Course Notes for Functional Analysis I, Math 655-601, Fall 2021

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Contents

Chapter 1

Some Basic Background

In this chapter we want to recall some important basic results from Functional Analysis most of which were already covered in the Real Analysis course Math607/608 and can be found in the textbooks [Fol] and [Roy].

1.1 Normed Linear Spaces, Banach Spaces

All our vectors spaces will be vector spaces over the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. In the case that the field is undetermined we denote it by $\mathbb{K}$.

Definition 1.1.1. [Normed linear spaces]
Let X be a vector space over $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A *semi norm* on X is a function $\|\cdot\| : X \to [0, \infty)$ satisfying the following properties for all $x, y \in X$ and $\lambda \in \mathbb{K}$

1. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality) and
2. $\|\lambda x\| = |\lambda| \cdot \|x\|$ (homogeneity),

and we call a semi norm $\|\cdot\|$ a *norm* if it also satisfies

3. $\|x\| = 0 \iff x = 0$, for all $x \in X$.

In that case we call $(X, \|\cdot\|)$, or simply X, a *normed space*. Sometimes we might denote the norm on X by $\|\cdot\|_X$ to distinguish it from some other norm $\|\cdot\|_Y$ defined on some other space Y.

For a normed space $(X, \|\cdot\|)$ the sets

$$B_X = \{x \in X : \|x\| \leq 1\} \text{ and } S_X = \{x \in X : \|x\| = 1\}$$

are called the *unit ball* and the *unit sphere of* X, respectively.

Note that a norm $\|\cdot\|$ on a vector space defines a metric $d(\cdot,\cdot)$ by

$$d(x,y) = \|x-y\|, \qquad x,y \in X,$$

and this metric defines a topology on X, also called the *strong topology.*

Definition 1.1.2. [Banach Spaces]
A normed space which is *complete*, i.e., in which every Cauchy sequence converges, is called a *Banach space.*

To verify that a certain norm defines a complete space it is enough, and sometimes easier to verify that absolutely converging series are converging:

Proposition 1.1.3. *Assume that X is a normed linear space so that for all sequences $(x_n) \subset X$ for which $\sum \|x\|_n < \infty$, the series $\sum x_n$ converges (i.e. $\lim_{n\to\infty} \sum_{j=1}^n x_j$ exists in X).*

Then X is complete.

Proposition 1.1.4. *A subspace of a Banach space is a Banach space if and only if it is closed.*

Proposition 1.1.5. [Completion of normed spaces]
If X is a normed space, then there is a Banach space $\tilde{X}$ so that:

There is an isometric embedding *I from X into $\tilde{X}$, meaning that $I : X \to \tilde{X}$ is linear and $\|I(x)\| = \|x\|$, for $x \in X$, so that the image of X under I is dense in $\tilde{X}$.*

Moreover $\tilde{X}$ is unique up to isometries, *meaning that whenever Y is a Banach space for which there is an isometric embedding $J : X \to Y$, with dense image, then there is an isometry $\tilde{J} : \tilde{X} \to Y$ (i.e. a linear bijection between $\tilde{X}$ and Y for which $\|\tilde{J}(\tilde{x})\| = \|\tilde{x}\|$ for all $\tilde{x} \in \tilde{X}$), so that $\tilde{J} \circ I(x) = J(x)$ for all $x \in X$.*

The space $\tilde{X}$ is called the completion of *X.*

Let us recall some examples of Banach spaces.

Examples 1.1.6. Let (Ω, Σ, μ) be a measure space, and let $1 \le p < \infty$, then put

$$\mathcal{L}_p(\mu) := \Big\{ f : \Omega \to \mathbb{K} \text{ measurable} : \int_\Omega |f|^p d\mu(x) < \infty \Big\}.$$

For $p = \infty$ we put

$$\mathcal{L}_\infty(\mu) := \big\{ f : \Omega \to \mathbb{K} \text{ mble} : \exists C \;\; \mu(\{\omega \in \Omega : |f(\omega)| > C\}) = 0 \big\}.$$

Then $\mathcal{L}_p(\mu)$ is a vector space, and the map

$$\|\cdot\|_p : \mathcal{L}_p(\mu) \to \mathbb{R}, \quad f \mapsto \left(\int_\Omega |f(\omega)|^p d\mu(\omega) \right)^{1/p},$$

if $1 \le p < \infty$, and

$$\|\cdot\|_\infty : \mathcal{L}_\infty(\mu) \to \mathbb{R}, \quad f \mapsto \sup\{C \ge 0 : \mu(\{\omega \in \Omega : |f(\omega)| \ge C\}) > 0\},$$

if $p = \infty$, is a seminorm on $\mathcal{L}_p(\mu)$.

For $f, g \in \mathcal{L}_p(\mu)$ define the equivalence relation by

$$f \sim g :\iff f(\omega) = g(\omega) \text{ for } \mu\text{-almost all } \omega \in \Omega.$$

Define $L_p(\mu)$ to be the quotient space $\mathcal{L}_p(\mu)/\sim$. Then $\|\cdot\|_p$ is well defined and a norm on $L_p(\mu)$, and turns $L_p(\mu)$ into a Banach space. Although, strictly speaking, elements of $L_p(\mu)$ are not functions but equivalence classes of functions, we treat the elements of $L_p(\mu)$ as functions, by picking a representative out of each equivalence class. Equality then means μ almost everywhere equality.

If $A \subset \mathbb{R}$, or $A \subset \mathbb{R}^d$, $d \in \mathbb{N}$, and μ is the Lebesgue measure on A we write $L_p(A)$ instead of $L_p(\mu)$. If Γ is a set and μ is the counting measure on Γ we write $\ell_p(\Gamma)$ instead of $L_p(\mu)$. Thus

$$\ell_p(\Gamma) = \Big\{ x_{(\cdot)} : \Gamma \to \mathbb{K} : \|x\|_p = \Big(\sum_{\gamma \in \Gamma} |x_\gamma|^p \Big)^{1/p} < \infty \Big\}, \text{ if } 1 \le p < \infty,$$

and

$$\ell_\infty(\Gamma) = \big\{ x_{(\cdot)} : \Gamma \to \mathbb{K} : \|x\|_\infty = \sup_{\gamma \in \Gamma} |x_\gamma| < \infty \big\}.$$

If $\Gamma = \mathbb{N}$ we write ℓ_p instead of $\ell_p(\mathbb{N})$ and if $\Gamma = \{1, 2, \ldots, n\}$, for some $n \in \mathbb{N}$ we write ℓ_p^n instead of $\ell_p(\{1, 2, \ldots, n\})$.

