(\mathcal{A}, X)$) There is a sequence of functions $(g_n)_{n=1}^\infty$ with $g_n : X^n \rightarrow X$ so that for any sequence $(z_n)_{n=1}^\infty$ the sequence (x_n) defined by

$$x_{2n-1} = z_n \text{ and } x_{2n} = f_n(z_1, z_2, \dots, z_n) \text{ whenever } n \in \mathbb{N}$$

is not in \mathcal{A} .

In the case that ($W_I(\mathcal{A}, X)$) or ($W_{II}(\mathcal{A}, X)$) holds we call $(f_n)_{n=0}^\infty$, respectively $(g_n)_{n=1}^\infty$ a *winning strategy for Player I*, respectively *Player II*.

Remark. It is easy to see that $(W_I(\mathcal{A}, X))$ and $(W_{II}(\mathcal{A}, X))$ cannot both hold. Indeed, assuming that maps $f_n : X^n \rightarrow X$, $n \in \mathbb{N}_0$, and $g_n : X^n \rightarrow X$, $n \in \mathbb{N}$, existed as in $(W_I(\mathcal{A}, X))$ and $(W_{II}(\mathcal{A}, X))$, we could choose by induction a sequence (x_n) as follows

$$x_1 = f_0 \text{ and } x_2 = g_1(x_1)$$

assuming x_1, x_2, \dots, x_{2n} have been chosen for some $n \in \mathbb{N}$, we put

$$x_{2n+1} = f_n(x_2, x_4, \dots, x_{2n}) \text{ and } x_{2n+2} = g_n(x_1, x_3, \dots, x_{2n+1}).$$

Then (x_n) satisfies the conditions in $(W_I(\mathcal{A}, X))$ and as well as in $(W_{II}(\mathcal{A}, X))$ which leads to a contradiction since that would imply that $(x_n) \in \mathcal{A}$ as well as $(x_n) \notin \mathcal{A}$.

It is not clear whether either $(W_I(\mathcal{A}, X))$ or $(W_{II}(\mathcal{A}, X))$ have to hold, i.e. whether or not the game is determined.

In the case that the game has finitely many steps, i.e. if there is an $n \in \mathbb{N}$, say n even, so that $\mathcal{A} \subset X^n$ then $(W_I(\mathcal{A}, X))$ is equivalent to

$$\exists x_1 \in X \quad \forall x_2 \in X \quad \exists x_3 \in X \quad \forall x_4 \in X \dots \exists x_{n-1} \in X \quad \forall x_n \in X \quad (x_i)_{i=1}^n \in \mathcal{A}.$$

and $(W_{II}(\mathcal{A}, X))$ is equivalent to

$$\forall x_1 \in X \quad \exists x_2 \in X \quad \forall x_3 \in X \quad \forall x_4 \in X \dots \forall x_{n-1} \in X \quad \exists x_n \in X \quad (x_i)_{i=1}^n \notin \mathcal{A}.$$

Since the negation can be pulled through finitely many quantifiers (replacing \exists by \forall and vice versa) we deduce that

$$\neg (W_I(\mathcal{A}, X)) \Rightarrow (W_{II}(\mathcal{A}, X)) \quad \neg (W_{II}(\mathcal{A}, X)) \Rightarrow (W_I(\mathcal{A}, X))$$

and, thus, that games of finite length are determined. In the case of infinite games the situation is different: one cannot always pull the negation sign through an infinite phrase. A good analogue to this problem in Logic is the fact known from Calculus that infinite sums $\frac{\partial}{\partial}$ and \sum do not always commute. The determinacy of infinite games will follow under some topological assumption on the set \mathcal{A} .

Notation. For $\mathcal{A} \subset X^\omega$ and $x = (x_1, x_2, \dots, x_n) \in X^{<\omega}$ we write:

$$\mathcal{A}(x_1, x_2, \dots, x_n) = \{(z_i) \in X^\omega : (x_1, x_2, \dots, x_n, z_1, z_2, \dots) \in \mathcal{A}\}.$$

We consider on X^ω the product topology of the discrete topology on X .

Proposition 1.2.3. *Assume $\mathcal{A} \subset X^\omega$. Then*

$$\mathcal{A} \text{ is open} \iff \exists \tilde{\mathcal{A}} \subset X^{<\omega} \quad \mathcal{A} = \bigcup_{(x_1, x_2, \dots, x_n) \in \tilde{\mathcal{A}}} \{(x_1, x_2, \dots, x_n)\} \times X^\omega.$$

$$\mathcal{A} \text{ is closed} \iff \forall (x_i) \in X^\omega [\forall n \in \mathbb{N} \quad \mathcal{A}(x_1, x_2, \dots, x_n) \neq \emptyset \Rightarrow (x_i) \in \mathcal{A}].$$

Theorem 1.2.4. [Ma] *If $\mathcal{A} \subset X^\omega$ is a Borel set then the game $G(\mathcal{A}, X)$ is determined,*

Proof. We will show Theorem 1.2.4 only in the case that \mathcal{A} is either closed or open. The following observations are intuitively clear and can easily be shown using the formal definitions of $(W_I(\mathcal{A}, X))$ and $(W_{II}(\mathcal{A}, X))$. For $\mathcal{A} \subset X^\omega$

$$(1.1) \quad (W_I(\mathcal{A}, X)) \iff \exists x \in X \forall y \in X \quad (W_I(\mathcal{A}(x, y), X))$$

$$(1.2) \quad (W_{II}(\mathcal{A}, X)) \iff \forall x \in X \exists y \in X \quad (W_{II}(\mathcal{A}(x, y), X))$$

Secondly,

$$(1.3) \quad \begin{aligned} \neg(W_{II}(\mathcal{A}, X)) &\iff \exists x \in X \forall y \in X \quad \neg(W_{II}(\mathcal{A}(x, y), X)) \\ &\implies \exists x \in X \forall y \in X \quad \mathcal{A}(x, y) \neq \emptyset \end{aligned}$$

$$(1.4) \quad \begin{aligned} \neg(W_I(\mathcal{A}, X)) &\iff \forall x \in X \exists y \in X \quad \neg(W_I(\mathcal{A}(x, y), X)) \\ &\implies \exists x \in X \forall y \in X \quad \mathcal{A}(x, y) \neq \emptyset \end{aligned}$$

Assume that $\neg(W_{II}(\mathcal{A}, X))$ holds. By induction on $n \in \mathbb{N}_0$ we can choose $f_n : X^n \rightarrow X$ so that

$$(1.5) \quad \begin{aligned} \forall (z_1, z_2, \dots, z_n, z_{n+1}) \in X^{n+1} \\ \neg(W_{II}(\mathcal{A}(f_0, z_1, f_1(z_1), z_2, f_2(z_1, z_2), \dots, f_n(z_1, z_2, \dots, z_n, z_{n+1}), X))) \end{aligned}$$

For $n = 0$ this follows from the assumption $\neg(W_{II}(\mathcal{A}, X))$. Assuming f_0, f_1, \dots, f_{n-1} have been chosen, we can apply for a given $(z_1, z_2, \dots, z_n) \in X^n$ our induction hypothesis to the game

$$G(\mathcal{A}(f_0, z_1, f_1(z_1), z_2, f_2(z_1, z_2), \dots, z_n), X)$$

and then apply (1.3) to the set

$$\mathcal{A}(f_0, z_1, f_1(z_1), z_2, \dots, f_{n-1}(z_1, z_2, \dots, z_{n-1}), z_n)$$

in order to get a y , which we define to be $f_n(z_1, \dots, z_n)$, so that,

$$(W_{II}(\mathcal{A}(f_0, z_1, f_1(z_1), z_2, f_2(z_1, z_2), \dots, f_n(z_1, z_2, \dots, z_n, z_{n+1}), X))) \text{ for all } z_{n+1} \in X.$$

This finishes the induction step.

We claim that if \mathcal{A} is closed (f_n) is a winning strategy for Player I. Indeed, let $(z_i) \in X^\omega$. From (1.3) (second implication) and (1.5) we deduce that for all $n \in \mathbb{N}$ the set $\mathcal{A}(f_0, z_1, f_1(z_1), z_2, \dots, z_{n+1})$ is not empty. Proposition (1.2.3) yields that the infinite sequence $(f_0, z_1, f_1(z_1), z_2, f_2(z_1, z_2), z_3, \dots)$ is in \mathcal{A} .

If \mathcal{A} is open we will assume that $\neg(W_I(\mathcal{A}, X))$ and prove in a similar way that $(W_{II}(\mathcal{A}, X))$ (see Exercise 1.2.5)

□

Exercise 1.2.5. Let $\mathcal{A} \subset X^\omega$ be open. Show that $\neg(W_I(\mathcal{A}, X))$ implies $(W_{II}(\mathcal{A}, X))$.

1.3 Ramsey's theorem for infinite sequences

Proposition 1.3.1. *Let X be a Polish space (completely metrizable and separable). For $A \subset X$ the following statements are equivalent.*

- A is the image of a Borel set $B \subset X \times X$ under the projection onto, say, the first coordinate.*
- There is a Polish space Y so that A is the image of a Borel set $B \subset X \times Y$ under the projection onto the first coordinate.*
- There is a continuous map $f : \mathbb{N}^\omega \rightarrow X$ (using the product of the discrete topology) whose image is A .*
- A is the image of a Borel set $B \subset X \times \mathbb{N}^\omega$ under the projection onto the first coordinate.*
- For every uncountable Polish space Y there is a G_δ -set (intersection of countably many open sets) $B \in X \times Y$ so that A is the image of B under the projection onto X .*

We call a set which satisfies above conditions analytic and we call the complement of an analytic set co-analytic.

Remark. The set of analytic subsets of a Polish space is not closed under complementation, in particular it is not a σ -algebra. Nevertheless it is easy to see that analytic sets are closed under countable unions.

We identify $\mathcal{P}(\mathbb{N})$ with the product $\{0, 1\}^\omega$ by

$$N \in [\mathbb{N}]^\omega \longleftrightarrow (\varepsilon_n), \text{ where } \varepsilon_n = \begin{cases} 1 & \text{if } n \in N \\ 0 & \text{if } n \notin N. \end{cases}$$

Therefore $\mathcal{P}(\mathbb{N})$ can be endowed with the product topology of the discrete topology on $\{0, 1\}$, which makes it a compact metric, and thus, Polish space, and we note that

$$\begin{aligned} \mathcal{U} &= \{ \{F \cup N : N \in \mathcal{P}(\mathbb{N}), \max F < \min N \text{ or } N = \emptyset\} : F \in [\mathbb{N}]^{<\omega} \} \\ &\equiv \{ \{(\varepsilon_i) \subset \{0, 1\} : \varepsilon_i = 1 \text{ if } i \in F \text{ and } \varepsilon_i = 0 \text{ if } i \leq \max F \text{ and } i \notin F\} : F \in [\mathbb{N}]^{<\omega} \} \end{aligned}$$

is a basis of the topology on $\mathcal{P}(\mathbb{N})$.

For $N_n, n < \omega$, and N in $\mathcal{P}(\mathbb{N})$,

$$N_n \rightarrow_{n \rightarrow \infty} N \iff \forall k \in \mathbb{N} \exists n_0 \in \mathbb{N} \forall n \geq n_0 \quad N_n \cap \{1, 2, 3 \dots k\} = N \cap \{1, 2, 3 \dots k\}.$$

On $[\mathbb{N}]^\omega$ we consider the relative topology. $[\mathbb{N}]^\omega$ is actually a dense \mathcal{G}_δ set in $\mathcal{P}(\mathbb{N})$, and therefore it is a Polish space itself.

Exercise 1.3.2. For $B \in [\mathbb{N}]^{<\omega}$ and $N \in \mathcal{P}(\mathbb{N})$ we say that N extends B if

$$B = N \cap \{1, 2, \dots, \max B\}.$$

Show that $\mathcal{A} \subset [\mathbb{N}]^\omega$ is clopen if and only if there is a $k \in \mathbb{N}$ and a $\mathcal{B} \subset [\mathbb{N}]^k$ so that

$$\mathcal{A} = \{N \in [\mathbb{N}]^\omega : \exists B \in \mathcal{B} \quad N \text{ extends } B\}.$$

We can now state the *Ramsey theorem for subsets of $[\mathbb{N}]^\omega$* .

Theorem 1.3.3. *Assume that $\mathcal{A} \subset [\mathbb{N}]^\omega$ is analytic (with respect to the product topology on $[\mathbb{N}]^\omega \equiv \{0, 1\}^\omega$).*

Then there is an $N \in [\mathbb{N}]^\omega$ so that either $[N]^\omega \subset \mathcal{A}$ or $[N]^\omega \cap \mathcal{A} = \emptyset$.

Remark. Note that from Exercise 1.3.2 it follows that Ramsey's theorem for finite sets, i.e. Theorem 1.1.1, coincides with the statement of Theorem 1.3.3 for clopen sets $\mathcal{A} \subset [\mathbb{N}]^\omega$.

Theorem 1.3.3 was shown by Ellentuck in 1974 [El]. For Borel sets it was proved by Galvin and Priky in 1973 [GP].

We will prove Theorem 1.3.3 first in the special case that $\mathcal{A} \subset [\mathbb{N}]^\omega$ is closed and then for all Borel sets.

Theorem 1.3.4. *Assume $\mathcal{A} \subset [\mathbb{N}]^\omega$ is closed.*

If there is an $N \in [\mathbb{N}]^\omega$ so that $[M]^\omega \cap \mathcal{A} \neq \emptyset$, for all $M \in [N]^\omega$, then there is an $N' \in [N]^\omega$ so that $[N']^\omega \subset \mathcal{A}$.

It is clear that Theorem 1.3.4 implies Theorem 1.3.3 in the case that \mathcal{A} is closed. For the proof we will need the following Lemma. Similar to the previous section we define for $F \in [\mathbb{N}]^{<\omega}$ and $\mathcal{A} \subset [\mathbb{N}]^\omega$

$$\mathcal{A}(F) = \{N \in [\mathbb{N}]^\omega : \max F < \min N \text{ and } F \cup N \in \mathcal{A}\}.$$

Lemma 1.3.5. *Let $\mathcal{A} \subset [\mathbb{N}]^\omega$ and $N \in [\mathbb{N}]^\omega$ and assume that for all $M \in [N]^\omega$ it follows that $\mathcal{A} \cap [M]^\omega \neq \emptyset$.*

Then for all $N' \in [N]^\omega$ there exists an $L \in [N']^\omega$ so that $\mathcal{A}(\{\ell\}) \cap [M]^\omega \neq \emptyset$ for all $\ell \in L$ and all $M \in [L]^\omega$.

Proof. Assume our claim were not true for some $N' \in [N]^\omega$. Then we could inductively choose $L_0 = N' \supset L_1 \supset L_2 \dots$ and $\ell_1 < \ell_2 < \dots$ so that $\ell_k < \min L_k$, $\ell_k \in L_{k-1}$ and $\mathcal{A}(\{\ell_k\}) \cap [L_k]^\omega = \emptyset$ for all $k \in \mathbb{N}$. But this would imply that $\mathcal{A} \cap [M]^\omega = \emptyset$ for $M = \{\ell_1, \ell_2, \dots\}$. \square

Proof of Theorem 1.3.4. Assume that there is an $N \in [\mathbb{N}]^\omega$ so that for all $M \in [N]^\omega$ there is an $L \in \mathcal{A} \cap [M]^\omega$. We need to show that there is an $N' \in [N]^\omega$ so that $[N']^\omega \subset \mathcal{A}$. We will choose recursively $N_0 = N \supset N_1 \supset N_2 \supset \dots$ and $n_1 < n_2 < \dots$ with $n_k \in N_{k-1}$, if $k \geq 1$, and so that for all $k \geq 0$, all $M \in [N_k]^\omega$ and all $F \subset \{n_1, \dots, n_k\}$ (if $k = 0$ only $F = \emptyset$ is possible)

$$\mathcal{A}(F) \cap [M]^\omega \neq \emptyset.$$

For $k = 0$, this is exactly our assumption. Assume that we have chosen $n_1 < n_2 < \dots, n_k$ and N_k . We order all subsets of $\{n_1, \dots, n_k\}$ into F_1, F_2, \dots, F_{2^k} and apply Lemma 1.3.5 successively to $\mathcal{B} = \mathcal{A}(F_i)$, $i = 1, 2, \dots, 2^k$ in order to get $L_0 = N_k \supset L_1 \supset L_2 \dots \supset L_{2^k}$ so that for all $i \leq 2^k$ and all $\ell \in L_i$ and all $M \in [L_i]^\omega$ the set $(\mathcal{A}(F_i))(\{\ell\}) \cap [M]^\omega = (\mathcal{A}(F_i \cup \{\ell\})) \cap [M]^\omega$ is not empty. Finally we choose $n_{k+1} = \min L_{2^k}$ and $N_{k+1} = L_{2^k} \setminus \{n_{k+1}\}$, which finishes our induction step.

Putting now $N' = \{n_1, n_2, \dots\}$ it follows that for all finite $F \subset N'$ the set $\mathcal{A}(F)$ is not empty. Since \mathcal{A} is closed we deduce from Proposition 1.2.3 that $[N']^\omega \subset \mathcal{A}$. \square

Later we want to extend results similar to Theorems 1.3.3 and 1.3.4 to Banach spaces and their closed subspaces. Unfortunately only part of the reasonings are transferable. We want therefor reinterpret the statements of Theorems 1.3.3 and 1.3.4.

We consider the following two person game of infinite length. Let $\mathcal{A} \subset [\mathbb{N}]^\omega$ be closed.

Player I : chooses $N_1 \in [\mathbb{N}]^\omega$

Player II: chooses $n_1 \in N_1$

Player I : chooses $N_2 \in [\mathbb{N}]^\omega$

Player II: chooses $n_2 \in N_2$

⋮

Player I wins if $\{n_1, n_2, \dots\} \in \mathcal{A}$

It follows from Theorem 1.2.4 that if \mathcal{A} Borel, then the game is determined i.e. one of the Players has a winning strategy.

We call above game $G(\mathcal{A})$. If $N \in [\mathbb{N}]^\omega$ and we demand that Player I can only choose subsets of N we call it the restriction of the $G(\mathcal{A})$ to N and denote it by $G(\mathcal{A}, N)$

Write $W_I(\mathcal{A}, N)$ ($W_I(\mathcal{A})$ if $N = \mathbb{N}$) if Player I has a winning strategy for the restriction of \mathcal{A} to N and we write $W_{II}(\mathcal{A}, N)$ ($W_{II}(\mathcal{A})$ if $N = \mathbb{N}$) if Player II has a winning strategy (which means that $\{n_1, n_2, \dots\} \notin \mathcal{A}$).

Remark. Using the formal definition (or our intuition) of $W_I(\mathcal{A}, N)$ and $W_{II}(\mathcal{A}, N)$ for $N \in [\mathbb{N}]^\omega$, it is clear that $W_I(\mathcal{A}, N)$ implies $\mathcal{A} \cap [N]^\omega \neq \emptyset$ and $W_{II}(\mathcal{A}, N)$ implies $\mathcal{A}^c \cap [N]^\omega \neq \emptyset$.

Therefore Theorem yields the following

Corollary 1.3.6. *Let $\mathcal{A} \subset [\mathbb{N}]^\omega$ be closed.*

a) *Either \mathcal{A} satisfies the condition*

$$\exists N \in [\mathbb{N}]^\omega \forall M \in [N]^\omega \quad W_I(\mathcal{A}, M),$$

then it follows

$$\exists N' \in [\mathbb{N}]^\omega \quad [N']^\omega \subset \mathcal{A}.$$

b) *Or \mathcal{A} satisfies*

$$\forall N \in [\mathbb{N}]^\omega \exists M \in [N]^\omega \quad W_{II}(\mathcal{A}, M)$$

which implies

$$\forall N \in [\mathbb{N}]^\omega \exists M \in [N]^\omega \quad \mathcal{A} \cap [M]^\omega = \emptyset.$$

Remark. Note that the first part of Corollary 1.3.6 means the following: If Player I has a winning strategy he/she has actually a very simple strategy: at step k Player I chooses $N' \cap \{n_{k-1} + 1, n_{k-1} + 2, \dots\}$, where n_{k-1} was the choice of Player II at step $k - 1$.

The second part can be interpreted as follows. If for all $N \in [\mathbb{N}]^\omega$ there is an M so that Player II has a winning strategy for the (\mathcal{A}, M) -game, then for any $N \in [\mathbb{N}]^\omega$ Player II can find an $M \in [N]^\omega$ so that in the (\mathcal{A}, M) -game he can each step choose any element out of the set Player I suggests, and win the game.

Proof of Corollary 1.3.6. (a) follows immediately from Theorem 1.3.4 and above mentioned remark.

(b) can be deduced as follows:

$$\begin{aligned}
\forall N \in [\mathbb{N}]^\omega \exists M \in [N]^\omega \quad W_{II}(\mathcal{A}, M) &\implies \forall N \in [\mathbb{N}]^\omega \exists M \in [N]^\omega \quad \mathcal{A}^c \cap [M]^\omega \neq \emptyset \\
&\iff \neg(\exists N \in [\mathbb{N}]^\omega \quad [N]^\omega \subset \mathcal{A}) \\
&\implies \neg(\exists N \in [\mathbb{N}]^\omega \forall M \in [N]^\omega \quad [M]^\omega \cap \mathcal{A} \neq \emptyset) \\
&\quad \text{[Using Theorem 1.3.4]} \\
&\iff \forall N \in [\mathbb{N}]^\omega \exists M \in [N]^\omega \quad [M]^\omega \cap \mathcal{A} = \emptyset.
\end{aligned}$$

□

We now want to show that the conclusion of Theorem 1.3.3 holds for all Borel sets in $[\mathbb{N}]^\omega$.

Definition 1.3.7. We call a set $\mathcal{A} \subset [\mathbb{N}]^\omega$ *Ramsey* if

$$(R) \quad \exists M \in [\mathbb{N}]^\omega \quad [M]^\omega \subset \mathcal{A} \text{ or } [M]^\omega \cap \mathcal{A} = \emptyset.$$

We say that \mathcal{A} is *hereditary Ramsey*

$$(RH) \quad \forall N \in [\mathbb{N}]^\omega \forall F \in [N]^{<\omega} \exists M \in [N]^\omega \quad [M]^\omega \subset \mathcal{A}(F) \text{ or } [M]^\omega \cap \mathcal{A}(F) = \emptyset.$$

Remark. If \mathcal{A} is closed, $N \in [\mathbb{N}]^\omega$ and $F \in [N]^{<\omega}$ then it follows that $\mathcal{A}(F) \cap [N]^\omega$ is also closed in $[N]^\omega$ and therefore it follows from Theorem 1.3.4 that \mathcal{A} is hereditary Ramsey.

Theorem 1.3.8. *The set of all subsets of $[\mathbb{N}]^\omega$ which are hereditary Ramsey forms a σ algebra. In particular, using above remark, all Borel sets on $[\mathbb{N}]^\omega$ are hereditary Ramsey.*

Proof of Theorem 1.3.8. Let \mathcal{R}_h be the subsets of $[\mathbb{N}]^\omega$ which are hereditary Ramsey. It is clear that \mathcal{R}_h is closed under taking complements. In order to show that it is closed under finite intersections let \mathcal{A} and \mathcal{B} be in \mathcal{R}_h and let $N \in [\mathbb{N}]^\omega$ and $F \in [N]^{<\omega}$. Then, by using (RH) for \mathcal{A} , there is an $M' \in [N]^\omega$ so that $[M']^\omega \subset \mathcal{A}(F)$ or $\mathcal{A}(F) \cap [M']^\omega = \emptyset$. Secondly we apply (RH) to \mathcal{B} and M' instead of N and find and $M \in [M']^\omega$ so that $[M]^\omega \subset \mathcal{B}(F)$ or $\mathcal{B}(F) \cap [M]^\omega = \emptyset$. Note that M still satisfies the same alternative with respect to $\mathcal{A}(F)$ as M' did. If M satisfies the first alternative with respect to $\mathcal{A}(F)$ as well as $\mathcal{B}(F)$ then, of course $[M]^\omega \subset \mathcal{A}(F) \cap \mathcal{B}(F) = (\mathcal{A} \cap \mathcal{B})(F)$. In all the other cases we deduce $[M]^\omega \cap (\mathcal{A} \cap \mathcal{B})(F) = \emptyset$. This shows that \mathcal{R}_h is an algebra and we deduce that in particular all open sets in $[\mathbb{N}]^\omega$ are hereditary Ramsey.

Therefore it is left to show that \mathcal{R}_h is closed under taking countable intersections.

Assume $\mathcal{A}_n \in \mathcal{R}_h$, for $n \in \mathbb{N}$, and let $F \in [N]^{<\omega}$ and $N \in [\mathbb{N}]^\omega$. We need to show that there is an $M \in [N]^\omega$ so that $[M]^\omega \subset \bigcap_{n \in \mathbb{N}} \mathcal{A}_n(F)$ or $[M]^\omega \cap \bigcap_{n \in \mathbb{N}} \mathcal{A}_n(F) = \emptyset$. We can assume that $\mathcal{A}_1 \supset \mathcal{A}_2 \dots$, otherwise replace \mathcal{A}_i by $\bigcap_{j \leq i} \mathcal{A}_j$. We also can assume that $F = \emptyset$, other wise replace \mathcal{A}_i by $\mathcal{A}_i(F)$.

Using our assumption that $\mathcal{A}_n \in \mathcal{R}_h$ we can choose by induction $k_1 < k_2 < \dots$ and infinite sets $N = L_0 \supset L_1 \supset L_2 \supset \dots$ so that for all $n \in \mathbb{N}$, $k_n = \min L_{n-1}$

$$\forall F \subset \{k_1, k_2, \dots, k_n\} \text{ either } [L_n]^\omega \subset \mathcal{A}_n(F) \text{ or } [L_n]^\omega \cap \mathcal{A}_n(F) = \emptyset.$$

Define $K = \{k_1, k_2, \dots\}$. Since the \mathcal{A}_n 's are decreasing we deduce that for each $F \in [K]^{<\omega}$, one and only one of the following alternatives can hold (let $i \in \mathbb{N}$ so that $k_i = \max F$):

$$\begin{aligned} \text{Either: (A1)} \quad & \forall n \geq i \quad [L_n]^\omega \subset \mathcal{A}_n(F) \\ \text{Or: (A2)} \quad & \exists n(F) \geq i \forall n \geq n(F) \quad [L_n]^\omega \cap \mathcal{A}_n(F) = \emptyset. \end{aligned}$$

We pass to a further subsequence $L = \{k_{\ell_1}, k_{\ell_2}, \dots\}$ of K so that the following holds:

$$(1.6) \quad \forall n \in \mathbb{N} \quad \ell_{n+1} \geq 1 + \max \{n(F) : F \subset \{k_{\ell_1}, k_{\ell_2}, \dots, k_{\ell_n}\}, F \text{ is (A2)}\}$$

We define \mathcal{F} to be all $F \in [L]^{<\omega}$ for which (A1) holds and put

$$\mathcal{B} = \bigcup_{F \in [L]^{<\omega} \setminus \mathcal{F}} \{F \cup M : M \in [\mathbb{N}]^\omega \text{ with } \min M > \max F\}.$$

\mathcal{B} is open in $[\mathbb{N}]^\omega$ and therefore we can choose an $M \in [L]^\omega$ so that either $[M]^\omega \subset \mathcal{B}$ or $[M]^\omega \cap \mathcal{B} = \emptyset$.

In the first case we can write any $M' \in [M]^\omega$ as

$$M' = \{k_{i(1)}, k_{i(2)}, \dots\}, \text{ with } i(1) < i(2) < i(3) \dots, \text{ in } \{\ell_1, \ell_2, \dots\}$$

and since $M' \in \mathcal{B}$ we find an initial segment F of M' , i.e. either $F = \emptyset$ or $F = \{k_{i(1)}, k_{i(2)}, \dots, k_{i(n)}\}$ for some $n \in \mathbb{N}$, so that $F \notin \mathcal{F}$ and, thus, by (1.6) $[L_{i(n+1)-1}]^\omega \cap \mathcal{A}_{i(n+1)-1}(F) = \emptyset$ which implies (recall that $M' \setminus F \in [L_{i(n+1)-1}]$) that $M' \notin \bigcap_n \mathcal{A}_n$. Since $M' \in [M]^\omega$ was arbitrary we deduce that $[M]^\omega \cap \bigcap_n \mathcal{A}_n = \emptyset$.

In the second case, let $M' \in [M]^\omega$ and write again M' as

$$M' = \{k_{i(1)}, k_{i(2)}, \dots\}, \text{ with } i(1) < i(2) < i(3) \dots \text{ in } \{\ell_1, \ell_2, \dots\}.$$

Let $n \in \mathbb{N}$ be arbitrary. We want to show that $M' \in \mathcal{A}_n$. In order to do so choose $j \in \mathbb{N}$ minimal so that $i(j) > n$, and let $F = \{k_{i(1)}, k_{i(2)}, \dots, k_{i(j-1)}\}$ (F could be empty). Since $M' \notin \mathcal{B}$ every initial segment of M' has to be in \mathcal{F} , in particular $F \in \mathcal{F}$, which implies that $M' \setminus F \in [L_n]^\omega \subset \mathcal{A}_n(F)$, and thus $M' \in \mathcal{A}_n$. Since $M' \in [M]^\omega$ and $n \in \mathbb{N}$ were arbitrary we deduce that $[M]^\omega \subset \bigcap_n \mathcal{A}_n$. \square

Remark. An example by Dan Freeman shows that the set \mathcal{R} of subsets of $[\mathbb{N}]^\omega$ which are Ramsey is not closed under taking finitely many intersections:

Let $\mathcal{A} \subset [\mathbb{N}]^\omega$ which is not Ramsey (in Example we will show that such an \mathcal{A} exists). Then take: $\mathcal{B}_1 = [2\mathbb{N}]^\omega \cup \mathcal{A}$ and $\mathcal{B}_2 = [2\mathbb{N} + 1]^\omega \cup \mathcal{A}$, then clearly \mathcal{B}_1 and \mathcal{B}_2 are Ramsey, but $\mathcal{A} = \mathcal{B}_1 \cap \mathcal{B}_2$ is not.

The Author of these notes does not know whether or not the set of all \mathcal{A} which satisfies the following *weak hereditary Ramsey* property (RWH) is a σ -algebra. \mathcal{A} is said to be *weak hereditary Ramsey* if

$$(RWH) \quad \forall N \in [\mathbb{N}]^\omega \exists M \in [M]^\omega \quad [M]^\omega \subset \mathcal{A} \text{ or } [M]^\omega \cap \mathcal{A} = \emptyset.$$

The next example shows that some conditions on $\mathcal{A} \subset [\mathbb{N}]^\omega$ are needed in order to derive the conclusion in Ramsey's Theorem for infinite sets. We will need transfinite recursion to do so. We denote the cardinal number of the continuum by ω_c and we consider on ω_c an order $<$ which has the property that for $\alpha \in \omega_c$ the set $\{\beta < \alpha\}$ is of strictly smaller cardinality than ω_c . Often instead of $\alpha \in \omega_c$ we write $\alpha < \omega_c$. Simply extend the wellorder of ω_c to $\omega_c \cup \{\omega_c\}$ and put and extend the well order onto $\omega_c \cup \{\omega_c\}$ by demanding that $\alpha < \omega_c$ for all $\alpha \in \omega_c$.

Remark. Let S be any set and let α be the cardinality of S . Then one can think of a well ordering of S as being a bijective map $\alpha \rightarrow S$, or equivalently as a family $(s_\beta)_{\beta < \alpha}$ with $s_\beta \neq s_\gamma$ for $\beta < \gamma < \alpha$, and $S = \{s_\beta : \beta < \alpha\}$.

Example 1.3.9. Let $(M_\alpha)_{\alpha < \omega_c}$ be a well-ordering of $[\mathbb{N}]^\omega$. By transfinite induction we will choose for each $\alpha < \omega_c$ a partition of M_α into to sets $M_\alpha^{(1)}$ and $M_\alpha^{(2)}$ so that

$$M_\alpha^{(1)}, M_\alpha^{(2)} \notin \{M_\beta^{(1)}, M_\beta^{(2)} : \beta < \alpha\}.$$

Indeed, assuming we have chosen $M_\beta^{(1)}$ and $M_\beta^{(2)}$ for all $\beta < \alpha$, it follows that the cardinality of $\{M_\beta^{(1)}, M_\beta^{(2)} : \beta < \alpha\}$ is strictly less than ω_c , while the cardinality of all partitions of M_α is ω_c .

Then we let

$$\mathcal{A} := \{M_\alpha^{(1)} : \alpha < \omega_c\}.$$

Let $M \in [\mathbb{N}]^\omega$, say $M = M_\alpha$, $\alpha < \omega_c$. Since, by choice, $M_\alpha^{(2)} \notin \mathcal{A}$ it follows that $[M]^\omega \not\subset \mathcal{A}$. Since $M_\alpha^{(1)} \in \mathcal{A}$ it also follows that $[M]^\omega \cap \mathcal{A} \neq \emptyset$.

Chapter 2

Application of Ramsey's theorem to Banach spaces

2.1 Bases of Banach spaces

Convention: All of our Banach spaces are considered to be vector spaces over \mathbb{R} .

Notation. If X is a Banach space with norm $\|\cdot\|$.

$$B_X = \{x \in X : \|x\| \leq 1\} \text{unit ball of } X$$

$$S_X = \{x \in X : \|x\| = 1\} \text{sphere of } X$$

$$X^* = \{f : X \rightarrow \mathbb{R} : f \text{ is linear and bounded}\} \text{dual space of } X$$

For $A \subset X$ we denote the linear span of A by $\text{span}(A)$ and the closed linear span by $\overline{\text{span}(A)}$.

If Y is a closed linear subspace of a Banach space X we write $Y \hookrightarrow X$.

Special spaces:

For $1 \leq p < \infty$

$$\ell_p = \{x = (\xi) \subset \mathbb{R} : \sum_{i=1}^{\infty} |\xi|^p < \infty\} \text{ with } \|x\|_p = \left(\sum_{i=1}^{\infty} |\xi|^p \right)^{1/p},$$

$$\ell_{\infty} = \{x = (\xi) \subset \mathbb{R} : \|x\|_{\infty} = \sup_{n \in \mathbb{N}} |\xi_n|\},$$

$$c_0 = \{x = (\xi) \in \ell_{\infty} : \lim_{n \rightarrow \infty} \xi_n = 0\},$$

For a measure space (ω, Σ, μ)

$$L_p(\mu) = \{f : \Omega \rightarrow \mathbb{R} \text{ measurable} \mid \int |f|^p d\mu < \infty\} \text{ with } \|f\|_p = \left(\int |f|^p d\mu \right)^{1/p},$$

$$L_{\infty}(\mu) = \{f : \Omega \rightarrow \mathbb{R} \text{ mble} \mid \|f\|_{\text{ess sup}} = \sup\{r > 0 : \mu(\{|f| \geq r\}) > 0\} < \infty\},$$

For a compact space K

$$C(K) = \{f : K \rightarrow \mathbb{R} \text{ continuous}\}, \text{ with } \|f\|_{\infty} = \sup_{\xi \in K} |f(\xi)|.$$

Definition 2.1.1. (Schauder basis)

Let X be a Banach space over \mathbb{R} . A sequence (x_i) is called (*Schauder*) *basis* of X if for every $x \in X$ there is a unique sequence $(a_i) \subset \mathbb{R}$ so that $x = \sum_{i=1}^{\infty} a_i x_i$.

We call $(x_i) \subset X$ *basic sequence* if it is a basis of its closed linear span.

Proposition 2.1.2. For $(x_i) \subset X$ be a Schauder basis. Define for $n \in \mathbb{N}$

$$x_n^* : X \rightarrow \mathbb{R}, \quad x = \sum a_i x_i \mapsto a_i$$

(we call the x_n^* the coordinate functionals with respect to (x_n)). and define

$$P_n : X \rightarrow X, \quad x \mapsto \left(\sum_{i=1}^n x_i \otimes x_i^* \right)(x) = \sum_{i=1}^n x_i \otimes x_i^*(x).$$

Then it follows

a) $x_n^* \in X^*$ for all $n \in \mathbb{N}$.

b) $C_b = \sup_{n \in \mathbb{N}} \|P_n\| < \infty$.

C is called the basis constant of (x_i) . We call (x_n) a *monotone basis* if $C = 1$.

Sketch of Proof: For $x = \sum_{i \in \mathbb{N}} a_i x_i \in X$ define

$$\| \|x\| \| = \sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^n a_i x_i \right\|.$$

Using the Principle of Uniform Boundedness it follows that $\| \| \cdot \| \|$ is an equivalent norm on X .

We then observe that

$$\sup_{n \in \mathbb{N}} \|P_n\| < \infty = \sup_{x \in B_X} \| \|x\| \| < \infty.$$

□

The converse of Proposition 2.1.2:

Proposition 2.1.3. Assume that $(x_i) \subset X$ is linear independent and let f_i the linear (but not necessarily bounded) coordinate functionals defined on the vector space $\text{span}(x_i : i \in \mathbb{N})$. For $n \in \mathbb{N}$ put

$$P_n = \sum_{i=1}^n x_i \otimes f_i : \text{span}(x_i : i \in \mathbb{N}) \rightarrow \text{span}(x_i : i \leq n), \quad x \mapsto \sum_{i=1}^n x_i \otimes f_i(x).$$

If P_n is a bounded operator for every n and $C = \sup \|P_n\| < \infty$, then (x_i) is a *basic sequence*.

Remark. Enflo [En] showed that there are separable infinite dimensional Banach spaces without a Schauder basis nevertheless the following result is easy to show

Proposition 2.1.4. *If X is an infinite dimensional Banach space. Then X contains a basic sequence (x_n) and if (y_n) is a given normalized weakly null sequence (x_n) can be chosen to be a subsequence of a given weakly null sequence.*

Moreover, in both cases, (x_n) can be chosen so that the basis constant is arbitrary close to 1.