The set

$$c_0 = \{(x_n : n \in \mathbb{N}) \subset \mathbb{K} : \lim_{n\to\infty} x_n = 0\}$$

is a linear closed subspace of ℓ_∞, and, thus, it is also a Banach space (with $\|\cdot\|_\infty$).

More generally, let S be a (topological) Hausdorff space, then

$$C_b(S) = \{f : S \to \mathbb{K} \text{ continuous and bounded}\}$$

is a closed subspace of $\ell_\infty(S)$, and, thus, $C_b(S)$ is a Banach space. If K is a compact space we will write $C(K)$ instead of $C_b(K)$ (since continuous functions on compact spaces are automatically bounded). If S is locally compact then

$$C_0(S) = \big\{f : S \to \mathbb{K} \text{ continuous and } \{|f| \ge c\} \text{ is compact for all } c > 0\big\}$$

is a closed subspace of $C_b(S)$, and, thus, it is a Banach space.

Let (Ω, Σ) be a measurable space and assume first that $\mathbb{K} = \mathbb{R}$. Recall that a *finite signed measure on* (Ω, Σ) is a map $\mu : \Sigma \to \mathbb{R}$ so that $\mu(\emptyset) = 0$, and so that

$$\mu(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mu(E_n), \text{ whenever } (E_n) \subset \Sigma \text{ is pairwise disjoint.}$$

The *Jordan Decomposition Theorem* says that such a signed measure can be uniquely written as the difference of two positive finite measure μ^+ and μ^- for which there is a partition (Ω^+, Ω^-) of Ω into two measurable sets so that $\mu^+(\Omega^-) = \mu^-(\Omega^+) = 0$.

If we let

$$\|\mu\|_v = \mu^+(\Omega) + \mu^-(\Omega) = \sup_{A,B\in\Sigma,\text{disjoint}} \mu(A) - \mu(B),$$

then $\|\cdot\|_v$ is a norm, the *variation norm,* on

$$M(\Sigma) = M_\mathbb{R}(\Sigma) := \{\mu : \Sigma \to \mathbb{R} : \text{ signed measure}\},$$

which turns $M(\Sigma)$ into a real Banach space.

If $\mathbb{K} = \mathbb{C}$, we define

$$M(\Sigma) = M_\mathbb{C}(M) = \big\{\mu + i\nu : \mu, \nu \in M_\mathbb{R}(\Sigma)\big\},$$

and define for $\mu + i\nu \in M_\mathbb{C}(\Sigma)$

$$\|\mu + i\nu\|_v = \sqrt{\|\mu\|_v^2 + \|\nu\|_v^2}.$$

Then $M_\mathbb{C}(\Sigma)$ is a complex Banach space.

Assume S is a topological space and $\mathcal{B}_S$ is the sigma-algebra of *Borel sets*, i.e. the σ-algebra generated by the open subsets of S. We call a (positive) measure on $\mathcal{B}_S$ a *Radon measure* if

1) $\mu(A) = \inf\{\mu(U) : U \subset S \text{ open and } A \subset U\}$ for all $A \in \mathcal{B}_S$, (outer regularity)

2) $\mu(U) = \sup\{\mu(C) : C \subset S \text{ compact and } C \subset U\}$ for all $U \subset S$, (inner regularity on open sets) and

3) it is finite on all compact subsets of S.

If $\mathbb{K} = \mathbb{R}$ a *signed Radon measure* is the difference of two finite positive Radon measure, and, as before, if $\mathbb{K} = \mathbb{C}$ then $\mu + i\nu$, where μ and ν are two real valued Radon measures, is a signed Radon measure

We denote the set of all signed Radon measures by $M(S)$. Then $M(S)$ is a closed linear subspace of $M(\mathcal{B}_S)$.

It can be shown (cf. [Fol, Proposition 7.5]) that a σ-finite Radon measure is inner regular on all Borel sets.

Proposition 1.1.7. [Fol, Theorem 7.8]

Let X be a locally compact space for which all open subsets are σ-compact (i.e. a countable union of compact sets). Then every Borel measure which is bounded on compact sets is a Radon Measure.

There are many ways to *combine Banach spaces* to new spaces.

Proposition 1.1.8. [Complemented sums of Banach spaces]

If X_i is a Banach space for all $i \in I$, I some index set, and $1 \le p \le \infty$, we let

$$\big(\oplus_{i\in I} X_i\big)_{\ell_p} := \Big\{(x_i)_{i\in I} : x_i \in X_i,\ \text{for } i\in I,\ \text{and } (\|x_i\| : i\in I) \in \ell_p(I)\Big\}.$$

We put for $x \in \big(\oplus_{i\in I} X_i\big)_{\ell_p}$

$$\|x\|_p := \big\|(\|x_i\| : i \in I)\big\|_p = \begin{cases} \Big(\sum_{i\in I} \|x_i\|_{X_i}^p\Big)^{1/p} & \text{if } 1 \le p < \infty, \\ \sup_{i\in I} \|x_i\|_{X_i} & \text{if } p = \infty. \end{cases}$$

Then $\|\cdot\|$ is a norm on $\big(\oplus_{i\in I} X_i\big)_{\ell_p}$ and $\big(\oplus_{i\in I} X_i\big)_{\ell_p}$ is a Banach space. We call $\big(\oplus_{i\in I} X_i\big)_{\ell_p}$ the ℓ_p sum of the *X_i, $i \in I$.*

Moreover,

$$\big(\oplus_{i\in I} X_i\big)_{c_0} := \big\{(x_i)_{i\in I} \in \big(\oplus_{i\in I} X_i\big)_{\ell_\infty} : \forall\, c > 0 \quad \{i \in I : \|x_i\| \ge c\} \text{ is finite}\big\}$$

is a closed linear subspace of $\big(\oplus_{i\in I} X_i\big)_{\ell_\infty}$, and, thus also a Banach space.

If all the spaces X_i are the same spaces in Proposition 1.1.8, say $X_i = X$, for $i \in I$ we write $\ell_p(I,X)$, and $c_0(I,X)$, instead of $\big(\oplus_{i\in I} X_i\big)_{\ell_p}$ or $\big(\oplus_{i\in I} X_i\big)_{c_0}$, respectively. We write $\ell_p(X)$, and $c_0(X)$ instead of $\ell_p(\mathbb{N}, X)$ and $c_0(\mathbb{N}, X)$, respectively, and $\ell_p^n(X)$, instead of $\ell_p(\{1, 2, \dots, n\}, X)$, for $n \in \mathbb{N}$.

Note that if I is finite then for any norm $\|\cdot\|$ on $\mathbb{R}^I$, the norm topology on $(\oplus X_i)_{\|\cdot\|}$ does not depend on $\|\cdot\|$. By $\oplus_{i\in I} X_i$ we mean therefore the norm product space, which is, up to isomorphism unique, for example in this case $(\oplus_{i\in I} X_i)_{\ell_\infty} \sim (\oplus X_{i\in I})_{\ell_1}$.

If X and Y are Banach space we often denote the product space $X \times Y$ also $X \oplus Y$.

1.2 Operators on Banach Spaces, Dual Spaces

If X and Y are two normed linear spaces, then for a linear map (we also say linear operator) $T : X \to Y$ the following are equivalent:

a) T is continuous,

b) T is continuous at 0,

c) T is *bounded*, i.e. $\|T\| = \sup_{x\in B_X} \|T(x)\| < \infty$.

In this case $\|\cdot\|$, as defined in (c), is a norm on

$$L(X, Y) = \{T : X \to Y \text{ linear and bounded}\}$$

which turns $L(X, Y)$ into a Banach space if Y is a Banach space, and we observe that

$$\|T(x)\| \leq \|T\| \cdot \|x\| \text{ for all } T \in L(X, Y) \text{ and } x \in X.$$

We call a bounded linear operator $T : X \to Y$ an *isomorphic embedding* if there is a number $c > 0$, so that $c\|x\| \leq \|T(x)\|$. This is equivalent to saying that the image $T(X)$ of T is a closed subspace of Y and T has an inverse $T^{-1} : T(X) \to Y$ which is also bounded.

An isomorphic embedding which is onto (we say also *surjective*) is called an *isomorphy* between X and Y. If $\|T(x)\| = \|x\|$ for all $x \in X$ we call T an *isometric embedding*, and call it an *isometry between X and Y* if T is surjective.

If there is an isometry between two spaces X and Y we write $X \simeq Y$. In that case X and Y can be identified for our purposes. If there is an isomorphism $T : X \to Y$ with $\|T\| \cdot \|T^{-1}\| \leq c$, for some number $c \geq 1$ we write $X \sim_c Y$ and we write $X \sim Y$ if there is a $c \geq 1$ so that $X \sim_c Y$.

If X and Y are two Banach spaces which are isomorphic (for example if both spaces are finite dimensional and have the same dimension), we define

$$d_{BM}(X, Y) = \inf\{\|T\| \cdot \|T^{-1}\| : T : X \to Y, T \text{ isomorphism}\},$$

and call it the *Banach Mazur distance between X and Y*. Note that always $d_{BM}(X, Y) \geq 1$.

Remark. If $(X, \|\cdot\|)$ is a finite dimensional Banach space over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and its dimension is $n \in \mathbb{N}$ we can, possibly after passing to an isometric image, assume that $X = \mathbb{K}^n$. Indeed, let $x_1, x_2, \dots x_n$ be a basis of X, and consider on $\mathbb{K}^n$ the norm given by:

$$\|(a_1, a_2, \dots, a_n)\|_X = \Big\| \sum_{j=1}^n a_j x_j \Big\|, \text{ for } (a_1, a_2, \dots a_n) \in \mathbb{K}^n.$$

Then

$$I : \mathbb{K}^n \to X, \quad (a_1, a_2, \dots, a_n) \mapsto \sum_{j=1}^n a_j x_j,$$

is an isometry. Therefore we can always assume that $X = (\mathbb{K}^n, \| \cdot \|_X)$. This means B_X is a closed and bounded subset of $\mathbb{K}^n$, which by the Theorem of Bolzano-Weierstraß, means that B_X is compact. In Theorem 1.5.4 we will deduce the converse and prove that a Banach space X, for which B_X is compact, must be finite dimensional.

Definition 1.2.1. [Dual space of X]
If $Y = \mathbb{K}$ and X is a normed linear space over $\mathbb{K}$, then we call $L(X, \mathbb{K})$ *the dual space of* X and denote it by X^*.

If $x^* \in X^*$ we often use $\langle \cdot, \cdot \rangle$ to denote the action of x^* on X, i.e. we write $\langle x^*, x \rangle$ instead of $x^*(x)$.

Theorem 1.2.2. [Representation of some Dual spaces]

1. *Assume that* $1 \le p < \infty$ *and* $1 < q \le \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *and assume that* (Ω, Σ, μ) *is a measure space without atoms of infinite measure. Then the following map is a well defined isometry between* $L_p^*(\mu)$ *and* $L_q(\mu)$.

$$\Psi : L_q(\mu) \to L_p^*(\mu), \quad \langle \Psi(g), f \rangle = \Psi(g)(f) := \int_\Omega f(\xi) g(\xi)\, d\mu(\xi),$$
$$\textit{for } g \in L_q(\mu), \textit{ and } f \in L_p(\mu).$$

2. *Assume that* S *is a locally compact Hausdorff space, then the map*

$$\Psi : M(S) \to C_0(S), \quad \langle \Psi(\mu), f \rangle = \Psi(\mu)(f) := \int_S f(\xi)\, d\mu(\xi)$$
$$\textit{for } \mu \in M(S) \textit{ and } f \in C_0(S),$$

is an isometry between $M(S)$ *and* $C_0^*(S)$.

Remark. If $p = \infty$ and $q = 1$ then the map Ψ in Theorem 1.2.2 part (1) is still an isometric embedding, but in general (i.e. if $L_\infty(\mu)$ is infinite dimensional) **not** onto.