Definition 2.1.5. We denote the vector space of all sequences $(a_i) \subset c_0$ which eventually vanish by c_{00} . We denote its usual unit vector space-basis by (e_i) , i.e.

$$e_i = (\underbrace{0, 0, \dots, 0}_{(i-1) \text{ times}}, 1, 0, 0, \dots).$$

For $x = \sum a_i e_i \in c_{00}$ we call

$$\text{supp}(x) = \{i \in \mathbb{N} : a_i \neq 0\}$$

the support of x .

Remark. Assume that (x_i) is a basis of X then it is easy to see that its *normalization* (e_i) with $e_i = x_i/\|x_i\|$, for $i \in \mathbb{N}$, is also a basis. Let e_i^* the coordinate functionals to (e_i) .

For $x = \sum a_i e_i \in X$ it follows that $\lim_{i \rightarrow \infty} a_i = 0$ and thus we can think of X being the completion of the vector space c_{00} under some norm $\|\cdot\|$.

Definition 2.1.6. Let (x_n) be a basis of a Banach space X . We call

$$C_p = \sup_{m \leq n} \|P_n - P_m\| = \sup_{m \leq n} \left\{ \left\| \sum_{i=m+1}^n a_i x_i \right\| : x = \sum_{i=1}^{\infty} a_i x_i \in B_X \right\},$$

the projection constant, and note that $C_p \leq 2C_b$. We call (x_n) *bimonotone* if $C_p = 1$.

Note that by putting

$$\|x\| = \sup_{m \leq n} \|(P_n - P_m)(x)\|,$$

we can always renorm X equivalently so that (x_i) becomes a bimonotone basis.

Definition 2.1.7. Let (x_n) and (y_n) be two basic sequences and $C \geq 1$. We say that (x_n) *C-dominates* (y_n) or that (y_n) is *C-dominated by* (x_n) if

$$\left\| \sum a_i y_i \right\| \leq C \left\| \sum a_i x_i \right\| \text{ for all } (a_i) \in c_{00}.$$

We say that (x_n) and (y_n) are *C-equivalent* if (x_n) C-dominates (y_n) and (y_n) C-dominates (x_n) i.e. if

$$\frac{1}{C} \left\| \sum a_i x_i \right\| \leq \left\| \sum a_i y_i \right\| \leq C \left\| \sum a_i x_i \right\| \text{ for all } (a_i) \in c_{00}.$$

Proposition 2.1.8. *For a sequence (z_n) in a Banach space Z and $z \in Z$ the following are equivalent:*

- a) *There is a $z \in Z$ so that for all $\varepsilon > 0$ there is a finite $F \subset \mathbb{N}$ so that $\|z - \sum_{n \in F'} z_n\| < \varepsilon$ whenever $F' \subset \mathbb{N}$ is finite with $F \subset F'$.*

- b) There is a z so that for every bijection $\pi : \mathbb{N} \rightarrow \mathbb{N}$ the series $\sum_{i=1}^{\infty} x_{\pi(i)}$ converges to $z \in Z$.
- c) For every $(\delta_i) \in \{-1, 1\}^{\omega}$ the series $\sum_{i=1}^{\infty} \delta_i x_i$ converges.
- d) For every $N \subset [N]^{\omega}$ the series $\sum_{i \in N} \delta_i x_i$ converges.

Definition 2.1.9. A basic sequence (x_n) is called *unconditional basis* if for all $x = \sum_{i=1}^{\infty} a_i x_i$ the series is unconditionally converging.

Proposition 2.1.10. Let (x_n) be a linearly independent sequence in a Banach space X and (x_n^*) its coordinate functionals defined on $\text{span}(x_i)$. For $A \subset \mathbb{N}$ finite define:

$$P_A : \text{span}(x_i : i \in \mathbb{N}) \rightarrow \text{span}(x_i : i \in A) \ni \sum_{i \in \mathbb{N}} a_i x_i \mapsto \sum_{i \in A} a_i x_i.$$

For $\sigma = (\sigma_i)_{i \in \mathbb{N}} \in \{-1, 1\}^{\omega}$ define:

$$T_{\sigma} : \text{span}(x_i : i \in \mathbb{N}) \rightarrow \text{span}(x_i : i \in \mathbb{N}) \ni \sum_{i \in \mathbb{N}} a_i x_i \mapsto \sum_{i \in \mathbb{N}} \sigma_i a_i x_i.$$

The following are equivalent:

- a) (x_n) is an unconditional basic sequence,
- b) $C_u = \sup_{\sigma = (\sigma_i)_{i \in \mathbb{N}} \in \{-1, 1\}^{\omega}} \|T_{\sigma}\| < \infty$
- c) $\tilde{C}_u = \sup_{A \in [N]^{<\omega}} \|P_A\| = \sup_{A \in [N]^{<\omega}} \sup_{x \in \text{span}(x_i : i \in \mathbb{N}), \|x\| \leq 1} \|P_A(x)\| < \infty$

We say that a basic sequence is c -unconditional if $C_u \leq c$ and suppression c -unconditional if $\tilde{C}_u \leq c$

Exercise 2.1.11. Show that always $\tilde{C}_u \leq C_u \leq 2\tilde{C}_u$.

Definition 2.1.12. Let (x_n) be a basic sequence. A *Block basis* of (x_n) is a sequence $(y_n) \subset X \setminus \{0\}$ of the form:

$$y_n = \sum_{i=k_{n-1}+1}^{k_n} a_i x_i, \text{ with } 0 = k_0 < k_1 < k_2 < \dots \text{ and } a_i \in \mathbb{R}, \text{ for } i = 1, 2, \dots$$

A subspace spanned by a block basis is called *block subspace*.

Remark. Note that every normalized block basis of the unit vector basis of ℓ_p , $1 \leq p < \infty$ or c_0 is isometrically equivalent to the unit vector basis of ℓ_p .

M. Zippin [Z2] proved that a normalized basic sequence (x_n) which is equivalent to all of its normalized block bases must be equivalent to the unit vector basis of ℓ_p , for some $1 \leq p < \infty$, or c_0

Exercise 2.1.13. Show that block bases of bases are basic sequences.

Proposition 2.1.14. *Let X be an infinite dimensional Banach space with a basis (x_n) and Y an infinite dimensional subspace. Given any sequence $(\varepsilon_n) \subset (0, 1)$ there is a normalized block sequence (y_n) of (x_n) and a sequence $(\tilde{y}_n) \subset Y$ of so that $\|y_n - \tilde{y}_n\| < \varepsilon_n$ for any $n \in \mathbb{N}$.*

Exercise 2.1.15. If (e_i) is a basis of X and the sequence of (e_i^*) its coordinate functionals is also a basic sequence (but not necessarily a basis of X^* which could for example be non separable) and (e_i^*) is unconditional if and only if (e_i) is.

Definition 2.1.16. A basic sequence (e_i) is called *shrinking* if the sequence of its coordinate functionals (e_i^*) is a basis of X^* , and (e_i) is called *boundedly complete* if the series $\sum a_i e_i$ converges whenever $\sup_{n \in \mathbb{N}} \|\sum_{i=1}^n e_i\| < \infty$.

The proofs of the following results on shrinking and boundedly complete bases can be found in [FHHMPZ]

Proposition 2.1.17. *For basis (e_i) of X the following are equivalent*

- a) (e_i) is shrinking
- b) For any $x^* \in X^*$ it follows that $\lim_{n \rightarrow \infty} \|x^*|_{\text{span}(e_i: i \geq n)}\| = 0$.

Proposition 2.1.18. *For basis (e_i) of X and (e_i^*) be the coordinate functionals. Then following are equivalent*

- a) (e_i) is boundedly complete
- b) Let $y = \overline{\text{span}(e_i^* : i \in \mathbb{N})}$ then the (canonical) map

$$T : X \mapsto Y^*, \quad x \mapsto \left[y = \sum b_i e_i^* \mapsto \sum b_i e_i^*(x) \right],$$

is an isometry onto Y^* . In particular X is a dual space.

Theorem 2.1.19. ([Ja2], see also [FHHMPZ, Theorem 6.11]) *Let X be a Banach space with basis (e_i) . Then X is reflexive if and only if (e_i) is shrinking and boundedly complete.*

Remark. Note that the unit vector basis of c_0 is shrinking but not boundedly complete, and that the unit vector basis of ℓ_1 is boundedly complete but not shrinking.

Definition 2.1.20. Let X be a Banach space. A closed $Y \hookrightarrow X$ is called *complemented in X* and we write $Y \overset{c}{\hookrightarrow} X$ if there is a *bounded projection from X onto Y* , i.e. a bounded map $P : X \rightarrow Y$ with $P(y) = y$ for all $y \in Y$.

Remark. If (e_i) is basis of a Banach space and Y be a subspace which is spanned by a subsequence of (e_i) then Y does not need to be complemented in X (see Exercise 2.1.22 below). Nevertheless, if (e_i) unconditional then it is easy to see that all closed subspaces spanned by subsequences are complemented.

Exercise 2.1.21. Show that every finite dimensional and every cofinite dimensional closed subspace of a Banach space X is complemented in X .

Exercise 2.1.22. Let J be James' space, and (e_i) its standard shrinking basis (c.f [FHHMPZ, Page 185]) Then $Y = \overline{\text{span}(e_i : i \in \mathbb{N}, \text{odd})}$ is not complemented in J .

Exercise 2.1.23. For a closed subspace Y of X the following statements are equivalent.

a) $Y \xrightarrow{c} X$

b) There is a closed subspace Z of X so that X is linearly isomorphic to the topological sum

$$Y \oplus Z = \{(y, z) : y \in Y, z \in Z\}$$

with the product topology (which is induced by a norm, for example by $\|(y, z)\| := \|y\| + \|z\|$).

2.2 Spreading models

The theory of spreading models is due to Brunel and Sucheston in the 70's [BS]. A consequence is that a normalized weakly null sequence admits a subsequence which is "asymptotically unconditional."

Let us first state a Corollary of Theorem 1.1.1 which is often called *the Ramsey Theorem for Analysts*. It shows that Ramsey's theorem can be seen as a generalization of the fact that in a compact metrizable space every sequence has a convergent subsequence.

Theorem 2.2.1. *Let (M, d) be a compact metric space, $k \in \mathbb{N}$,*

For any map $F : [\mathbb{N}]^k \rightarrow M$ and any sequence (ε_j) there is an $N = \{n_1, n_2, \dots\} \in [\mathbb{N}]^\omega$, with $n_1 < n_2 < \dots$, and an $m_0 \in M$ so that

$$d(F(n_{i_1}, n_{i_2}, \dots, n_{i_k}), m_0) < \varepsilon_j \text{ whenever } j \leq i_1 < i_2 < \dots < i_k.$$

In particular,

$$\lim_{i_1 \rightarrow \infty} \lim_{i_2 \rightarrow \infty} \dots \lim_{i_k \rightarrow \infty} F(i_1, i_2, \dots, i_k) = m_0.$$

Proof. For $j \in \mathbb{N}$ choose a finite covering $(U_\ell^{(j)})_{\ell=1}^{L_j}$ of M so that, the diameter of each $U_\ell^{(j)}$ does not exceed $\varepsilon_j/2$, and so that for any $j \in \mathbb{N}$, $j \geq 2$, and $1 \leq \ell \leq L_j$ there is an $1 \leq \ell' \leq L_{j-1}$ so that $U_\ell^{(j)} \subset U_{\ell'}^{(j-1)}$.

Choose a sequence $\mathbb{N} = N_0 \supset N_1 \supset N_2 \dots$ by induction as follows. If $j \in \mathbb{N}$ and N_{j-1} has been chosen, let $(\xi_\ell^j)_{\ell=1}^{L_j}$ be finite ε_j -net of M , then apply Corollary 1.1.3 to the sets $\mathcal{A}_\ell \subset [N_{j-1}]^\omega$, where

$$\mathcal{A}_\ell = \{(n_1, n_2, \dots, n_k) \in [N_{j-1}]^k : F(n_1, n_2, \dots, n_k) \in U_\ell^{(j)}\}, \text{ for } \ell = 1, 2, \dots, L,$$

in order to get an infinite $N_j \subset N_{j-1}$ and an ℓ_j so that $[N_j]^k \subset \mathcal{A}_{\ell_j}$.

It follows that $m_0 \in \bigcap_j \overline{U_{\ell_j}^{(j)}}$ is unique and that the claim follows if we let N be a diagonal sequence of the N_j 's. \square

Theorem 2.2.2. *Let (x_n) be a normalized basic sequence in X and let $\varepsilon_n \downarrow 0$. There exists a subsequence (y_n) of (x_n) and a normalized basis (e_n) for a Banach space E with the following property. If $n \in \mathbb{N}$ and $(a_i)_{i=1}^n \in [-1, 1]^n$ then*

$$(2.1) \quad \left| \left\| \sum_{i=1}^n a_i y_{k_i} \right\| - \left\| \sum_{i=1}^n a_i e_i \right\| \right| < \varepsilon_n \text{ whenever } n \leq k_1 < \dots < k_n$$

Exercise 2.2.3. Proof Theorem 2.2.2

Hint: Use the fact that

$$\mathcal{M}_n = \{ \|\cdot\| : \mathbb{R}^n \rightarrow [0, \infty) : \|\cdot\| \text{ is a norm on } \mathbb{R}^n \text{ and } \|e_1\| = \|e_2\| = \dots = \|e_n\| = 1 \},$$

with the metric

$$d_n(\|\cdot\|_1, \|\cdot\|_2) = \sup_{\xi \in [-1, 1]^n} \left| \|\xi\|_1 - \|\xi\|_2 \right|,$$

is a compact space.

Definition 2.2.4. Let (x_n) be a basic normalized sequence in Banach space X . A Banach space E with a basis (e_i) and norm denoted by $\|\cdot\|_E$ is called *the spreading model of (x_n)* if for every $k \in \mathbb{N}$ and every $(a_i)_{i=1}^k \in \mathbb{R}^k$

$$\left\| \sum_{i=1}^k a_i e_i \right\|_E = \lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \dots \lim_{n_k \rightarrow \infty} \left\| \sum_{i=1}^k a_i x_{n_i} \right\|.$$

Remark. Note that Theorem 2.2.2 implies that every basic sequence in a Banach space has a subsequence which has a spreading model, and that, moreover, one can pass to a further subsequence for which the stronger statement, more quantitative, (2.1) holds.

Theorem 2.2.5. *Assume the space E with a normalized basis (e_i) is the spreading model of a normalized weakly null sequence (x_n) in the Banach space X .*

Then (e_i) is suppression-1-unconditional.

We will first prove the following

Lemma 2.2.6. *Let (x_i) be a weakly null sequence.*

Then for any $\varepsilon > 0$, $m \in \mathbb{N}$ there exists an $n \in \mathbb{N}$, $n > m$ so that

$$(2.2) \quad \forall x^* \in B_{X^*} \exists i \in (m, n] \quad |x^*(x_i)| < \varepsilon.$$

Proof. Assume our claim were not true then we could choose for any $n \in \mathbb{N}$ an $x_n^* \in B_{X^*}$ so that $|x_n^*(x_i)| \geq \varepsilon$ for all $i \in (m, n]$. We can pass to a subsequence $x_{n_k}^*$ which converges in w^* to some $x^* \in B_{X^*}$. This means that for all $n \in (m, \infty)$

$$|x^*(x_n)| = \lim_{k \rightarrow \infty} |x_{n_k}^*(x_n)| \geq \varepsilon.$$

But this contradicts the assumption that (x_n) is weakly null. □

Iterating Lemma 2.2.6 yields

Corollary 2.2.7. *If (x_n) is a weakly null sequence in X and $(\varepsilon_\ell) \subset (0, 1)$ there is an increasing subsequence $(n_\ell) \subset \mathbb{N}$ so that for all for all ℓ (put $n_0 = 0$)*

$$(2.3) \quad \forall x^* \in B_{X^*} \exists i \in (n_{\ell-1}, n_\ell] \quad |x^*(x_i)| < \varepsilon_\ell.$$

Proof of Theorem ??. Let $(\varepsilon_n) \subset (0, 1)$ decrease to 0. According to Theorem 2.2.2 we can assume, after passing to an appropriate subsequence of (x_n) , that

$$(2.4) \quad \left| \left\| \sum_{i=1}^k a_i x_{n_i} \right\| - \left\| \sum_{i=1}^k a_i e_i \right\| \right| < \varepsilon_k$$

whenever $k \in \mathbb{N}$, $(a_i) \subset [-1, 1]^n$ and $k \leq n_1 < n_2 < n_k$.

Let $k \in \mathbb{N}$, $(a_i)_{i=1}^k \in [-1, 1]$, and $i_0 \in \{1, 2, \dots, n\}$. It is enough to show that

$$\left\| \sum_{i=1, i \neq i_0}^k a_i e_i \right\| \leq \left\| \sum_{i=1}^k a_i e_i \right\|.$$

Choose (n_i) like in Corollary 2.2.7, then we deduce for any $m \in \mathbb{N}$

$$\begin{aligned}
\left\| \sum_{i=1, i \neq i_0}^k a_i e_i \right\| &\leq \varepsilon_m + \left\| \sum_{i=1, i \neq i_0}^k a_i x_{\ell_i} \right\| \\
&\text{[By (2.4)]} \\
&\text{where } \ell_i \in (n_{m+i-1}, n_{m+i}] \text{ for all } i \in \{1, 2, \dots, k\} \setminus \{i_0\} \\
&= \varepsilon_m + \sum_{i=1, i \neq i_0}^k a_i x^*(x_{\ell_i}) \\
&\text{[For appropriate } x^* \in S_{X^*} \text{ by Hahn Banach]} \\
&\leq 2\varepsilon_m + \sum_{i=1}^k a_i x^*(x_{\ell_i}) \\
&\text{[For appropriate } \ell_{i_0} \in (n_{m+i_0-1}, n_{m+i_0}], \text{ by Corollary 2.2.7]} \\
&\leq 2\varepsilon_m + \left\| \sum_{i=1}^k a_i x_{\ell_i} \right\| \\
&\leq 3\varepsilon_m + \left\| \sum_{i=1}^k a_i e_i \right\| \\
&\text{[By (2.4)]}
\end{aligned}$$

□

The following example shows that the spreading model can be quite different from the underlying basis.

Example 2.2.8. The *Schreier space*

Define first the *Schreier sets* $\mathcal{S}_1 \subset [\mathbb{N}]^{<\omega}$ as

$$\mathcal{S}_1 = \{E \in [\mathbb{N}]^\omega : \#E \leq \min E\}.$$

For example $\{3, 7, 1000\} \in \mathcal{S}_1$ but $\{3, 7, 17, 20\} \notin \mathcal{S}_1$. Then define for $x = (x_i) \in c_{00}$:

$$\|x\|_S = \max_{E \in \mathcal{S}_1} \sum_{i \in E} |x_i|.$$

Let S be the completion of c_{00} under $\|\cdot\|_S$.

It follows for any $k \in \mathbb{N}$ and any $(a_i)_{i=1}^k$ that

$$\lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \dots \lim_{n_k \rightarrow \infty} \left\| \sum_{i=1}^k a_i e_{n_i} \right\|_S = \sum_{i=1}^k |a_i|.$$

Thus, the spreading model of (e_i) in S is the ℓ_1 unit vector basis.

On the other hand S is hereditarily c_0 (see next exercise).

Exercise 2.2.9. Show that every block subspace of S has a further block which is equivalent to the c_0 unit vector basis.

Hint: let (x_i) be a normalized block in S , say:

$$x_i = \sum_{j=n_{i-1}+1}^{n_i} a_j^{(i)} e_j, \text{ with } 0 = n_0 < n_1 < n_2 < \dots$$

Then we distinguish between two cases:

If we have

$$\liminf_{i \rightarrow \infty} \|x_i\|_\infty = \liminf_{i \rightarrow \infty} \max_{j \in (n_{i-1}, n_i]} |a_j^{(i)}| = 0.$$

then one can find a subsequence which is $(1 + \varepsilon)$ -equivalent to the c_0 -unit basis for any $\varepsilon > 0$.

Other wise one can find a normalized block of x_i for which the first case applies.

Remark. Rosenthal's ℓ_1 Theorem (Theorem 2.3.1 below) yields that every X contains either an isomorph of ℓ_1 or a normalized weakly null sequence. Thus every X always admits an unconditional spreading model. There are a number of known results and open problems concerning spreading models. They take on one of two forms. Given X , perhaps with certain properties, what can one say about the class of spreading models $S(X)$ of X . Or given specific information about $S(X)$ what can one say about X ? For example it can be shown that X is not reflexive if X admits a spreading model that is not unconditional. On the other hand care must be taken since there exists a reflexive space T (Tsirelson's space) all of whose spreading models are isomorphic to ℓ_1 . Other known results are as follows. If $X \subseteq L_p$ ($1 < p < \infty$) and all spreading models of X are isomorphic to ℓ_p then X embeds into ℓ_p . There exists a space X so that no spreading model contains an isomorph of c_0 or ℓ_p ($1 \leq p < \infty$). It is not known if for every X there exists n and a chain of successive spreading models $\tilde{X}_1, \dots, \tilde{X}_n$ with \tilde{X}_n isomorphic to c_0 or ℓ_p for some $1 \leq p < \infty$. In [S2] it was shown that if $S(X)$ admits two quite different spreading models of weakly null sequences then one can construct a "nontrivial" operator $T : W \subseteq X \rightarrow X$ ($T \neq \lambda I + K$, where K is a compact operator, and I the inclusion operator). Recent results and more questions on spreading models can be found in [AOST].

Let us mention some interesting problems. We denote the set of all spreading models generated by normalized weakly null sequences in X by $SP_w(X)$. We identify sequences in $SP_w(X)$ which are equivalent.

We say $SP_w(X)$ is *stabilized* if $SP_w(Y) = SP_w(X)$ for all infinite-dimensional subspaces Y of X .

Question. Does every reflexive Banach space X have an infinite-dimensional subspace Z for which $SP_w(Z)$ is stabilized?

It is easy to see that if $SP_w(X)$ is countable, then the answer is positive. It is also not hard to see that one can always *stabilize with respect to cardinality*, to wit, there is an infinite-dimensional subspace Z such that $\text{card}SP_w(Z) = \text{card}SP_w(Y)$ for every infinite-dimensional subspace Y of Z . But very little is known about which values $\text{card}SP_w(Z)$ can assume in this case.

Question. Assume that $SP_w(X)$ is stabilized with respect to cardinality. What are the possible values for $\text{card}SP_w(X)$?

In the case in which $\text{card}SP_w(X) = 1$ the following question is open.

Question. Assume that $\text{card}SP_w(X) = 1$, i.e. that all spreading models are equivalent. Does it follow that the spreading models are equivalent to the ℓ_p unit-vector basis for some p , $1 \leq p < \infty$, or to the unit basis in c_0 ?

Remark. In [AOST] it is shown that the last question has a positive answer provided there is a constant $C \geq 1$ so that all spreading models are C -equivalent.

Of course $\text{card}SP_w(\ell_p) = 1$. For the space S constructed by the author in [S1] we find that $\text{card}SP_w(S) = \omega_c$, the cardinality of the continuum. No other possible values for $\text{card}SP_w(X)$ are known (assuming of course that $\text{card}SP_w(X)$ is stabilized). On the other hand, it is not known if any cardinality between 1 and ω_c can be excluded from being the stabilized cardinality of $SP_w(X)$ for some reflexive space X . For example, it is not known if there is a reflexive space X for which $SP_w(X)$ is stabilized and for which $\text{card}SP_w(X) = 2$.

We finish the section with an example by Maurey and Rosenthal [MR] which exhibits a weakly null basic sequence none of whose subsequences are unconditional. It is one of the building stones towards constructing a Banach space which does not have a subspace with unconditional basis Gowers and Maurey [GM1].

We start with an exercise

Exercise 2.2.10. In c_0 let $s_n = \sum_{i=1}^n e_i = (1, 1, \dots, 1, 0, 0, \dots)$. Show that (s_n) is a monotone basis for c_0 which is *conditional* (i.e., not unconditional).

(s_n) is called the *summing basis* for c_0 .

Example 2.2.11. [MR] We will construct a certain family $\mathcal{F} \subset c_0$ of real sequences and define a norm $\|\cdot\|_{\mathcal{F}}$ on c_{00} by setting

$$\|x\|_{\mathcal{F}} = \sup\{|\langle f, x \rangle| : f \in \mathcal{F}\} \text{ whenever } x \in c_{00},$$

where we put for $f = (f_i) \in c_0$ and $x = (x_i) \in c_{00}$

$$\langle f, x \rangle = \sum_i x_i f_i.$$

We then let $X_{\mathcal{F}}$ be the completion of $(c_{00}, \|\cdot\|_{\mathcal{F}})$. Note, for example, that ℓ_p could be defined this way by taking $\mathcal{F} = B_{\ell_q} \cap c_{00}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

If $E, F \in [\mathbb{N}]^{<\omega}$ we write “ $E < F$ ” if $\max E < \min F$. Let (m_i) be a certain lacunary subsequence of \mathbb{N} (we will specify what we mean later), with $m_1 = 1$. Let

$$\vec{F} = \{(F_i)_1^n : n \in \mathbb{N}, F_1 < F_2 < \dots < F_n \text{ and } F_i \in [\mathbb{N}]^{<\omega} \text{ for } i \in \mathbb{N}\}$$

Let $\phi : \vec{F} \rightarrow \{m_i\}_{i=1}^{\infty}$ be one-to-one (ϕ is called a coding). We define

$$\mathcal{F} = \left\{ f = \sum_{i=1}^{\infty} \frac{1_{E_i}}{\sqrt{\#E_i}} : E_1 < E_2 < \dots, \#E_1 = 1 \text{ and } \#E_{i+1} = \phi(E_1, \dots, E_i) \right\}$$

Note that if $\#E = m_i$, $\#F = m_j$ and, say $i < j$, then

$$\left\langle \frac{1_E}{\sqrt{\#E}}, \frac{1_F}{\sqrt{\#F}} \right\rangle \leq \frac{m_i}{\sqrt{m_i}\sqrt{m_j}} = \frac{\sqrt{m_i}}{\sqrt{m_j}} < \varepsilon_j,$$

where $\varepsilon_j \downarrow 0$ rapidly can be chosen in advance, which determines how (m_i) is chosen.

Claim 1: (e_n) is a normalized monotone basis for $X_{\mathcal{F}}$.

(Exercise)

Claim 2: (e_n) is weakly null.

We think of \mathcal{F} being a subset of the compact space $[0, 1]^{\omega}$ and can therefore think of X being (isometrically) a subspace of $C(K)$ where $K = \overline{\mathcal{F}}$.

Then note that in fact

$$(2.5) \quad K = \{1_{[1,n]}f : n \in \mathbb{N} \text{ and } f \in \mathcal{F}\} \cup \mathcal{F}.$$

Indeed, let $f_k \in \mathcal{F}$ for $k \in \mathbb{N}$, say

$$f_k = \sum_{i=1}^{\infty} \frac{1_{E_i^{(k)}}}{\sqrt{\#E_i^{(k)}}}.$$

We can assume also that $f_k \neq f_{k'}$, for $k \neq k'$ in \mathbb{N} .

By passing to a subsequence, we can assume that there is an $\ell \geq 1$ so that

$$E_i^{(k)} = E_i^{(k')} =: E_i \text{ whenever } i < \ell, \text{ and } k, k' \in \mathbb{N}$$

$$E_{\ell}^{(k)} \neq E_{\ell}^{(k')} \text{ whenever } k \neq k' \text{ are } \mathbb{N}$$

Secondly we can, after passing to a subsequence assume that $E_{\ell}^{(k)}$ can be written as a union

$$E_{\ell}^{(k)} = A \cup \tilde{E}_{\ell}^{(k)}, \text{ with } \min \tilde{E}_{\ell}^{(k)} \rightarrow \infty.$$

and, thus, f_n converges pointwise to

$$f = \sum_{i=1}^{\ell-1} \frac{1_{E_i}}{\sqrt{\#E_i}} + \frac{1_A}{\sqrt{\phi(E_1, E_2, \dots, E_{\ell-1})}}.$$

which shows (2.5).

In order to finish the proof of claim 1, we simply observe that for any $f \in K$ we have $e_n(f) \rightarrow 0$ for $n \rightarrow \infty$ and recall the fact that in a $C(K)$ -space, where K a compact, weak convergence and point wise convergence of sequences are equivalent.

Claim 3: Every subsequence of (e_i) has a block which is equivalent to the summing basis of c_0 .

Let $M \in [\mathbb{N}]^{\omega}$. Then there exists (E_i) so that $f = \sum_1^{\infty} (1_{E_i}/\sqrt{|E_i|}) \in \mathcal{F}$ for all and $\text{supp} f \subseteq M$.

We claim that $(1_{E_i}/\sqrt{|E_i|})_{i=1}^{\infty} \subseteq X_{\mathcal{F}}$ is equivalent to the summing basis.

Indeed, it suffices to show that for $(a_i)_1^m \subseteq \mathbb{R}$

$$\sup_{n \leq m} \left| \sum_1^n a_i \right| \leq \left\| \sum_1^m a_i \frac{1_{E_i}}{\sqrt{|E_i|}} \right\| \leq 3 \sup_{n \leq m} \left| \sum_1^n a_i \right|.$$

The left hand estimate is easy. To see the right hand estimate let $f = \sum_1^\infty \frac{1_{F_i}}{\sqrt{|F_i|}} \in \mathcal{F}$. Let $i_0 = \max\{i : F_i = E_i\}$. Then

$$\begin{aligned} \left| \left\langle f, \sum_{i=1}^m a_i \frac{1_{E_i}}{\sqrt{|E_i|}} \right\rangle \right| &\leq \left| \sum_{i=1}^{i_0} a_i \right| + \left| \left\langle \frac{1_{F_{i_0+1}}}{\sqrt{|F_{i_0+1}|}}, \sum_{i=i_0+1}^m a_i \frac{1_{E_i}}{\sqrt{|E_i|}} \right\rangle \right| \\ &\quad + \sum_{j=i_0+2}^\infty \sum_{i=i_0+1}^m |a_i| \left\langle \frac{1_{F_j}}{\sqrt{|F_j|}}, \frac{1_{E_i}}{\sqrt{|E_i|}} \right\rangle. \end{aligned}$$

We estimate the second term as follows.

$$\begin{aligned} &\left| \left\langle \frac{1_{F_{i_0+1}}}{\sqrt{|F_{i_0+1}|}}, \sum_1^m a_i \frac{1_{E_i}}{\sqrt{|E_i|}} \right\rangle \right| \\ &\leq |a_{i_0+1}| \left| \left\langle \frac{1_{F_{i_0+1}}}{\sqrt{|F_{i_0+1}|}}, \frac{1_{E_{i_0+1}}}{\sqrt{|E_{i_0+1}|}} \right\rangle \right| + \left| \left\langle \frac{1_{F_{i_0+1}}}{\sqrt{|F_{i_0+1}|}}, \sum_{i=i_0+2}^m a_i \frac{1_{E_i}}{\sqrt{|E_i|}} \right\rangle \right| \\ &\leq |a_{i_0+1}| + \sum_{i=i_0+2}^m |a_i| \varepsilon_i. \end{aligned}$$

The third term can be esitimated as follows

$$\begin{aligned} &\sum_{j=i_0+2}^\infty \sum_{i=i_0+1}^m |a_i| \left\langle \frac{1_{F_j}}{\sqrt{|F_j|}}, \frac{1_{E_i}}{\sqrt{|E_i|}} \right\rangle \\ &\leq \sum_{j=i_0+1}^\infty \sum_{i=i_0+1}^m |a_i| \varepsilon_{\max(i,j)} \end{aligned}$$

[Note that for $i \in \mathbb{N}$ $\#E_i = m_{i'}$ for some $i' \geq i$. Similar observation holds for $\#F_j$. Also note that $\#E_i \neq \#F_j$ whenever $i = i_0 + 1, 2, \dots$ and $j = i_0 + 2, i_0 + 3, \dots$]

We deduce the claim, assuming we have chosen ε_i fast enough decreasing to 0.

2.3 Rosenthal's ℓ_1 theorem

One of the prettiest theorems in Banach space theory, *Rosenthal's ℓ_1 theorem* [Ro], can also be proved using Ramsey theory at a certain point. Let us place the theorem in context. In the search for a “nice” infinite dimensional subspace of a given Banach space X , one conjecture was: Every Banach space X contains a subspace Y which is either reflexive or isomorphic to c_0 or to ℓ_1 . (This was ultimately shown to be false by W.T. Gowers [Go1].) Now if X is not reflexive then B_X is not weakly compact and so by the Eberlein-Smulian theorem there exists $(x_n) \subseteq S_X$ with no weakly convergent subsequence. This splits into two cases:

- (I) (x_n) has no *weak Cauchy subsequence*: for all $(y_n) \subseteq (x_n)$ there exists $x^* \in X^*$ with $(x^*(y_n))_n$ being divergent.
- (II) (x_n) has a weak Cauchy subsequence (y_n) but there is no $y \in X$ so that $x^*(y_n) \rightarrow x^*(y)$ for all x^* (X is not *weakly sequentially complete*).

H. Rosenthal proved that (I) yields $\ell_1 \hookrightarrow X$.

Theorem 2.3.1. [Ro] *Let (x_n) be normalized with no weak Cauchy subsequence. Then there exists $(y_n) \subseteq (x_n)$ which is equivalent to the unit basis of ℓ_1 .*

Definition 2.3.2. A sequence of pairs of sets (A_n, B_n) is called *Boolean independent* if for any finite and disjoint subsets I_1 and I_2 of \mathbb{N}

$$\bigcap_{n \in I_1} A_n \cap \bigcap_{n \in I_2} B_n \neq \emptyset$$

The following Lemma is easy:

Lemma 2.3.3. *Let S be a set and let $f_n : S \rightarrow [-1, 1]$ and let $A_n = \{f_n = 1\}$, $B_n = \{f_n = -1\}$ then $(A_n, B_n)_{n=1}^\infty$ are Boolean independent*

Then $\|\sum_1^k a_i f_i\|_\infty = \sum_1^k |a_i|$ for all k and $(a_i)_1^k \subseteq \mathbb{R}$, i.e. f_i as sequence in $\ell_\infty(S)$ is isometrically equivalent to the ℓ_1 -unit vector basis.

The following is an *isomorphic version* of Lemma 2.3.3.

Lemma 2.3.4. *Let S be a set, $f_n : S \rightarrow [-1, 1]$, $r \in \mathbb{R}$, $\delta > 0$, $A_n = \{f_n > r + \delta\}$ and $B_n = \{f_n < r\}$. Assume that $(A_n, B_n)_{n=1}^\infty$ are Boolean independent. Then $(f_n) \subset \ell_\infty(S)$ is equivalent to the unit vector basis of ℓ_1 .*

Proof. Let $(a_i)_1^k \subseteq \mathbb{R}$. It suffices to show that there exists $s \in S$ with

$$\left| \sum_1^k a_i f_i(s) \right| \geq \frac{\delta}{2} \sum_1^k |a_i|.$$

Let $I_1 = \{i : a_i \geq 0\}$ and $I_2 = \{i : a_i < 0\}$. Let $s_1 \in \bigcap_{I_1} A_i \cap \bigcap_{I_2} B_i$ and $s_2 \in \bigcap_{I_2} A_i \cap \bigcap_{I_1} B_i$. then it follows that

$$\sum_{i=1}^k a_i (f_i(s_1) - f_i(s_2)) \geq \delta \sum_{i=1}^k |a_i|,$$

and, thus,

$$\max \left(\left| \sum_{i=1}^k a_i(f_i(s_1)) \right|, \left| \sum_{i=1}^k a_i(f_i(s_2)) \right| \right) \geq \frac{\delta}{2} \sum_{i=1}^k |a_i|.$$

[Note that by choice of s_1 and s_2 it follows that $a_i(f_i(s_1) - f_i(s_2)) > |a_i|\delta$] \square

Definition 2.3.5. A sequence of pairs of disjoint sets (A_n, B_n) is said to have no *convergent subsequence* if for all $M \in [\mathbb{N}]^\omega$ there exists $s \in S$ so that s belongs to infinitely many A_n 's, $n \in M$ and to infinitely many B_n 's, $n \in M$.

Lemma 2.3.6. Let $A_n \cap B_n = \emptyset$, $A_n, B_n \subseteq S$, for $n \in \mathbb{N}$ and assume that (A_n, B_n) has no convergent subsequence.

Then there is a subsequence of (A_n, B_n) which is Boolean independent.

Proof. Let

$$\mathcal{A} = \left\{ L = (\ell_i)_{i=1}^\infty \in [\mathbb{N}]^\omega : \bigcap_{i=1}^k A_{\ell_{2i-1}} \cap \bigcap_{i=1}^k B_{\ell_{2i}} \neq \emptyset \text{ for all } k \in \mathbb{N} \right\}.$$

Since

$$\mathcal{A} = \bigcap_{k \in \mathbb{N}} \left\{ L = (\ell_i)_{i=1}^\infty \in [\mathbb{N}]^\omega : \bigcap_{i=1}^k A_{\ell_{2i-1}} \cap \bigcap_{i=1}^k B_{\ell_{2i}} \neq \emptyset \text{ for all } k \in \mathbb{N} \text{ with } \ell_{2k} < \ell \right\},$$

\mathcal{A} is closed in $[\mathbb{N}]^\omega$ and therefore we can apply Ramsey's Theorem 1.3.3. From the assumption that (A_n, B_n) has no convergent subsequence it follows that there is an $L \in [\mathbb{N}]^\omega$ with $[L]^\omega \subseteq \mathcal{A}$. Finally take $M = (\ell_2, \ell_4, \ell_6, \dots)$. \square

Prove of Theorem 2.3.1. Let $(x_n) \subset B_X$ have no w-convergent subsequence. Let $S = B_{X^*}$ and regard x_n as functions in $L_\infty(S)$.