Example 1.2.3. $c_0^* \simeq \ell_1$ (by Theorem 1.2.2 part (2)) and $\ell_1^* \simeq \ell_\infty$ (by Theorem 1.2.2 part (1)) .

1.3 Baire Category Theorem and its Consequences

The following result is a fundamental Theorem in Topology and leads to several useful properties of Banach spaces.

Theorem 1.3.1. (The Baire Category Theorem, c.f [Fol, Theorem 5.4]) *Assume that (S, d) is a complete metric space. If (U_n) is a sequence of open and dense subsets of S then $\bigcap_{n=1}^{\infty} U_n$ is also dense in S.*

Often we will use the Baire Category Theorem in the following equivalent restatement.

Corollary 1.3.2. *If (C_n) is a sequence of closed subsets of a complete metric space (S, d) whose union is all of S, then there must be an $n \in \mathbb{N}$, so that C_n°, the open interior of C_n, is not empty, and thus there is an $x \in C_n$ and an $\varepsilon > 0$ so that $B(x, \varepsilon) = \{z \in S : d(z, x) < \varepsilon\} \subset C_n$.*

Proof. Assume our conclusion were not true. Let $U_n = S \setminus C_n$, for $n \in \mathbb{N}$. Then U_n is open and dense in S. Thus $\bigcap_{n\in\mathbb{N}} U_n$ is also dense, in particular not empty. But this is in contradiction to the assumption that $\bigcup_{n\in\mathbb{N}} C_n = S$. $\square$

The following results are important applications of the Baire Category Theorem to Banach spaces.

Theorem 1.3.3. (The Open Mapping Theorem, cf [Fol, Theorem 5.10]) Let X and Y be Banach spaces and let $T \in L(X, Y)$ be surjective. Then T is also open (the image of every open set in X under T is open in Y).

Corollary 1.3.4. *Let X and Y be Banach spaces and $T \in L(X, Y)$ be a bijection. Then its inverse T^{-1} is also bounded, and thus T is an isomorphism.*

Theorem 1.3.5. (Closed Graph Theorem, c.f. [Fol, Theorem 5.12]) Let X and Y be Banach spaces and $T : X \to Y$ be linear. If T has a closed graph (i.e $\Gamma(T) = \{(x, T(x)) : x \in X\}$ is closed with respect to the product topology in $X \times Y$), then T is bounded.

Often the Closed Graph Theorem is used in the following way:

Corollary 1.3.6. *Assume that $T : X \to Y$ is a bounded, linear and bijective operator between two Banach spaces X and Y. Then T is an isomorphism.*

Theorem 1.3.7. (Uniform Boundedness Principle, c.f. [Fol, Theorem 5.13]) *Let X and Y be Banach spaces and let $\mathcal{A} \subset L(X, Y)$. If for all $x \in X$ $\sup_{T \in \mathcal{A}} \|T(x)\| < \infty$ then $\mathcal{A}$ is bounded in $L(X, Y)$, i.e.*

$$\sup_{T \in \mathcal{A}} \|T\| = \sup_{x \in B_X} \sup_{T \in \mathcal{A}} \|T(x)\| < \infty.$$

An important consequence of the Uniform Boundedness Principle is the following

Theorem 1.3.8. [Theorem of Banach-Steinhaus]

a) *If $A \subset X$, and $\sup_{x \in A} |\langle x^*, x \rangle| < \infty$, for all $x^* \in X^*$, then A is (norm) bounded.*

b) *If $A \subset X^*$, and $\sup_{x^* \in A} |\langle x^*, x \rangle| < \infty$, for all $x \in X$, then A is (norm) bounded.*

In particular, weak compact subsets of X and weak* compact subsets of X^* are norm bounded.

Proposition 1.3.9. (Quotient spaces)
Assume that X is a Banach space and that $Y \subset X$ is a closed subspace. Consider the quotient space

$$X/Y = \{x + Y : x \in X\}$$

(with usual addition and multiplication by scalars). For $x \in X$ put $\overline{x} = x + Y \in X/Y$ and define

$$\|\overline{x}\|_{X/Y} = \inf_{z \in \overline{x}} \|z\|_X = \inf_{y \in Y} \|x + y\|_X = \operatorname{dist}(x, Y).$$

Then $\| \cdot \|_{X/Y}$ is norm on X/Y which turns X/Y into a Banach space.

Proof. For x_1, x_2 in X and $\lambda \in \mathbb{K}$ we compute

$$\begin{aligned}
\|\overline{x}_1 + \overline{x}_2\|_{X/Y} &= \inf_{y \in Y} \|x_1 + x_2 + y\| \\
&= \inf_{y_1, y_2 \in Y} \|x_1 + y_1 + x_2 + y_2\| \\
&\leq \inf_{y_1, y_2 \in Y} \big(\|x_1 + y_1\| + \|x_2 + y_2\|\big) = \|\overline{x}_1\|_{X/Y} + \|\overline{x}_2\|_{X/Y}
\end{aligned}$$

and

$$\begin{aligned}
&\|\lambda\overline{x}_1\|_{X/Y} \\
&= \inf_{y\in Y} \|\lambda x_1 + y\| \\
&= \inf_{y\in Y} \|\lambda(x_1 + y)\| = |\lambda| \cdot \inf_{y\in Y} \|x_1 + y\| = |\lambda| \cdot \|\overline{x}_1\|_{X/Y}.
\end{aligned}$$

Moreover, if $\|\overline{x}\|_{X/Y} = 0$, it follows that there is a sequence (y_n) in Y, for which $\lim_{n\to\infty} \|x - y_n\| = 0$, which implies, since Y is closed that $x = \lim_{n\to\infty} y_n \in Y$ and thus $\overline{x} = \overline{0}$ (the zero element in X/Y). This proves that $(X/Y, \|\cdot\|_{X/Y})$ is a normed linear space. In order to show that X/Y is complete let $x_n \in X$ with $\sum_{n\in\mathbb{N}} \|\overline{x}_n\|_{X/Y} < \infty$. It follows that there are $y_n \in Y$, $n \in \mathbb{N}$, so that

$$\sum_{n=1}^{\infty} \|x_n + y_n\|_X < \infty,$$

and thus, since X is a Banach space,

$$x = \sum_{n=1}^{\infty} (x_n + y_n),$$

exists in X and we observe that

$$\Big\|\overline{x} - \sum_{j=1}^{n} \overline{x}_j\Big\| \le \Big\|x - \sum_{j=1}^{n} (x_j + y_j)\Big\| \le \sum_{j=n+1}^{\infty} \|x_j + y_j\| \to_{n\to\infty} 0,$$

which verifies that X/Y is complete. □

From Corollary 1.3.4 we deduce

Corollary 1.3.10. *If X and Y are two Banach spaces and $T : X \to Y$ is a linear, bounded and surjective operator, it follows that $X/\mathcal{N}(T)$ and Y are isomorphic, where $\mathcal{N}(T)$ is the null space of T.*