For $r \in [-1, 1]$ $\delta > 0$ and $n \in \mathbb{N}$ we put

$$A_n(r, \delta) = \{s \in S : x_n(s) > r + \delta\} \text{ and } B_n(r, \delta) = \{s \in S : x_n(s) < r\}$$

and claim that for some choice of $r \in [-1, 1]$ and $\delta > 0$ there is an infinite M so that $(A_m(r, \delta), B_m(r, \delta))_{m \in M}$ has no convergent subsequence. Indeed, if this were not true, let $(r_i, \delta_i)_{i=1}^\infty$ be dense in $[-1, 1] \times (0, 1]$ and inductively choose $M_{i+1} \in [M_i]^\omega$ so that $(A_n(r_i, \delta_i), B_n(r_i, \delta_i))_{n \in M_i}$ "converges" (every s belongs to at most finitely many of the A_n 's or of the B_n 's). If M is a diagonal of the M_i 's then $(x_n)_M$ is point wise convergent on S .

The claim of the Theorem follows now by applying Lemmas 2.3.3, 2.3.4 and 2.3.6. \square

2.4 Partial Unconditionality

Given a weakly null, normalized sequence in a Banach space X , can we pass to a subsequence that is a basic sequence and is in some sense close to being unconditional (recall that Example 2.2.11 shows that not every weakly null sequence may have an unconditional subsequence)?

There are various ways in which one can make this vague question precise, and in many situations one has a positive answer. There are important cases, however, for which the corresponding question is still open. In this section we want to present some of the known results.

The following result of John Elton uses the idea of preserving the positive ℓ_1 mass, $\sum_I y^*(y_i)^+$, on an arbitrary set I to obtain a weakened form of unconditionality.

Theorem 2.4.1. *For each $\delta > 0$ there exists $K(\delta) < \infty$ with the following property. Let (x_n) be a normalized weakly null sequence in a Banach space. There exists a subsequence (y_n) of (x_n) such that if $(a_i)_{i=1}^\infty \subseteq [-1, 1]$ and $I \subseteq \{i : |a_i| \geq \delta\}$ then $\|\sum_I a_i y_i\| \leq K(\delta) \|\sum_i a_i y_i\|$.*

Remark. It is an open problem whether or not K in Theorem 2.4.1 could be chosen independently of $\delta > 0$. At first sight this might seem like trying to prove that every weakly null sequence has a subsequence which is unconditional (which is contradicted by Example 2.2.11). But note that we still would have to choose the subsequence (y_n) depending on the given δ . For example in [DKK] it was shown that $K \sim 3$ if we assume that (x_i) does not have a spreading model which is equivalent to the c_0 -unit vector basis. More about this problem as well as further partial answers can be found in [DOSZ].

Proof. We first note that we are trying to bound

$$\inf_{(y_i) \subseteq (x_i)} \sup \left\{ \frac{\|\sum_I a_i y_i\|}{\|\sum a_i y_i\|} : |a_i| \leq 1 \text{ for all } i \text{ and } I \subseteq \{i : |a_i| \geq \delta\} \right\}$$

independently of the original sequence (x_i) . Let $x^* \in B_{X^*}$ with

$$x^*\left(\sum_I a_i y_i\right) = \left\| \sum_I a_i y_i \right\|$$

then split I into 4 sets $I_{++}, I_{+-}, I_{-+}, I_{--}$ where for example

$$I_{+-} = \{i \in I : a_i > 0 \text{ and } x^*(y_i) \leq 0\}.$$

We see that for at least one of these sets, say I_{+-} , we have $|\sum_{I_{+-}} a_i x^*(y_i)| \geq \|\sum_I a_i y_i\|/4$. It follows (e.g. replace x^* by $-x^*$ in this case) that we can find $x^* \in B_{X^*}$ so that

$$\left\| \sum_I a_i y_i \right\| \leq 4 \sum_I a_i^+ x^*(y_i)^+ = 4 \sum_{I_{++}} a_i x^*(y_i)$$

(where $b^+ = b \vee 0$ for $b \in \mathbb{R}$).

Let us suppose that, given $\varepsilon > 0$, we can find $(y_n) \subseteq (x_n)$ so that in this case there exists $y^* \in B_{X^*}$ with

$$(2.6) \quad \sum_{I_{++}} y^*(y_i)^+ \geq \sum_{i \in I_{++}} x^*(y_i) - \varepsilon \quad \text{and} \quad \sum_{\{i: i \notin I_{++} \text{ or } y^*(y_i) < 0\}} |y^*(y_i)| \leq \varepsilon \sum_{I_{++}} y^*(y_i)^+.$$

Then we can easily derive our claim. Indeed, we then deduce that

$$\begin{aligned} \left\| \sum a_i y_i \right\| &\geq y^* \left(\sum a_i y_i \right) \\ &\geq -\varepsilon \sum_{I_{++}} y^*(y_i)^+ + \sum_{I_{++}} a_i y^*(y_i)^+ \\ &\geq (\delta - \varepsilon) \sum_{I_{++}} y^*(y_i)^+ \\ &\geq (\delta - \varepsilon) \sum_{I_{++}} x^*(y_i) - \delta \varepsilon \\ &\geq \frac{(\delta - \varepsilon)}{C} \sum_{I_{++}} a_i x^*(y_i) - \delta \varepsilon \\ &\quad [\text{where } C \text{ is the projection constant of } (y_i)] \\ &\geq \frac{(\delta - \varepsilon)}{C4} \left\| \sum_{i \in I} a - i y_i \right\|. \end{aligned}$$

Since (x_n) is weakly null we can assume, after dropping to a subsequence if necessary, that the basis constant of (x_n) is close to 1, and, thus, projection constant is close to 2. We therefore obtain the result is with $K(\delta) \lesssim 8/\delta$. Thus, it will suffice to prove the following. \square

Lemma 2.4.2. *Let $B > 0$ and $\varepsilon > 0$. There exists a subsequence (y_i) of (x_i) so that for all A*

$$\begin{aligned} \exists x^* \in B_{X^*} \text{ with } \sum_{j \in A} x^*(y_j)^+ &\geq B \\ \implies \exists y^* \in B_{X^*} \text{ with } \sum_{j \in A} y^*(y_j)^+ &\geq B - \varepsilon \text{ and } \sum_{\substack{j \in \mathbb{N} \setminus J \text{ or} \\ j \in A, y^*(y_j) < 0}} |y^*(y_j)| < \varepsilon. \end{aligned}$$

Proof. Let $(\varepsilon_i) \subset (0, 1)$ so that $\sum \varepsilon_i < \varepsilon$. By induction we choose $n_1 < n_2 < n_3 < \dots$ and $\mathbb{N} = N_0 \supset N_1 \supset N_3 \supset \dots$ so that for $k \in \mathbb{N}$

- a) $n_k = \min(N_{k-1})$ and $n_k < \min N_k$
- b) For all $F \subset \{n_1, n_2, \dots, n_k\}$, all $L = \{\ell_0, \ell_1, \ell_2, \dots\} \subset N_k$ and all $m \in \mathbb{N}$

$$(*) \quad \exists x^* \in B_{X^*} \text{ with } \sum_{j \in F \cup \{\ell_1, \ell_2, \dots, \ell_m\}} x^*(x_j)^+ > B$$

$$\begin{aligned} \implies \exists y^* \in B_{X^*} \text{ with } & \sum_{j \in F \cup \{\ell_1, \ell_2, \dots, \ell_m\}} y^*(y_j)^+ > B - \varepsilon \\ & |y^*(x_j)| < \varepsilon_i \text{ for } j = n_i \in \{n_1, n_2, \dots, n_k\} \setminus F \\ & |y^*(x_j)| < \varepsilon_i \text{ or } j = n_i \in F \text{ with } y^*(x_j) < 0 \\ & \text{and } |y^*(x_{\ell_0})| < \varepsilon_{k+1} \end{aligned}$$

Assume that $n_1 < n_2 < \dots < n_{k-1}$ and $\mathbb{N} = N_0 \supset N_1 \supset \dots \supset N_{k-1}$ have been chosen. Let $n_k = \min(N_{k-1})$ and fix an $F \subset \{n_1, n_2, \dots, n_k\}$ and an $N \in [N_{k-1}]^\omega$. Define

$$\mathcal{A}_F = \{L = \{\ell_0, \ell_1, \dots\} \in [N]^\omega : \forall m \in \mathbb{N} (*) \text{ holds}\}.$$

Since

$$\mathcal{A}_F = \bigcap_m \mathcal{A}_{F,m}, \text{ where}$$

$$\mathcal{A}_{F,m} = \{L = \{\ell_0, \ell_1, \dots\} \in [N_{k-1}]^\omega : (*) \text{ holds}\},$$

and containance of an L in $\mathcal{A}_{F,m}$ only depends on the first $m+1$ elements of L it follows that we can apply the Theorem of Ramsey 1.3.3.

We claim that we are in Case 1 of Theorem 1.3.3 (meaning that we can choose an $L_F \in [N]^\omega$ so that $[L]^\omega \subset \mathcal{A}$). Indeed, assume that there is an $L \in [N]^\omega$ so that for all $[L]^\omega \cap \mathcal{A} = \emptyset$, say $L = \{\ell_1, \ell_2, \ell_3, \dots\}$, $\ell_1 < \ell_2 < \ell_3 < \dots$.

Fix an $n \in \mathbb{N}$ and define for $i = 1, 2, \dots$

$$L^{(i)} = \{\ell_i, \ell_{n+1}, \ell_{n+2}, \ell_{n+3}, \dots\}.$$

Since $L^{(i)} \notin \mathcal{A}$ we can find for $i = 1, 2, \dots, n$, a number $m_i \in \mathbb{N}$, and an $x_i^* \in B_{X^*}$ so that

$$(2.7) \quad \sum_{j \in F \cup \{\ell_{n+1}, \ell_{n+2}, \dots, \ell_{n+m_i}\}} x_i^*(x_j)^+ > B$$

$$(2.8) \quad \forall y^* \in B_{X^*} \text{ with } \sum_{j \in F \cup \{\ell_{n+1}, \ell_{n+2}, \dots, \ell_{n+m_i}\}} y^*(y_j)^+ > B - \varepsilon$$

one of the following **does not hold**

$$\begin{aligned} & |y^*(x_j)| < \varepsilon_i \text{ for } j = n_i \in \{n_1, n_2, \dots, n_k\} \setminus F \\ & |y^*(x_j)| < \varepsilon_i \text{ for } j = n_i \in F \text{ with } y^*(x_j) < 0 \\ & \text{and } |y^*(x_{\ell_i})| < \varepsilon_{k+1} \end{aligned}$$

For each $i \in \{1, \dots, n\}$ an $y_i^* \in B_{X^*}$ we define

$$j(i) = \min\{0 \leq j \leq k : \forall j'. j n_j \in F \text{ and } x_i^*(x_{j'}) > 0\},$$

and put $y_i^* = x_i^*$ if $j(i) = 0$. Other wise we apply our induction hypothesis to $F' = \{1, 2, \dots, n_{j(i)-1}\} \cap F$ and to $L' = \{n_{j(i)}, n_{j(i)+1}, \dots, n_k, \ell_{n+1}, \ell_{n+2}, \dots\} \subset N_{j(i)-1}$ to find a y_i^* so that

$$(2.9) \quad \sum_{j \in F \cup \{\ell_{n+1}, \ell_{n+2}, \dots, \ell_{n+m_i}\}} y_i^*(y_j)^+ > B - \varepsilon$$

$$\begin{aligned} |y_i^*(x_j)| &< \varepsilon_i \text{ for } j = n_i \in \{n_1, n_2, \dots, n_k\} \setminus F \\ |y_i^*(x_j)| &< \varepsilon_i \text{ for } j = n_i \in F \text{ with } y_i^*(x_j) < 0. \end{aligned}$$

Note that (2.9) is also satisfied if $j(i) = 0$. From (2.8) we deduce therefore for each $i = 1, 2, \dots, n$ that

$$|y_i^*(x_{\ell_i})| \geq \varepsilon_{k+1}.$$

Letting $m = \max_{i \leq n} m_i$ and choosing $i_0 \leq n$ so that $m_{i_0} = m$

$$(2.10) \quad \begin{aligned} \sum_{j \in F \cup \{\ell_{n+1}, \ell_{n+2}, \dots, \ell_{n+m}\}} y_{i_0}^*(x_j)^+ &> B - \varepsilon \\ |y_{i_0}^*(x_j)| &< \varepsilon_i \text{ for } j = n_i \in \{n_1, n_2, \dots, n_k\} \setminus F \\ |y_{i_0}^*(x_j)| &< \varepsilon_i \text{ for } j = n_i \in F \text{ with } y^*(x_j) < 0. \end{aligned}$$

which implies that for all $i \in \{1, 2, \dots, n\}$

$$(2.11) \quad |y_{i_0}^*(x_{\ell_i})| \geq \varepsilon_{k+1}.$$

We ended up proving the following: for any $n \in \mathbb{N}$ there is a $z_n^* \in B_{X^*}$ (namely $z_n^* = y_{i_0}^*$ from (2.11)) so that $z_n^*(x_{\ell_i}) \geq \varepsilon_{k+1}$ for all $i \leq n$. Let z^* be an accumulation point of the sequence (z_n^*) then it follows that $z^*(x_i) \geq \varepsilon_k$ for all $i \in \mathbb{N}$. But this is a contradiction of the assumption that (x_i) is a weak null sequence, and finishes our claim.

Write $\mathcal{P}(\{n_1, \dots, n_k\})$ as $\{F_1, F_2, \dots, F_{2^k}\}$ we can apply our proven claim to all $F = F_i$ successively and get $N_{k-1} \supset L_1 \supset L_2 \supset \dots \supset L_{2^k}$ so that $[L_i]^\omega \subset \mathcal{A}_{F_i}$. Finally we chose $N_k = L_{2^k}$ which finishes our induction step.

The claim of our Lemma now follows if we take the subsequence $(y_i) = (x_{n_i})$. \square

A second type of *partial unconditionality* is the following result due to E. Odell. We first need the following Definition

Theorem 2.4.3. [O2] (*Schreier Unconditionality*) *Let (x_i) be a normalized weakly null sequence in a Banach space X and let $\varepsilon > 0$.*

Then there exists a basic subsequence (y_i) of (x_i) which is $(2+\varepsilon)$ Schreier unconditional, which means the projections:

$$P_A : \overline{\text{span}(y_i)} \ni \sum a_i y_i \mapsto \sum_{i \in A} a_i y_i,$$

are of norm not exceeding $2 + \varepsilon$ provided $A \in \mathcal{S}_1$. \mathcal{S}_1 denotes the Schreier sets in $[\mathbb{N}]^{<\omega}$ introduced in Example 2.2.8.

The proof is sketched in the following Exercise.

Exercise 2.4.4. Let (x_i) be a normalized weakly null sequence in X

- a) Let $\varepsilon > 0$, $k \in \mathbb{N}$ and I_1, I_2, \dots, I_k be subintervals of $[-1, 1]$. Show that there is a subsequence (y_i) of (x_i) so that

- (*) If there exists an $x^* \in B_{X^*}$ so that $x^*(y_{m_i}) \in I_i$, for $i = 1, 2, \dots, k$, for some choice of $m_1 < m_2 < \dots < m_k$ in \mathbb{N} then there is also a $y^* \in B_{X^*}$ so that $y^*(y_{m_i}) \in I_i$, for $i = 1, 2, \dots, k$, and $\sum_{j \in \mathbb{N} \setminus \{m_1, m_2, \dots, m_k\}} |y^*(y_{m_j})| < \varepsilon$.

Hint: Let $(\varepsilon_m) \subset (0, 1)$, with $\sum \varepsilon_m < \varepsilon$. By induction choose $n_1 < n_2 < \dots$ and $\mathbb{N} = N_0 \supset N_1 \supset N_2$ so that for all $m \in \mathbb{N}$
 $n_m = \min N_{m-1}$ and
 for all $L = \{\ell_0, \ell_1, \ell_2, \dots\} \subset N_m$, all $k' \leq k$ and all $F = \{f_1, f_2, \dots, f_{k'}\} \subset \{n_1, n_2, \dots, n_m\}$
 the following implication holds:

$$\begin{aligned} & \exists x^* \in B_{X^*} \\ & (x^*(x_{f_1}), x^*(x_{f_2}), \dots, x^*(x_{f_{k'}}), x^*(x_{\ell_1}), x^*(x_{\ell_2}), \dots, x^*(x_{\ell_{k-k'}})) \in I_1 \times I_2 \times \dots \times I_k \\ & \Rightarrow \\ & \exists y^* \in B_{X^*} \\ & (y^*(x_{f_1}), y^*(x_{f_2}), \dots, y^*(x_{f_{k'}}), y^*(x_{\ell_1}), y^*(x_{\ell_2}), \dots, y^*(x_{\ell_{k-k'}})) \in I_1 \times I_2 \times \dots \times I_k \\ & |y^*(x_{n_i})| < \varepsilon_i \text{ if } n_i \in \{n_1, n_2, \dots, n_m\} \setminus F \text{ and } |y^*(x_{\ell_0})| < \varepsilon_{m+1} \end{aligned}$$

- b) Let $k \in \mathbb{N}$ and $\varepsilon > 0$. Show that there is a subsequence (y_i) of (x_i) so that for all $(a_i) \subset c_{00}$ and all $I \subset \mathbb{N}$, with $\#I \leq k$:

$$\left\| \sum_{i \in I} a_i y_i \right\| \leq (1 + \varepsilon) \left\| \sum a_i y_i \right\|.$$

- c) Prove Theorem 2.4.3.

Chapter 3

Distortion of Banach spaces

3.1 Introduction

We are interested in the following type of problem:

Let X be an infinite dimensional and separable Banach space and let $(A_i)_{i=1}^r$ be an r -coloring of the sphere S_X . Can we choose an infinite dimensional subspace Y so that S_Y is monochromatic?

As stated above it is easy to find a counterexample:

Example 3.1.1. Let X be a space with a normalized basis (x_i) and (x_i^*) the corresponding coordinate functionals. For each $x = \sum_{i=1}^{\infty} x_i^*(x)x_i \in S_X$ let $i_0(x) \in \mathbb{N}$ be the minimum of all i 's in \mathbb{N} for which $|x_i^*(x)| = \max_{j \in \mathbb{N}} |x_j^*(x)|$.

Then let

$$A = \{x \in S_X : x_{i_0(x)}^*(x) > 0\} \text{ and } B = \{x \in S_X : x_{i_0(x)}^*(x) < 0\}.$$

Then X does not contain an infinite dimensional (actually not even a one dimensional) subspace which is monochromatic with respect to the coloring (A, B) .

As usual in Analysis, one has to allow arbitrary small perturbations in order to come to the right question.

Problem 3.1.2. Let X be Banach space and let $(A_i)_{i=1}^r$ be a coloring of S_X . Does there exist for any $\varepsilon > 0$ an infinite dimensional subspace Y and an $i \leq r$ so that

$$S_Y \subset (A_i)_{\varepsilon} := \{x \in S_X : \text{dist}(x, A_i) < \varepsilon\}?$$

Definition 3.1.3. Let $f : S_X \rightarrow \mathbb{R}$. We say that f stabilizes on infinite dimensional subspaces if for all infinite dimensional $Y \hookrightarrow X$ and $\varepsilon > 0$ there exists an infinite dimensional $Z \hookrightarrow Y$ so that

$$\text{osc}(f, S_Z) \equiv \sup\{f(z_1) - f(z_2) : z_1, z_2 \in S_Z\} < \varepsilon.$$

Closely connected to the the notion of stabilization is the notion of distortion of norms.

Definition 3.1.4. Let X be an infinite dimensional and separable Banach space and Let $\lambda > 1$. X is λ -*distortable* if there exists an equivalent norm $|\cdot|$ on X so that for all $Y \subseteq X$,

$$\sup \left\{ \frac{|y_1|}{|y_2|} : y_1, y_2 \in S_{(Y, \|\cdot\|)} \right\} \geq \lambda .$$

In that case we call $|\cdot|$ a λ -distortion.

X is *distortable* if X is λ -distortable for some $\lambda > 1$. X is *arbitrarily distortable* if X is λ -distortable for all $\lambda > 1$.

Exercise 3.1.5. Let the Banach space X have a basis (x_n) and let $f : S_X \rightarrow \mathbb{R}$ be Lipschitz continuous.

Prove that f stabilizes if and only if for every block subspace Y of X and for every $\varepsilon > 0$. there is a blocksubspace Z of Y so that $\text{osc}(f, S_Z) < \varepsilon$.

Hint: Proposition 2.1.14

Exercise 3.1.6. Define for $x = (\xi_i) \in \ell_2$

$$\|x\| = \left(\sum |\xi_i|^2 \right)^{1/2} + \max_{i \in \mathbb{N}} |\xi_i|.$$

Show that $\|\cdot\|$ is an equivalent norm on ℓ_2 , but not a distortion of the usual norm.

Proposition 3.1.7. For a separable Banach space X the following are equivalent.

- Every Lipschitz function $f : S_X \rightarrow \mathbb{R}$ stabilizes.
- Every symmetric Lipschitz function $f : S_X \rightarrow \mathbb{R}$ ($f(x) = f(-x)$ for $x \in S_X$) stabilizes.
- For any r -coloring $(A_i)_{i=1}^r$, any infinite dimensional $Y \hookrightarrow X$ and any $\varepsilon > 0$ there is an infinite dimensional $Z \hookrightarrow Y$ and an $i \leq r$ so that $S_Y \subset (A_i)_\varepsilon$
- For any covering $(A_i)_{i=1}^r$ with symmetric sets of S_X any infinite dimensional $Y \hookrightarrow X$ and any $\varepsilon > 0$ there is an infinite dimensional $Z \hookrightarrow Y$ and an $i \leq r$ so that $S_Y \subset (A_i)_\varepsilon$.

Proof. (a) \Rightarrow (b) clear.

(b) \Rightarrow (a) Note that a Lipschitz function $f : S_X \rightarrow \mathbb{R}$ can be written as

$$f = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)).$$

If (b) holds then $\frac{1}{2}(f(x) + f(-x))$ as well $|\frac{1}{2}(f(x) - f(-x))|$ stabilizes. But $|\frac{1}{2}(f(x) - f(-x))|$ can only stabilize at 0, i.e. for all infinite dimensional subspaces $Y \hookrightarrow X$ and all $\varepsilon > 0$ there exists an infinite dimensional $Z \hookrightarrow Y$ so that $|\frac{1}{2}(f(x) - f(-x))| < \varepsilon$ and we can find a further infinite dimensional $U \hookrightarrow Z$ so that $\text{osc}(\frac{1}{2}(f(x) + f(-x)), S_U) < \varepsilon$, which yields that

$$\text{osc}(f, S_U) \leq \text{osc}\left(\frac{1}{2}(f(x) + f(-x)), S_U\right) + \text{osc}\left(\frac{1}{2}(f(x) - f(-x)), S_U\right) < 2\varepsilon.$$

(a) \Rightarrow (c) Let $(A_i)_{i=1}^r$, and apply (a) to the Lipschitz functions $f_i(x) = \text{dist}(A_i, x)$ and note that $\min f_i = 0$.

(c) \Rightarrow (d) clear

(d) \Rightarrow (b) Let $f : S_X \rightarrow [0, 1]$ (range of f is bounded) be symmetric and Lipschitz and $\varepsilon > 0$ and apply (d) to the sets

$$A_i = [i - 1)\varepsilon, i\varepsilon], \text{ for } i = 1, 2, \dots, \lceil \frac{1}{\varepsilon} \rceil.$$

□

Definition 3.1.8. A Banach space is called *uniformly convex* if for all ε there exists a $\delta = \delta(\varepsilon)$ so that

$$\forall x, y \in S_X \quad \|x - y\| \geq \varepsilon \Rightarrow \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

In that case we call

$$\delta_X : (0, 1) \rightarrow (0, 1), \quad \delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in S_X, \|x - y\| \geq \varepsilon \right\},$$

the *modulus of uniform convexity* of X .

Lemma 3.1.9. Assume that X is uniform convex. Then there exists for every $0 < \varepsilon < 1$ a $\eta = \eta(\varepsilon)$ so that for any $x \in S_X$

$$\text{dist}(x, \text{co}(B_X \setminus B_\varepsilon(x))) \geq \eta(\varepsilon),$$

where $\text{co}(A)$ for $A \subset X$ denotes the convex hull A , i.e. the set

$$\text{co}(A) = \left\{ \sum_{i=1}^n \alpha_i x_i : x_i \in A, \alpha_i \geq 0, \sum \alpha_i = 1 \right\} = \bigcap \{C \subset X : C \text{ convex and } A \subset C\}.$$

Proof. Assume our claim is not true for some $\varepsilon > 0$. Then we can find sequences (x_n) and (z_n) with $z_n \in \text{co}(B_\varepsilon(x_n))$, for $n \in \mathbb{N}$, so that $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$. Write each z_n as

$$z_n = \sum_{i=1}^{k_n} \alpha_i^{(n)} y_i^{(n)}, \text{ with } k_n \in \mathbb{N}, \alpha_i^{(n)} \geq 0, \sum_{i=1}^{k_n} \alpha_i^{(n)} = 1, \text{ and } y_i^{(n)} \in B_X \setminus B_\varepsilon(z),$$

and choose for $n \in \mathbb{N}$ $x_n^* \in S_{X^*}$ with $x_n^*(x_n) = 1$. It follows that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} \alpha_i^{(n)} x_n^*(y_i^{(n)}) = 1,$$

and thus we can choose for $n \in \mathbb{N}$ an $i_n \in \{1, 2, \dots, k_n\}$ so that

$$\lim_{n \rightarrow \infty} x_n^*(y_{i_n}^{(n)}) = 1.$$

But this is a contradiction to the assumed uniform convexity since on one hand $\|x_n - y_{i_n}^{(n)}\| \geq \varepsilon$ for each $n \in \mathbb{N}$ and on the other hand

$$\limsup_{n \in \mathbb{N}} \|x_n + y_{i_n}^{(n)}\| \geq \limsup_{n \in \mathbb{N}} x_n^*(x_n + y_{i_n}^{(n)}) = 2.$$

□

Proposition 3.1.10. *If there is a Lipschitz function f on S_X which does not stabilize, then there is a distortion on an infinite dimensional subspace Y of X .*

Proof. By Proposition 3.1.7. there is a symmetric set $A \subset S_X$, an infinite dimensional subspace Y of X and an $0 < \varepsilon < 1/4$ so that for every infinite dimensional subspace Z of Y

$$(3.1) \quad A \cap S_Z \neq \emptyset \text{ and } S_Z \setminus A_\varepsilon \neq \emptyset.$$

Now define B to be the closed convex hull of $A \cup \frac{1}{4}B_{X, \|\cdot\|}$ and let $\|\cdot\|$ be the Minkowski functional with respect to $\|\cdot\|$, i.e.

$$\|x\| = \inf \{r : x \in rB\} \text{ for } x \in X.$$

Since A is symmetric it follows that $\|\cdot\|$ is a norm and, since $\frac{1}{4}B_{X, \|\cdot\|} \subset B \subset B_{X, \|\cdot\|}$ it follows for any $x \in X$,

$$\|x\| \leq \|x\| \leq 4\|x\|.$$

We claim that $\|\cdot\|$ is a distortion on the space Y . Indeed, let $Z \hookrightarrow Y$ be infinite dimensional. By the first part of (3.1) there is an $z_1 \in S_Z \cap A$, and thus $\|z_1\| = \|z_1\| = 1$. By the second part of (3.1) there is an $z_2 \in S_Z \setminus A_\varepsilon$. It follows that (let $\eta(\varepsilon)$ chosen as in Lemma

$$\begin{aligned} \|z_2\| &= \inf\{r : z_2 \in rB\} \text{ for } x \in X \\ &\geq \inf\{r : z_2 \in r\text{co}(B_X \setminus B_\varepsilon(z_2))\} \text{ for } x \in X \\ &[\text{Since } A \cup \frac{1}{4}B_X \subset B_X \setminus B_\varepsilon(z_2)] \\ &\geq 1 + \frac{1}{2}\eta(\varepsilon) \\ &[\text{Otherwise } z_2 \in (1 + \frac{3}{4}\eta(\varepsilon))\text{co}(B_X \setminus B_\varepsilon(z_2))] \\ &\quad \text{and, thus, } \text{dist}(z_2, \text{co}(B_X \setminus B_\varepsilon(z_2))) < \eta(\varepsilon)]. \end{aligned}$$

It follows that $\|\cdot\|$ is a $(1 + \eta(\varepsilon))$ distortion. \square

Theorem 3.1.11. [Ja2] ℓ_1 and c_0 are not distortable.

Proof. We first consider ℓ_1 and denote the usual norm on ℓ_1 by $\|\cdot\|$.

Let $\|\cdot\|$ be an equivalent norm on ℓ_1 .

Put

$$r = \lim_{n \rightarrow \infty} r_n \text{ with } r_n = \inf \{ \|x\| : x \in \text{span}(e_i : i \geq n) : \|x\| = 1 \} \text{ for } n \in \mathbb{N},$$

and let $0 < \varepsilon < r/10$. Then choose $n \in \mathbb{N}$ so that $r - \varepsilon/2 \leq r_n \leq r$ and then choose a block basis (x_i) of $(e_i)_{i \geq n}$ which is normalized with respect to $\|\cdot\|$ and so that $r - \varepsilon \leq \|x_i\| \leq r + \varepsilon$,

Then it follows for any $(a_i) \in c_{00}$ with $\sum |a_i| = 1$, by the triangular inequality

$$\left\| \sum a_i x_i \right\| \leq \sum |a_i| \|x_i\| \leq r + \varepsilon.$$

On the other hand

$$\left\| \sum a_i x_i \right\| \geq \inf \{ \|x\| : x \in \text{span}(e_i : i \geq n) : \|x\| = 1 \} = r_n \geq r - \varepsilon/2,$$

thus for all $y_1, y_2 \in \text{span}(x_i : i \in \mathbb{N})$, $\|y_1\| = \|y_2\| = 1$ it follows that

$$\frac{\| \|y_1\| \| \|y_2\| \|}{\| \|y_2\| \|} \leq \frac{r + \varepsilon}{r - \varepsilon/2},$$

which can be made arbitrary close to 1 providing we choose $\varepsilon > 0$ small enough.

Secondly we consider the space c_0 and denote its usual norm by $\|\cdot\|$.

Let $\| \cdot \|$ be an equivalent norm on c_0 , define

$$r = \lim r_n \text{ with } r_n = \sup \{ \| \|x\| \| : x \in \text{span}(e_i : i \geq n) : \|x\| = 1 \} \text{ for } n \in \mathbb{N},$$

and let $\varepsilon > 0$.

Choose $n \in \mathbb{N}$ large enough so that $r \leq r_n \leq r + \varepsilon$, and choose (x_i) to be a block sequence of $(e_i)_{i \geq n}$, which is normalized with respect to $\|\cdot\|$, so that

$$r - \varepsilon \leq \| \|x_i\| \| \leq r_n \leq r + \varepsilon.$$

Let $(a_i)c_{00}$ with $1 = |a_{i_0}| = \max |a_i|$. Then

$$\| \left\| \sum a_i x_i \right\| \| \leq r_n \leq r + \varepsilon = (r + \varepsilon) \left\| \sum a_i x_i \right\|.$$

On the other hand

$$\| \left\| \sum a_i x_i \right\| \| + \left\| \left\| \sum_{i \neq i_0} a_i x_i - a_{i_0} x_{i_0} \right\| \right\| \geq 2 \| \|x_{i_0}\| \| \geq 2(r - \varepsilon),$$

and thus

$$\left\| \left\| \sum a_i x_i \right\| \right\| \geq 2(r - \varepsilon) - \left\| \left\| \sum_{i \neq i_0} a_i x_i - a_{i_0} x_{i_0} \right\| \right\| \geq r - 3\varepsilon = (r - 3\varepsilon) \left\| \sum a_i x_i \right\|.$$

Since $\varepsilon > 0$ is arbitrary small □

Exercise 3.1.12. Show the following finite version of James' Theorem for ℓ_1 .

Let $n, k \in \mathbb{N}$ and $C > 1$ and assume that $(f_i)_{i=1}^{n^k}$ is a normalized sequence which is C -equivalent to the unit vector basis of $\ell_1^{n^k}$. Show that there is a normalized block of $(f_i)_{i=1}^{n^k}$ of length n which is $C^{1/k}$ -equivalent to the unit vector basis of ℓ_1^n .

The following consequence of Dvoretzky's Theorem is due to Milman and Schechtman.

Theorem 3.1.13. (see [MS, p.6]) *For every $\varepsilon > 0$ and any $k \in \mathbb{N}$ there is an $n = n(\varepsilon, k)$ so that if E is an n dimensional $f : S_X \rightarrow \mathbb{R}$ is Lipschitz with constant 1, there is a k dimensional subspace $F \hookrightarrow E$ so that $\text{osc}(f, S_F) \leq \varepsilon$*

Let us finish this section by giving giving a short overview about the history of distortion:

(1964) James [Ja2] showed that ℓ_1 and c_0 are not distortable.

(1971) At this time it was not yet known whether or not there existed an infinite dimensional Banach space which does not contain ℓ_p , for some $1 \leq p < \infty$, or c_0 . Milman [M] showed:

An infinite dimensional Banach space X which does not contain ℓ_p , for some $1 \leq p < \infty$, must have a distortable infinite dimensional subspace.

(1974) Tsirelson [T] constructed the first known space not containing ℓ_p for some $1 \leq p < \infty$, or c_0 .

(1974) Figiel and Johnson [FJ] discription of the dual of Tsirelson's space and construction of a uniform convex space not containing ℓ_p for some $1 < p < \infty$.

(1990) Haydon, Odell, Rosenthal and Schlumprecht (see [OS3]) proved the following refinement of Milman's result: If X is a separable space not containing ℓ_p or c_0 , then there exists a $z \in Z$ so that the following norm $\|\cdot\|_z$ distorts an infinite dimensional subspace:

$$\|\cdot\|_z : X \rightarrow [0, \infty), \quad x \mapsto \|x\|_z := \|z\|x\| + \|x\| + \|z\|x\| - \|x\|.$$

(1991) Gowers [Go1] showed that all Lipschitz functions on the sphere of c_0 stabilize.

(1991) First arbitrarily distortable Banach space was constructed by author of these notes [S1]

(1993) Gowers and Maurey: space without any unconditional basic sequence.

(1994) By Odell and the author of these notes [OS1] it was shown that all ℓ_p 's, $1 < p < \infty$ are arbitrarily distortable.

(1995) Maurey [Mau] showed that every asymptotic ℓ_p space has an arbitrarily distortable subspace.

Still Open: Is Tsirelson space arbitrarily distortable? More generally, is every distortable space arbitrarily distortable?

3.2 Spaces not containing ℓ_p or c_0

In 1974 Tsirelson [T] constructed the first known Banach space which did not contain any subspace isomorphic to ℓ_p , $1 \leq p < \infty$, or c_0 in this notes we will the dual of Tsirelson's original construction, a space which constructed by Figiel and Johnson [FJ], and which does not contain copies ℓ_p , $1 \leq p < \infty$, or c_0 , either.

Notation. For $A, B \in [\mathbb{N}]^{<\omega}$ we will write $A < B$ if $\max A < \min B$. For $n \in \mathbb{N}$ we write $n \leq A$ if $n \leq \min A$.

We introduce the convention that $\min \emptyset = \infty$ and $\max \emptyset = 0$, which implies that $\emptyset < A$ and $A > \emptyset$ for any $A \in [\mathbb{N}]^{<\omega}$.

For $x = \sum_{i=1}^{\infty} a_i e_i \in c_{00}$ and $E \in [\mathbb{N}]^{<\omega}$ we write

$$E(x) = \sum_{i \in E} a_i e_i.$$

Definition 3.2.1. Recall the *Schreier sets*

$$\mathcal{S}_1 := \{A \subset [\mathbb{N}]^{<\omega} : \#A \leq \min A\}.$$

A finite sequence $(E_i)_{i=1}^n \subset [\mathbb{N}]^{<\omega}$ is called \mathcal{S}_1 -admissible if $E_1 < E_2 < \dots < E_n$ and if $\{\min E_i : i = 1, 2, \dots, n\} \in \mathcal{S}_1$, i.e. if $n \leq \min E_1$.

Definition 3.2.2. For each $n \in \mathbb{N}_0$ we define by induction a norm $|\cdot|_n$ on c_{00} .

$$|x|_0 = \|x\|_{\infty} \text{ for } x \in c_{00}$$

and assuming $|\cdot|_{n-1}$ has been defined for some $n \in \mathbb{N}$

$$|x|_n = \max \left(|x|_{n-1}, \max_{(E_i)_{i=1}^n \in \mathcal{S}_1\text{-admissible}} \frac{1}{2} \sum_{i=1}^n |E_i(x)|_{n-1} \right).$$

Note that for any $x \in c_{00}$ the sequence $(|x|_n)$ is increasing and must become constant, and we put

$$\|x\| = \max_n |x|_n.$$

T (sometimes also denoted by $T_{1/2}$) is then the completion of under the norm $\|\cdot\|$.

First some easy observations.

Proposition 3.2.3. *The unit vector basis is a 1-unconditional basis (e_i) of T .*

Proposition 3.2.4. *The norm on T satisfies the following implicit equation. For $x = (\xi_i) \in T$*

$$(3.2) \quad \|x\| = \max \left(\|x\|_{\infty}, \sup_{n \in \mathbb{N}, (E_i)_{i=1}^n \in \mathcal{S}_1\text{-admissible}} \frac{1}{2} \sum_{i=1}^n \|E_i(x)\| \right)$$

(if $x \in c_{00}$ then above "sup" can be replaced by "max").

Proof. We only need to show the claim for $x \in c_{00}$ since for general $x \in T$ it follows by density of c_{00} in T .

Let $x \in c_{00}$. If $\|x\| = \|x\|_\infty$ then clearly the right side of (3.2) is at least as large as the right side. Otherwise there is an $k \in \mathbb{N}$, and an $\mathcal{S} - 1$ -admissible collection $(E_i)_{i=1}^n \subset [\mathbb{N}]^{<\omega}$ so that

$$\|x\| = |x|_n = \frac{1}{2} \sum_{i=1}^n |E_i(x)|_{k-1} \leq \frac{1}{2} \sum_{i=1}^n \|E_i(x)\|.$$

To show the converse let $(E_i)_{i=1}^n$ be \mathcal{S}_1 admissible. We can choose appropriate k_1, k_2, \dots, k_n so that

$$\frac{1}{2} \sum_{i=1}^n \|E_i(x)\| = \frac{1}{2} \sum_{i=1}^n |E_i(x)|_{k_i} \leq \frac{1}{2} \sum_{i=1}^n |E_i(x)|_{\max_{i \leq n} (k_i)} \leq |x|_{1+\max_{i \leq n} (k_i)} \leq \|x\|.$$

□

The following is an alternate way of constructing T , via normalising functionals

Proposition 3.2.5. *Define by induction on $n = 0, 1 \dots$ the following sets $A_n \subset c_{00}$:*

$$A_0 = \{\pm e_i : i \in \mathbb{N}\},$$

and assuming A_{n-1} has been defined for some $n \in \mathbb{N}$ we put

$$A_n = A_{n-1} \cup \left\{ \frac{1}{2} \sum_{i=1}^k f_i : k \in \mathbb{N}, f_i \in A_{n-1}, \text{ for } i \leq k, (\text{supp}(f_i))_{i=1}^k \text{ is } \mathcal{S}_1\text{-admissible} \right\}$$

Finally put $A = \bigcup A_i$. Let $\|\cdot\|$ be the norm on T then

$$(3.3) \quad \|x\| = \sup_{f \in A} \langle f, x \rangle.$$

Proof. First we proof by induction that $A_n \subset B_{T^*}$. For $n = 0$ this is clear and, assuming $A_{n-1} \subset B_{T^*}$ and

$$f = \frac{1}{2} \sum_{i=1}^k f_i \in A_n,$$

then for $x \in T$

$$|\langle f, x \rangle| = \frac{1}{2} \sum_{i=1}^k \langle f_i, x \rangle \leq \frac{1}{2} \sum_{i=1}^k \|\text{supp}(f_i)(x)\| \leq \|x\|.$$

Thus, we showed for $x \in c_{00}$. that $\|x\| \geq \sup_{f \in A} |\langle f, x \rangle|$

The proof of \leq will be shown by induction on the cardinality of $\text{supp}(x)$. For $k = \#\text{supp}(x) = 1$ the claim is trivial.

If our claim is true for k and if $x \in c_{00}$ with $\text{supp}(x) = k + 1$ then either $\|x\| = \|x\|_\infty$, in which case we are done. Other wise we find sets E_1, E_2, \dots, E_n in $[\mathbb{N}]^{<\omega}$ with $n \leq E_1 < E_2 \dots E_n$ so that

$$\|x\| = \frac{1}{2} \sum_{i=1}^n \|E_i(x)\|$$

$$\leq \frac{1}{2} \sum_{i=1}^n f_i(x)$$

[By induction for appropriate $f_1, \dots, f_n \in A$, and, since (e_i) is 1-unconditional basis we can assume that $n \leq \text{supp}(f_1) < \dots < \text{supp}(f_n)$]

$$\leq f(x)$$

$$[\text{with } f = \frac{1}{2} \sum_{i=1}^n f_i].$$

□

Proposition 3.2.6. *For any normalized block basis (x_i) in T it follows that*

$$\left\| \sum_{i=1}^n x_i \right\| \geq \frac{n}{4}.$$

In particular, T cannot contain any copy of ℓ_p , with $1 < p < \infty$, or c_0 .

Theorem 3.2.7. *For every $\lambda < 2$ the space T is λ -distortable.*

Remark. It is not known whether or not T is λ -distortable for some $\lambda > 2$.

Proof. For $k \in \mathbb{N}$ define

$$\|\cdot\|_k : T \rightarrow [0, \infty) : x \mapsto \sup_{\substack{(E_i)_{i=1}^j \text{ is } \mathcal{S}_1\text{-admissible, and } 1 \leq j \leq k}} \frac{1}{2} \sum_{i=1}^j \|E_i(x)\|.$$

It is clear that

$$(3.4) \quad \frac{1}{2} \|x\| \leq \|x\|_k \leq \|x\| \text{ and}$$

$$(3.5) \quad \|x\| = \max(\|x\|_\infty, \sup_{k \in \mathbb{N}} \|x\|_k) \text{ for all } k \in \mathbb{N} \text{ and } x \in c_{00}.$$

We will show that $\|\cdot\|_k$ is a λ_k -distortion of T , with $\lim_{k \rightarrow \infty} \lambda_k = 2$.

Step 1: ℓ_1 is *finitely block represented in every blocks pace of T* . This means that for any normalized block (x_i) , any $\varepsilon > 0$ and any $n \in \mathbb{N}$ there is a normalized block $(y_i)_{i=1}^n$ of (x_i) so that $(y_i)_{i=1}^n$ is $(1 + \varepsilon)$ -equivalent to the ℓ_1^n -unit vector basis.

This follows from the fact that any block sequence $(y_i)_{i=1}^\ell$, with $\ell \leq \text{supp}(y_1)$, is 2-equivalent to the ℓ_1^n -unit vector basis and then we can choose k so that $2^{1/k} < 1 + \varepsilon$, $\ell = n^k$ and apply exercise 3.1.12.

For $\varepsilon > 0$ and $n \in \mathbb{N}$ we will call $z \in T$ an ε - ℓ_1^n -average if

$$z = \frac{1}{n} \sum_{i=1}^n z_i$$

where (z_i) is a normalized block in T which is $(1 + \varepsilon)$ -equivalent to the ℓ_1^n unit vector basis.

Note that in particular it follows for an $\varepsilon\text{-}\ell_1^n$ -average z that $\|z\| \geq \frac{1}{1+\varepsilon}$.
 Step 2: For $n, k \in \mathbb{N}$ and any normalized block $(z_i)_{i=1}^n$

$$(3.6) \quad \left\| \frac{1}{n} \sum_{i=1}^n z_i \right\|_k \leq \frac{1}{2} \frac{n+k}{n}$$

Indeed write $z = \frac{1}{n} \sum_{i=1}^n z_i$, let $m \leq k$ and $m \leq E_1 < E_2 < \dots < E_m$ in $[\mathbb{N}]^{<\omega}$ so that

$$\|z\|_k = \frac{1}{2} \sum_{j=1}^m \|E_j(z)\|.$$

For $j = 1, 2, \dots, m$ put $\ell_j = \max\{1 \leq i \leq n : \min \text{supp}(z_i) \leq \max E_j\}$ (putting $\max \emptyset = 0$) and put $\ell_0 = 1$. Since $(z_i)_{i=1}^n$ is a block basis it follows

$$\{i \in \{1, 2, \dots, n\} : \text{supp}(z_i) \cap E_j \neq \emptyset\} \subset [\ell_{j-1}, \ell_j],$$

since the unit vector basis in T is 1-unconditional we deduce that

$$\sum_{j=1}^m \|E_j(z)\| \leq \frac{1}{n} \sum_{j=1}^m \left\| \sum_{i=\ell_{j-1}}^{\ell_j} z_i \right\| \leq \frac{1}{n} \sum_{j=1}^m (\ell_j - \ell_{j-1} + 1) = \frac{n+m}{n} \leq \frac{n+k}{n},$$

which implies our claim.

Step 3: Let $\varepsilon > 0$ and ℓ . We call a block $(y_i)_{i=1}^\ell$ a ε -rapidly increasing sequence of length ℓ (RIS) if for each $i = 1, \dots, \ell$ y_i is a $\varepsilon\text{-}\ell_1^{n_i}$ average with $(n_i) \subset \mathbb{N}$ satisfying:

$$(3.7) \quad \ell < \varepsilon n_1$$

$$(3.8) \quad l \leq \min(\text{supp}(y_1)) \text{ and } \max \text{supp}(y_{i-1}) < n_i \text{ and } n_{i-1} < \varepsilon n_i \text{ for } i = 2, 3, \dots, \ell$$

Let $\varepsilon > 0$ and let $\ell \in \mathbb{N}$, $\ell \geq 1$. If $(y_i)_{i=1}^\ell$ is ε -rapidly increasing sequence of length ℓ then

$$(3.9) \quad \left\| \frac{1}{\ell} \sum_{i=1}^\ell y_i \right\| \leq \frac{2}{\ell} + \frac{1}{2}(1 + \varepsilon)$$

Indeed let $k \in \mathbb{N}$ for $y = \frac{1}{\ell} \sum_{i=1}^\ell y_i$ $\|y\| = \|y\|_k$ (note that $\|y\| \neq \|y\|_\infty$).
 If $k \leq \varepsilon n_1$ we deduce from (3.6) that

$$(3.10) \quad \|y\|_k \leq \frac{1}{\ell} \sum_{i=1}^\ell \|y_i\|_k \leq \frac{1}{\ell} \sum_{i=1}^\ell \frac{1}{2} \frac{n_i + k}{n_i} \leq \frac{1}{2} \frac{n_1 + k}{n_1} \frac{1}{2} (1 + \varepsilon)$$

Choose $l \leq k$ and $l \leq E_1 < E_2 < \dots < E_l$, so that

$$\|y\| = \|y\|_k = \frac{1}{2} \sum_{j=1}^l \|E_j(y)\|,$$

Put $n_{\ell+1} := \infty$ and $n_0 := \lfloor \varepsilon n_1 \rfloor$, $n_{\ell+1} := +\infty$ and choose $i_0 \in \{0, 1, 2, \dots\}$ so that $k \in (n_{i_0}, n_{i_0+1}]$. Then for it follows for $i = 1, 2, \dots, i_0 - 1$ and $j = 1, 2, \dots, k$ (note that by (3.8)

$\max(\text{supp}(x_i)) \leq n_{i_0} < k \leq E_1$) $E_j(y_i) = 0$. For $i = i_0 + 2, i_0 + 3, \dots, \ell$ it follows from (3.6) and (3.8) that

$$\|y_i\|_k \leq \frac{1}{2} \frac{n_i + k}{n_i} < \frac{1}{2} \frac{n_i + n_{i_0+1}}{n_i} \leq \frac{1}{2}(1 + \varepsilon),$$

and thus we deduce in this case that

$$\|y\|_k \leq \frac{1}{\ell} \sum_{i=i_0}^{\ell} \|y_i\|_k \leq \frac{1}{\ell} \left[2 + \sum_{i=i_0+2}^{\ell} \|y_i\|_k \right] \leq \frac{2}{\ell} + \frac{1}{2}(1 + \varepsilon),$$

which implies our claim of Step 3.

We will show that for $k \in \mathbb{N}$ the equivalent norm $\|\cdot\|$ is an $\frac{2k}{k+20}$ -distortion, which will finish the proof of our theorem.

Let Y be an arbitrary block subspace. If $n > k^2$ and if x_1 is an $\frac{1}{n}$ - ℓ_1^n average we deduce for $z_1 = x_1/\|x_1\|$ that

$$\|z_1\|_k = \frac{1}{\|x_1\|} \|x\|_k \leq \frac{1}{2} \left(1 + \frac{1}{n}\right) \frac{n+k}{n} \leq \frac{1}{2} \left(1 + \frac{1}{k}\right)^2$$

If we let and choose a

$x_2 = \frac{1}{k} \sum y_i$ and (y_i) is a $\frac{1}{k}$ -rapidly increasing sequence of length k and we put

$$z_2 = x_2/\|x_2\|$$

Then, obviously,

$$\|z_2\|_k \geq \frac{1}{\|x_2\|} \frac{1}{2} \frac{1}{k} \sum_{i=1}^k \|y_i\| \geq \frac{1}{2} \frac{1}{\|x_2\|} \frac{1}{1 + \frac{1}{k}},$$

and (3.9) of Step 3 yields that

$$\|x_2\| \leq \frac{2}{k} + \frac{1}{2} \left(1 + \frac{1}{k}\right) = \frac{1}{2} \left(1 + \frac{5}{k}\right).$$

Thus

$$\|z_2\|_k \geq \frac{1}{1 + \frac{1}{k}} \frac{1}{1 + \frac{5}{k}},$$

and

$$\frac{\|z_2\|_k}{\|z_1\|_k} \geq 2 \frac{1}{\left(1 + \frac{1}{k}\right)^2} \frac{1}{1 + \frac{5}{k}} > \frac{2k}{k+20}.$$

□

Since by James' theorem 3.1.11 ℓ_1 is not distortable deduce the following

Corollary 3.2.8. *T does not contain a copy of ℓ_1 .*

Remark. In the definition (see Definition 3.2.2) of the norms $|\cdot|_n$ one can replace $\frac{1}{2}$ by some $\gamma \in (0, 1)$, and put $\|\cdot\|_\gamma = \max |\cdot|_n$ and denote the completion of c_{00} with respect to $\|\cdot\|_\gamma$ by T_γ . Using the same arguments as in the proof of Theorem 3.2.7 one shows that T_γ is $\frac{1}{\gamma}$ -distortable. The next construction is a space which is arbitrarily distortable.

Definition 3.2.9. [S1] For $s \in [1, \infty)$ define $f(s) = \log_2(s + 1)$. For $n \in \mathbb{N}_0$ and $x \in c_{00}$ define by induction as follows.

$$|x|_0 = \|x\|_\infty$$

and, assuming that $|x|_{n-1}$ has been defined for all $x \in c_{00}$ we put

$$|x|_n = \max \left(|x|_{n-1}, \max_{\ell \in \mathbb{N}, E_1 < E_2 < \dots < E_\ell} \frac{1}{f(\ell)} \sum_{i=1}^{\ell} |E_i(x)|_{n-1} \right).$$

Then put for $x \in c_{00}$

$$\|x\| = \sup_{n \in \mathbb{N}_0} |x|_n.$$

$\|\cdot\|$ is a norm on c_{00} and we denote the completion of c_{00} with respect to $\|\cdot\|$ by S .

The following observations can be shown easily.

Proposition 3.2.10. a) *The unit vector basis is a 1-subsymmetric basis of S*

b) *The norm $\|\cdot\|$ on S satisfies the following implicit equation:*

$$(3.11) \quad \|x\| = \max \left(\|x\|_\infty, \sup_{\ell \geq 2, E_1 < E_2 < \dots < E_\ell} \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|E_i(x)\| \right)$$

whenever $x \in S$.

c) *For every normalized block $(y_i)_{i=1}^{\ell}$ in S and any $(a_i)_{i=1}^{\ell} \subset \mathbb{R}$*

$$\left\| \sum_{i=1}^{\ell} a_i y_i \right\| \geq \frac{1}{\ell} \sum_{i=1}^{\ell} |a_i|$$

d) ℓ_1 *is finitely block represented in every block subspace of S .*

(Use (c) and the finite version of James' Theorem, Exercise 3.1.12).

d) *The norm on S can also be obtained by describing its normalizing functionals in the following way.*

Define $A_0 = \{\pm e_i : i \in \mathbb{N}\}$ and assuming A_{n-1} has been defined for some $n \in \mathbb{N}$ we put

$$A_n = \left\{ \frac{1}{f(\ell)} \sum_{i=1}^{\ell} f_i : \ell \in \mathbb{N}, (f_i)_{i=1}^{\ell} \subset A_{n-1} \text{ is block sequence} \right\} \cup A_{n-1}.$$

Let $A = \bigcup_{n=1}^{\infty} A_n$ Then it follows for $x \in S$ that

$$\|x\| = \sup_{f \in A} \langle f, x \rangle.$$

Similar to the space T we define the following equivalent norms $\|\cdot\|_\ell$ for $\ell \in \mathbb{N}$. For $x \in S$ define

$$(3.12) \quad \|x\|_\ell = \frac{1}{f(\ell)} \sup_{E_1 < E_2 < \dots < E_\ell} \sum_{i=1}^{\ell} \|E_i(x)\|.$$

Clearly,

$$(3.13) \quad \frac{1}{f(\ell)} \|x\| \leq \|x\|_\ell \leq \|\cdot\| \text{ and}$$

$$(3.14) \quad \|x\| = \max \left(\|x\|_\infty, \sup_{\ell \in \mathbb{N}, \ell \geq 2} \|x\|_\ell \right)$$

Theorem 3.2.11. *For every $\ell \in \mathbb{N}$ the norm $\|\cdot\|_\ell$ is an $cf(\ell)$ -distortion of S for some fixed constant c*

Proof. For $\varepsilon > 0$ and $n \in \mathbb{N}$ use definition for ε - ℓ_1^n averages introduced in Step 1 of the proof of Theorem 3.2.7 and note that by Proposition 3.2.10 (d) every block subspace of S contains for any $\varepsilon > 0$ and $n \in \mathbb{N}$ an ε - ℓ_1^n average.

Following similar arguments one can show that if z is an ε - ℓ^n -average in S , and thus $\|z\| \geq \frac{1}{1+\varepsilon}$, and $\ell \in \mathbb{N}$ it follows that

$$(3.15) \quad \|z\|_\ell \leq \frac{1}{f(\ell)} \frac{\ell}{n + \ell},$$

which is close to $\frac{1}{f(\ell)}$ if $n \gg \ell$.

In order to show that for any $\ell \in \mathbb{N}$ there are vectors in each block subspace for which the $\|\cdot\|_\ell$ and $\|\cdot\|$ norm are approximately the same we have to redefine what we mean by a rapidly increasing sequence in S .

We call a sequence $(y_i)_{i=1}^k$ a *rapidly increasing sequence (RIS)* if there are $\varepsilon_i > 0$ and $n_i \in \mathbb{N}$ for $i = 1, \dots, k$ so that

$$(3.16) \quad \sum_{i=1}^k \varepsilon_i < 1$$

$$(3.17) \quad y_i \text{ is an } \varepsilon_i - \ell_1^{n_i} \text{ average}$$

$$(3.18) \quad n_1 > 2k$$

$$(3.19) \quad f(n_i) \geq \max \text{supp}(y_{i-1}) \text{ and } \frac{n_{i-1}}{n_i} < \varepsilon_i \text{ for } i = 2, 3, 4, \dots, k$$

and we will show that there is a constant $C \geq 1$ so that if $(y_i)_{i=1}^k$ is a rapidly increasing sequence it follows that

$$(3.20) \quad \left\| \sum_{i=1}^k y_i \right\| \leq \frac{Ck}{f(k)}.$$

Since clearly (use for the second \leq inequality (3.15))

$$\left\| \sum_{i=1}^k y_i \right\|_k \geq \frac{1}{f(k)} \sum_{i=1}^k \|y_i\| \geq \frac{k}{f(k)}$$

it follows together with (3.15) that $\|\cdot\|_k$ is a $\frac{f(k)}{C}$ -distortion.

In order to proof (3.20) we first need a little Calculus lemma

Lemma 3.2.12. *For $\ell \in \mathbb{N}$ and $D > \ell$ it follows that*

$$\max \left\{ \sum_{i=1}^{\ell} \frac{\xi_i}{f(\xi_i)} : \xi_i \geq 1 \text{ for } i = 1, \dots, \ell \text{ and } \sum \xi_i = D \right\} = \frac{D}{f(D\ell)}.$$

(i.e. maximum is achieved when $\xi_1 = \xi_2 = \dots = \xi_\ell = D/\ell$)

Proof. $[1, \infty) \ni \xi \mapsto \frac{\xi}{f(\xi)}$ is convex. □

Continuation of Proof of Theorem 3.2.11. We are choosing $C > 1$ large enough to satisfy the following conditions

$$(3.21) \quad C \geq 5 \text{ and } C \geq 2 \frac{f(k)}{f(k/2)} \text{ if } k \geq 2 \text{ (note that } \lim_{k \rightarrow \infty} \frac{f(k)}{f(k/2)} = 1 \text{)}$$

$$(3.22) \quad C \geq 2 \frac{f(k/\ell)f(\ell)}{f(\ell)f(k/\ell) - f(k)} \text{ whenever } \ell, k \in \mathbb{N} \text{ with } 2 \leq \ell \leq k/2$$

For the fulfilment of (3.22) note that for $k \in \mathbb{N}$, $k \geq 4$ that the function

$$g_k : [2, k/2] \rightarrow \mathbb{R}, \quad x \mapsto \frac{f(k)}{f(x)f(k/x)},$$

achieves its maximal value at each of the endpoints and note that

$$\sup_{k \in \mathbb{N}, 4 \geq 4} g_k(2) = \sup \frac{f(k)}{f(2)f(k/2)} = \frac{4}{f(2)f(2)} < 1,$$

Thus

$$\sup_{k \in \mathbb{N}, 2 \leq \ell \leq k/2} \frac{f(k/\ell)f(\ell)}{f(\ell)f(k/\ell) - f(k)} = \sup_{k \in \mathbb{N}, 2 \leq \ell \leq k/2} \frac{1}{1 - \frac{f(k)}{f(\ell)f(k/\ell)}} < \infty.$$

We will show by induction on $k \in \mathbb{N}$ that (3.15) hold, or equivalently that for a rapidly increasing sequence $(y_i)_{i=1}^k$ of length k and for all $\ell \geq 2$ we have

$$(3.23) \quad \left\| \sum_{i=1}^k y_i \right\|_\ell \leq \frac{kC}{f(k)}.$$

For $k = 1$ the claim is trivial. Assume that the claim is true for all $1 \leq k' < k$, let $(y_i)_{i=1}^k$ be rapidly increasing sequence of length k , put $y = \sum_{i=1}^k y_i$ and let $\ell \in \mathbb{N}$.

We consider three different cases.

Case 1: $\ell \in [2, \frac{k}{2}]$ (in particular $k \geq 4$). Choose $E_1 < E_2 < \dots < E_\ell$ so that

$$\|y\|_\ell = \frac{1}{f(\ell)} \sum_{j=1}^{\ell} \|E_j(y)\|.$$

Using the 1-unconditionality of the unit vector basis of (e_i) in S we can assume that $(E_j)_{j=1}^\ell$ is a partition of the the interval $[\min \text{supp}(y_i), \max \text{supp}(y_k)]$ into ℓ intervals and choose for $j = 1, 2, \dots, \ell$ choose $i(j) \in \{1, 2, \dots, k\}$ such that

$$\max E_j \in (\max \text{supp} y_{i(j)-1}, \max \text{supp} y_{i(j)}].$$

(with $y_0 = 0$ and $\max \text{supp}(0) = 0$). We observe that (for the first inequality it is recommended to draw a picture)

$$\left\| \sum_{i=1}^k y_i \right\|_\ell \leq \frac{1}{f(\ell)} \left[\sum_{j=1}^\ell \left\| \sum_{i=i(j-1)+1}^{i(j)-1} y_i \right\| + \sum_{j=1}^\ell \sum_{s=1}^\ell \|E_s(y_{i(j)})\| \right]$$

[If $i_{j-1} = i(j) - 1$ the first \sum vanishes, and let $i(0) = 0$]

$$\leq \frac{1}{f(\ell)} \sum_{j=1}^\ell C \frac{i(j) - i(j-1) - 1}{f(i(j) - i(j-1) - 1)} + \frac{\ell}{f(\ell)} \frac{n_1 + \ell}{n_1}$$

[For first part note that subsequences of rapidly increasing sequence are

rapidly increasing and use induction hypothesis, and let $\frac{0}{0} := 0$]

[For second part use (3.15)]

$$\leq C \frac{k}{f(\ell)f(k/\ell)} + \frac{2\ell}{f(\ell)}$$

[By Lemma 3.2.12 and by (3.18)]

$$= C \left[\frac{k}{f(\ell)f(k/\ell)} + \frac{\ell}{f(\ell)} \frac{f(\ell)f(k/\ell) - f(k)}{f(k/\ell)f(\ell)} \right]$$

[Using (3.22)]

$$= C \left[\frac{k}{f(\ell)f(k/\ell)} + \frac{k}{f(k)} \frac{f(\ell)f(k/\ell) - f(k)}{f(k/\ell)f(\ell)} \right] = C \frac{k}{f(k)}.$$

[The function $[1, \infty) \ni x \mapsto f(x)/x$ is increasing]

Case 2: $k/2 \leq \ell \leq n_1$.

In this case we use (3.15) and observe that

$$\begin{aligned} \left\| \sum_{i=1}^k y_i \right\|_\ell &\leq \sum_{i=1}^k \|y_i\|_\ell \\ &\leq \frac{1}{f(\ell)} \sum_{i=1}^k \frac{n_i + \ell}{n_i} \\ &\leq \frac{2k}{f(\ell)} \leq \frac{2k}{f(k/2)} \leq \frac{Ck}{f(k)} \text{ [By (3.19) and (3.21)]} \end{aligned}$$

Case 3: $\ell > n_1$ Choose $i_0 \in \{1, 2, \dots, k\}$ so that $\ell \in (n_{i_0}, n_{i_0+1}]$, with $n_{k+1} := \infty$.

Then we deduce that

$$\left\| \sum_{i=1}^k y_i \right\|_\ell \leq \left\| \sum_{i=1}^{i_0-1} y_i \right\|_\ell + 2 + \sum_{i=i_0+2}^k \|y_i\|_\ell$$

$$\begin{aligned}
&\leq \frac{\max \operatorname{supp}(y_{i_0} - 1)}{f(n_{i_0})} + 2 + \frac{2}{f(\ell)} \\
&[\text{By (3.15), (3.16) and (3.19) (part 2)}] \\
&\leq 5 \leq \frac{Ck}{f(k)} [\text{By (3.19) (part 1) and (3.21)}]
\end{aligned}$$

which handles the third case and finishes the proof of the induction step. \square

Remark. With more careful estimation one can actually prove that the constant C in the proof of Theorem 3.2.11 is 1. It was actually shown in [S1] that if (y_i) is a sequence of increasing ℓ_1 -averages (i.e. for some $\varepsilon > 0$ y_i is an $\varepsilon - \ell_1^{n_i}$ -average for $i \in \mathbb{N}$ and $n_i \nearrow \infty$) then (y_i) has a subsequence (\tilde{y}_i) whose spreading model is isometrically equivalent to the unit vector basis (e_i) in S .

This implies that

$$\lim_{n_k > n_{k-1} > \dots > n_1 \rightarrow \infty} \left\| \sum_{i=1}^k \tilde{y}_{n_i} \right\| = \left\| \sum_{i=1}^k e_i \right\| = \frac{k}{f(k)},$$

where the last equality follows easily from Lemma 3.2.12 by induction on $k \in \mathbb{N}$. This implies, using the arguments of the proof of 3.2.11 that $\|\cdot\|_\ell$ is a $f(\ell)$ -distortion.

In [AS] a stronger result was shown: Using a strong enough definition of what rapidly increasing sequences are (but so that nevertheless every block subspace contains such sequences) it was shown that rapidly increasing sequences are equivalent to the unit vector basis. From that it was deduced that S is complementably minimal, every infinite dimensional subspace of S has a further subspace which is isomorphic to S . It is unknown whether or not S is prime, i.e. whether or not every infinite dimensional complemented subspace of S is isomorphic to S .

Remark. From our arguments in the proof of Theorem 3.2.11 we can deduce the following observations which will be important for the construction of the space of Gowers and Maurey.

Let $(y_i)_{i=1}^k$ be an RIS of length k and $\ell \in \mathbb{N}$.

If $\ell \leq k/2$ we have shown (third inequality of handling case 1) that

$$\left\| \frac{f(k)}{k} \sum_{j=1}^k y_j \right\|_\ell \leq C \frac{f(k)}{f(\ell)f(k/\ell)} + 2 \frac{\ell f(k)}{k f(\ell)}.$$

This implies for some constant C_1 (using that f is a logarithmic function) that

$$\left\| \frac{f(k)}{k} \sum_{j=1}^k y_j \right\|_\ell \leq \frac{C_1}{f(\ell)} \text{ if } \ell \leq \sqrt{k},$$

and

$$\left\| \frac{f(k)}{k} \sum_{j=1}^k y_j \right\|_\ell \leq C \leq C_1 \text{ if } \sqrt{k} \leq \ell \leq k^2.$$

If $k^2 \leq \ell \leq n_1$ we deduce from the computation done in Cases 2 that

$$\left\| \frac{f(k)}{k} \sum_{j=1}^k y_j \right\|_{\ell} \leq C \frac{f(k)}{f(\ell)},$$

and if $\ell > n_1$ we deduce from our inequalities in case Case 3 that

$$\left\| \frac{f(k)}{k} \sum_{j=1}^k y_j \right\|_{\ell} \leq C \frac{f(k)}{k}.$$

So combining all these inequalities we finally get for some universal constant C' and all $k, \ell \in \mathbb{N}$ that

$$(3.24) \quad \left\| \frac{f(k)}{k} \sum_{j=1}^k y_j \right\|_{\ell} \leq C' \min \left(\frac{f(\ell)}{f(k)}, \max \left(\frac{f(k)}{f(\ell)}, \frac{f(k)}{k} \right) \right).$$

From Remark 3.2 we deduce the following:

Corollary 3.2.13. *Let $\varepsilon > 0$ For $m \in \mathbb{N}$ define:*

$$A_m = \left\{ \frac{f(m)}{m} \sum_{i=1}^m y_i : (y_i)_{i=1}^m \text{ is an RIS of length } m \right\}$$

$$A_m^* = \left\{ \frac{f(m)}{m} \sum_{i=1}^m y_i^* : (y_i)_{i=1}^m \text{ is a block in } S_{S^*} \right\}$$

Note that $A_m \subset CB_S$ and that $A_m^* \subset B_{S^*}$ and that for and $x \in A_m$ there is a $x^* \in A_m^*$ so that $x^*(x) \geq 1/2$ (note that $\|y_i\| \geq 1/2$ for all elements of an RIS (y_i)). Also note that

$$\|x\|_m = \sup_{x^* \in A_m^*} |x^*(x)| \text{ for all } x \in S.$$

Let $\varepsilon_i \searrow 0$. Using Remark 3.2, we can find a lacunary enough sequence $M = (m_i) \in [\mathbb{N}]^\omega$ So that

$$(3.25) \quad \forall i, j \in \mathbb{N} \forall x \in A_{m_i}, x^* \in A_{m_j}^* \quad |\langle x^*, x \rangle| \leq \varepsilon_{\min(i,j)}.$$

Definition 3.2.14. A Banach space X is *biorthogonally distortable* If there is a sequence of subsets (B_n) of B_X and a sequence of subsets (B_n^*) of B_{X^*} so that

- For all $n \in \mathbb{N}$ the set B_n is *asymptotic*, i.e. for all infinite dimensional subspaces $Y \subset X$, $\text{dist}(Y, B_n) = 0$.
- There is a $c > 0$ so that for all $n \in \mathbb{N}$ and all $x \in B_n$ there is an $x^* \in B_n^*$ so that $x^*(x) \geq c$.
- There is a sequence $\varepsilon_n \nearrow 0$ so that for all $n \neq m$ in \mathbb{N}

$$\langle B_m^*, B_n \rangle = \sup_{x^* \in B_m^*, x \in B_n} |x^*(x)| \leq \varepsilon_{\min(m,n)}.$$

In this case we call (B_n, B_n^*) *nearly biorthogonal sequences* in X .

Exercise 3.2.15. Assume X separable and has a normalized 1-unconditional basis (e_i) (so we assume it is the completion of c_{00} under the norm $\|\cdot\|$) and assume that X is biorthogonally distortable. Let (B_n) and (B_n^*) be given as in Definition 3.2.14 after passing to subsequences we can assume that (ε_n) , as given in c), is summable and $\sum_{n=1}^{\infty} \varepsilon_n \leq 1$. We can also assume that for $n \in \mathbb{N}$ $B_n \subset c_{00}$ and that B_n^* is countable and $B_n^* \subset \text{span}(e_i^*) \equiv c_{00}$ (pass to a perturbation of a subset for which (b) is still satisfied, with $c/2$ instead of c). Let $B^* = \cup B_n^*$ and let $\sigma : B^{*\omega} \rightarrow \mathbb{N}$ be injective. For $\ell \in \mathbb{N}$ define

$$\Gamma_\ell = \left\{ \sum_{i=1}^{\ell} f_i : f_i \in B^*, \text{ for } 1 \leq i \leq \ell \text{ and } f_i \in B_{\sigma(f_1, f_2, \dots, f_{i-1})}^* \right\},$$

and for $x \in c_{00}$

$$\| \|x\| \|_\ell = \|x\| + \sup_{f \in \Gamma_\ell} |f(x)|.$$

Show that $\| \cdot \|_\ell$ is an equivalent norm on X , and that there is a sequence (λ_ℓ) with $\lambda_\ell \nearrow \infty$, so that (Use the arguments in the construction of Maurey and Rosenthal in Example 2.2.11).

We can also construct a Banach space which does not contain any unconditional basic sequence

Definition 3.2.16. (The space of Gowers and Maurey, slightly modified)
We first consider a set $Q \subset c_{00} \cap [-1, 1]^\omega$ with the following properties:

- a) $c_{00} \cap ([-1, 1] \cap \mathbb{Q})^\omega \subset Q$,
- b) If $(x_1, x_2, \dots, x_\ell) \subset Q$ is a finite block sequence then

$$\frac{1}{f(\ell)} \sum_{i=1}^{\ell} x_i, \frac{1}{\sqrt{f(\ell)}} \sum_{i=1}^{\ell} x_i \in Q$$

- c) Q is countable.

Let $(\varepsilon_n) \subset (0, 1)$ summable such that

$$(3.26) \quad \sum_{i=1}^{\infty} 2\varepsilon_i \leq \sqrt{f(2)} - 1$$

Let $M \subset \mathbb{N}$ be subset which is lacunary enough satisfying the following condition:

$$(3.27) \quad \frac{k}{f(\ell)} < \varepsilon_k \text{ whenever } k, \ell \in M \text{ and } k < \ell.$$

Let $\sigma : Q^{<\omega} = \{(z_i^*)_{i=1}^{\ell} : \ell \in \mathbb{N}, z_i^* \in Q \text{ for } i \leq \ell\} \rightarrow M$ be injective.

By induction on $n \in \mathbb{N}_0$ we define a set $F_n \subset c_{00}$ as follows:

For $n = 0$ we put as usual $F_0 = \{\pm e_i i \in \mathbb{N}\}$ and assume that F_{n-1} , has been defined for some $n \in \mathbb{N}$. The define for $k \in \mathbb{N}$,

$$U(n, k) = \left\{ \frac{1}{f(k)} \sum_{i=1}^k z_i^* : k \in \mathbb{N}, (z_i^*)_{i=1}^k \subset F_{n-1} \text{ finite block} \right\}$$

and put $U_n = \bigcup_{k=1}^{\infty} U_{k=1}(n, k)$. Then define for $\ell \in \mathbb{N}$,

$$C(n, \ell) = \left\{ \frac{1}{\sqrt{f(\ell)}} \sum_{i=1}^{\ell} z_i^* : (z_i^*)_{i=1}^{\ell} \subset U_n \text{ block and } z_i^* \in U(n, \sigma(z_1^*, \dots, z_{i-1}^*)), \text{ if } i > 1 \right\}.$$

Then let

$$F_n = \{[m, n](x^*) : x^* \in \bigcup_{\ell \in \mathbb{N}} C(n, \ell)\}.$$

For $x \in c_{00}$ we finally define

$$\|x\|_{GM} = \sup_{z^* \in \bigcup F_n} |\langle f, x \rangle|.$$

And we let GM be the completion of c_{00} under $\|\cdot\|_{GM}$.

Theorem 3.2.17. [GM1] GM does not contain any unconditional basis sequence.

Sketch of a proof. An application of the finite version of James' Theorem, Exercise 3.1.12) shows that every infinite dimensional blockspace space contains ℓ_1 finitely block represented, this implies every infinite dimensional block space contains Rapidly increasing sequences defined as in the space S .

For $k \in \mathbb{N}$ we put $U(k) = \bigcup_{n \in \mathbb{N}} U(n, k)$ and $C(k) = \bigcup_{n \in \mathbb{N}} C(n, k)$, and define as in S :

$$\|x\|_k = \sup_{x^* \in \bigcup U_k} |x^*(x)| \text{ for } x \in c_{00}.$$

Let $(x_i)_{i=1}^n \subset GM$ be a normalized block in GM and $\varepsilon > 0$. We call $(x_i)_{i=1}^n$ ℓ_1^{+n} average if

$$\left\| \frac{1}{n} \sum_{i=1}^n x_i \right\| \geq \frac{1}{1 + \varepsilon}.$$

We define RIS as in S , using ℓ_1^+ averages.