Proof. Since T is continuous $\mathcal{N}(T)$ is a closed subspace of X. We put

$$\overline{T} : X/\mathcal{N}(T) \to Y, \quad x + \mathcal{N}(T) \mapsto T(x).$$

Then $\overline{T}$ is well defined, linear, and bijective (linear Algebra), moreover, for $x \in X$

$$\|\overline{T}(x + \mathcal{N}(T))\| = \inf_{z\in\mathcal{N}(T)} \|T(x + z)\|$$

$$\leq \|T\| \inf_{z \in \mathcal{N}(T)} \|x + z\| = \|T\| \cdot \|x + \mathcal{N}(T)\|_{X/\mathcal{N}(T)}.$$

Thus, $\overline{T}$ is bounded and our claim follows from Corollary 1.3.4. □

Proposition 1.3.11. *For a bounded linear operator $T : X \to Y$ between two Banach spaces X and Y the following statements are equivalent:*

1. *The range $T(X)$ is closed.*
2. *The operator $\overline{T} : X/\mathcal{N}(T) \to Y$, $\overline{x} \mapsto T(x)$ is an isomorphic embedding,*
3. *There is a number $C > 0$, so that* $\mathrm{dist}(x, \mathcal{N}(T)) = \inf_{y \in \mathcal{N}} \|x - y\| \leq C\|T(x)\|$.

1.4 The Hahn Banach Theorem

Definition 1.4.1. Suppose that V is a vector space over $\mathbb{K}$. A real-valued function p on V, satisfying

- $p(0) = 0$,
- $p(x + y) \leq p(x) + p(y)$, and
- $p(\lambda x) = \lambda p(x)$ for $\lambda > 0$,

is called a *sublinear functional* on V.

Note that $0 = p(0) \leq p(x) + p(-x)$, and, thus, $p(-x) \geq -p(x)$.

Theorem 1.4.2. (The analytic Hahn-Banach Theorem, real version, c.f. [Fol, Theorem 5.6])
Suppose that p is a sublinear functional on a real vector space V, that W is a linear subspace of V and that f is a linear functional on W satisfying $f(y) \leq p(y)$ for all $y \in W$. Then there exists a linear functional g on V such that $g(x) = f(x)$ for all $x \in W$ (g extends f) and such that $g(y) \leq p(y)$ for all $y \in V$ (control is maintained).

Theorem 1.4.3. (The analytic Hahn-Banach Theorem, complex version, c.f. [Fol, Theorem 5.7])
Suppose that p is a seminorm on a complex vector space V, that W is a linear subspace of V and that f is a linear functional on W satisfying $|f(x)| \leq p(x)$ for all $x \in W$. Then there exists a linear functional g on V such that $g(x) = f(x)$ for all $x \in W$ (g extends f) and such that $|g(y)| \leq p(y)$ for all $y \in V$ (control is maintained).

Corollary 1.4.4. *Let X be a normed linear space Y a subspace and $y^* \in Y^*$. Then there exists an extension x^* of y^* to an element in X^* with $\|x^*\| = \|y^*\|$.*

Proof. Put $p(x) = \|y^*\|\|x\|$. □

Corollary 1.4.5. *Let X be a normed linear space, Y a subspace of X, and $x \in X$ with $h = \mathrm{dist}(x, Y) > 0$. Then there exists an $x^* \in X^*$, with $x^*|_Y \equiv 0$, $\|x^*\|$ and $x^*(x) = h$.*

Proof. Consider $Z = \{y + ax : y \in Y \text{ and } a \in \mathbb{K}\}$. Note that every $z \in Z$ has a unique representation $z = y + ax$, with $y \in Y$ and $a \in \mathbb{K}$. Indeed, if $y_1 + a_1 x = y_2 + a_2 x$, with $y_1, y_2 \in Y$ and $a_1, a_2 \in \mathbb{K}$, then we observe that $a_1 = a_2$, because otherwise $x = (y_1 - y_2)/(a_1 - a_2) \in Y$. Thus also $y_1 = y_2$.

We define $f : Z \to \mathbb{K}$, $y + ax \mapsto ah$. The unique representation of each $z \in Z$ implies that f is linear, and it follows for $a \neq 0$ and $y \in Y$ that

$$|f(y + ax)| = |a|\delta \leq |a|\|a^{-1}y + x\| = \|y + ax\|.$$

Thus $\|f\| \leq 1$ We can therefore apply the Hahn-Banach Theorem 1.4.2 to the linear functional f on Z and the norm $p(x) = \|x\|$. and extend it to an $x^* \in X^*$, with $\|x^*\| = 1$ □

Corollary 1.4.6. *Let X be a normed linear space and $x \in X$. Then there is an $x^* \in X^*$, $\|x^*\| = 1$, so that $\langle x^*, x\rangle = \|x\|$.*

Proof. Let $p(x) = \|x\|$ and $f(\alpha x) = \alpha\|x\|$, for $\alpha x \in \text{span}(x) = \{ax : a \in \mathbb{K}\}$. □

Definition 1.4.7. (The Canonical Embedding, Reflexive spaces)
For a Banach space we put $X^{**} = (X^*)^*$ (the dual space of the dual space of X).

Consider the map

$$\chi : X \to X^{**}, \text{ with } \chi(x) : X^* \to \mathbb{K}, \langle \chi(x), x^*\rangle = \langle x^*, x\rangle, \text{ for } x \in X.$$

The map χ is well defined (i.e. $\chi(x) \in X^{**}$ for $x \in X$), and since for $x \in X$

$$\|\chi(x)\|_{X^{**}} = \sup_{x^* \in B_{X^*}} |\langle x^*, x\rangle| \leq \|x\|,$$

it follows that $\|\chi\|_{L(X,X^{**})} \leq 1$. By Corollary 1.4.6 we can find for each $x \in X$ an element $x^* \in B_{X^*}$ with $\langle x^*, x\rangle = \|x\|$, and thus $\|\chi(x)\|_{X^{**}} = \|x\|_X$.