We call a sequence $(y_i)_{i=1}^k$ a *rapidly increasing sequence (RIS)* in GM if there are $\varepsilon_i > 0$ and $n_i \in \mathbb{N}$ for $i = 1, \dots, k$ so that

$$(3.28) \quad \sum_{i=1}^k \varepsilon_i < 1$$

$$(3.29) \quad y_i \text{ is an } \varepsilon_i - \ell_1^{+n_i} \text{ average}$$

$$(3.30) \quad n_1 > 2k$$

$$(3.31) \quad f(n_i) \geq \max \text{supp}(y_{i-1}) \text{ and } \frac{n_{i-1}}{n_i} < \varepsilon_i \text{ for } i = 2, 3, 4, \dots, k$$

Step 1: There is a constant $C > 0$ so that for any (y_i) be an RIS and any $\ell \in \mathbb{N}$

$$(3.32) \quad \left\| \frac{f(\ell)}{\ell} \sum_{i=1}^{\ell} y_i \right\|_k \leq C \min \left(\frac{f(\ell)}{f(k)}, \frac{f(k)}{f(\ell)} \right)$$

$$(3.33) \quad \left\| \frac{f(\ell)}{\ell} \sum_{i=1}^{\ell} y_i \right\| \leq C.$$

We show this claim by induction on ℓ for the C chosen as in Theorem 3.2.11. For $\ell = 1$ this is trivial and assuming our claim is true for $\ell - 1$ and let

$$y = \frac{1}{f(\ell)} \sum_{i=1}^{\ell} y_i.$$

By following the same arguments as in the proof of Theorem 3.2.11 we deduce (3.32). In order to show (3.33) we still need to show that for any $\ell \geq 2$ and $x^* \in C(\ell)$ it follows that $|x^*(y)| \leq C$. But this follows from (3.32), the condition (3.27), and from the lacunarity condition (3.27) on M

Note: If $m \ll k$ and if $(y_i)_{i=1}^{mk}$ is an RIS of length mk . Define for $i = 1, 2, \dots, m$

$$z_i = \frac{f(k)}{k} \sum_{j=1}^k y_{j+(i-1)m}.$$

Then $\|z_i\| \sim 1$, for $i = 1, \dots, m$ and

$$\left\| \sum_{i=1}^m z_i \right\| = \frac{f(k)}{k} \|y_i\| \sim \frac{f(k)}{k} \frac{km}{f(km)} \sim m.$$

Thus $\frac{km}{f(km)} \sum_{i=1}^m y_i$ is close to an ℓ_1^{+m} average.

Step 2: GM has no unconditional basic sequence.

Let Y be a block subspace let $k \in \mathbb{N}$ We define by induction on $\ell \in \{1, 2, \dots, k\}$ the following elements $n_\ell \in \mathbb{N}$, $x_\ell \in CB_Y$, with $\text{supp}(x_{\ell-1}) < \text{supp}(x_\ell)$, if $\ell > 1$, and $x^* \in U_{n_\ell}$. $n_1 = 1$, $x_1 \in B_X \cap c_{00}$ arbitrary, and $x^* \in U_1^*$ with $x^*(x) \geq 1/2$ (note that U_1 is normalizing).

Assume $n_{\ell-1}$, $x_{\ell-1}$ and $x_{\ell-1}^*$, have been chosen, we put $n_\ell = \sigma(x_1^*, \dots, x_{\ell-1}^*)$ and we choose x_ℓ , with $\text{supp}(x_\ell) > \text{supp}(x_{\ell-1})$ to be of the form

$$x_\ell = \frac{1}{f(n_\ell)} \sum_{i=1}^{n_\ell} y_i,$$

where (y_i) is an RIS with $\text{supp}(y_1) > \text{supp}(x_{\ell-1})$. Then we choose $x_\ell^* \in U^*(n_\ell)$ with $\text{supp}(x_\ell^*) > \text{supp}(x_{\ell-1}^*)$ with $x_\ell^*(x_\ell) \geq 1/2$.

Also note: At the moment we choose x_ℓ^* we have infinitely many choices (i.e. arbitrary small perturbations of x_ℓ^*) this means we can, assure that $n_{\ell+1} = \sigma(x_1^*, x_2^*, \dots, x_\ell^*)$ can be made arbitrarily large, depending on our previous choice. This implies together with the note after step 1, we can assure that $(x_i)_{i=1}^k$ can be assumed to be an RIS itself.

From our construction it follows that for

$$\frac{1}{\sqrt{f(\ell)}} \sum_{i=1}^{\ell} x_i^* \in C_\ell \subset B_{GM^*}.$$

and, thus,

$$\left\| \sum_{i=1}^{\ell} x_i \right\| \geq \frac{1}{2} \frac{\ell}{\sqrt{f(\ell)}}.$$

Using the same arguments as in Example 2.2.11 (Maurey-Rosenthal) we can show that there is a constant C' so that

$$\left\| \sum_{i=1}^{\ell} (-1)^i x_i \right\| \leq C' \frac{\ell}{\sqrt{f(\ell)}}.$$

Since Y was an arbitrary block space this implies that GM cannot contain an unconditional basic sequence. \square

3.3 Hilbert space is arbitrarily distortable

The main goal of this section is to show the following

Theorem 3.3.1. [OS1] ℓ_p , $1 < p < \infty$ is biorthogonally distortable.

We will achieve the distortion of ℓ_p by “transporting” the nearly biorthogonal sequences of sets (A_n) and (A_n^*) from the space S and S^* to sequences of sets (B_n) and (B_n^*) in ℓ_p and ℓ_q^* . This will be done via the *generalized Mazur map* which might be of independent interest.

Exercise 3.3.2. Let $1 < p < \infty$. Then the mapping

$$M_p : S_{\ell_1} \rightarrow S_{\ell_p}, \quad (\xi_i)_{i \in \mathbb{N}} \mapsto (\text{sign}(\xi_i) |\xi_i|^{1/p})_{i \in \mathbb{N}}.$$

Is a uniform homeomorphism from S_{ℓ_p} onto S_{ℓ_1} .

Definition 3.3.3. The generalization is as follows. Let X have a 1-unconditional normalized basis (e_i) By the *positive cone of X* we mean the set

$$X^+ = \left\{ \sum \xi_i e_i : \in X : \xi_i \geq 0, \text{ for } i \in \mathbb{N} \right\}.$$

We put $S_X^+ = S_X \cap X^+$ and $B_X^+ = B_X \cap X^+$.

If $\sigma = (\sigma_i) \in \{-1, 1\}^\omega$

$$X^\sigma = \left\{ \sum \xi_i e_i : \in X : \sigma_i \xi_i \geq 0, \text{ for } i \in \mathbb{N} \right\}.$$

is the cone for σ in X .

The *entropy function E* is defined by

$$E : \ell_1 \times X \rightarrow [-\infty, \infty), \quad ((h_i), (\xi_i)) \mapsto \sum_{i=1}^{\infty} |h_i| \log |\xi_i|,$$

with the convention that $0 \log 0 = 0$

Remark. If $h = (h_i) \in \ell_1^+$ with $\|h\|_{\ell_1} = 1$ one can think of (h) being a probability on \mathbb{N} , and for $x \in X^+$ $E(h, x)$ would then be the entropy of the random variable x .

Proposition 3.3.4. Let $h = (h_i) \in \ell_1 \cap c_{00}$ and put $B = \text{supp}(h)$. Then there exists a unique $x = (\xi_i) \in S_X$ so that

- a) $E(h, x) = \max_{y \in S_X} E(h, y)$,
- b) $\text{supp} h = \text{supp} x = B$
- c) $\text{sign}(\xi_i) = \text{sign}(h_i)$ for $i \in B$.

This unique x is then denoted by $F_X(h)$ and we call $F_X : S_{\ell_1} \rightarrow S_X$ the *generalized Mazur map* from S_{ℓ_1} to S_X . Moreover we put $E_X(h) = E(h, F_X(h)) = \max_{y \in S_X} E(h, y)$.

Proof. For $h \in \ell_1 \cap c_{00}$, $B = \text{supp}(h)$ we consider the restriction

$$E(h, \cdot) : \{x \in S_X^+ : \text{supp}(x) \subset B\} \rightarrow [-\infty, 0].$$

Then $E(h, \cdot)$ is continuous taking real values for $x \in S_X^+$ with $\text{supp}(x) = B$ and taking the value $-\infty$ if $\text{supp}(x) \subsetneq B$, and thus there is an $\tilde{x} = (\tilde{\xi}_i) \in S_X^+$, with $\text{supp}(\tilde{x}) = B$ for which

$$E(h, \tilde{x}) = \max \{E(h, z) : z \in S_X^+ : \text{supp}(z) \subset B\}.$$

From the strict monotonicity and strict concavity of the $\log(\cdot)$ we deduce that for $z_1, z_2 \in S_X^+$

$$E\left(h, \frac{z_1 + z_2}{\|z_1 + z_2\|}\right) \geq E\left(h, \frac{1}{2}(z_1 + z_2)\right) > \frac{E(h, z_1) + E(h, z_2)}{2},$$

and thus that the \tilde{x} is unique. By letting $x = (\xi_i)$ with $\xi_i = \text{sign}(h_i)\tilde{\xi}_i$ we satisfy (a), (b) and (c). \square

Exercise 3.3.5. Justify the name “generalized Mazur map” and show that $F_X = M_p$ if $X = \ell_p$.

For the next result we need to define the following mapping $\Psi : (0, 1) \rightarrow (0, 1)$. First note that the function

$$g : (0, \infty) \rightarrow (0, \infty), \quad a \mapsto a + \frac{1}{a},$$

has a minimum at $a = 1$, is strictly decreasing on $(0, 1)$ and strictly increasing on $(1, \infty)$. Therefore we put

$$(3.34) \quad \eta : (0, 1) \rightarrow (0, 1), \quad \eta(\varepsilon) = \varepsilon \inf \left\{ \log \left(\frac{1}{2} \left(\sqrt{a} + \frac{1}{\sqrt{a}} \right) \right) : |a - 1| > \varepsilon \right\},$$

$$(3.35) \quad \Psi : (0, 1) \rightarrow (0, 1), \quad \Psi(\varepsilon) = \varepsilon \eta(\varepsilon)$$

and note that $\Psi(\varepsilon) > 0$ for $\varepsilon > 0$ and $\Psi(\varepsilon) \searrow 0$ if $\varepsilon \searrow 0$.

Proposition 3.3.6. *Let X have a 1-unconditional basis (e_i) .*

a) *Let $h \in S_{\ell_1}^+ \cap c_{00}$, $v \in B_X^+$, and $\varepsilon > 0$ so that*

$$E(h, v) \geq E_X(h) - \Psi(\varepsilon),$$

i.e. $E(h, v)$ is close to its maximum $E(h, \tilde{v})$ of over all $v \in B_X^+$, and let $u = F_X(v)$. Then, there is an $A \subset \text{supp}(h)$ so that

$$(3.36) \quad \|Ah\| \geq 1 - \varepsilon \text{ and } (1 - \varepsilon)u_i \leq v_i \leq (1 + \varepsilon)u_i \text{ for } i \in A.$$

b) *Let $h_1, h_2 \in S_{\ell_1}^+ c_{00}$ with $\|h_1 - h_2\| \leq \varepsilon$ and let $u_i = F_X(h_i)$, $i = 1, 2$. Then*

$$\left\| \frac{1}{2}(u_1 + u_2) \right\| \geq 1 - \sqrt{\|h_1 - h_2\|}$$

(Recall that if X is uniform convex this means that the closer the value of $\|\frac{1}{2}(u_1 + u_2)\|$ is to 1 the closer u_1 and u_2 are to each other).

Proof. We have $B := \text{supp}(h) = \text{supp}(u)$ and if $E(h, v) \geq F_X(h) - \Psi(\varepsilon)$ we must have $\text{supp}(v) \subset B$. After passing to $\tilde{v} = B(v)$ we might assume that $\text{supp}(v) = B$.

We observe that

$$\begin{aligned}
\Psi(\varepsilon) &\geq E_X(h) - E(h, v) \\
&\geq E\left(\frac{1}{2}(u + v)\right) - E(h, v) \\
&[\text{Maximality of } E_X(h)] \\
&= \sum_{i \in B} h_i [\log(\frac{1}{2}(u_i + v_i)) - \log v_i] \\
&= \sum_{i \in B} h_i [\frac{1}{2} \log u_i + \frac{1}{2} \log v_i + \log(\frac{1}{2}(u_i + v_i)) - \log \sqrt{u_i v_i} - \log v_i] \\
&= \frac{1}{2} \sum_{i \in B} h_i (\log u_i - \log v_i) + \sum_{i \in B} \log \frac{1}{2} \left(\sqrt{\frac{v_i}{u_i}} + \sqrt{\frac{u_i}{v_i}} \right) \geq \sum_{i \in B} \log \frac{1}{2} \left(\sqrt{\frac{v_i}{u_i}} + \sqrt{\frac{u_i}{v_i}} \right)
\end{aligned}$$

Define

$$A = \left\{ i \in B : \left| \frac{v_i}{u_i} - 1 \right| \leq \varepsilon \right\}.$$

First note that $i \in A \iff (1 - \varepsilon)u_i \leq v_i \leq (1 + \varepsilon)u_i$, and by (3.34) if $i \notin A$ then

$$\eta(\varepsilon) \leq \log \frac{1}{2} \left(\sqrt{\frac{u_i}{v_i}} + \sqrt{\frac{v_i}{u_i}} \right)$$

Thus we conclude that

$$\sum_{i \in B \setminus A} h_i \leq \sum_{i \in B \setminus A} h_i \frac{1}{\eta(\varepsilon)} \log \frac{1}{2} \left(\sqrt{\frac{u_i}{v_i}} + \sqrt{\frac{v_i}{u_i}} \right) \leq \frac{\Psi(\varepsilon)}{\eta(\varepsilon)} = \varepsilon.$$

In order to show (b) let $h_1, h_2 \in S_{\ell_1}^+ \cap c_{00}$ and $u_i = F_X(h_i)$ and define $\varepsilon \geq 0$ by $\|\frac{1}{2}(u_1 + u_2)\| = 1 - 2\varepsilon$. W.l.o.g we assume that $\varepsilon > 0$. Let $\tilde{u}_1 = u_1 + \varepsilon u_2$ and $\tilde{u}_2 = u_2 + \varepsilon u_1$. Thus $\text{supp}(\tilde{u}_1) = \text{supp}(\tilde{u}_2) = \text{supp} h_1 \cup \text{supp} h_2$, $\|\frac{1}{2}(\tilde{u}_1 + \tilde{u}_2)\| \leq 1 - \varepsilon$ and, thus,

$$\begin{aligned}
E(h_1, \tilde{u}_1) &\geq E(h_1, u_1) \\
&\geq E\left(h_1, \frac{\tilde{u}_1 + \tilde{u}_2}{2(1 - \varepsilon)}\right) \\
&[\text{Maximality of } E(h_1, u_1)] \\
&= E\left(h_1, \frac{\tilde{u}_1 + \tilde{u}_2}{2}\right) + |\log(1 - \varepsilon)| \geq \frac{1}{2}E(h_1, \tilde{u}_1) + \frac{1}{2}E(h_1, \tilde{u}_2) + |\log(1 - \varepsilon)|
\end{aligned}$$

and, thus,

$$|\log(1 - \varepsilon)| \leq \frac{1}{2}E(h_1, \tilde{u}_1) - \frac{1}{2}E(h_1, \tilde{u}_2)$$

and similarly we can show

$$|\log(1 - \varepsilon)| \leq \frac{1}{2}E(h_2, \tilde{u}_2) - \frac{1}{2}E(h_2, \tilde{u}_1).$$

Since for $j \in \text{supp}(\tilde{x}_1)$

$$|\log \tilde{x}_{1,j} - \log \tilde{x}_{2,j}| = \left| \log \left(\frac{x_{1,j} + \varepsilon x_{2,j}}{x_{2,j} + \varepsilon x_{1,j}} \right) \right| \leq |\log \varepsilon|,$$

we obtain

$$\begin{aligned} \varepsilon \leq |\log(1 - \varepsilon)| &\leq \frac{1}{4} (E(h_1, \tilde{u}_1) - E(h_1, \tilde{u}_2) - E(h_2, \tilde{u}_1) + E(h_2, \tilde{x}_2)) \\ &\leq \frac{1}{4} \sum_{j \in \text{supp}(\tilde{x}_1)} (h_{1,j} - h_{2,j})(\log \tilde{x}_{1,j} - \log \tilde{x}_{2,j}) \\ &\leq \frac{1}{4} \|h_1 - h_2\|_{\ell_1} |\log \varepsilon| \leq \frac{1}{4} \|h_1 - h_2\|_{\ell_1} \varepsilon^{-1}, \end{aligned}$$

and, thus, $\varepsilon \leq \frac{1}{2} \|h_1 - h_2\|^{1/2}$, hence

$$\left\| \frac{u_1 + u_2}{2} \right\| = 1 - 2\varepsilon \geq \|h_1 - h_2\|^{1/2}.$$

□

Remark. Proposition 3.3.6 implies that if X is uniform convex the map F_X is uniformly continuous on S_X^+ and thus on any cone of X . More over the modulus of uniform continuity only depends on the modulus of uniform convexity.

Indeed, recall that the modulus of uniform convexity:

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x - y\| \geq \varepsilon \right\}.$$

So, if $h_1, h_2 \in S_{\ell_1}^+ c_{00}$ with $\|h_1 - h_2\| < \delta^2(\varepsilon)$ then it follows from Proposition 3.3.6 that $1 - \left\| \frac{F_X(h_1) + F_X(h_2)}{2} \right\| < \delta(\varepsilon)$ which implies that $\|F_X(h_1) - F_X(h_2)\| < \varepsilon$.

In our next step we extend that remark to all of S_X

Proposition 3.3.7. *Let X be a uniform convex Banach space with a 1-unconditional basis. The map $F_X : S_{\ell_1} \cap c_{00} \rightarrow S(X)$ is uniformly continuous and the modulus of uniform continuity solely depends on the modulus of uniform convexity of X .*

Proof. Let $g(\varepsilon)$ be the modulus of uniform continuity of $F_X|_{S_{\ell_1}^+}$. We first note that if $h \in S_{\ell_1}^+$ and $u = F_X(h)$ and $I \subset \text{supp}(h)$ so that $\sum_{i \in I} h_i \leq \varepsilon$, then $\left\| \sum_{i \in I} u_i e_i \right\| \leq g(2\varepsilon)$. Indeed,

$$\begin{aligned} &\left\| h - \left(\sum_{i \notin I} h_i e_i \right) / \left\| \sum_{i \notin I} h_i e_i \right\| \right\|_{\ell_1} \\ &\leq \left\| \sum_{i \in I} h_i e_i \right\|_{\ell_1} + \left\| \sum_{i \notin I} h_i e_i - \left(\sum_{i \notin I} h_i e_i \right) / \left\| \sum_{i \notin I} h_i e_i \right\|_{\ell_1} \right\|_{\ell_1} \leq 2\varepsilon \end{aligned}$$

Since the coordinates in I of the vector $u = F_X(h)$ and the vector

$$F_X(h) - F_X \left(\left(\sum_{i \notin I} h_i e_i \right) / \left\| \sum_{i \notin I} h_i e_i \right\|_{\ell_1} \right)$$

coincide it follows that

$$\left\| \sum_{i \in I} u_i e_i \right\| = \left\| F_X(h) - F_X \left(\left(\sum_{i \notin I} h_i e_i \right) / \left\| \sum_{i \notin I} h_i e_i \right\| \right) \right\|_{\ell_1} \leq g(2\varepsilon).$$

Let $h_1, h_2 \in S_{\ell_1}$ and $u_i = F_X(h_i)$, for $i = 1, 2$ and put $\varepsilon = \|h_1 - h_2\|$. Then $\text{sign}(u_{i,j}) = \text{sign}(h_{i,j})$ for $i = 1, 2$ and $j \in \text{supp}(h_i)$. Letting

$$I = \{j \in \text{supp}(h_1) \cup \text{supp}(h_2) : \text{sign}(h_{1,j}) \neq \text{sign}(h_{2,j})\}$$

and $h_i = (h_{i,j})_{j \in \mathbb{N}} \in \ell_1^+$, and, similarly, $|u_i| = F_X(|h_i|) = (u_{i,j})_{j \in \mathbb{N}} \in X^+$, for $i = 1, 2$.

$$\begin{aligned} \|u_1 - u_2\| &\leq \left\| |u_1| - |u_2| \right\| + \left\| \sum_{j \in I} (|u_{1,j}| + |u_{2,j}|) e_j \right\| \\ &\leq g(\| |h_1| - |h_2| \|_{\ell_1}) + \left\| \sum_{j \in I} |u_{1,j}| e_j \right\| + \left\| \sum_{j \in I} |u_{2,j}| e_j \right\| \leq g(\varepsilon) + 2g(2\varepsilon). \end{aligned}$$

$$[\text{Note that } \sum_{j \in I} |h_{i,j}| \leq \|h_1 - h_2\| = \varepsilon, \text{ for } i = 1, 2]$$

□

Definition 3.3.8. A Banach space X is uniformly smooth If for each $x \in S_X$ there is a unique $x^* \in S_{X^*}$ so that $x^*(x) = 1$ and if the mapping $\Phi : S_X \rightarrow S_{X^*}$, $x \rightarrow x^*$ is uniformly continuous.

The map Φ is called *support map*.

Proposition 3.3.9. Let X be uniform smooth. Extend the support homogenous map $(\cdot)^* : S_X \rightarrow S_{X^*}$ to a map $(\cdot)^* : X \rightarrow X^*$, i.e for $x \in X \setminus \{0\}$, $x^* = \|x\| \cdot \left(\frac{x^*}{\|x\|} \right)$.

Then the norm function is differentiable on $X \setminus \{0\}$ with

$$\frac{\partial}{\partial x} \|x\| = \nabla \|x\| = x^*,$$

Proof. We will show that for $x, y \in S_X$, and $\lambda > 0$

$$x^*(y) \leq \frac{\|x + \lambda y\| - \|x\|}{\lambda} \leq \frac{(x + \lambda y)^*(y)}{\|x + \lambda y\|},$$

which implies our claim after letting $\lambda \searrow 0$.

$$\begin{aligned} x^*(y) &= \frac{x^*(\lambda y)}{\lambda} = \frac{x^*(x + \lambda y) - 1}{\lambda} \\ &\leq \frac{\|x + \lambda y\| - \|x\|}{\lambda} \\ &= \frac{\|x + \lambda y\|^2 - \|x\| \cdot \|x + \lambda y\|}{\lambda \|x + \lambda y\|} \\ &\leq \frac{(x + \lambda y)^*(x + \lambda y) - (x + \lambda y)^*(x)}{\lambda \|x + \lambda y\|} \\ &= \frac{(x + \lambda y)^*(\lambda y)}{\lambda \|x + \lambda y\|} = \frac{(x + \lambda y)^*(y)}{\|x + \lambda y\|} \end{aligned}$$

□

Theorem 3.3.10. (Pisier) *Every uniformly convex Banach space admits an equivalent norm which is also uniformly smooth and every uniformly smooth Banach space admits an equivalent norm which is also uniform convex.*

Proposition 3.3.11. *Let X be a unifrom smooth and uniform convex Banach space with a 1-unconditional basis (e_i) Then $F_X : S_{\ell_1} \rightarrow S_X$ is invertible and F^{-1} is also uniformly continuous, with the modulus of continuity solely on the modulus of uniform smoothness of X .*

Moreover

$$F_X^{-1} : S_X \rightarrow \ell_1, x = \sum_{i=1}^{\infty} x_i e_i \mapsto \sum_{i=1}^{\infty} \text{sign}(x_i) x_i x_i^* e_i,$$

where $x^* = \sum_{i=1}^{\infty} x_i^* e_i$ normalizes x .

Proof. The biorthogonal functionals (e_i^*) are a 1-unconditional basis for X^* and thus we can express for $x \in S_X$, x^* as

$$x^* = \sum_{i \in \text{supp}(x)} x_i^* e_i <$$

with $\text{sign}(x_i^*) = \text{sign}(x_i)$ and $x^* \circ x := (x_i^* \cdot x_i) \in S_{\ell_1}^+$.

Let $G(x) = |x^*| \circ x$. We claim that $G : S_X \rightarrow S_{\ell_1}^+$ is uniform continuous. Indeed, for $x, y \in S_X$ and $u = G(x)$ and $v = G(y)$ we deduce that

$$\begin{aligned} \|u - v\|_{\ell_1} &= \left\| |x^*| \circ x - |y^*| \circ y \right\| \\ &\leq \left\| |x^*| \circ (x - y) \right\| + \left\| |x^* - y^*| \circ (y) \right\| \leq \|x - y\| + \|x^* - y^*\|, \end{aligned}$$

which proves that G is uniform continuous and that the modulus of uniform continuity solely depends on the modulus of uniform continuity of the map $S_X \ni x \rightarrow x^* \in S_{X^*}$.

It remains to show that $G = F^{-1}X$. Since $G(x) = \text{sign}(x) \circ G(|x|)$, and $F_X(x) = \text{sign}(x) \circ F(|x|)$ it is enough to show that for $h \in S_{\ell_1}^+ \cap c_{00}$ and $x \in S_X^+ \cap c_{00}$ we have $G(F(h)) = h$ and $F(G(x)) = x$.

If $h \in S_{\ell_1}^+ \cap c_{00}$ and $x = F(h)$ then, by the Lagrange multiplier method, for some $\lambda > 0$

$$(h_i/x_i)_{i \in \text{supp} h} = \nabla_x E(h, x) = \lambda \nabla \|\cdot\|(x) = x^*,$$

Thus $h = \lambda x \circ x^* = \lambda G(x)$. It follows (take norm on both sides) that $\lambda = 1$ and that $G(F(h)) = G(x) = h$.

In order to show that $F(G(x)) = x$, it is enough to show that G injective. So assume that $x, y \in S_X^+$ and $h = x^* \circ |x| = y^* \circ |y|$. Let $U = \{u \in X^+ : \text{supp}(u) = \text{supp}(h)\}$ and for $u \in U$ put $f(u) := \|u\| - E(h, u)$. f is strictly convex on U , and therefore there is at most one point $z \in U$ for which $\nabla f(z) = 0$. But

$$\nabla f(z) = 0 \iff z^* - \sum_{i \in \text{supp}(h)} \frac{h_i}{z_i} = 0 \iff h = z \circ z^*.$$

□

Now we will sketch how to show that ℓ_2 is biorthogonally distortable. We consider the nearly biorthogonal sets (A_m) and (A_m^*) in S and S^* , respectively, as constructed in Corollary 3.2.13

$$A_m = \left\{ \frac{f(m)}{m} \sum_{i=1}^m y_i : (y_i)_{i=1}^m \text{ is an RIS of length } m \right\}$$

$$A_m^* = \left\{ \frac{f(m)}{m} \sum_{i=1}^m y_i^* : (y_i)_{i=1}^m \text{ is a block in } S_{S^*} \right\}$$

Let $\varepsilon_k \searrow 0$. As we have shown in Corollary 3.2.13 we can pass to subsequence (B_m, B_m^*) of (A_m, A_m^*) so that for some constant C :

$$(3.37) \quad \forall m \in \mathbb{N} \forall x \in B_m \exists x^* \in B_m^* \quad |x^*(x)| \geq (1 - \varepsilon_m), 1 \leq \|x\| \leq C,$$

$$(3.38) \quad \forall m \neq k, m, k \in \mathbb{N} \forall x \in B_m, x^* \in B_m^* \quad |x^*(x)| \leq \varepsilon_{\min(m,k)}$$

$$(3.39) \quad \forall m \in \mathbb{N} \forall Y \hookrightarrow X, \dim(Y) = \infty \quad \text{dist}(B_m, Y) = 0.$$

Since (e_i) is an unconditional and subsymmetric basis of S , it follows also that

$$(3.40) \quad \forall m \in \mathbb{N} \quad B_m \text{ and } B_m^* \text{ are unconditional and spreading}$$

($A \subset X$ unconditional and spreading means that if $x = \sum a_i e_i \in A$ then $\sum \sigma_i a_i e_{n_i} \in A$, for and $(\sigma_i) \subset \{+1, -1\}$ and increasing $(n_i) \subset \mathbb{N}$).

Theorem 3.3.12. *For $m \in \mathbb{N}$ define*

$$C_m = \{x^* \circ |x|, \text{ with } x^* \in B_m^*, x \in B_m \text{ and } x^*(x) \geq 1/2\}.$$

and for $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ we put

$$C_m^{(p)} = M_p(C_m) = \{(\text{sign}|x_i|^{1/p}) : x \in C_m\} \text{ and } C_m^{*(p)} = M_q(C_m) = \{(\text{sign}|x_i|^{1/q}) : x \in C_m\}$$

Then $(C_m^{(p)})$ and $(C_m^{*(p)})$ are nearly biorthogonal in ℓ_p .

Lemma 3.3.13. [OS1][Lemma 3.5] *Let Y be a block subspace of ℓ_1 and $\varepsilon > 0$ and $m \in \mathbb{N}$. Then there is a vector $u \in S$ which is an $(1 + \varepsilon)$ - ℓ_1^m average and a $u^* \in S_{S^*}$ so that $u^*(u) > 1 - \varepsilon$ and $\text{dist}(u^* \circ |u|, Y) < \varepsilon$.*

We skip the quite technical proof of Lemma 3.3.13 which makes use of Proposition 3.3.6 (a).

Proof of Theorem 3.3.12. We will show our claim for $p = 2$, and put $D_m = C_m^{(2)}$ (note $C_m^{*(2)} = C_m^{(2)}$).

Step 1: For $m \in \mathbb{N}$ C_m is asymptotic in ℓ_1 . Indeed, let Y be a block subspace of ℓ_1 . Then we can use Lemma 3.3.13 to find an RIS $(x_i)_{i=1}^m$ in S and $(x_i^*) \subset S_{S^*}$ so that $x_i^*(x_i) > 1 - \varepsilon$ and $\text{dist}(|x_i| \circ x_i^*, Y)$ then choose

$$y = \frac{f(m)}{m} \sum_{i=1}^m x_i \in A_m \text{ and } y^* = \frac{1}{f(m)} \sum_{i=1}^m x_i^* \in A_m^*,$$

and

$$\text{dist}(y^* \circ |y|) Y \leq \frac{1}{m} \sum_{i=1}^m \text{dist}(x_i^* \circ |x_i|) < \varepsilon.$$

Step 2: If $A \subset \ell_1$ is asymptotic then $M_2(A)$ is asymptotic in ℓ_2 . This follows from the fact that if $Y \subset \ell_1$ is a block subspace then $M_2(Y \cap S_Y)$ is also a sphere of a block subspace of ℓ_2 .

Step 3: (C_m) is nearly biorthogonal (to itself). Let $k \neq \ell$, and $v \in C_k$, and $w \in C_\ell$. We write

$$v = (v_i), \text{ with } v_i = \text{sign}(x_i) \sqrt{|x_i| \cdot |x_i^*|} \text{ and } x \in B_k, x^* \in B_k^*$$

$$w = (w_i), \text{ with } w_i = \text{sign}(y_i) \sqrt{|y_i| \cdot |y_i^*|} \text{ and } y \in B_\ell, y^* \in B_\ell^*$$

Then

$$\begin{aligned} |\langle v, w \rangle| &\leq \sum_{i \in \mathbb{N}} |v_i| \cdot |w_i| \\ &= \sum (|x_i| \cdot |x_i^*| \cdot |y_i| \cdot |y_i^*|)^{1/2} \\ &\leq \left(\sum |x_i^*| \cdot |y_i| \right)^{1/2} \left(\sum |y_i^*| \cdot |x_i| \right)^{1/2} \leq \varepsilon_{\min\{k, \ell\}}. \end{aligned}$$

(Since B_ℓ and B_k are unconditional)

□

Chapter 4

Versions of Ramsey's theorem in Banach spaces

4.1 Gowers' game on blocks and the dichotomy theorem

In this section we present a special case of Gowers' Ramsey theorem on block spaces of a Banach space with a basis. Throughout this section let X be a separable Banach space with a normalized and bimonotone basis (e_i) . The assumed bimonotonicity of (e_i) is not really necessary but will simplify the arguments.

As usual we think of X being the completion of c_{00} under some norm. For $x, y \in c_{00}$ we write $x < y$ if $\max \text{supp}(x) < \min \text{supp}(y)$. We adopt the convention that $0 < x$ and $0 > x$ for any $x \in c_{00}$.

If Y, Z are block subspaces of X we write $Y \preceq Z$ or $Z \succeq Y$ if Y is a block subspace of Z . For $x \in c_{00}$ and $Y \preceq X$ we write $x < Y$ if $x < y$ for all $y \in Y$ of Z .

We also assume that our space satisfies the following technical condition:

- (*) For all $x \in c_{00}$ there is an $\varepsilon = \varepsilon(\text{supp}(x))$ so that for all $y \in c_{00}$, with $x > y$ it follows that

$$\|x + y\| \geq \|x\| + \varepsilon\|y\|.$$

Exercise 4.1.1. In order to show that condition (*) holds for an arbitrary small perturbation of a norm we define the following norm on c_{00} . Let $c > 1$. For $x = (x_i) \in c_{00}$ put

$$\|x\| = \sup \left\{ \sum_{i=0}^{\infty} c^{-i} x_{n_i} : n_0 < n_1 < n_2 < \dots \right\}.$$

Show that $\|\cdot\|$ is an equivalent norm on c_0 which satisfies (*)

Let \mathcal{B}^∞ be the set of all infinite normalized block sequences in X and \mathcal{B}^f the set of all finite normalized block sequences in X . We also consider \emptyset to be an element of \mathcal{B}^f (block of length 0). For $Y \preceq X$ we denote the set of all normalized infinite or finite block sequences in Y (these are exactly the normalized block of the block basis generating Y) by $\mathcal{B}^\infty(Y)$ and $\mathcal{B}^f(Y)$.

If $\bar{x} = (x_1, x_2, \dots, x_m) \in \mathcal{B}^f$ and $\bar{y} = (y_1, x_2, \dots, y_m) \in \mathcal{B}^f$ or $\bar{y} = (y_i) \in \mathcal{B}^\infty$ we write $\bar{x} < \bar{y}$ if $x_m < y_1$ and in this case we denote the *concatenation of x and y* by (x, y) , i.e.

$$(x, y) = \begin{cases} (x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) & \text{if } \bar{y} \in \mathcal{B}^f \\ (x_1, x_2, \dots, x_m, y_1, y_2, \dots) & \text{if } \bar{y} \in \mathcal{B}^\infty. \end{cases}$$

We denote the length of \bar{x} by $|\bar{x}|$.

Definition 4.1.2. We consider on \mathcal{B}^∞ the product topology of the discrete topology on $S_X \cap c_{00}$, i.e. if $\bar{y}^{(n)} = (y_i^{(n)})_{i=1}^\infty \in \mathcal{B}^\infty$ for $n \in \mathbb{N}$ and $\bar{y} = (y_i)_{i=1}^\infty \in \mathcal{B}^\infty$ then $\bar{y}^{(n)}$ converges to \bar{y} if and only if for any $m \in \mathbb{N}$ there is an $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$

$$y_i^{(n)} = y_i \text{ whenever } n \geq n_0 \text{ and } i = 1, 2, \dots, m.$$

For $\mathcal{A} \subset \mathcal{B}^\infty$ we denote the closure in the product topology of the discrete topology on $S_X \cap c_{00}$ by $\overline{\mathcal{A}}$.

For $\bar{\varepsilon} = (\varepsilon_i) \subset [0, \infty)$ and $\mathcal{A} \subset \mathcal{B}^\infty$ we define the $\bar{\varepsilon}$ *fattening of $\mathcal{A} \subset \mathcal{B}^\infty$* by

$$(4.1) \quad \mathcal{A}_{\bar{\varepsilon}} = \{(z_i) \in \mathcal{B}^\infty : \exists (x_i) \in \mathcal{A} \quad \|x_i - z_i\| \leq \varepsilon_i \text{ for } i = 1, 2, \dots\}.$$

If $\bar{\varepsilon} = (\varepsilon_i)_{i=1}^n$ is a finite sequence we understand by $\mathcal{A}_{\bar{\varepsilon}}$ the set $\mathcal{A}_{\bar{\eta}}$ with

$$\bar{\eta} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, 0, 0, 0, \dots).$$

For $\mathcal{A} \subset \mathcal{B}^\infty$ and $Y \prec X$ we consider now the following infinite game between two players

Player I : chooses $Y_1 \preceq Y$
 Player II: chooses $y_1 \in S_{Y_1} \cap c_{00}$
 Player I : chooses $Y_2 \preceq Y$
 Player II: chooses $y_2 \in S_{Y_2} \cap c_{00}$
 ⋮

Player I wins if the resulting sequence (y_1, y_2, \dots) lies in \mathcal{A} .

It follows from Theorem 1.2.4 that the game is determined if \mathcal{A} is Borel with respect to the product of the discrete topology on S_X , i.e. that one of the Players has a winning strategy.

Write $W_I(\mathcal{A}, Y)$ if Player I has a winning strategy for the (\mathcal{A}, Y) -game and we write $W_{II}(\mathcal{A}, Y)$ if Player II has a winning strategy.

The main result of this section is the following.

Theorem 4.1.3. [Go3]. *Assume $\mathcal{A} \subset \mathcal{B}^\infty$ is closed. Then the following are equivalent.*

1. *For all $\bar{\varepsilon} = (\varepsilon_i) \subset (0, 1]$ and for all $Y \preceq X$ it follows $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}}, Y)$*
2. *For all $\bar{\varepsilon} = (\varepsilon_i) \subset (0, 1]$ and for all $Y \preceq X$ there exists $Z \preceq Y$ so that every normalized block sequence (z_n) in Z is in $\overline{\mathcal{A}_{\bar{\varepsilon}}}$.*

Remark. In [Go3] above result was proved for coanalytic sets $\mathcal{A} \subset \mathcal{B}^\infty$

Proof. Needs to be rewritten. □

Theorem 4.1.3 was a central part for Gowers to prove his *dichotomy result* which lead to the solution of the *homogeneous Banach space problem*.

The Homogeneous Banach space problem.

Let X be an infinite dimensional separable Banach space which is isomorphic to all of its infinite dimensional closed subspace (we call such a space *homogeneous*).

Does it follow that X is isomorphic to Hilbert space?

In order to outline the solution we need *Gowers' dichotomy Theorem*.

Definition 4.1.4. *Hereditary Indecomposable Spaces.*

An infinite dimensional Banach space X is called *indecomposable* if it is not isomorphic the complemented sum of two infinite dimensional Banach spaces Z_1 and Z_2 .

X is called *hereditary indecomposable* (HI) if no infinite dimensional closed subspace is decomposable.

Remark. Note that an (HI) space cannot contain an unconditional basic sequence. The existence of (HI) spaces was shown in [GM1] and [GM2].

Assume that Y and Z are two closed subspaces of a Banach space X in order for

$$Y + Z = \{y + z : y \in Y \text{ and } z \in Z\}$$

to be the complemented sum of Y and Z it is necessary and sufficient that the map

$$Y \times Z \rightarrow Y + Z, \quad (y, z) \mapsto (y + z)$$

is an isomorphism Here $Y \times Z$ is the topological product of Y and Z which can be endowed for example with the norm $\|(y, z)\| = \|y\| + \|z\|$.

Thus, for $Y + Z$ not being the complemented sum of Y and Z , it is necessary and equivalent that there are sequences $(y_n) \subset S_Y$ and $(z_n) \subset S_Z$ for which $\lim_{n \rightarrow \infty} \|y_n - z_n\| = 0$. Therefore we observed the following proposition.

Proposition 4.1.5. *For space X the following are equivalent*

- a) X is (HI)
- b) $\text{dist}(S_Y, S_Z) = \inf_{y \in S_Y, z \in S_Z} \|y - z\| = 0$ for any two infinite dimensional subspaces $Y, Z \hookrightarrow X$.

Theorem 4.1.6. *Gowers' Dichotomy* [Go3]

If X is an infinite dimensional Banach space. Then either it contains an unconditional basic sequence or it has a subspace which is (HI).

Proof. W.l.o.g. X has a normalized basis (e_i) and let $C > 1$.

Note that a normalized block sequence (x_n) is C -unconditional if and only if for all normalized blocks (y_n) , all $n \in \mathbb{N}$, and all $(\lambda_i) \in [0, 1]^n$

$$\left\| \sum_{i=1}^n (-1)^i \lambda_i x_i \right\| \leq C \left\| \sum_{i=1}^n \lambda_i x_i \right\|.$$

Let X be a Banach space with normalized basis (e_i) . For a fixed $C > 1$ we define

$$\begin{aligned} \mathcal{A}^{(C)} &= \left\{ (x_n) : \forall n \in \mathbb{N} \forall (\lambda_i)_{i=1}^n \subset [0, 1] \left\| \sum_{i=1}^n (-1)^i \lambda_i x_i \right\| \leq C \left\| \sum_{i=1}^n \lambda_i x_i \right\| \right\} \\ &= \bigcap_{n \in \mathbb{N}, (\lambda_i)_{i=1}^n \subset [0, 1]} \left\{ (x_n) \subset S_X \text{ block} : \sum_{i=1}^n (-1)^i \lambda_i x_i \right\| \leq C \left\| \sum_{i=1}^n \lambda_i x_i \right\| \right\}. \end{aligned}$$

Note that $\mathcal{A}^{(C)}$ is closed in the product topology of the discrete topology, and that therefore the $\mathcal{A}^{(C)}$ -game is determined and that Theorem 4.1.3 applies.

Let $\varepsilon > 0$. Now, either Player I has a winning strategy for $(\mathcal{A}^{(C)})_{(2^{-n\varepsilon})} \subset \mathcal{A}^{(C+\varepsilon)}$ on every closed $Y \hookrightarrow X$. Then we deduce from Theorem 4.1.3 that there is a normalized blocksequence (y_n) all of its normalized blocks are in $\mathcal{A}^{(C+\varepsilon)}$ and it follows that (y_n) is $(C + \varepsilon)$ -unconditional. Or Player II has a winning strategy on some closed block subspace Y of X . He in particular has a winning strategy if Player I chooses $Z_1, Z_2, Z_1, Z_2, \dots$ for any given block subspaces Z_1 and Z_2 of Y . Therefore Player II can choose a normalized block (z_i) in Y so that $z_{2i-1} \in Z_1$ and $z_{2i} \in Z_2$ for which there is an $n \in \mathbb{N}$ and a sequence $(\lambda_i)_{i=1}^n$ in $[-1, 1]$ So that

$$\left\| \sum_{i=1}^{2n} (-1)^i \lambda_i z_i \right\| > C \left\| \sum_{i=1}^{2n} \lambda_i z_i \right\|.$$

We choose

$$\begin{aligned} y_1 &= \sum_{i=1}^n \lambda_{2i-1} z_{2i-1} \\ y_2 &= \sum_{i=1}^n \lambda_{2i} z_{2i} \end{aligned}$$

and assume w.l.o.g that $\|y_1\| = 1$ (otherwise divide y_1 and y_2 by $\|y_1\|$) and that $\|y_2\| \leq \|y_1\|$ (otherwise swap the roles of y_1 and y_2).

It follows that

$$1 \geq \|y_2\| \geq 1 - \|y_1 + y_2\| \geq 1 - \frac{1}{C} \|y_1 - y_2\| \geq 1 - \frac{2}{C} = \frac{C-2}{C}$$

and thus

$$\left\| y_1 + \frac{y_2}{\|y_2\|} \right\| \leq \|y_1 + y_2\| + \|y_2\| \left(1 - \frac{C-2}{C} \right) \leq \frac{4}{C}.$$

It follows that for any two infinite dimensional subspaces Z_1 and Z_2 of Y $\text{dist}(S_{Z_1} S_{Z_2}) \leq \frac{5}{C}$

It follows that

$$\|y_1 + y_2\| \leq \frac{1}{C} \|y_1 - y_2\| \leq \frac{2}{C}$$

which yields $\text{dist}(Z_1, Z_2) \leq \frac{2}{C}$.

Now we consider the a sequence $C_n \nearrow \infty$ and apply the argument successively for each C_n . We will get block spaces $X = Y_0 \succ Y_1 \succ Y_2$ so that either for some n Y_n has a $C_n + 1$ -unconditional subsequence or for all n and all closed subspaces $Z_1, Z_2 \subset Y_n$ it follows that $\text{dist}(S_{Z_1} S_{Z_2}) \leq \frac{5}{C_n}$.

In the second case we take Z to be a diagonal space of the Y_n 's (i.e. $Z = \text{span}(z_i)$ with $z_1 \prec z_2 \prec \dots$ and $z_n \in Y_n$ for $n \in \mathbb{N}$) and deduce that Z is (HI). \square

Theorem 4.1.7. [GM2] *A hereditary indecomposable space is not isomorphic to any proper subspaces.*

In particular Theorems 4.1.6 and 4.1.7 yield that if X is an infinite dimensional Banach space which is isomorphic to all of its subspaces, it must contain an unconditional basic sequence and, thus, have an unconditional basis and all its infinite dimensional subspaces have unconditional bases.

But this only happens in Hilbert space as the following result by Komorowski and Tomczak-Jaegermann shows.

Theorem 4.1.8. [KT] *Let X be a homogenous Banach space not containing ℓ_2 . Then X has an infinite dimensional closed subspace without unconditional basis.*

4.2 Trees and Branches in Banach spaces and embedding theorems

In this section we present a second version of Ramsey like theorems in Banach space and consider a similar game to the one introduced in Section 4.1. But this time Player I will only be able to choose cofinite dimensional subspaces of a given space.

We first introduce some notation.

Definition 4.2.1. Let Z be a Banach space and $E = (E_n)$ be a sequence of finite dimensional subspaces of Z . We call (E_n) a *finite dimensional decomposition of Z* , and we abbreviate it by FDD, if for every $x \in X$ there is a unique sequence $(x_n) \subset X$, with $x_n \in E_n$, for $n \in \mathbb{N}$, so that $x = \sum_{n=1}^{\infty} x_n$.

Note that FDD's can be thought of as a generalization of bases. Indeed, if $\dim(E_n) = 1$, for all $n \in \mathbb{N}$ for an FDD (E_n) then (x_n) , where $x_n \in E_n \setminus \{0\}$ for $n \in \mathbb{N}$, is a basis. Many basic observations on bases can be extended to FDDs.

Let Z be a Banach space with an FDD $E = (E_n)$. For $n \in \mathbb{N}$ we denote the n -th *coordinate projection* by P_n^E , i.e. $P_n^E : Z \rightarrow E_n$, $\sum z_i \mapsto z_n$. For finite $A \subset \mathbb{N}$ we put $P_A^E = \sum_{n \in A} P_n^E$. As in the case of bases one can show that the *projection constant* of (E_n) (in Z)

$$K = K(E, Z) = \sup_{m \leq n} \|P_{[m,n]}^E\|$$

is finite.

As in the case of bases we call (E_i) *bimonotone (in Z)* if $K = 1$. By passing to the equivalent norm

$$\|\cdot\| : Z \rightarrow \mathbb{R}, \quad z \mapsto \sup_{m \leq n} \|P_{[m,n]}^E(z)\|,$$

we can always renorm Z so that $K = 1$.

(E_i) is called a C -unconditional FDD of Z if for all $(x_i)_{i=1}^n \subset Z$, with $x_i \in E_i$, for $i \in \mathbb{N}$, it follows that

$$\left\| \sum_{i=1}^n \varepsilon_i x_i \right\| \leq \left\| \sum_{i=1}^n x_i \right\| \text{ for all } (\varepsilon_i) \subset \{+1, -1\},$$

and it is called a *suppression C -unconditional FDD of Z* if

$$\left\| \sum_{i \in A} x_i \right\| \leq \left\| \sum_{i=1}^n x_i \right\| \text{ for all } A \subset \{1, 2, \dots, n\}.$$

As in the case of bases, C -unconditionality implies *suppression C -unconditionality*, and *suppression C -unconditionality* implies C -unconditionality. We call (E_i) *unconditional* or *suppression unconditional* if there is a $C \geq 1$ for which (E_i) unconditional or *suppression unconditional*, respectively.

In important example are the ℓ_p -sums of finite dimensional spaces (E_i) .

$$(\oplus_{i=1}^{\infty} E_i)_{\ell_p} = \{(x_i) : x_i \in E_i, \text{ for } i \in \mathbb{N}, \text{ and } \sum_i \|x_i\|^p < \infty\},$$

$$(\oplus_{i=1}^{\infty} E_i)_{c_0} = \{(x_i) : x_i \in E_i, \text{ for } i \in \mathbb{N}, \text{ and } \lim_{i \rightarrow \infty} \|x_i\| = 0\}.$$

For a sequence (E_i) of finite dimensional spaces we define the vector space

$$c_{00}(\oplus_{i=1}^{\infty} E_i) = \{(z_i) : z_i \in E_i, \text{ for } i \in \mathbb{N}, \text{ and } \{i \in \mathbb{N} : z_i \neq 0\} \text{ is finite}\},$$

which is dense in each Banach space for which (E_n) is an FDD. For $A \subset \mathbb{N}$ we denote by $\oplus_{i \in A} E_i$ the linear subspace of $c_{00}(\oplus E_i)$ generated by the elements of $(E_i)_{i \in A}$ and we denote its closure in Z by $(\oplus E_i)_Z$. As usual we denote the vector space of sequences in \mathbb{R} which are eventually zero by c_{00} and its unit vector basis by (e_i) .

The vector space $c_{00}(\oplus_{i=1}^{\infty} E_i^*)$, where E_i^* is the dual space of E_i , for $i \in \mathbb{N}$, is a w^* -dense subspace of Z^* . (More precisely E_i^* is the subspace of Z^* generated by all elements z^* for which $z^*|_{E_n} = 0$ if $n \neq i$. E_i^* is uniformly isomorphic to the dual space of E_i and is isometric to it if $K(E, Z) = 1$.) We denote the norm closure of $c_{00}(\oplus_{i=1}^{\infty} E_i^*)$ in Z^* by $Z^{(*)}$. $Z^{(*)}$ is w^* -dense in Z^* , the unit ball $B_{Z^{(*)}}$ norms Z and (E_i^*) is an FDD of $Z^{(*)}$ having a projection constant not exceeding $K(E, Z)$. If $K(E, Z) = 1$ then $B_{Z^{(*)}}$ is 1-norming and $Z^{(*)}(\ast) = Z$.

For $z \in c_{00}(\oplus E_i)$ we define the E -support of z by

$$\text{supp}_E(z) = \{i \in \mathbb{N} : P_i^E(z) \neq 0\}.$$

A non-zero sequence (finite or infinite) $(z_j) \subset c_{00}(\oplus E_i)$ is called a *block sequence of (E_i)* if

$$\max \text{supp}_E(z_n) < \min \text{supp}_E(z_{n+1}), \text{ whenever } n \in \mathbb{N} \text{ (or } n < \text{length}(z_j)),$$

and it is called a *skipped block sequence of (E_i)* if $1 < \min \text{supp}_E(z_1)$ and

$$\max \text{supp}_E(z_n) < \min \text{supp}_E(z_{n+1}) - 1, \text{ whenever } n \in \mathbb{N} \text{ (or } n < \text{length}(z_i)).$$

Let $\bar{\delta} = (\delta_n) \subset (0, 1]$. A (finite or infinite) sequence $(z_j) \subset S_Z = \{z \in Z : \|z\| = 1\}$ is called a $\bar{\delta}$ -*block sequence of (E_n)* or a $\bar{\delta}$ -*skipped block sequence of (E_n)* if there are $1 \leq k_1 < \ell_1 < k_2 < \ell_2 < \dots$ in \mathbb{N} so that

$$\|z_n - P_{[k_n, \ell_n]}^E(z_n)\| < \delta_n, \text{ or } \|z_n - P_{(k_n, \ell_n]}^E(z_n)\| < \delta_n, \text{ respectively,}$$

for all $n \in \mathbb{N}$ (or $n \leq \text{length}(z_j)$). Of course one could generalize the notion of $\bar{\delta}$ -block and $\bar{\delta}$ -skipped block sequences to more general sequences, but we prefer to introduce this notion only for normalized sequences. It is important to note that in the definition of $\bar{\delta}$ -skipped block sequences $k_1 \geq 1$, and that therefore the E_1 -coordinate of z_1 is small (depending on δ_1).

A sequence of finite-dimensional spaces (G_n) is called a *blocking of (E_n)* if there are $0 = k_0 < k_1 < k_2 < \dots$ in \mathbb{N} so that $G_n = \oplus_{i=k_{n-1}+1}^{k_n} E_i$, for $n = 1, 2, \dots$

Definition 4.2.2. An FDD (E_i) of a Banach space is called *shrinking* if the sequence of its coordinate functionals (E_i^*) is an FDD of X^* , and (E_i) is called *boundedly complete* if the series $\sum x_i$, $x_i \in E_i$, for $i \in \mathbb{N}$ converges whenever $\sup_{n \in \mathbb{N}} \|\sum_{i=1}^n x_i\| < \infty$.

(F_i) is called an *unconditional finite dimensional decomposition (UFDD)*, if for all $x \in X$, the representation as $x = \sum_{i=1}^{\infty} x_i$, with $x_i \in E_i$, for $i \in \mathbb{N}$, converges unconditionally.

Let $\mathcal{A} \subset S_Z^\omega$ and $\mathcal{B} = \prod_{i=1}^\infty B_i$, where $B_n \subset S_Z$ for $n \in \mathbb{N}$.

We consider the following $(\mathcal{A}, \mathcal{B})$ -game between two players: Assume that $E = (E_i)$ is an FDD for Z .

Player I chooses $n_1 \in \mathbb{N}$,

Player II chooses $z_1 \in c_{00}(\oplus_{i=n_1+1}^\infty E_i) \cap B_1$,

Player I chooses $n_2 \in \mathbb{N}$,

Player II chooses $z_2 \in c_{00}(\oplus_{i=n_2+1}^\infty E_i) \cap B_2$,

\vdots

Player I wins the $(\mathcal{A}, \mathcal{B})$ -game if the resulting sequence (z_n) lies in \mathcal{A} . If Player I has a winning strategy (forcing the sequence (z_i) to be in \mathcal{A}) we will write $W_I(\mathcal{A}, \mathcal{B})$ and if Player II has a winning strategy (being able to choose (z_i) outside of \mathcal{A}) we write $W_{II}(\mathcal{A}, \mathcal{B})$. If \mathcal{A} is a Borel set with respect to the product of the discrete topology on S_Z^ω (note that \mathcal{B} is always closed in the product of the discrete topology on S_Z^ω), it follows from Theorem 1.2.4 that the game is determined, i.e., either $W_I(\mathcal{A}, \mathcal{B})$ or $W_{II}(\mathcal{A}, \mathcal{B})$.

Let us define $W_{II}(\mathcal{A}, \mathcal{B})$ slightly differently from the definition provided in Section 1.1 and use *trees in Banach spaces*.

We define

$$T_\infty = \bigcup_{\ell \in \mathbb{N}} \{(n_1, n_2, \dots, n_\ell) : n_1 < n_2 < \dots < n_\ell \text{ are in } \mathbb{N}\}.$$

If $\alpha = (m_1, m_2, \dots, m_\ell) \in T_\infty$, we call ℓ the *length* of α and denote it by $|\alpha|$, and $\beta = (n_1, n_2, \dots, n_k) \in T_\infty$ is called an *extension* of α , or α is called a *restriction* of β , if $k \geq \ell$ and $n_i = m_i$, for $i = 1, 2, \dots, \ell$. We then write $\alpha \leq \beta$ and with this order (T_∞, \leq) is a tree.

In this section *trees* in a Banach space X are families in X indexed by T_∞ , thus they are countable infinitely branching trees of countably infinite length.

For a tree $(x_\alpha)_{\alpha \in T_\infty}$ in a Banach space X , and $\alpha = (n_1, n_2, \dots, n_\ell) \in T_\infty \cup \{\emptyset\}$ we call the sequences of the form $(x_{(\alpha, n)})_{n > n_\ell}$ *nodes* of $(x_\alpha)_{\alpha \in T_\infty}$. The sequences (y_n) , with $y_i = x_{(n_1, n_2, \dots, n_i)}$, for $i \in \mathbb{N}$, for some strictly increasing sequence $(n_i) \subset \mathbb{N}$, are called *branches* of $(x_\alpha)_{\alpha \in T_\infty}$. Thus, branches of a tree $(x_\alpha)_{\alpha \in T_\infty}$ are sequences of the form (x_{α_n}) where (α_n) is a maximal linearly ordered (with respect to extension) subset of T_∞ .

If $(x_\alpha)_{\alpha \in T_\infty}$ is a tree in X and if $T' \subset T_\infty$ is closed under taking restrictions so that for each $\alpha \in T' \cup \{\emptyset\}$ infinitely many direct successors of α are also in T' then we call $(x_\alpha)_{\alpha \in T'}$ a *full subtree* of $(x_\alpha)_{\alpha \in T_\infty}$. Note that $(x_\alpha)_{\alpha \in T'}$ could then be relabeled to a family indexed by T_∞ and note that the branches of $(x_\alpha)_{\alpha \in T'}$ are branches of $(x_\alpha)_{\alpha \in T_\infty}$ and that the nodes of $(x_\alpha)_{\alpha \in T'}$ are subsequences of certain nodes of $(x_\alpha)_{\alpha \in T_\infty}$.

We call a tree $(x_\alpha)_{\alpha \in T_\infty}$ in a Banach space X *normalized* if $\|x_\alpha\| = 1$, for all $\alpha \in T_\infty$ and *weakly null* if every node is weakly null. More generally if \mathcal{T} is a topology on X and a tree $(x_\alpha)_{\alpha \in T_\infty}$ in a Banach space X is called \mathcal{T} -*null* if every node converges to 0 with respect to \mathcal{T} .

If $(x_\alpha)_{\alpha \in T_\infty}$ is a tree in a Banach space Z which has an FDD (E_n) we call it a *block tree* of (E_n) if every node is a block sequence of (E_n) .

We will also need to consider trees of finite length. For $\ell \in \mathbb{N}$ we call a family $(x_\alpha)_{\alpha \in T_\infty, |\alpha| \leq \ell}$ in X a *tree of length ℓ* . Note that the notions nodes, branches, \mathcal{T} -null and block trees can be defined analogously for trees of finite length.

Remark. Assume that (x_n) is a sequence in X . We can define *the associated tree* as follows: For $(n_1, n_2, \dots, n_k) \in T_\infty$ we put $x_{(n_1, n_2, \dots, n_k)} := x_{n_k}$.

Note that then every subsequence of (x_n) is a branch of $(x_\alpha)_{\alpha \in T_\infty}$ and vice versa, and that $(x_\alpha)_{\alpha \in T_\infty}$ is normalized, weakly null or a block tree, if (x_n) is a normalized, weakly null or a block sequence.

Using the formal definition of winning strategies as introduced in chapter 1 we can easily derive the following Proposition.

Proposition 4.2.3. *Assume that Z is a Banach space with an FDD (E_i) , $\mathcal{A} \subset S_Z^\omega$ and $\mathcal{B} = \prod_{i=1}^\infty B_i$, with $B_i \subset S_Z$ for $i \in \mathbb{N}$.*

Then Player II has a winning strategy for the $(\mathcal{A}, \mathcal{B})$ -game if and only if

$(W_{II}(\mathcal{A}, \mathcal{B}))$ *There exists a block tree $(x_\alpha)_{\alpha \in T_\infty}$ of (E_i) in S_Z all of whose branches are in \mathcal{B} but none of its branches are in \mathcal{A} .*

In case that the $(\mathcal{A}, \mathcal{B})$ -game is determined $W_I(\mathcal{A}, \mathcal{B})$ can be therefore stated as follows.

$(W_I(\mathcal{A}, \mathcal{B}))$ *Every block tree $(x_\alpha)_{\alpha \in T_\infty}$ of (E_i) in S_X , all of whose branches are in \mathcal{B} , has a branch in \mathcal{A} .*

The proof of the following Proposition is easy.

Proposition 4.2.4. *Let $\mathcal{A}, \tilde{\mathcal{A}} \subset S_Z^\omega$, $\mathcal{B} = \prod_{i=1}^\infty B_i$, with $B_i \subset S_Z$ for $i \in \mathbb{N}$. Assume that the $(\mathcal{A}, \mathcal{B})$ -game and the $(\tilde{\mathcal{A}}, \mathcal{B})$ -game are determined.*

a) *If $\mathcal{A} \subset \tilde{\mathcal{A}}$, then*

$$W_I(\mathcal{A}, \mathcal{B}) \Rightarrow W_I(\tilde{\mathcal{A}}, \mathcal{B}) \text{ and } W_{II}(\tilde{\mathcal{A}}, \mathcal{B}) \Rightarrow W_{II}(\mathcal{A}, \mathcal{B}).$$

b) $W_I(\mathcal{A}, \mathcal{B}) \iff \exists n \in \mathbb{N} \forall x \in (\bigoplus_{i=n+1}^\infty E_i) \cap B_1 \quad W_I(\mathcal{A}(x), \prod_{i=2}^\infty B_i)$

c) *If $\ell \in \mathbb{N}$, $\bar{\varepsilon} = (\varepsilon_i)_{i=1}^\ell \subset [0, \infty)$ and $x_i, y_i \in B_i$ with $\|x_i - y_i\| \leq \varepsilon_i$ for $i = 1, 2, \dots, \ell$ then*

$$W_I\left(\mathcal{A}(x_1, x_2, \dots, x_\ell), \prod_{i=\ell+1}^\infty B_i\right) \Rightarrow W_I\left(\mathcal{A}_{\bar{\varepsilon}}(y_1, y_2, \dots, y_\ell), \prod_{i=\ell+1}^\infty B_i\right).$$

Here $\mathcal{A}(x_1, x_2, \dots, x_n)$, for $x_1, \dots, x_n \in S_Z$, and $\mathcal{A}_{\bar{\varepsilon}}$, the $\bar{\varepsilon}$ -fattening of \mathcal{A} , was defined in (4.1) of Section 4.1

Now we can state one of our main combinatorial principles.

Theorem 4.2.5. *Let Z have an FDD (E_i) and let $B_i \subset S_Z$, for $i = 1, 2, \dots$. Put $\mathcal{B} = \prod_{i=1}^\infty B_i$ and let $\mathcal{A} \subset S_Z^\omega$.*

Assume that for all $\bar{\varepsilon} = (\varepsilon_i) \subset (0, 1]$ we have $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}}, \mathcal{B})$.

Then for all $\bar{\varepsilon} = (\varepsilon_i) \subset (0, 1]$ there exists a blocking (G_i) of (E_i) so that every skipped block sequence (z_i) of (G_i) , with $z_i \in B_i$, for $i \in \mathbb{N}$, is in $\overline{\mathcal{A}_{\bar{\varepsilon}}}$.

Proof. Let $\bar{\varepsilon} = (\varepsilon_i) \subset (0, 1]$ be given. W.l.o.g assume that $W_I(\mathcal{A}, \mathcal{B})$ (otherwise replace \mathcal{A} by $\overline{\mathcal{A}_{\bar{\varepsilon}/2}}$ and $\bar{\varepsilon}$ by $\bar{\varepsilon}/2$).

For $k = 0, 1, 2, \dots$ put $\bar{\varepsilon}^{(k)} = (\varepsilon_i^{(k)})$ with $\varepsilon_i^{(k)} = \varepsilon_i(1 - 2^{-1-k})$ for $i \in \mathbb{N}$.

For $\ell \in \mathbb{N}$ we write $\mathcal{B}^{(\ell)} = \prod_{i=\ell+1}^{\infty} B_i$.

By induction we choose for $k \in \mathbb{N}$ numbers $n_k \in \mathbb{N}$ so that $0 = n_0 < n_1 < n_2 < \dots$, and so that for any $k \in \mathbb{N}$, if $G_k = \bigoplus_{i=n_{k-1}+1}^{n_k} E_i$,

(4.2) $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}^{(k)}}}(\sigma, x), \mathcal{B}^{(\ell+1)})$ for any $0 \leq \ell < k$ and any normalized skipped block

$$\sigma = (x_1, x_2, \dots, x_\ell) \in \prod_{i=1}^{\ell} B_i \text{ of } (G_i)_{i=1}^{k-1} \text{ } (\sigma = \emptyset \text{ if } \ell = 0)$$

and any $x \in S_{\bigoplus_{i=n_k+1}^{\infty} E_i} \cap B_{\ell+1}$

(4.3) $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}^{(k)}}}(\sigma), \mathcal{B}^{(\ell)})$ for any $0 \leq \ell < k$ and any normalized skipped block

$$\sigma = (x_1, x_2, \dots, x_\ell) \in \prod_{i=1}^{\ell} B_i \text{ of } (G_i)_{i=1}^k$$

For $k = 1$ we deduce from Proposition 4.2.4 (b), the fact that and the hypothesis that there is an $n_1 \in \mathbb{N}$ so that $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}^{(1)}}}(x), \mathcal{B}^{(1)})$ for any $x \in S_{\bigoplus_{i=n_1+1}^{\infty} E_i} \cap B_1$. This implies (4.2) and (4.3) (note that for $k = 1$ σ can only be chosen to be \emptyset in (4.2) and (4.3)).

Assume $n_1 < n_2 < \dots < n_k$ have been chosen for some $k \in \mathbb{N}$. We will first choose n_{k+1} so that (4.2) is satisfied. In the case that $k = 1$ we simply choose $n_2 = n_1 + 1$ and note that (4.2) for $k = 2$ follows from (4.2) for $k = 1$ since in both cases $\sigma = \emptyset$ is the only choice. If $k > 1$ we can use the compactness of the sphere of a finite dimensional space and choose a finite set \mathcal{F} of normalized skipped blocks $(x_1, x_2, \dots, x_\ell) \in \prod_{i=1}^{\ell} B_i$, of $(G_i)_{i=1}^k$ so that for any $\ell \leq k$ and any normalized skipped block with length ℓ , $\sigma = (x_1, x_2, \dots, x_\ell) \in \prod_{i=1}^{\ell} B_i$ of $(G_i)_{i=1}^k$, there is a $\sigma' = (x'_1, x'_2, \dots, x'_\ell) \in \mathcal{F}$ with $\|x_i - x'_i\| < \varepsilon_i 2^{-k-2}$, for $i = 1, 2, \dots, \ell$. Then, using the induction hypothesis (4.3) for k , and Proposition 4.2.4 (b), we choose $n_{k+1} \in \mathbb{N}$ large enough so that $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}^{(k)}}}(\sigma, x), \mathcal{B}^{(\ell+1)})$ for any $\sigma \in \mathcal{F}$ and $x \in S_{\bigoplus_{i=n_{k+1}+1}^{\infty} E_i} \cap B_{\ell+1}$.

From Proposition 4.2.4 (c) and our choice of \mathcal{F} we deduce $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}^{(k+1)}}}(\sigma, x), \mathcal{B}^{(\ell+1)})$ for any $0 \leq \ell < k$, any normalized skipped block σ of $(G_i)_{i=1}^k$ of length ℓ in $\prod_{i=1}^{\ell} B_i$ and any $x \in S_{\bigoplus_{i=n_{k+1}+1}^{\infty} E_i} \cap B_{\ell+1}$, and, thus, (using the induction hypothesis for $\sigma = \emptyset$) we deduce (4.2) for $k + 1$.

In order to verify (4.3) let $\sigma = (x_1, x_2, \dots, x_\ell) \in \prod_{i=1}^{\ell} B_i$ be a normalized skipped block of $(G_i)_{i=1}^{k+1}$ (the case $\sigma = \emptyset$ follows from the induction hypothesis). Then $\sigma' = (x_1, x_2, \dots, x_{\ell-1})$ is empty or a normalized skipped block sequence of $(G_i)_{i=1}^{k-1}$ in $\prod_{i=1}^{\ell-1} B_i$. In the second case $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}^{(k+1)}}}(\sigma), \mathcal{B}^{(\ell)}) = W_I(\overline{\mathcal{A}_{\bar{\varepsilon}^{(k+1)}}}(\sigma', x_\ell), \mathcal{B}^{(\ell)})$ follows from (4.2) for k and from Proposition 4.2.4 (a). This finishes the recursive definition of the n_k 's and G_k 's.

Let (z_n) any normalized skipped block sequence of (G_i) which lies in \mathcal{B} . For any $n \in \mathbb{N}$ it follows from (4.3) for $\sigma = (z_i)_{i=1}^n$ that $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}/2}}(\sigma), \mathcal{B})$, and, thus, $\mathcal{A}_{\bar{\varepsilon}}(\sigma) \neq \emptyset$, which means that σ is extendable to a sequence in $\mathcal{A}_{\bar{\varepsilon}}$ (note that $\lim_{n \rightarrow \infty} \varepsilon_i^{(n)} = \varepsilon_i$). Thus, any normalized skipped block sequence which is element of \mathcal{B} lies in $\overline{\mathcal{A}_{\bar{\varepsilon}}}$. \square

Now let X be a closed subspace of Z having an FDD (E_i) and $\mathcal{A} \subset S_X^\omega$. We consider

the following game

Player I chooses $n_1 \in \mathbb{N}$,
 Player II chooses $x_1 \in \left(\bigoplus_{i=n_1+1}^{\infty} E_i \right)_Z \cap X$, $\|x_1\| = 1$,
 Player I chooses $n_2 \in \mathbb{N}$,
 Player II chooses $x_2 \in \left(\bigoplus_{i=n_2+1}^{\infty} E_i \right)_Z \cap X$, $\|x_2\| = 1$,
 ⋮

As before, Player I wins if $(x_i) \in \mathcal{A}$. Since the game does not only depend on \mathcal{A} but on the superspace Z in which X is embedded and its FDD (E_i) we call this the (\mathcal{A}, Z) -game.

Definition 4.2.6. Assume that X is a subspace of a space Z which has an FDD (E_i) and that $\mathcal{A} \subset S_X^\omega$. Define for $n \in \mathbb{N}$

$$X_n = X \cap \left(\bigoplus_{i=n+1}^{\infty} E_i \right)_Z = \{x \in X : \forall z^* \in \bigoplus_{i=1}^n E_i^* \quad z^*(x) = 0\} ,$$

a closed subspace of finite codimension in X .

We say that *Player II has a winning strategy in the (\mathcal{A}, Z) -game if*

$W_{II}(\mathcal{A}, Z)$ there is a tree $(x_\alpha)_{\alpha \in T_\infty}$ in S_X so that for any $\alpha = (n_1, \dots, n_\ell) \in T_\infty \cup \emptyset$
 $x_{(\alpha, n)} \in X_n$ whenever $n > n_\ell$, and so that no branch lies in \mathcal{A} .

In the case that the (\mathcal{A}, Z) -game is determined *Player I has a winning strategy in the (\mathcal{A}, Z) -game if the negation of $W_{II}(\mathcal{A}, Z)$ is true and thus*

$W_I(\mathcal{A}, Z)$ for any tree $(x_\alpha)_{\alpha \in T_\infty}$ in S_X so that for any $\alpha = (n_1, \dots, n_\ell) \in T_\infty \cup \emptyset$
 $x_{(\alpha, n)} \in X_n$ whenever $n > n_\ell$, there is branch in \mathcal{A} .

For $\mathcal{A} \subset S_X^\omega \subset S_Z^\omega$ and a sequence $\bar{\varepsilon} = (\varepsilon_i)$ in $[0, \infty)$ we understand by $\mathcal{A}_{\bar{\varepsilon}}$ the $\bar{\varepsilon}$ -fattening of \mathcal{A} as a subset of S_Z^ω . In case we want to restrict ourselves to S_X we write $\mathcal{A}_{\bar{\varepsilon}}^X$, i.e.

$$\mathcal{A}_{\bar{\varepsilon}}^X = \mathcal{A}_{\bar{\varepsilon}} \cap S_X^\omega = \{(x_i) \in S_X^\omega : \exists (y_i) \in \mathcal{A} \quad \|x_i - y_i\| \leq \varepsilon_i \text{ for all } i \in \mathbb{N}\} .$$

Since S_X^ω is closed in S_Z^ω with respect to the product of the discrete topology, we deduce that $\overline{\mathcal{A}}^X = \overline{\mathcal{A}}^X$ for $\mathcal{A} \subset S_X^\omega$.

The following Proposition reduces the (\mathcal{A}, Z) -game to a game we treated before. In order to be able to do so we need some technical assumption on the embedding of X into Z (see condition (4.4) below).

Proposition 4.2.7. *Let $X \subset Z$, a space with an FDD (E_i) . Assume the following condition on X , Z and the embedding of X into Z is satisfied:*

(4.4) *There is a $C > 0$ so that for all $m \in \mathbb{N}$ and $\varepsilon > 0$ there is an $n = n(\varepsilon, m) \geq m$*

$$\|x\|_{X/X_m} \leq C [\|P_{[1, n]}^E(x)\| + \varepsilon] \text{ whenever } x \in S_X .$$

Assume that $\mathcal{A} \subset S_X^\omega$ and that for all null sequences $\bar{\varepsilon} \subset (0, 1]$ we have $W_I(\overline{\mathcal{A}}_{\bar{\varepsilon}}^X, Z)$.

Then it follows for all null sequences $\bar{\varepsilon} = (\varepsilon_i) \subset (0, 1]$ that $W_I(\overline{\mathcal{A}}_{\bar{\varepsilon}}, (S_X^\omega)_{\bar{\delta}})$ holds, where $\bar{\delta} = (\delta_i)$ with $\delta_i = \varepsilon_i/28CK$ for $i \in \mathbb{N}$, with C satisfying (4.4) and K being the projection constant of (E_i) in Z .

Question. (open) Is the technical condition (4.4) necessary to derive the conclusion of Proposition 4.2.7

Proof. Let $\mathcal{A} \subset S_X^\omega$ and assume that $W_I(\overline{\mathcal{A}_\eta^X}, Z)$ is satisfied for all null sequences $\bar{\eta} = (\eta_i) \subset (0, 1]$. For a null sequence $\bar{\varepsilon} = (\varepsilon_i) \subset (0, 1]$ we need to verify $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}}, (S_X^\omega)_{\bar{\delta}})$ (with $\delta_i = \varepsilon_i/28KC$ for $i \in \mathbb{N}$) and so we let $(z_\alpha)_{\alpha \in T_\infty}$ be a block tree of (E_i) in S_Z all of whose branches lie in $(S_X^\omega)_{\bar{\delta}} = \{(z_i) \in S_Z^\omega : \text{dist}(z_i, S_X) \leq \delta_i \text{ for } i = 1, 2, \dots\}$.

After passing to a full subtree of (z_α) we can assume that for any $\alpha = (m_1, \dots, m_\ell)$ in T_∞

$$(4.5) \quad z_\alpha \in \bigoplus_{j=1+n(\delta_\ell, m_\ell)}^\infty E_j$$

(where $n(\varepsilon, m)$ is chosen as in (4.4)).