It follows therefore that χ is an isometric embedding of X into X^{**}. We call χ the *canonical embedding of X into X^{**}*.

We say that X is *reflexive* if χ is onto.

Remark. There are Banach spaces X for which X and X^{**} are isometrically isomorphic, but not via the canonical embedding. An Example by R. C. James will be covered in Chapter 3.

Definition 1.4.8. (The adjoint of an operator)

Assume that X and Y are Banach spaces and $T : X \to Y$ a linear and bounded operator. Then *adjoint of T* is the operator

$$T^* : Y^* \mapsto X^*, \quad y^* \mapsto y^* \circ T,$$

(i.e. $\langle T^*(y^*), x\rangle = \langle y^* \circ T, x\rangle = \langle y^*, T(x)\rangle$ for $y^* \in Y^*$ and $x \in X$).

Proposition 1.4.9. *Assume X and Y are Banach spaces and $T : X \to Y$ a linear and bounded operator. Then T^* is a bounded linear operator from Y^* to X^*, and $\|T^*\| = \|T\|$.*

Moreover if T is surjective T^* is an isomorphic embedding, and if T is an isomorphic embedding T^* is surjective.

Proof. Since for $y^* \in Y^*$, we have that $T^*(y^*)$ is the composition $y^* \circ T$ it follows that $T^*(y^*) \in X^*$ and $\|T^*(y^*)\| \leq \|T^*\| \cdot \|y^*\|$, and thus $\|T^*\| \leq \|T\|$. Conversely, for an arbitrary small $\varepsilon > 0$ we can find $x \in B_X$, so that $\|T(x)\| \geq \|T\| - \varepsilon$. Then, by the Hahn Banach Theorem, we can choose $y^* \in S_{Y^*}$, so that $|y^*(T(x))| \geq \|T(x)\|$, and, thus $\|T^*\| \geq \|T(y^*)\| \geq |y^*(T(x))| \geq \|T\| - \varepsilon$, which implies that $\|T^*\| \geq \|T\|$, since $\varepsilon > 0$ was arbitrary.

If $T : X \to Y$ is surjective, we can, by the Open Mapping Theorem (Corollary 1.3.3), find an $\rho > 0$ so that $\rho B_Y \subset T(B_X)$, and thus it follows for $y^* \in Y^*$, that

$$\|T^*(y^*)\| = \sup_{x \in B_X} |y^*(T(x))| = \sup_{y \in T(B_X)} |y^*(y)| \geq \sup_{y \in \rho B_Y} |y^*(y)| = \rho \|y^*\|,$$

which shows that T^* is an isomorphic embedding.

If $T : X \to Y$ is an isomorphic embedding, and $x^* \in X^*$ we can define $z^* : T(X) \to \mathbb{K}$ by $z^*(T(x)) := x^*(x)$ (i.e. $z^* = x^* \circ T^{-1}$). Then we use the Hahn Banach Theorem to extend z^* to an element $y^* \in Y^*$. For all $x \in X$ it follows that

$$\langle T^*(y^*), x \rangle = \langle y^*, T(x) \rangle = \langle z^*, T(x) \rangle = x^*(x).$$

Since $x^* \in X^*$ was arbitrary, this shows that T^* is surjective.

□

1.5 Finite Dimensional Banach Spaces

Theorem 1.5.1. (Auerbach bases)
If $X = (\mathbb{K}^n, \|\cdot\|)$ is an n-dimensional Banach space, then X has a basis $x_1, x_2, \ldots x_n$ for which there are functionals $x_1^*, \ldots x_n^* \in X^*$, so that

a) $\|x_j\| = \|x_j^*\| = 1$ for all $j = 1, 2, \ldots, n$,

b) for all $i, j = 1, 2, \ldots, n$

$$\langle x_i^*, x_j \rangle = \delta_{(i,j)} = \begin{cases} \text{if } i = j, \\ \text{if } i \neq j. \end{cases} \quad .$$

We call in this case (x_j, x_j^*) an *Auerbach basis of* X.

Proof. We consider the function

$$\begin{aligned} \text{Det} : X^n = \underbrace{X \times X \times X}_{n \text{ times}} \to \mathbb{K}, \\ (u_1, u_2, \ldots u_n) \mapsto \det(u_1, u_2, \ldots u_n). \end{aligned}$$

Thus, we consider $u_i \in \mathbb{K}^n$, to be column vectors and take for $u_1, u_2, \ldots u_n \in \mathbb{K}^n$ the determinant of the matrix which is formed by vectors u_i, for $i = 1, 2, \ldots n$. Since $(B_X)^n$ is a compact subset of X^n with respect to the product topology, and since Det is a continuous function on X^n we can choose $x_1, x_2, \ldots x_n$ in B_X so that

$$\big|\text{Det}(x_1, x_2, \ldots x_n)\big| = \max_{u_1, u_2, \ldots u_n \in B_X} \big|\text{Det}(u_1, u_2, \ldots u_n)\big|.$$

By multiplying x_1 by the appropriate number $\alpha \in \mathbb{K}$, with $|\alpha| = 1$, we can assume that

$$\text{Det}(x_1, x_2, \ldots x_n) \in \mathbb{R} \text{ and } \text{Det}(x_1, x_2, \ldots x_n) > 0.$$

Define for $i = 1, \ldots n$

$$x_i^* : X \to \mathbb{K}, \quad x \mapsto \frac{\text{Det}(x_1, \ldots x_{i-1}, x, x_{i+1}, \ldots, x_n)}{\text{Det}(x_1, x_2, \ldots x_n)},$$

It follows that x_i^* is a linear functional on X (taking determinants is linear in each column), and

$\langle x_i^*, x_i \rangle = 1,$

$$\|x_i^*\| = \sup_{x\in B_X} |\langle x_i^*, x\rangle| = \sup_{x\in B_X} \left|\frac{\mathrm{Det}(x_1, \dots x_{i-1}, x, x_{i+1}, \dots, x_n)}{\mathrm{Det}(x_1, x_2, \dots x_n)}\right| = 1$$

(by the maximality of $\mathrm{Det}(x_1, x_2, \dots x_n)$ on $(B_X)^n$),

$$\langle x_i^*, x_j\rangle = \frac{\mathrm{Det}(x_1, \dots x_{i-1}, x_j, x_{i+1}, \dots, x_n)}{\mathrm{Det}(x_1, x_2, \dots x_n)} = 0 \text{ if } i \neq j,\ i, j \in \{1, 2, \dots, n\}$$

(by linear dependence of columns)

which finishes our proof. □

Corollary 1.5.2. *For any two n-dimensional Banach spaces X and Y it follows that*

$$d_{BM}(X, Y) \leq n^2.$$

Remark. Corollary 1.5.2 Is not the best result one can get. Indeed from the following Theorem of John (1948) it is possible to deduce that for any two n-dimensional Banach spaces X and Y it follows that

$$d_{BM}(X, Y) \leq n.$$

Theorem 1.5.3. (John's theorem)
Let $X = (\mathbb{K}^n, \|\cdot\|)$ be an n-dimensional Banach space. Then there is an invertible matrix T so that

$$B_{\ell_2} \subset T(B_X) \subset \sqrt{n} B_{\ell_2}.$$

Theorem 1.5.4. *For any Banach space X*

$$X \text{ is finite dimensional} \iff B_X \text{ is compact.}$$

Proof. The implication "$\Rightarrow$" was already noted in the remark in Section 6.2 the implication "$\Leftarrow$" will follow from the following Proposition. □

Proposition 1.5.5. *The unit ball of every infinite dimensional Banach space X contains a 1-separated infinite sequence.*

Proof. By induction we choose for each $n \in \mathbb{N}$ an element $x_n \in B_x$, so that $\|x_j - x_n\| \geq 1$, for $j = 1, 2, \dots, n-1$. Choose an arbitrary $x_1 \in S_X$. Assuming $x_1, x_2, \dots, x_{n-1}$ have been chosen, let $F = \mathrm{span}(x_1, \dots, x_{n-1})$, (the linear space generated by x_j, $j = 1, 2, \dots, n-1$). X/F is infinite dimensional, thus there is a $z \in X$ so that

$$1 = \|\overline{z}\|_{X/F} = \inf_{y\in F} \|z + y\| = \inf_{y\in F, \|y\|\leq 1+\|z\|} \|z + y\| = \min_{y\in F, \|y\|\leq 1+\|z\|} \|z + y\|,$$

where the last equality follows from the assumed compactness of the unit ball. We can therefore choose $x_n = z + y$ so that $y \in F$ and

$$\|z + y\| = \min_{\tilde{y} \in F, \|\tilde{y}\| \leq 1 + \|z\|} \|z + \tilde{y}\| = 1,$$

it follows that

$$1 = \|\overline{x}_n\|_{X/F} \leq \|x_n - x_j\| \text{ for all } j = 1, 2, \ldots, n - 1.$$

$\square$

Remark. With little bit more work (see Exercise in Homeowrk) one can find in the unit ball of each infinite dimensional Banach space X a sequence (x_n) with $\|x_m - x_n\| > 1$, for all $m \neq n$ in $\mathbb{N}$. This is a result of Kottman [Kot].

A much deeper result by J. Elton and E. Odell (see [EO]) says that for each infinite dimensional Banach space X there is a $\varepsilon > 0$ and a sequence $(x_n) \subset B_X$ with $\|x_m - x_n\| \geq 1 + \varepsilon$, for all $m \neq n$ in $\mathbb{N}$.

Definition 1.5.6. An operator $T : X \to Y$ is called a finite rank operator if $T(X)$ is finite dimensional. In this case we call $\dim(T(X))$ the *rank of* T and denote it by $\mathrm{rk}(T)$.

For $y \in Y$ and $x^* \in X^*$ we denote the operator

$$X \to Y, \quad x \mapsto y\langle x^*, x\rangle$$

by $y \otimes x^*$. Clearly, $y \otimes x^*$ is of rank one.

Proposition 1.5.7. *Assume that X and Y are Banach spaces and that $T : X \to Y$ is a linear bounded operator of finite rank n. Then there are $x_1^*, x_2^*, \ldots, x_n^* \in X$ and $y_1,\, y_2, \ldots, y_n$ in Y so that*

$$T = \sum_{j=1}^{n} y_j \otimes x_j^*.$$

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Chapter 2

Weak Topologies and Reflexivity

2.1 Topological Vector Spaces and Locally Convex Spaces

Definition 2.1.1. [Topological Vector Spaces and Locally Convex Spaces]
Let E be a vector space over $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $\mathcal{T}$ be a topology on E. We call $(E, \mathcal{T})$ (or simply E, if there cannot be a confusion), a *topological vector space*, if the addition:

$$+ : E \times E \to E, \quad (x, y) \mapsto x + y,$$

and the multiplication by scalars

$$\cdot : \mathbb{K} \times E \to E, \quad (\lambda, x) \mapsto \lambda x,$$

are continuous functions. A topological vector space is called *locally convex* if 0 (and thus any point $x \in E$) has a neighbourhood basis consisting of convex sets.

Remark. Topological vector spaces are in general not metrizable. Thus, continuity, closedeness, and compactness etc, cannot be described by sequences. We will need *nets*.

Assume that $(I, \leq)$ is a *directed set* . This means

- (reflexivity) $i \leq i$, for all $i \in I$,
- (transitivity) if for $i, j, k \in I$ we have $i \leq j$ and $j \leq k$, then $i \leq k$, and

- (existence of upper bounds) for any $i, j \in I$ there is a $k \in I$, so that $i \leq k$ and $j \leq k$.

A *net* is a family $(x_i : i \in I)$ indexed over a directed set $(I, \leq)$.

A *subnet* of a net $(x_i : i \in I)$ is a net $(y_j : j \in J)$, together with a map $j \mapsto i_j$ from J to I, so that $x_{i_j} = y_j$, for all $j \in J$, and for all $i_0 \in I$ there is a $j_0 \in J$, so that $i_j \geq i_0$ for all $j \geq j_0$.