For $\alpha = (m_1, m_2, \dots, m_\ell) \in T_\infty$ we choose $y_\alpha \in S_X$ with $\|y_\alpha - z_\alpha\| < 2\delta_\ell$ and, thus, by (4.5)

$$\|P_{[1, n(\delta_\ell, m_\ell)]}^E(y_\alpha)\| = \|P_{[1, n(\delta_\ell, m_\ell)]}^E(y_\alpha - z_\alpha)\| \leq 2K\delta_\ell.$$

Using (4.4) we can therefore choose an $x'_\alpha \in X_{m_\ell}$ so that

$$\|x'_\alpha - y_\alpha\| \leq C(2K\delta_\ell + \delta_\ell) \leq 3CK\delta_\ell,$$

and thus

$$1 - 3CK\delta_\ell \leq \|x'_\alpha\| \leq 1 + 3CK\delta_\ell.$$

Letting $x_\alpha = x'_\alpha/\|x'_\alpha\|$ we deduce that

$$\begin{aligned} \|y_\alpha - x_\alpha\| &\leq \|y_\alpha - x'_\alpha\| + \|x'_\alpha - x_\alpha\| \\ &\leq 3CK\delta_\ell + (1 + 3CK\delta_\ell)3CK\delta_\ell/(1 - 3CK\delta_\ell) \leq 12CK\delta_\ell \end{aligned}$$

(the last inequality follows from the fact that $(1 + 3CK\delta_\ell)/(1 - 3CK\delta_\ell) \leq 3$) and, thus,

$$\|z_\alpha - x_\alpha\| \leq 14CK\delta_\ell = \varepsilon_\ell/2.$$

Using $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}/2}^X}, Z)$ and noting that $x_\alpha \in X_{m_\ell}$, for $\alpha = (m_1, m_2, \dots, m_\ell) \in T_\infty$ we can choose a branch of (x_α) which is in $\overline{\mathcal{A}_{\bar{\varepsilon}/2}^X}$. Thus, the corresponding branch of (z_α) lies in $\overline{\mathcal{A}_{\bar{\varepsilon}}}$. \square

From [OS1, Lemma 3.1] it follows that every separable Banach space X is a subspace of a space Z with an FDD satisfying the condition (4.4) (with $n(m) = m$). The following Proposition exhibits two general situations in which (4.4) is automatically satisfied.

Proposition 4.2.8. *Assume X is a subspace of a space Z having an FDD (E_i) . In the following two cases (4.4) holds:*

- a) *If (E_i) is a shrinking FDD for Z . In that case C in (4.4) can be chosen arbitrarily close to 1.*
- b) *If (E_i) is boundedly complete for Z (i.e., Z is the dual of $Z^{(*)}$) and the ball of X is a w^* -closed subset of Z . In that case C can be chosen to be the projection constant K of (E_i) in Z .*

Proof. In order to prove (a) we will show that for any $m \in \mathbb{N}$ and any $0 < \varepsilon < 1$ there is an $n = n(\varepsilon, m)$ so that

$$\|x\|_{X/X_m} \leq (1 + \varepsilon) [\|P_{[1,n]}^E(x)\| + \varepsilon], \text{ whenever } x \in S_X$$

(i.e., C in (4.4) can be chosen arbitrarily close to 1).

Since X/X_m is finite dimensional and

$$(X/X_m)^* = X_m^\perp = \{x^* \in X^* : x^*|_{X_m} \equiv 0\},$$

we can choose a finite set $A_m \subset S_{X_m^\perp} \subset S_{X^*}$ for which

$$\|x\|_{X/X_m} \leq (1 + \varepsilon) \max_{f \in A_m} |f(x)| \text{ whenever } x \in X.$$

By the Theorem of Hahn Banach we can extend each $f \in A_m$ to an element $g \in S_{Z^*}$. Let us denote the set of all of these extensions B_m . Since B_m is finite and since (E_i^*) is an FDD of Z^* we can choose an $n = n(\varepsilon, m)$ so that $\|P_{[1,n(m)]}^{E^*}(g) - g\| < \varepsilon$ for all $g \in B_m$. Since $P_{[1,n(m)]}^{E^*}$ is the adjoint operator of $P_{[1,n(m)]}^E$ (consider $P_{[1,n(m)]}^{E^*}$ to be an operator from Z^* to Z^* and $P_{[1,n(m)]}^E$ to be an operator from Z to Z), it follows for $x \in S_X$, that

$$\begin{aligned} \|x\|_{X/X_m} &\leq (1 + \varepsilon) \max_{g \in B_m} |g(x)| \\ &\leq (1 + \varepsilon) \max_{g \in B_m} [|P_{[1,n(m)]}^{E^*}(g)(x)| + \|P_{[1,n(m)]}^{E^*}(g) - g\|] \\ &\leq (1 + \varepsilon) [\max_{g \in B_m} |g(P_{[1,n(m)]}^E(x))| + \varepsilon] \leq (1 + \varepsilon) [\|P_{[1,n(m)]}^E(x)\| + \varepsilon], \end{aligned}$$

which proves our claim and finishes the proof of part (a).

In order to show (b) we assume that X is a subspace of a space Z which has a boundedly complete FDD (E_i) and the unit ball of X is a w^* -closed subset of Z , which is the dual of Z^* .

For $m \in \mathbb{N}$ and $\varepsilon > 0$ we will show that the inequality in (4.4) holds for some n and $C = K$. Assuming that this was not true we could choose a sequence $(x_n) \subset S_X$ so that for any $n \in \mathbb{N}$

$$\|x_n\|_{X/X_m} > K [\|P_{[1,n]}^E(x_n)\| + \varepsilon].$$

By the compactness of B_X in the w^* topology we can choose a subsequence x_{n_k} which converges w^* to some $x \in B_X$. For fixed ℓ it follows that $(P_{[1,\ell]}^E(x_{n_k}))$ converges in norm to $P_{[1,\ell]}^E(x)$. Secondly, since X/X_m is finite dimensional it follows that $\lim_{k \rightarrow \infty} \|x_{n_k}\|_{X/X_m} = \|x\|_{X/X_m}$, and, thus, it follows that

$$\begin{aligned} \|x\| &= \lim_{\ell \rightarrow \infty} \|P_{[1,\ell]}^E(x)\| \\ &= \lim_{\ell \rightarrow \infty} \lim_{k \rightarrow \infty} \|P_{[1,\ell]}^E(x_{n_k})\| \\ &\leq K \limsup_{k \rightarrow \infty} \|P_{[1,n_k]}^E(x_{n_k})\| \\ &\leq \limsup_{k \rightarrow \infty} \|x_{n_k}\|_{X/X_m} - K\varepsilon = \|x\|_{X/X_m} - K\varepsilon, \end{aligned}$$

which is a contradiction since $\|x\| \geq \|x\|_{X/X_m}$. \square

By combining Theorem 4.2.5 and Proposition 4.2.7 we deduce

Corollary 4.2.9. *Let X be a subspace of a space Z with an FDD (E_i) and assume that this embedding satisfies condition (4.4). Let $K \geq 1$ be the projection constant of (E_i) in Z and let $C \geq 1$ be chosen so that (4.4) holds.*

For $\mathcal{A} \subset S_X^\omega$ the following conditions are equivalent

- a) *For all null sequences $\bar{\varepsilon} = (\varepsilon_n) \subset (0, 1]$, $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}}, Z)$ holds.*
- b) *For all null sequences $\bar{\varepsilon} = (\varepsilon_n) \subset (0, 1]$ there exists a blocking (G_n) of (F_n) so that every $\bar{\varepsilon}/420CK$ -skipped block sequence $(z_n) \subset X$ of (G_n) is in $\overline{\mathcal{A}_{\bar{\varepsilon}}}$.*

In the case that X has a separable dual (a) and (b) are equivalent to the following condition

- c) *For all null sequences $\bar{\varepsilon} = (\varepsilon_n) \subset (0, 1]$ every weakly null tree in S_X has a branch in $\overline{\mathcal{A}_{\bar{\varepsilon}}}$.*

In the case that (E_i) is a boundedly complete FDD of Z and B_X is w^ -closed in $Z = (Z^{(*)})^*$ the conditions (a) and (b) are equivalent to*

- d) *For all null sequences $\bar{\varepsilon} = (\varepsilon_n) \subset (0, 1]$ every w^* -null tree in S_X has a branch in $\overline{\mathcal{A}_{\bar{\varepsilon}}}$*

Proof. (a) \Rightarrow (b) Let $\bar{\varepsilon} = (\varepsilon_i) \subset (0, 1]$ be a null sequence, choose $\bar{\eta} = (\eta_i)$ with $\eta_i = \varepsilon_i/3$, for $i \in \mathbb{N}$, and $\bar{\delta} = (\delta_i)$ with $\delta_i = \eta_i/140CK = \varepsilon_i/420CK$.

We deduce from Proposition 4.2.7 that $W_I(\overline{\mathcal{A}_{\bar{\eta}}}, (S_X^\omega)_{5\bar{\delta}})$ holds. Using Theorem 4.2.5 we can block (E_i) into (G_i) so that every skipped block of (G_i) in $(S_X^\omega)_{5\bar{\delta}}$ (as a subset of S_Z) is in $\overline{\mathcal{A}_{2\bar{\eta}}}$ (actually we are using the quantified result given by the proof of Theorem 4.2.5).

Assume $(x_i) \subset S_X$ is a $\bar{\delta}$ -skipped block sequence of (G_i) and let $1 \leq k_1 < \ell_1 < k_2 < \ell_2 < \dots$ in \mathbb{N} so that

$$\|x_n - P_{(k_n, \ell_n]}^E(x_n)\| < \delta_n, \text{ for all } n \in \mathbb{N}.$$

The sequence (z_n) with $z_n = P_{(k_n, \ell_n]}^E(x_n)/\|P_{(k_n, \ell_n]}^E(x_n)\|$, for $n \in \mathbb{N}$, is a skipped block sequence of S_Z and we deduce that

$$\begin{aligned} \|x_n - z_n\| &\leq \|x_n - P_{(k_n, \ell_n]}^E(x_n)\| + \|P_{(k_n, \ell_n]}^E(x_n)\| \left| 1 - \frac{1}{\|P_{(k_n, \ell_n]}^E(x_n)\|} \right| \\ &\leq \delta_n + (1 + \delta_n) \frac{\delta_n}{1 - \delta_n} \leq 5\delta_n. \end{aligned}$$

This implies that $(z_n) \in \overline{\mathcal{A}_{2\bar{\eta}}}$ and thus by our choice of $\bar{\eta}$,

$$(x_i) \in \overline{(\overline{\mathcal{A}_{2\bar{\eta}}})_{\bar{\eta}}} \subset \overline{\mathcal{A}_{\bar{\varepsilon}}}$$

which finishes the verification of (b).

(b) \Rightarrow (a) is clear since for any blocking (G_i) of (E_i) and any null sequence $\bar{\delta} = (\delta_i) \subset (0, 1]$ every tree $(x_\alpha)_{\alpha \in T_\infty}$ in S_X with the property that $x_{(\alpha, n)} \in X_n$, whenever $n > n_\ell$ and $\alpha = (n_1, \dots, n_\ell) \in T_\infty \cup \emptyset$ has a full subtree all of whose branches are $\bar{\delta}$ -skipped block sequences of (G_i) .

Now assume that X has a separable dual, or (E_i) is a boundedly complete FDD of Z and B_X in Z w^* -closed.

It is clear that (c) or (d), respectively, imply (a). Secondly, since for any null sequence $\bar{\delta} = (\delta_i) \subset (0, 1]$ and any blocking (G_i) every weakly null tree in S_X (in the case that X , has a separable dual) or every w^* null tree (in the boundedly complete case) has a full subtree all of whose branches are $\bar{\delta}$ -skipped block sequences of (G_i) we deduce that (b) implies (c) or (d) respectively. \square

Motivated by the asymptotic structure of a Banach space we introduce the following “coordinate-free” variant of our games. Again let X be a separable Banach space and for $\mathcal{A} \subset S_X^\omega$ we consider the following *coordinate-free \mathcal{A} -game*.

Player I chooses $X_1 \in \text{cof}(X)$,
 Player II chooses $x_1 \in X_1$, $\|x_1\| = 1$,
 Player I chooses $X_2 \in \text{cof}(X)$,
 Player II chooses $x_2 \in X_2$, $\|x_2\| = 1$,
 \vdots

As before, Player I wins if $(x_i) \in \mathcal{A}$. We will show that X can be embedded into a space Z with an FDD so that for all $\bar{\varepsilon} = (\varepsilon_i) \subset (0, 1]$ Player I has a winning strategy in the coordinate-free $\overline{\mathcal{A}_{\bar{\varepsilon}}}$ -game, which we will denote by $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}}, \text{cof}(X))$, if and only if for all $\bar{\varepsilon} \subset (0, 1]$ he has a winning strategy for the $(\overline{\mathcal{A}_{\bar{\varepsilon}}}, Z)$ -game.

First note that since we only considering fattened sets and their closures, Player II has a winning strategy if and only if he has a winning strategy choosing his vectors out of a dense and countable subset of S_X determined before the game starts. But this implies that there is countable set of cofinite dimensional subspaces, say $\{Y_n : n \in \mathbb{N}\}$ from which player I can choose if he has a winning strategy. Moreover if we consider a countable set \mathcal{B} of coordinate free games, there is a countable set $\{Y_n : n \in \mathbb{N}\}$ so that for all $\mathcal{A} \in \mathcal{B}$

$$(4.6) \quad \forall \bar{\varepsilon} \subset (0, 1] \quad W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}}, \text{cof}(X)) \iff \forall \bar{\varepsilon} \subset (0, 1) \quad W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}}, \{Y_n : n \in \mathbb{N}\}) ,$$

where we write $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}}, \{Y_n : n \in \mathbb{N}\})$, if player I has a winning strategy for the coordinate-free \mathcal{A} -game, even if he can only choose his spaces out of the set $\{Y_n : n \in \mathbb{N}\}$. Note that by passing to $(\bigcap_{i=1}^n Y_i)$ we can always assume that the Y_n 's are decreasing in $n \in \mathbb{N}$. In case that X has a separable dual and we let (x_n^*) be a dense subset of X^* , we can put for $n \in \mathbb{N}$

$$Y_n = \{x_1^*, x_2^*, \dots, x_n^*\}^\perp = \{x \in X : \forall i \leq n \ x_i^*(x) = 0\} ,$$

and observe that (4.6). holds for all $\mathcal{A} \subset S_Z^\omega$.

The following result was shown in [OS1, Lemma 3.1] and its proof was based on techniques and results of W.B. Johnson, H. Rosenthal and M. Zippin [JRZ].

Lemma 4.2.10. *Let (Y_n) be a decreasing sequence of closed subspaces of X , each having finite codimension. Then X is isometrically embeddable into a space Z having an FDD (E_i) so that (we identify X with its isometric image in Z)*

a) $c_{00}(\bigoplus_{i=1}^\infty E_i) \cap X$ is dense in X .

- b) For every $n \in \mathbb{N}$ the finite codimensional subspace $X_n = \bigoplus_{i=n+1}^{\infty} E_i \cap X$ is contained in Y_n .
- c) There is a $c > 0$, so that for every $n \in \mathbb{N}$ there is a finite set $D_n \subset S_{\bigoplus_{i=1}^n E_i^*}$ such that whenever $x \in X$

$$(4.7) \quad \|x\|_{X/Y_n} = \inf_{y \in Y_n} \|x - y\| \leq c \max_{w^* \in D_n} w^*(x).$$

From (a) it follows that $c_{00}(\bigoplus_{i=n+1}^{\infty} E_i) \cap X$ is a dense linear subspace of X_n .

Moreover if X has a separable dual (E_i) can be chosen to be shrinking (every normalized block sequence in Z with respect to (E_i) converges weakly to 0, or, equivalently, $Z^* = \bigoplus_{i=1}^{\infty} E_i^*$), and if X is reflexive Z can also be chosen to be reflexive.

So assume that for a countable set \mathcal{B} of games that (Y_n) is a sequence of decreasing finite codimensional closed spaces satisfying the equivalences of (4.6).. We then use Lemma 4.2.10 to embed X into a space Z with an FDD (E_i) .

Note that b) of Lemma 4.2.10 implies that for all $\mathcal{A} \in \mathcal{B}$

$$\forall \bar{\varepsilon} \subset (0, 1) \quad W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}}, \text{cof}(X)) \iff \forall \bar{\varepsilon} \subset (0, 1) \quad W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}}, Z).$$

Using the embedding of X given by Lemma 4.2.10 a result similar to Proposition 4.2.7 can be shown. The proof is very similar, therefore we will only present a sketch.

Proposition 4.2.11. *Assume that X is a Banach space and $\{Y_n : n \in \mathbb{N}\}$ a decreasing sequence of cofinite dimensional subspaces. Let Z be a space with an FDD (E_i) which satisfies the conclusion of Lemma 4.2.10.*

Assume that $\mathcal{A} \subset S_X^{\omega}$ such that we have $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}}, \{Y_n : n \in \mathbb{N}\})$ for all null sequences $\bar{\varepsilon} \subset (0, 1]$.

Then for all null sequences $\bar{\varepsilon} = (\varepsilon_i) \subset (0, 1]$, $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}}, (S_X^{\omega})\bar{\delta})$ holds, where $\bar{\delta} = (\delta_i) = (\varepsilon_i/28cK)$, with c as in Lemma 4.2.10, K is the projection constant of (E_i) in Z , and where the fattenings $\overline{\mathcal{A}_{\bar{\varepsilon}}}$ and $(S_X^{\omega})\bar{\delta}$ are taken in Z .

Sketch of proof. Note that instead of condition (4.4) the following condition is satisfied.

$$(4.8) \quad \text{There is a } C > 0 \text{ so that for all } m \in \mathbb{N} \\ \|x\|_{X/Y_m} \leq C \|P_{[1,m]}^E(x)\| \text{ whenever } x \in S_X.$$

Also note that $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}}, \{Y_n : n \in \mathbb{N}\})$ means that every tree $(x_{\alpha}) \subset S_X$, with the property that for $\alpha = (m_1, m_2, \dots, m_{\ell}) \in T_{\infty}$ we have that $x_{\alpha} \in Y_{m_{\ell}}$, has a branch in $\overline{\mathcal{A}_{\bar{\varepsilon}}}$.

We follow the proof of Proposition 4.2.7 until choosing the x_{α} 's which we will not choose in $X_{m_{\ell}}$ but in $Y_{m_{\ell}}$ instead. Then the proof continues as the proof of Proposition 4.2.7. \square

Using Proposition 4.2.11 and Theorem 4.2.5 we deduce the following.

Corollary 4.2.12. *Let $\mathcal{A} \subset S_X^{\omega}$ and assume that Z is a space with an FDD (E_i) which contains X and satisfies the conclusion of Lemma 4.2.10.*

Then the following conditions are equivalent:

- a) For all null sequences $\bar{\varepsilon} = (\varepsilon_n) \subset (0, 1]$, $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}^X}, \text{cof})$ holds.
- b) For all null sequences $\bar{\varepsilon} = (\varepsilon_n) \subset (0, 1]$, $W_I(\overline{\mathcal{A}_{\bar{\varepsilon}}^X}, Z)$ holds.
- c) For all null sequences $\bar{\varepsilon} = (\varepsilon_n) \subset (0, 1]$, there exists a blocking (G_n) of (E_n) so that every $\bar{\varepsilon}/420CK$ -skipped block sequence $(z_n) \subseteq X$ of (G_n) is in $\mathcal{A}_{\bar{\varepsilon}}$.

In the case that X has a separable dual (a), (b) and (c) are equivalent to the following condition (which is independent of the choice of Z).

- d) For all null sequences $\bar{\varepsilon} = (\varepsilon_n) \subset (0, 1]$ every weakly null tree in S_X has a branch in $\overline{\mathcal{A}_{\bar{\varepsilon}}}$.

Moreover, in the case that X has a separable dual we deduce from the remarks after the equivalence (4.6), Corollary 4.2.9 and Proposition 4.2.8 that above equivalences hold for any embedding of X into a space Z having a shrinking FDD.

4.3 Embedding and Universality Theorems

We will use the main Theorems of section 4.2 to derive results of the following type:

- a) Given a Banach space of a certain class, can it be embedded into a Banach space of the same class, or some closely related class, which has a basis or an FDD?
- b) Given a class \mathcal{C} of spaces. Does there exist a Banach space X of this class \mathcal{C} , or of a closely related class $\tilde{\mathcal{C}}$, which is universal for \mathcal{C} , i.e. contains an isomorphic copy of every element of \mathcal{C} ?

The fact that every separable infinite dimensional Banach space X embeds into $C[0, 1]$, and, thus that $C[0, 1]$ is universal for all separable Banach spaces, dates back to the early days of Banach space theory [Ba, Théorème 9, page 185]. Pełczyński [Pe] showed that there is a Banach space having a basis/unconditional basis, which is complementably universal for all Banach spaces having basis/unconditional bases.

An example for the second type of question was answered by Zippin.

Theorem 4.3.1. [Z2] *Every Banach space with a separable dual is embeddable into a Banach space with shrinking basis.*

Every reflexive Banach space is embeddable into a reflexive space with basis.

In this section we interested in characterizing the property of a Banach space (which or may not have an FDD) into spaces with an FDD satisfying C -(p, q)-estimates.

Definition 4.3.2. Let $1 \leq q \leq p \leq \infty$ and $C < \infty$. A (finite or infinite) FDD (E_i) for a Banach space Z is said to satisfy C -(p, q)-estimates if for all $n \in \mathbb{N}$ and block sequences $(x_i)_{i=1}^n$ w.r.t. (E_i) ,

$$C^{-1} \left(\sum_1^n \|x_i\|^p \right)^{1/p} \leq \left\| \sum_1^n x_i \right\| \leq C \left(\sum_1^n \|x_i\|^q \right)^{1/q}.$$

The *coordinate free version* of the C -(p, q)-estimates is the following

Definition 4.3.3. Let $1 \leq q \leq p \leq \infty$ and $C < \infty$. A space X satisfies C -(p, q)-tree estimates if for all weakly null trees in S_X there exist branches $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$ satisfying for all $(a_i) \in c_{00}$,

$$(4.9) \quad C^{-1} \left(\sum |a_i|^p \right)^{1/p} \leq \left\| \sum a_i x_i \right\| \quad \text{and} \quad \left\| \sum a_i y_i \right\| \leq C \left(\sum |a_i|^q \right)^{1/q}.$$

If $X \subseteq Y^*$, a separable dual space, we say that X satisfies C -(p, q)- w^* -tree estimates if each w^* null tree in S_X admits branches (x_i) and (y_i) satisfying (4.9).

We will say that X satisfies (p, q) -tree estimates if it satisfies C -(p, q)-tree estimates for some $C < \infty$ and similarly for (p, q) - w^* tree estimates.

In the definition of q -upper and p -lower tree estimates it is actually not necessary to assume that C exists uniformly for all trees as the following proposition shows.

Proposition 4.3.4. [OSZ, Proposition 1.2] *Let $1 \leq q \leq p \leq \infty$. Assume that X is a Banach space with the property that every normalized, weakly null tree in X has a branch which dominates the ℓ_p -unit vector basis and a branch which is dominated by the ℓ_q -unit vector basis. Then X satisfies (p, q) -tree estimates.*

Proof. For $C \geq 1$ define

$$\mathcal{A}^{(C)} = \left\{ (x_n) \subset S_X : \begin{array}{l} (x_n) \text{ is } C\text{-basic and for all } (a_i) \in c_{00} \\ (\sum |a_i|^p)^{1/p} \leq \|\sum_{i=1}^{\infty} x_i\| \leq (\sum |a_i|^q)^{1/q} \end{array} \right\}.$$

We consider the $(\mathcal{A}^{(C)}, \text{cof})$ -game on X . Assuming that for every weakly null tree there is a C and a branch which is in $\mathcal{A}^{(C)}$, we need to show that there is a uniform C working for all trees.

Assume that this is not true.

By Corollary 4.2.12 we conclude that for any $C > 1$ Player II has a winning strategy for the $(\mathcal{A}^{(C)}, \text{cof})$ -game (since $\mathcal{A}^{(C)}$ is closed under the product of the discrete topology on S_X , this game is determined by Theorem 1.2.4, and also note that for $\varepsilon > 0$ there are nullsequences $\bar{\delta} = (\delta_i) \subset (0, 1)$ and $\bar{\eta} = (\eta_i) \subset (0, 1)$ so that $\overline{\mathcal{A}_{\bar{\delta}}^{(C)}} \subset \mathcal{A}^{(C+\varepsilon)} \subset \overline{\mathcal{A}_{\bar{\eta}}^{(C)}}$).

Player II could choose a sequence (C_n) in \mathbb{R}^+ which increases to ∞ and could play the following strategy: first he follows his strategy for achieving a sequence (x_n) outside of $\mathcal{A}^{(2C_1)}$ and after finitely many steps s_1 he must have chosen a sequence x_1, x_2, \dots, x_{s_2} which is either not C_1 -basic or does satisfy one of the two required inequalities for some $a = (a_i)_{i=1}^{s_1} \in R^{s_1}$. Then Player II follows his strategy for getting a sequence outside of $\mathcal{A}^{(2C_2)}$, and continues that way using C_3, C_4 etc. It follows that the infinite sequence (x_n) , which is obtained by Player II cannot be in any $\mathcal{A}^{(C)}$. Therefore Player II has a winning strategy for choosing a sequence outside of $\bigcup_{C \geq 1} \mathcal{A}^{(C)}$ which means that there are weakly null tree (z_α) none of whose branches is in $\bigcup_{C \geq 1} \mathcal{A}^{(C)}$. \square

The following result is a typical *Embedding Theorem* one can show using the results of Section 4.2

Theorem 4.3.5. [OS3] *Let X be a reflexive and separable Banach space and let $1 < p < \infty$. Assume that every weakly null tree in S_X has a branch which is equivalent to the unit vector basis of ℓ_p*

Then there is a sequence of finite dimensional spaces (G_n) so that X can be isomorphically embedded into $(\oplus G_n)_{\ell_p}$.

Remark. In [Jo1] it was shown that a subspace X of L_p can be embedded into $(\oplus G_n)_{\ell_p}$ if and only if there is a $C \geq 1$ every normalized weakly null sequence has a subsequence which is C -equivalent to the ℓ_p -unit vector basis.

This latter condition is in general (i.e. for reflexive spaces which are not subspaces of L_p) not sufficient to imply embedability into $(\oplus G_n)_{\ell_p}$. For an example see [OS3].

We will show the following generalization of Theorem 4.3.5

Theorem 4.3.6. *Let X and Y be Banach spaces, assume that X is reflexive, let V be a Banach space with a sub-symmetric and normalized basis (v_i) and let $T : X \rightarrow Y$ be linear and bounded.*

Assume that for some $C \geq 1$ every normalized weakly null tree of X has a branch (x_n) so that

$$(4.10) \quad \left\| \sum_{i=1}^{\infty} a_n x_n \right\|_X \sim_C \left\| \sum_{i=1}^{\infty} a_n v_n \right\|_V \vee \left\| T \left(\sum_{i=1}^{\infty} a_n x_n \right) \right\|_Y \text{ whenever } (a_i) \in c_{00}.$$

Then there is a sequence of finite dimensional spaces (G_i) so that X is isomorphic to a subspace of $(\oplus_{i=1}^{\infty} G_i)_V \oplus Y$.

More precisely, if Z is any reflexive space with an FDD (E_i) which contains a copy of X (such a space Z always exists [Z2]) and if $S : X \rightarrow Z$ is an isomorphic embedding, then there is a blocking (G_i) so that S is a bounded linear operator from X to $(\oplus_{i=1}^{\infty} G_i)_V$ and the operator

$$(S, T) : X \rightarrow (\oplus_{i=1}^{\infty} G_i)_V \oplus Y, \quad x \mapsto (S(x), T(x)),$$

is an isomorphic embedding.

We will need the following Lemma which uses a blocking trick of Johnson [Jo1].

Lemma 4.3.7. *Let X be a subspace of a space Z having a boundedly complete FDD (E_i) with projection constant K with B_X being a w^* -closed subset of Z . Let $\delta_i \downarrow 0$. Then there exists a blocking (F_i) of (E_i) given by $F_i = \bigoplus_{j=N_{i-1}+1}^{N_i} E_j$ for some $0 = N_0 < N_1 < \dots$ with the following properties. For all $x \in S_X$ there exists $(x_i)_{i=1}^{\infty} \subseteq X$ and for all $i \in \mathbb{N}$ there exists $t_i \in (N_{i-1}, N_i)$ satisfying ($t_0 = 1$ and $t_1 > 1$)*

- a) $x = \sum_{j=1}^{\infty} x_j$,
- b) $\|x_i\| < \delta_i$ or $\|P_{(t_{i-1}, t_i)}^E x_i - x_i\| < \delta_i \|x_i\|$,
- c) $\|P_{(t_{i-1}, t_i)}^E x - x_i\| < \delta_i$,
- d) $\|x_i\| < K + 1$,
- e) $\|P_{t_i}^E x\| < \delta_i$.

Moreover, the above hold for any blocking of (F_i) (which would redefine the N_i 's).

Proof. We observe that for all $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists $n > N$ such that if $x \in B_X$, $x = \sum y_i$ with $y_i \in E_i$ for all i , then there exists $t \in (N, n)$ with

$$\|y_t\| < \varepsilon \text{ and } \text{dist} \left(\sum_{i=1}^{t-1} y_i, X \right) < \varepsilon.$$

Indeed, if this was not true for any $n > N$ we can find $y^{(n)} \in B_X$ failing the conclusion for $t \in (N, n)$. Choose a subsequence of $(y^{(n)})$ converging w^* to $y \in X$ and choose $t > N$ so that $\|P_{[t, \infty)}^E y\| < \varepsilon/2K$. Then choose $y^{(n)}$ from the subsequence so that $t < n$ and $\|P_{[1, t]}^E (y - y^{(n)})\| < \varepsilon/2K$. Thus

$$\|P_{[1, t]}^E y^{(n)} - y\| \leq \|P_{[1, t]}^E (y^{(n)} - y)\| + \|P_{[t, \infty)}^E y\| < \frac{\varepsilon}{2K} + \frac{\varepsilon}{2K} < \varepsilon.$$

Also

$$\|P_t^E y^{(n)}\| \leq \|P_t^E (y^{(n)} - y)\| + \|P_t^E y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

This contradicts our choice of $y^{(n)}$.

Let $\varepsilon_i \downarrow 0$ and by the observation choose $0 = N_0 < N_1 < \dots$ so that for all $x \in S_X$ there exists $t_i \in (N_{i-1}, N_i)$ and $z_i \in X$ with $\|P_{t_i}^E x\| < \varepsilon_i$ and $\|P_{[1, t_i-1]}^E x - z_i\| < \varepsilon_i$ for all $i \in \mathbb{N}$. Set $x_1 = z_1$ and $x_i = z_i - z_{i-1}$ for $i > 1$. Thus $\sum_{i=1}^n x_i = z_n \rightarrow x$ so a) holds. Also

$$\|P_{(t_{i-1}, t_i)}^E x - x_i\| \leq \|P_{[1, t_i]}^E x - z_i\| + \|P_{[1, t_{i-1}]}^E x - z_{i-1}\| < \varepsilon_i + 2\varepsilon_{i-1} ,$$

and

$$\|P_{(t_{i-1}, t_i)}^E x_i - x_i\| = \|(I - P_{(t_{i-1}, t_i)}^E)(x_i - P_{(t_{i-1}, t_i)}^E x)\| < (K + 1)(\varepsilon_i + 2\varepsilon_{i-1}) .$$

From these inequalities b), c) and d) follow if we take (ε_i) so that $(K + 1)(\varepsilon_i + 2\varepsilon_{i-1}) < \delta_i^2$. \square

Proof of Theorem 4.3.6. By Zippin's theorem [Z2] we can assume that X is the subspace of a reflexive space Z with an FDD $E = (E_i)$. After renorming we can assume that the projection constant $K = \sup_{m \leq n} \|P_{[m, n]}^E\| = 1$. We also assume without loss of generality that $\|T\| \leq 1$.

For a sequence $\bar{x} = (x_i) \in S_X$ and $a = \sum a_i e_i \in c_{00}$ we define

$$\left\| \sum a_i e_i \right\|_{(V, T, \bar{x})} = \left\| \sum a_i v_i \right\|_V \vee \left\| T \left(\sum a_i x_i \right) \right\|_Y .$$

Then $\|\cdot\|_{(V, T, \bar{x})}$ is a norm on c_{00} and we denote the completion of c_{00} with respect to $\|\cdot\|_{(V, T, \bar{x})}$ by $X(V, T, \bar{x})$.

Define

$$\mathcal{A} = \left\{ \bar{x} = (x_n) \subset S_X : \begin{array}{l} \bar{x} \text{ is } \frac{3}{2}\text{-basic and } \frac{3}{2}C\text{-equivalent} \\ \text{to } (e_i) \text{ in } X(V, T, \bar{x}) \end{array} \right\} .$$

By Corollary 4.2.12 applied to an appropriate sequence $\bar{\varepsilon} = (\varepsilon_i) \subset (0, 1)$ we can find a blocking $F = (F_i)$ of (E_i) and a sequence $(\bar{\delta}) \subset (0, 1)$, so that every $\bar{\delta}$ -skipped block $(x_i) \subset S_X$ of (F_i) is 2-basic and $2C$ -equivalent to (e_i) in $X(V, T, \bar{x})$. Now we apply Lemma 4.3.7 to get a further blocking (G_i) , $G_i = \bigoplus_{j=N_{i-1}+1}^{N_i} F_j$, for $i \in \mathbb{N}$ and some sequence $0 = N_0 < N_1 < N_2 \dots$, so that for every $x \in S_X$ there is a sequence $(t_i) \subset \mathbb{N}$, with $t_i \in (N_{i-1}, N_i)$ for $i \in \mathbb{N}$, and a sequence (x_i) satisfying (a)-(e).

We also may assume that $\sum_{i=1}^{\infty} \delta_i < 1/36C$ and will show that for every $x \in X$

$$(4.11) \quad \|x\|_X \sim_{36C} \left(\left\| \sum_{i=1}^{\infty} \|P_i^G(x)\| v_i \right\|_V \right) \vee \|T(x)\|_Y .$$

This implies that the map $X \rightarrow (\bigoplus G_i)_V \oplus Y$, $x \mapsto ((P_i^G(x)), T(x))$, is an isomorphic embedding.

Let $x \in S_X$ and choose $(t_i) \subset \mathbb{N}$ and $(x_i) \subset X$ as prescript in Lemma 4.3.7. Letting $B = \{i \geq 2 : \|P_{(t_{i-1}, t_i)}^F(x_i) - x_i\| \leq \delta_i \|x_i\|\}$ it follows that $(x_i)_{i \in B}$ is a $\bar{\delta}$ -skipped block and therefore

$$(4.12) \quad \left\| \sum_{i \in B} x_i \right\|_X \sim_{2C} \left\| \sum_{i \in B} \|x_i\| v_i \right\|_V \vee \left\| T \left(\sum_{i \in B} x_i \right) \right\|.$$

If $\|x_1\| \geq 1/8C$ then we deduce that

$$(4.13) \quad \begin{aligned} \frac{1}{8C} &\leq \|x_1\| \leq \left\| \sum_{i=1}^{\infty} \|x_i\| v_i \right\|_V \vee \|T(x)\|_V \\ &\leq \left(\left\| \sum_{i \in B} \|x_i\| v_i \right\|_V + \|x_1\| + \sum_{i \notin B} \delta_i \right) \vee \|T(x)\|_Y \\ &\leq 2C \left\| \sum_{i \in B} x_i \right\| + 2 + \sum \delta_i \quad [\text{By (4.12) and (d) of Lemma 4.3.7}] \\ &\leq 2C\|x\| + 2C\|x_1\| + 3C \sum \delta_i + 2C \leq 9C \end{aligned}$$

If $\|x_1\| < 1/8C$ then

$$(4.14) \quad \begin{aligned} \frac{1}{4C} &\leq \left\| \sum_{i \in B} \|x_i\| v_i \right\|_V \vee \left(\left\| T \left(\sum_{i \in B} x_i \right) \right\|_Y - \frac{1}{4C} \right) \quad [\text{By (4.12)}] \\ &\leq \left\| \sum_{i \in B} \|x_i\| v_i \right\|_V \vee \|T(x)\|_Y \quad [\text{Since } \|T\| \leq 1] \\ &\leq \left\| \sum_{i=1}^{\infty} \|x_i\| v_i \right\|_V \vee \|T(x)\|_Y \\ &\leq \left\| \sum_{i \in B} \|x_i\| v_i \right\|_V \vee \left\| T \left(\sum_{i \in B} x_i \right) \right\|_Y + \frac{1}{4C} \\ &\leq 2C \left\| \sum_{i \in B} x_i \right\| + \frac{1}{4C} \quad [\text{By (4.12)}] \\ &\leq 2C\|x\| + 2C\|x_1\| + 2C \sum \delta_i + \frac{1}{4C} \leq 4C. \end{aligned}$$

(4.13) and (4.14) imply that

$$(4.15) \quad 1 \sim_{9C} \left\| \sum_{i=1}^{\infty} \|x_i\| v_i \right\|_V \vee \|T(x)\|.$$

For $n \in \mathbb{N}$ define $y_n = P_{(t_{n-1}, t_n]}^F(x)$. From Lemma 4.3.7 (c) and (e) it follows that $\|y_n - x_n\| \leq \|P_{(t_{n-1}, t_n]}^F(x) - x_n\| + \|P_{t_n}^F(x)\| \leq 2\delta_n$ and thus $\sum \|y_n - x_n\| \leq 1/18C$ which implies by (4.15) that

$$(4.16) \quad 1 \sim_{18C} \left\| \sum_{i=1}^{\infty} \|y_i\| v_i \right\|_V \vee \|T(x)\|.$$

Since for $n \in \mathbb{N}$ we have $(N_{n-1}, N_n] \subset (t_{n-1}, t_{n+1})$ and $(t_{n-1}, t_n] \subset (N_{n-2}, N_n)$ (put $N_1 = N_0 = 0$ and $P_0^H = 0$) it follows from the assumed sub-symmetry of (v_n) and the assumed bi-monotonicity of (E_i) in Z that

$$\begin{aligned} \frac{1}{2} \left\| \sum_{n \in \mathbb{N}} \|y_n\| v_n \right\|_V &\leq \frac{1}{2} \left\| \sum_{n \in \mathbb{N}} (\|P_{n-1}^G(x)\| + \|P_n^G(x)\|) v_n \right\|_V \\ &\leq \left\| \sum_{n \in \mathbb{N}} \|P_n^G(x)\| v_n \right\|_V \\ &\leq \left\| \sum_{n \in \mathbb{N}} \|P_{(t_{n-1}, t_{n+1})}^F(x)\| v_n \right\|_V \\ &\leq \left\| \sum_{n \in \mathbb{N}} (\|y_n\| + \|y_{n+1}\|) v_n \right\|_V \leq 2 \left\| \sum_{n \in \mathbb{N}} \|y_n\| v_n \right\|_V, \end{aligned}$$

which implies with (4.16) that

$$1 \sim_{36C} \left\| \sum_{i=1}^{\infty} \|x_i\| v_i \right\|_V \vee \|T(x)\|.$$

and finishes the proof of our claim. \square

Example 4.3.8. Let $1 < p < \infty$. There exists a reflexive space X with an unconditional basis so that X satisfies: for all $\varepsilon > 0$ every normalized weakly null sequence in X admits a subsequence $(1 + \varepsilon)$ -equivalent to the unit vector basis of ℓ_p . Yet X is not a subspace of an ℓ_p -sum of finite dimensional spaces.

Proof. Fix $1 < q < p$. We define $X = (\sum X_n)_p$ where each X_n is given as follows. X_n will be the completion of $c_{00}([\mathbb{N}]^{\leq n})$ under the norm

$$\|x\|_n = \sup \left\{ \left(\sum_{i=1}^m \|x|_{\beta_i}\|_q^p \right)^{1/p} : (\beta_i)_1^m \text{ are disjoint segments in } [\mathbb{N}]^{\leq n} \right\}.$$

By a *segment* we mean a sequence $(A_i)_{i=1}^k \in [\mathbb{N}]^{\leq n}$ with $A_1 = \{n_1, n_2, \dots, n_\ell\}$, $A_2 = \{n_1, n_2, \dots, n_\ell, n_{\ell+1}\} \dots A_k = \{n_1, n_2, \dots, n_\ell, n_{\ell+1} \dots, n_{\ell+k-1}\}$, for some $n_1 < n_2 < \dots, n_{\ell+k-1}$. Thus a segment can be seen as an interval of a branch (with respect to the usual partial order in $[\mathbb{N}]^{\leq n}$), while a branch is a maximal segment.

Clearly the *node basis* $(e_A^{(n)})_{A \in [\mathbb{N}]^{\leq n}}$ given by $e_A^{(n)}(B) = \delta_{(A,B)}$ is a 1-unconditional basis for X_n . Furthermore the unit vector basis of ℓ_q^n is 1-equivalent to $(e_{A_i}^{(n)})_{i=1}^n$, if $(A_i)_{i=1}^n$ is any branch of $[\mathbb{N}]^{\leq n}$.

Thus no extension of the tree $(e_A^{(n)})_{A \in [\mathbb{N}]^{\leq n}}$ to a weakly null tree of infinite length in S_X has a branch whose basis distance to the ℓ_p -unit vector basis is closer than $\text{dist}_b(\ell_p^{(n)}, \ell_q^{(n)}) = n^{\frac{1}{q} - \frac{1}{p}} \rightarrow \infty$ for $n \rightarrow \infty$. Since it is clear that in every subspace Y of an ℓ_p sum of finite dimensional spaces every weakly null tree in S_Y must have a branch equivalent (for a fixed constant) to the unit vector basis of ℓ_p it follows that X cannot be embedded into a subspace of an ℓ_p -sum of finite dimensional spaces.

Also each X_n is isomorphic to ℓ_p and thus X is reflexive.

It remains to show that if (x_j) is a normalized weakly null sequence in X and $\varepsilon > 0$ then a subsequence is $(1 + \varepsilon)$ -equivalent to the unit vector basis of ℓ_p . This will essentially follow from the following lemma.

We say that a Banach space has property $(*)$ is

$$(*) \quad \forall (x_n) \subset S_X, w\text{-null, and } \varepsilon > 0 \exists (x'_n) \subset (x_n) \quad (x'_n) \text{ is } (1 + \varepsilon)\text{-equiv. to } \ell_p\text{-u.v.b.}$$

Lemma 4.3.9. *If Y_n has property $(*)$ for all $n \in \mathbb{N}$ then $(\oplus Y_n)_{\ell_p}$ has property $(*)$.*

Firstly we use Lemma 4.3.9 to reduce our claim that X has property $(*)$ to the claim that each X_n has property $(*)$. Then we show by induction for each $n \in \mathbb{N}$ that X_n has property $(*)$. For $n = 1$ this is trivial. Assuming we showed that X_n satisfies $(*)$, we observe that X_{n+1} is the ℓ_p -sum of spaces Y_n where Y_n isometrically isomorphic to a 1-dimensional extension of X_n , say $Y_n \equiv \mathbb{R} \oplus X_n$. Since every weakly null sequence (z_k) in Y_n is up to passing to subsequence and small perturbation in X_n , it follows that each Y_n has property $(*)$, we apply Lemma 4.3.9 again to conclude that X_{n+1} has property $(*)$. \square

Proof of Lemma 4.3.9. Let y_n be a normalized sequence in $Y = \oplus_k Y_k$. For $k \in \mathbb{N}$ denote by P_k the canonical projection of Y onto Y_k and put $y(n, k) = P_k(y_n)$. For $A \subset \mathbb{N}$ and $z \in Y$ define $P_A(z) = \sum_{k \in A} P_k(z)$

After passing to a subsequence and an arbitrary perturbation we may assume that for each $k \in \mathbb{N}$ $a_k = \lim_{n \rightarrow \infty} \|y(n, k)\|$ exists and that for some sequence (k_i) in \mathbb{N} $y_n = P_{[1, k_n]}(y_n)$ for all $n \in \mathbb{N}$.

Note that $\alpha = \sum_{i=1}^{\infty} |a_k|^p \leq 1$. If $\alpha = 0$, we can find a subsequence (y'_n) which is an arbitrary perturbation of a sequence (z_n) for which the sets $A_n = \{k : P_k(z_n) \neq 0\}$ are disjoint. If $\alpha > 0$ we proceed as follows:

Let $\varepsilon_i \searrow 0$ fast enough (to be determined later). First choose $\ell_1 \in \mathbb{N}$ so that $\sum_{k > \ell_1} \alpha_k^p < \varepsilon_1$ and a subsequence $N_1 \subset \mathbb{N}$ so that

$$\left(\frac{P_{[1, \ell_1]}(y_n)}{\|P_{[1, \ell_1]}(y_n)\|} : n \in N_1 \right)$$

is $(1 + \varepsilon_1)$ -equivalent to the unit vector basis in ℓ_p and choose some $n_1 \in N_1$.

Then choose $\ell_2 \geq k_{n_1}$ so that $\sum_{k > \ell_2} \alpha_k^p < \varepsilon_2$ and choose subsequence $N_2 \subset N_1$ so that

$$\left(\frac{P_{[1, \ell_2]}(y_n)}{\|P_{[1, \ell_2]}(y_n)\|} : n \in N_2 \right),$$

is $(1 + \varepsilon_2)$ -equivalent to the unit vector basis in ℓ_p and so that $\|P_{(\ell_2, k_{n_1})}(y_n)\| \leq \varepsilon_1$ for $n \in N_2$. and choose some $n_2 \in N_2$.

Continuing this way we deduce that (y_{n_i}) is $1 - \varepsilon$ -equivalent to the unit vector basis in ℓ_p with ε depending on $\sum_{i=1}^{\infty} \varepsilon_i$. \square

Theorem 4.3.10. *Let X be a reflexive Banach space and let $1 \leq q \leq p \leq \infty$. The following are equivalent.*

a) X satisfies (p, q) -tree estimates.

- b) X is isomorphic to a subspace of a reflexive space Z having an FDD which satisfies (p, q) -estimates.
- c) X is isomorphic to a quotient of a reflexive space Z having an FDD which satisfies (p, q) -estimates.

Sketch of a proof of (a) \Rightarrow (b). .

Step 1: Embed X in a reflexive space Z with FDD (E_i) . Using the same proof as in Theorem 4.3.6 we can proof:

There is a blocking $F = (F_i)$ of (E_i) so that X embeds in $Z(F, p)$, which is the completion of $c(\oplus_{i=1}^{\infty} F_i)$ under the following norm:

$$\|x\|_{(Z,p)} = \sup_{0=n_0 < n_1 < \dots < n_k, k \in \mathbb{N}} \left(\sum \|P_{(n_{i-1}, n_i)}^F(x)\|_Z^p \right)^{1/p}.$$

Step 2: The condition X satisfies upper q -tree estimates dualizes, i.e. X^* satisfies lower q' -tree estimates.

Step 3: Consider quotient map $Q : Z_p^* \searrow X^*$.

For $i \in \mathbb{N}$ let \tilde{F}_i be the quotient space of F_i^* determined by Q . Thus if $z \in F_i^*$, the norm on \tilde{z} (the equivalence class of z in F_i) is $|||\tilde{z}||| = \|Qz\|$. We may assume $\tilde{F}_i \neq \{0\}$ for all i . More generally for $\tilde{z} = \sum \tilde{z}_i \in c_{00}(\oplus_{i=1}^{\infty} \tilde{F}_i)$ with $\tilde{z}_i \in \tilde{F}_i$ for every i , we set

$$|||\tilde{z}||| = \sup_{m \leq n} \left\| \sum_{i=m}^n Qz_i \right\| = \sup_{m \leq n} \|QP_{[m,n]}^F z\|.$$

We let \tilde{Z} be the completion of $(c_{00}(\oplus_{i=1}^{\infty} \tilde{F}_i), |||\cdot|||)$. Note that if $\tilde{z} = \sum \tilde{z}_i \in c_{00}(\oplus_{i=1}^{\infty} \tilde{F}_i)$ then setting $\tilde{Q}\tilde{z} \equiv \sum \tilde{Q}\tilde{z}_i \equiv \sum Qz_i$, we have $\|Q\tilde{z}\| \leq |||\tilde{z}|||$. Thus \tilde{Q} extends to a norm one map from \tilde{Z} into X .

- a) (\tilde{E}_i) is a bimonotone shrinking FDD for \tilde{Z} .
- b) \tilde{Q} is a quotient map from \tilde{Z} onto X . More precisely if $x \in X$ and $z \in Z$ with $Qz = x$, $\|z\| = \|x\|$, and $z = \sum z_i$ with $z_i \in F_i^*$, then $\tilde{z} = \sum \tilde{z}_i \in \tilde{Z}$, $|||\tilde{z}||| = \|z\|$ and $\tilde{Q}\tilde{z} = x$.
- c) Let (\tilde{z}_i) be a block sequence of (\tilde{F}_i) in $B_{\tilde{Z}}$ and assume that $(\tilde{Q}\tilde{z}_i)$ is a basic sequence with projection constant \bar{K} and $a = \inf_i \|\tilde{Q}\tilde{z}_i\| > 0$. Then for all scalars (a_i) we have

$$\left\| \sum a_i \tilde{Q}(\tilde{z}_i) \right\| \leq ||| \sum a_i \tilde{z}_i ||| \leq \frac{3\bar{K}}{a} \left\| \sum a_i \tilde{Q}\tilde{z}_i \right\|.$$

Step 4: There is a blocking \tilde{H}_i of \tilde{F}_i so that \tilde{Q} is still a quotient map from $\tilde{Z}(q', \tilde{H})$ onto X^*

Step 5: X embeds into $Z^*(q', \tilde{H})$ and $H^* = (\tilde{H}_i^*)$ is a an FDD of $Z^*(q', H)$ which satisfies upper q -estimates.

Step 6: apply Step 1 again to $Z^*(q', H)$.

□

In [Bo] Bourgain asked whether or not there is a reflexive space universal for all *super reflexive spaces*.

Definition 4.3.11. Let E and X be two Banach spaces. We say that E is *finitely represented in X* if there is a $C \geq 1$ for every finite dimensional subspace F of E there is a finite dimensional subspace F' in X , so that the Banach mazur distance between F and F' is not larger than C .

X is called *super reflexive* if every space E which is finitely represented in X is reflexive.

Remark. Since every Banach space X is finitely represented in it self, every super reflexive space must be reflexive. On the other hand Tsireson space is reflexive but not super reflexive.

In order to solve Bourgain's question we also need the following two results.

Theorem 4.3.12. [Pr](*Solution of Bourgain's problem for spaces with FDD*)
There exists a reflexive Banach space X which is universal for all spaces with a finite dimensional decomposition (FDD) which satisfy (p, q) -estimates for some $1 < q \leq p < \infty$.

Theorem 4.3.13. [Ja3]. *Let $c \geq 1$. For every super reflexive Banach space X there are $1 < q \leq p < \infty$ and $C \geq 1$, so that very normalized basic sequence (x_n) , whose basis constant does not exceed c , satisfies (p, q) estimates.*

Since every weakly null sequence has a subsequence whose basis constant is not larger than, say, 2. It follows that for every super reflexive Banach space X there are $1 < q \leq p < \infty$ so that X satisfies (p, q) -tree estimates.

Remark. From the arguments in [Ja3] it follows that separable Banach space which satisfies (p, q) -tree estimates for some $1 < p \leq q$ is reflexive.

But such a space does not need to be super reflexive as the example $(\oplus_{n=1}^{\infty} \ell_1^n)_{\ell_2}$ shows. There fore the following question is still open

Question. Does a separable super reflexive space embed in a super reflexive space with a basis, or with an FDD?

We will deduce Theorem 4.3.10 from Corollary 4.2.9 only in the spacial case that $p = q$. In this case it was already shown and answered a problem of Johnson, asking for an intrinsic characterization of the property of a Banach space being a subspace of an ℓ_p -sum of finite dimensional Banach spaces.

Chapter 5

Ordinal numbers

5.1 Definition of ordinal numbers

Definition 5.1.1. A *well order* on a set S is a relation $<$ on S which has the following properties.

- (WO1) $(A, <)$ is a linear order, i.e. for any $a, b \in S$, one and only one of the following cases occur: Either $a < b$ or $b < a$ or $a = b$.
- (WO2) Every nonempty subset A of S has a minimum, i.e. there is an $a_0 \in A$ so that for all $a \in A$ either $a_0 < a$ or $a_0 = a$.

In that case the pair $(S, <)$ is called a well ordering and we introduce the following notations:

For $a, b \in S$ we write $a \leq b$ if $a < b$ or $a = b$.

For $a, b \in S$, with $a \leq b$ we introduce the following *intervals*:

$$[a, b] = \{x \in S : a \leq x \leq b\}$$

$$[a, b) = \{x \in S : a \leq x < b\}$$

$$(a, b] = \{x \in S : a < x \leq b\}$$

$$(a, b) = \{x \in S : a < x < b\}$$

For a none empty $A \subset S$ we denote the (by (OW2) existing) minimal element of A by $\min(A)$, and write $0_S = \min S$. If $a \in S$ is not a maximal element then the set of *successors* of a , namely the set

$$\text{Succ}(a) = \{x \in S : x > a\},$$

must have a minimal element which we call the *direct successor* of a and denote it by a^+ .

The following theorem is an easy consequence of the Hausdorff Maximal Principle.

Theorem 5.1.2. *Every set can be well ordered.*

Proof. Let S be a set. Define

$$\mathcal{W} = \{(A, <_A) : A \subset S \text{ and } <_A \text{ well order on } A\}.$$

For $(A, <_A)$ and $(\tilde{A}, <_{\tilde{A}})$ in \mathcal{W} we write $(A, <_A) \prec (\tilde{A}, <_{\tilde{A}})$ if $A \subset \tilde{A}$ and the restriction of $<_{\tilde{A}}$ to A is $<_A$. Then it is easy to see that for \prec every linear ordered subset in \mathcal{W} has an upper bound. By the Hausdorff Maximal Principle there is a maximal element $(A, <_A)$ in \mathcal{W} , and since one could easily extend $<_A$ to one more element if $A \neq S$, it follows that $A = S$. \square

Definition 5.1.3. Two $(S, <)$ and (S', \prec) are called *order isomorphic* if there exists a bijective (1 – 1 and onto) map: $\Phi : S \rightarrow S'$ which is *order preserving* i.e.

$$\forall a, b \in S \quad (a < b \iff \phi(a) < \phi(b)).$$

In that case we call ϕ an *order isomorphism between S and S'* .

Proposition 5.1.4. *If $(S, <)$ and (S', \prec) are order isomorphic well orderings, then the order isomorphism between them is unique.*

Proof. W.l.o.g. $S, S' \neq \emptyset$. Let $\Psi, \Phi : S \rightarrow S'$ be two order isomorphism, and assuming that $\Psi \neq \Phi$ we can define:

$$a = \min\{x \in S : \Psi(x) \neq \Phi(x)\}.$$

Since $\Psi(a) \neq \Phi(a)$ we may w.l.o.g. assume that $\Psi(a) < \Phi(a)$. But now it follows that

$$\Phi(S) \subset [0_{S'}, \Psi(a)) \cup \{x' \in S' : x' \succ \Phi(a)\},$$

(Indeed: $x < a \Rightarrow \Psi(x) = \Phi(x) \prec \Phi(a)$, and $x \geq a \Rightarrow \Psi(x) \succeq \Psi(a) \succ \Phi(a)$), i.e. $\Phi(a) \notin \Psi(S)$, which contradicts that Ψ is surjective. \square

Proposition 5.1.5. *If $(S, <)$ is a well ordering and $a, b \in S$. Then $[0_S, a]$ and $[0_S, b]$ are order isomorph if and only if $a = b$.*

Proof. Let $\Phi : [0_S, a] \rightarrow [0_S, b]$ be an order isomorphism and consider

$$A = \{x \in [0_S, a] : \Phi(x) \neq x\}.$$

If this set was not empty we could choose $x_0 = \min(A)$, and, using a similar argumentation as in the proof of Proposition 5.1.4 we would get contradiction to the assumed surjectivity of Φ . If A is empty, it follows that Φ is the identity on $[0_S, a]$, and, since Φ is supposed to be surjective it follows that $a = b$. \square

Corollary 5.1.6. *If $(S, <)$ and (S', \prec) are two well orderings and there is an $a' \in S$ so that $(S, <)$ is order isomorph to $[0_{S'}, a')$. Then such an $a' \in S'$ is unique.*

Theorem 5.1.7. *Let $(S, <)$ and (S', \prec) be two well orderings. Then one and only one of the following cases occurs.*

Case 1. $(S, <)$ and (S', \prec) are order isomorphic.

Case 2. There is an $a \in S$ so that $[0_S, a]$ and S' are order isomorphic.

Case 3. There is an $a' \in S'$ so that $[0_{S'}, a')$ and S are order isomorphic.

Proof. Define:

$$A = \{a \in S : \exists a' \in S' \quad [0_S, a] \text{ and } [0_{S'}, a'] \text{ are order isomorphic}\}.$$

Let us first assume that $A = S$. If S has a maximal element a it follows that there exists an order isomorphism between $S = [0_S, a]$ and some interval $[0_{S'}, b']$ (which might or might not be all of S') which is order isomorphic to S . This means that we are either in Case 1 or in Case 3 (choose $a' = b'^+$). If S has not a maximal element we choose for $a \in S$, an $a' \in S'$ and an order isomorphism $\Phi_a : [0_S, a] \rightarrow [0_{S'}, a']$ (note that by Proposition 5.1.4 and Corollary 5.1.6 a' and Φ_a are unique for every $a \in S$) then choose

$$\Phi : S \rightarrow S', a \mapsto \Phi_a(a).$$

From the uniqueness of the Φ_a 's it follows for $a < b$ in S that $\Phi_a(a) = \Phi_b(a)$. Moreover, if $\Phi(S) = S'$ it follows that Φ is an order isomorphism between S and S' , and if $\Phi(S) \neq S'$ it follows that Φ is an order isomorphism between S and a' , where $a' = \min\{x' \in S' : x' \notin \Phi(S)\}$.

If $A \neq S$ and we put $a_0 = \min S \setminus A$. For $a < a_0$ we choose (the uniquely existing) $a' \in S'$ and $\Phi_a : [0_S, a] \rightarrow [0_{S'}, a']$, where Φ_a is an order isomorphism and claim that $S' = \{a' : a \in S\}$. Indeed if this were not so, we could put $a'_0 = \min S' \setminus \{a' : a \in S\}$ and define

$$\Phi_{a_0} : [0, a_0] \rightarrow [0, a'_0], a \mapsto \begin{cases} \Phi_a(a) & \text{if } a < a_0 \\ a'_0 & \text{if } a = a_0 \end{cases}$$

and deduce from the uniqueness of the a' and Φ_a for $a \in S$ that Φ_{a_0} is an order isomorphism between $[0, a_0]$ and $[0, a'_0]$, which contradicts our definition of a_0 .

Therefore it follows that

$$A' = \{a' \in S' : \exists a \in S \quad [0_S, a] \text{ and } [0_{S'}, a'] \text{ are order isomorphic}\} = S',$$

and using our previous arguments we can show that we are either in Case 1 or Case 2.

Moreover, only one of the three cases can happen. Indeed, assume Cases 1 and 2 hold. This would imply that for some $a \in S$ the sets S and $[0_S, a]$ were order isomorphic. Thus, let $\Phi : S \rightarrow [0_S, a]$ be an order isomorphism. But this yields that $\Phi|_{[0_S, a]} : [0_S, a] \rightarrow [0, S, \Phi(a)]$ is an order isomorphism, which implies that $\Phi(a) = a$ which is a contradiction. Similarly we can show that Cases 1 and 3, as well as Cases 2 and 3 cannot hold at the same time. \square

Our next step will be, roughly speaking, the following: Call two well orderings equivalent if they are order isomorphic. We choose out of each equivalence class of well orderings one representant which we will call *ordinal number*. Since these *classes* are not sets we will not be able to use the Axiom of Choice to do so.

We can proceed as follows. Note that “ \in ” can be thought of a (non symmetric, since $A \in A$ never holds) relation between sets.

Definition 5.1.8. An *ordinal number* is a set α which has the following two properties:

(Or1) α is well ordered by the relation \in , i.e. the relation $<$ on α defined by

$$\beta < \gamma \iff \beta \in \gamma \text{ for } \beta, \gamma \in \alpha$$

is a well order on α .

(Or2) Every element of α is also a subset of it.

If α is an ordinal we write $\alpha \in \text{Ord}$ (although Ord is actually not a set but a *class*).

Example 5.1.9. Let us write down the first “couple of ordinals”:

$0 := \emptyset$ (Note that the empty set is always well ordered, no matter how you define $<$)

$1 := \{\emptyset\}$

$2 := \{\emptyset, \{\emptyset\}\}$

$3 := \{\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$

\vdots

$\omega := \{0, 1, 2, 3, 4, \dots\} = \bigcup_{n=0}^{\infty} n$

$\omega + 1 := \omega \cup \{\omega\}$

$\omega + 2 := \omega \cup \{\omega\}$

\vdots

$\omega \cdot 2 := \bigcup_{n=1}^{\infty} \omega + n$

$\omega \cdot 2 + 1 := \omega \cdot 2 \cup \{\omega \cdot 2\}$

\vdots

Proposition 5.1.10. *Let $\alpha \in \text{Ord}$.*

a) *For all $\gamma \in \alpha$ it follows that $\gamma \in \text{Ord}$ and $\gamma = [0, \gamma)$.*

b) *If $\alpha \neq \emptyset$ then $\emptyset \in \alpha$ and $\emptyset = 0_\alpha = \min \alpha$. Therefore we will write instead of 0_α from now on simply 0 .*

Proof. For $\gamma \in \alpha$ it follows from (Or2) that:

$$[0, \gamma) = \{\beta \in \alpha : \beta \in \gamma\} = \{\beta : \beta \in \gamma\} = \gamma.$$

Since intervals of the form $[0_A, a)$, for a well ordering $(A, <)$ and $a \in A$ is also a well ordering it follows that $(\gamma, \in) = ([0_\alpha, \gamma), \in)$ is a well ordering. For $\beta \in \gamma = [0_\alpha, \gamma) \subset \alpha$ it follows that

$$\beta = [0_\alpha, \beta) \subset [0_\alpha, \gamma) = \gamma,$$

thus γ also satisfies (Or2), which finishes the proof of (a). Since

$$\emptyset = [0_\alpha, 0_\alpha) = 0_\alpha,$$

we deduce (b). □

In order to prove that the class of ordinal numbers is well ordered itself, we will need the following *Well Foundedness Principle* which is needed to build up Set theory.

The Well Foundedness Principle. Every nonempty set A contains an element a for which $A \cap a = \emptyset$.

The next Proposition that

Theorem 5.1.11. *The class of ordinals itself is well ordered by \in .*

Proof. Let $\alpha, \beta \in \text{Ord}$. We need to show that either $\alpha \in \beta$ or $\beta \in \alpha$ or $\alpha = \beta$. By Theorem 5.1.7 we can w.l.o.g assume that there is an injective and order preserving embedding map $\Phi : \alpha \rightarrow \beta$ whose image is $[0, \tilde{\beta})$ for some $\tilde{\beta} \in \beta$ or $\tilde{\beta} = \beta$ (in the later case we define $[0, \beta) = \beta$). We need to show that $\alpha = \tilde{\beta}$ and $\Phi(\gamma) = \gamma$ for all $\gamma \in \alpha$. Assuming that this is not true we could choose

$$\gamma_0 = \min\{\gamma \in \alpha : \Phi(\gamma) \neq \gamma\},$$

and deduce from Proposition

$$\Phi(\gamma_0) = [0, \Phi(\gamma_0)) = [0, \gamma_0) = \gamma_0,$$

which is a contradiction. So we deduce (WO1).

In order to show (WO2), let $A \subset \text{Ord}$ a non empty subset of the (class) Ord. By the well foundedness principle there is an $\alpha \in A$ which is disjoint from A (as sets). Therefore it must follow that $\alpha \in \beta$ for all $\beta \in A \setminus \{\alpha\}$. Indeed, otherwise it would follow from (WO1) that there is a $\beta \in \alpha \cap A$. Thus, we showed (WO2).

To show our last claim, we let $(S, <)$ be a well ordering and assume that there is no $\alpha \in \text{Ord}$ which is order isomorphic to S . We can there for define \square

Corollary 5.1.12. *If $\alpha \in \text{Ord}$ then $\alpha^+ = \alpha \cup \alpha$ is also an ordinal.*

If A is a set of ordinals then,

$$\sup(A) = \bigcup_{\alpha} \alpha^+$$

is also an ordinal and is the smallest ordinal which contains all elements of A .

Proof. The first claim is clear. The claim that $\sup(A)$ is an ordinal follows immediately from Theorem 5.1.11 \square

Theorem 5.1.13. *For any well ordering $(S, <)$ there is a (unique) ordinal α which is order isomorphic to S .*

5.2 Arithmetic of ordinals

5.3 Classification of countable compacts by the Cantor Bendixson index

Proposition 5.3.1. *Every countable compact space is metrizable.*

Lemma 5.3.2. *In a complete separable metric space there is no strictly descending chain of closed subsets indexed by ω_1 .*

Lemma 5.3.3. *Every non empty, countable and complete metric space has isolated points*

Proof. Baire category theorem. □

Lemma 5.3.4. *If X is a compact space and has infinitely many points then it has accumulation points.*

Lemma above allows us to define the Cantor Bendixson index for compact

Bibliography

- [AA] D. Alspach and S.A. Argyros, *Complexity of weakly null sequences*, Dissertationes Mathematicae, **321** (1992), 1–44.
- [AOST] G. Androulakis, N. Tomczak-Jaegermann, E. Odell, and Th. Schlumprecht, *On the structure of the spreading models of a Banach space*, Canad. J. Math. **57** (2005) 673–707.
- [AS] G. Androulakis and Th. Schlumprecht, *The Banach space S is complementably minimal and subsequentially prime*. Studia Math. **156** (2003), no. 3, 227–242.
- [Ba] S. Banach, *Théorie des opérations linéaires*, Z Subwencji Funduszu Kultury Narodowej, Warszawa (1932).
- [Bo] J. Bourgain, *On separable Banach spaces, universal for all separable reflexive spaces*, Proc. Amer. Math. Soc. **79** (1980), no.2, 241–246.
- [BS] A. Brunel and L. Sucheston, *B -convex Banach spaces*, Math. Systems Theory, **7** (1974)., 294–299.
- [DKK] S. Dilworth, N. Kalton, and D. Kutzarova, *On the existence of almost greedy bases in Banach spaces*, Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday. Studia Math. **159** (2003), no. 1, 67–101.
- [DFOS] S.J. Dilworth, D. Freeman and E. Odell, and Th. Schlumprecht, *Greedy bases for Besov spaces*. Constr. Approx. **34** (2011), no. 2, 281 – 296.
- [DKSTW] S. Dilworth, D. Kutzarova, K. Shuman, V. Temlyakov, and P. Wojtaszczyk, *Weak convergence of greedy algorithms in Banach spaces*. J. Fourier Anal. Appl. **14** (2008), no. 5-6, 609 - - 628.
- [DOSZ] S. Dilworth, E. Odell, Th. Schlumprecht and A. Zsak, *Partial Unconditionality*, Houston J. Math. **35** (2009), no. 4, 1251 – 1311.
- [DOSZ1] S. Dilworth, E. Odell, Th. Schlumprecht, and A. Zsak, *Renormings and symmetry properties of one-greedy bases*, J. Approx. Theory **163** (2011), no. 9, 1049 – 1075.
- [DOSZ2] S. Dilworth, E. Odell, Th. Schlumprecht, and A. Zsak, *Partial Unconditionality*

- [DOSZ3] S. Dilworth, E. Odell, Th. Schlumprecht, and A. Zsak, *On the convergence of greedy algorithms for initial segments of the Haar basis*. Math. Proc. Cambridge Philos. Soc. **148** (2010), no. 3, 519 – 529.
- [DKOSW] S. Dilworth, D. Kutzarova, E. Odell, Th. Schlumprecht, and P. Wojtaszczyk, *Weak thresholding greedy algorithms in Banach spaces*. J. Funct. Anal. **263** (2012), no. 12, 3900– 3921.
- [DKOSZ] S. Dilworth, D. Kutzarova, E. Odell, Th. Schlumprecht, and A. Zsak, *Renorming spaces with greedy bases*, J. Approx. Theory **188** (2014), 39 – 56.
- [El] Ellentuck, E. E. (1974). A new proof that analytic sets are Ramsey. *J. Symbolic Logic*, **39**, 163–165.
- [En] P. Enflo, *A counterexample to the approximation property in Banach spaces*, **130** (1973), 309–317.
- [FHHMPZ] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucia, J. Pelant and V. Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 8. Springer-Verlag, New York, (2001).
- [FJ] T. Figiel and W.B. Johnson, *A uniformly convex Banach space which contains no ℓ_p* , Compositio Math. **29** (1974), 179–190.
- [GP] F. Galvin and K. Prikry, *Borel sets and Ramsey’s theorem*, J. Symbolic Logic, **38**, (1973)193–198.
- [Ga] I. Gasparis, *A dichotomy theorem for subsets of the power set of the natural numbers*, Proc. Amer. Math. Soc. **129**, no. 3, 759–764.
- [Go1] W.T. Gowers, *Lipschitz functions on classical spaces*, European J. Combin. **13** (1992), 141–151.
- [Go2] Gowers, W. T. (1994). A space not containing c_0 , ℓ_1 or a reflexive subspace. *Trans. Amer. Math. Soc.*, **344**, 407–420.
- [Go3] W. T. Gowers, *An infinite Ramsey theorem and some Banach-space dichotomies*, Annals of Mathematics **156** (2002), 797 – 833.
- [GM1] W. T. Gowers, W. T. and B. Maurey, *The unconditional basic sequence problem*. J. Amer. Math. Soc., **6** (1993) 851–874.
- [GM2] W. T. Gowers, and B. Maurey, *Banach spaces with small spaces of operators*, Math. Ann. **307** no. 4, (1997) 543–568.
- [Hi] Hindman, N. (1974). Finite sums from sequences within cells of a partition on \mathbb{N} . *J. Combin. Theory, (A)* **17**, 1–11.
- [HORS] R. Haydon, E. Odell, H. Rosenthal and Th. Schlumprecht, *On distorted norms in Banach spaces and the existence of ℓ_p types*, unpublished manuscript.

- [Ja1] R. C. James, *Bases and reflexivity in Banach spaces*, Ann. Math. **52** (1950) 518–527.
- [Ja2] R. C. James, *Uniformly nonsquare Banach spaces*, Ann. of Math. (2) **80** (1964) 542–550.
- [Ja3] R.C. James, *Super-reflexive spaces with bases*, Pacific J. Math. **41** (1972), 409–419.
- [Jo1] W.B. Johnson, *On quotients of L_p which are quotients of ℓ_p* , Compositio Math. **34** (1977), 69–89.
- [JRZ] W.B. Johnson, H. Rosenthal, and M. Zippin, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. **9** (1971), 488–506.
- [KT] R. Komorowski and N. Tomczak-Jaegermann, *Banach spaces without local unconditional structure*, Israel J. Math. **89** no. 1-3,(1995) 205–226.
- [Ma] D.A. Martin, *Borel determinacy*, Annals of Math. **102** (1975), 363–371.
- [Mau] B. Maurey, *A remark about distortion*, Oper. Theory: Adv. Appl. **77** (1995), 131–142.
- [MR] B. Maurey and H. Rosenthal, (1977) *Normalized weakly null sequences with no unconditional subsequences*. *Studia Math.*, **61** (1977) 77–98.
- [Mi] Milliken, K. (1975). Ramsey’s theorem with sums or unions. *J. Combin. Theory*, (A) **18**, 276–290.
- [M] V.D. Milman, *Geometric theory of Banach spaces II, geometry of the unit sphere*, Russian Math. Survey **26** (1971), 79–163, (trans. from Russian).
- [MS] V.D. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Math., vol. 1200, Springer-Verlag, Berlin and New York, 1986, 156 pp.
- [MT] V.D. Milman and N. Tomczak-Jaegermann, *Asymptotic ℓ_p spaces and bounded distortions*, (Bor-Luh Lin and W.B. Johnson, eds.), Contemp. Math. **144** (1993), 173–195.
- [MW] V.D. Milman and R. Wagner, *Asymptotic versions of operators and operator ideals*, Convex geometric analysis (Berkeley, CA, 1996), 165-169, Cambridge Univ. Press (1999).
- [O1] E. Odell, *On subspaces asymptotic structure and Distortion of Banach spaces; Connection with logic*, in Analysis and Logic, by W. Henson, J. Iovino, A. Kechris and E.Odell, London Mathematical Society Lecture Note Series **262** .
- [O2] Odell, E. *On Schreier unconditional sequences*. Banach spaces (Mérida, 1992), 197–201, Contemp. Math., **144**, Amer. Math. Soc., Providence, RI, 1993.

- [OS1] E. Odell and Th. Schlumprecht, *The distortion problem*, Acta Math. **173** (1994), 259–281.
- [OS2] E. Odell and Th. Schlumprecht, *Distortion and asymptotic structure*, Handbook of the geometry of Banach spaces, North-Holland, Amsterdam, Vol. 2, (2003) 1333–1360.
- [OS3] E. Odell, and Th. Schlumprecht, *Trees and branches in Banach spaces*, Trans. Amer. Math. Soc. **354** (2002), no. 10, 4085–4108.
- [OS4] E. Odell and Th. Schlumprecht, *A universal reflexive space for the class of uniformly convex Banach spaces*, preprint.
- [OS5] E. Odell, and Th. Schlumprecht, *Embedding into Banach spaces with finite dimensional decompositions*, preprint.
- [OSZ] E. Odell, Th. Schlumprecht, and M. Zsak, *On the structure of asymptotic ℓ_p spaces*, preprint.
- [Pe] A. Pełczyński, *Universal bases*, Studia Math. **32** 1969 247–268.
- [Pr] S. Prus, *Finite-dimensional decompositions with p -estimates and universal Banach spaces* (Russian summary), Bull. Polish Acad. Sci. Math. **31** (1983), no.5–8, 281–288 (1984).
- [PR] P. Pudlák and V. Rödl, *Partition Theorems for systems of finite subsets of integers*, Discrete Mathematics , **39** 1, 1982, 67–73.
- [Ra] Ramsey, F. P. (1930). On a problem of formal logic. *Proc. London Math. Soc.*, (2) **30**, 264–286.
- [Ro] Rosenthal, H. (1974). A characterization of Banach spaces containing ℓ_1 . *Proc. Nat. Acad. Sci. U.S.A.*, **71**, 2411–2413.
1–37.
- [S1] Th. Schlumprecht, *An arbitrarily distortable Banach space*, Israel J. Math. **76** (1991), 81–95.
- [S2] Th. Schlumprecht, *How many operators do exist on a Banach space?* Trends in Banach spaces and operator theory (Memphis, TN, 2001), 295–333, Contemp. Math., **321**, Amer. Math. Soc., Providence, RI, 2003.
- [S3] Th. Schlumprecht, Notes on trees and and indices, notes.
- [S4] Th. Schlumprecht, Course Notes for Functional Analysis, Fall 2012, http://www.math.tamu.edu/~schlump/course_notes_FA2012.pdf
- [S5] Th. Schlumprecht, *On Zippin's embedding Theorem of Banach spaces into Banach spaces with bases*. Adv. Math. **274** (2015), 833 – 880
- [T] *Not every Banach space contains ℓ_p or c_0* , Functional Anal. Appl. **8** (1974), 138–141.

- [W1] R. Wagner, Gowers' dichotomy for asymptotic structure, *Proc. Amer. Math. Soc.* **124** no.10 (1996) 3089 – 3095.
- [W2] R. Wagner, Finite high-order games and an inductive approach towards Gowers' dichotomy, *Annals of pure and applied Logic* **111** (2001) 39 – 60.
- [Z1] M. Zippin, *On perfectly homogeneous bases in Banach spaces*, Israel J. Math. **4** (1966) 265–272.
- [Z2] M. Zippin, *Banach spaces with separable duals*, Trans. AMS, **310**, Nr. **1** (1988), 371–379.