Definition 2.1.2. In a topological space $(T, \mathcal{T})$, we say that a net $(x_i : i \in I)$ *converges to* x, if for all open sets U with $x \in U$ there is an $i_0 \in I$, so that $x_i \in U$ for all $i \geq i_0$. If $(T, \mathcal{T})$ is Hausdorff x is unique and we denote it by $\lim_{i \in I} x_i$.

Using nets we can describe continuity, closeness, and compactness in arbitrary topological spaces:

a) A map between two topological spaces is continuous if and only if the image of converging nets are converging.

b) A subset A of a topological space S is closed if and only if the limit point of every converging net in A is in A.

c) A topological space S is compact if and only if every net has a convergent subnet.

Note: A subnet of a sequence is not necessarily a subsequence.

Example 2.1.3. An important example of directed sets and nets indexed by them, are neighborhood bases:

Let $(T, \mathcal{T})$ be a topological space, $x \in T$, and $\mathcal{U}_x$ a *neighborhood basis of* x, *i.e.* $\mathcal{U}_x \subset \mathcal{P}(T)$, with

1. $x \in U^\circ$, for all $U \in \mathcal{U}_x$,

2. For each open $V \subset T$, with $x \in V$, there is a $U \in \mathcal{U}_x$, for which $U \subset V$,

3. For any $U_1, U_2 \in \mathcal{U}_x$, there is $U \in \mathcal{U}_x$, with $U \subset U_1 \cap U_2$.

Then $\mathcal{U}_x$ is a directed set, with respect to reverse inclusion.

Pick $y(U) \in U$, for each $U \in \mathcal{U}_x$, then $(y(U) : U \in \mathcal{U}_x)$ is a net which converges to x (exercise).

Assume that T is compact (in particular Hausdorff) and let $(x_n)_{n \in \mathbb{N}} \subset T$ be a sequence in T, of pairwise distinct elements. Then (x_n) may not have a convergent subsequence. Nevertheless it has a convergent subnet, which can be defined as follows: Let $x \in T$ be an accumulation point of (x_n) (exercise: there is an accumulation point) which means that $\{n \in \mathbb{N} : x_n \in U\}$ is infinite for each open $U \subset T$ which contains x. Let $\mathcal{U}_x$ be a neighborhood basis of x. Then put for each $U \in \mathcal{U}_x$, $x_U = x_{\min\{n \in \mathbb{N} : x_n \in U\}}$.

It follows (exercise) that (x_U) is a subnet of (x_n) which converges to x.

In order to define a topology on a vector space E which turns E into a topological vector space we (only) need to define an appropriate neighborhood basis of 0.

Proposition 2.1.4. *Assume that $(E, \mathcal{T})$ is a topological vector space. And let*

$$\mathcal{U}_0 = \{U \in \mathcal{T}, 0 \in U\}.$$

Then

a) For all $x \in E$, $x + \mathcal{U}_0 = \{x + U : U \in \mathcal{U}_0\}$ is a neighborhood basis of x,

b) for all $U \in \mathcal{U}_0$ there is a $V \in \mathcal{U}_0$ so that $V + V \subset U$,

c) for all $U \in \mathcal{U}_0$ and all $R > 0$ there is a $V \in \mathcal{U}_0$, so that

$$\{\lambda \in \mathbb{K} : |\lambda| < R\} \cdot V \subset U,$$

d) for all $U \in \mathcal{U}_0$ and $x \in E$ there is an $\varepsilon > 0$, so that $\lambda x \in U$, for all $\lambda \in \mathbb{K}$ with $|\lambda| < \varepsilon$,

e) if $(E, \mathcal{T})$ is Hausdorff, then for every $x \in E$, $x \neq 0$, there is a $U \in \mathcal{U}_0$ with $x \notin U$,

f) if E is locally convex, then for all $U \in \mathcal{U}_0$ there is a convex $V \in \mathcal{T}$, with $V \subset U$.

Conversely, if E is a vector space over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and

$$\mathcal{U}_0 \subset \{U \in \mathcal{P}(E) : 0 \in U\}$$

is non empty and is downwards directed, *i.e. if for any $U, V \in \mathcal{U}_0$, there is a $W \in \mathcal{U}_0$, with $W \subset U \cap V$ and satisfies (b), (c) and (d), then*

$$\mathcal{T} = \{V \subset E : \forall x \in V\, \exists U \in \mathcal{U} : \ x + U \subset V\},$$

defines a topological vector space for which $\mathcal{U}_0$ is a neighborhood basis of 0. $(E, \mathcal{T})$ is Hausdorff if $\mathcal{U}$ also satisfies (e) and locally convex if it satisfies (f).

Proof. Assume $(E,\mathcal{T})$ is a topological vector space and $\mathcal{U}_0$ is defined as above.

We observe that for all $x\in E$ the linear operator $T_x : E\to E$, $z\mapsto z+x$ is continuous. Since also $T_x\circ T_{-x}=T_{-x}\circ T_x = Id$, it follows that T_x is an homeomorphism, and thus maps open neighborhoods of 0 to open neighborhoods of x, which implies (a). Property (b) follows from the continuity of addition at 0. Indeed, we first observe that $\mathcal{U}_{0,0}=\{V\times V : V\in\mathcal{U}_0\}$ is a neighborhood basis of $(0,0)$ in $E\times E$, and thus, if $U\in\mathcal{U}_0$, then there exists a $V\in\mathcal{U}_0$ so that

$$V\times V\subset(\cdot+\cdot)^{-1}(U)=\{(x,y)\in E\times E : x+y\in U\},$$

and this translates to $V+V\subset U$.

The claims (c) and (d) follow similarly from the continuity of scalar multiplication at 0. If E is Hausdorff then $\mathcal{U}_0$ clearly satisfies (e) and it clearly satisfies (f) if E is locally convex.

Now assume that $\mathcal{U}_0\subset\{U\in\mathcal{P}(E) : 0\in U\}$ is non empty and downwards directed, that for any $U,V\in\mathcal{U}_0$, there is a $W\in\mathcal{U}_0$, with $W\subset U\cap V$, and that $\mathcal{U}_0$ satisfies (b), (c) and (d). Then

$$\mathcal{T}=\{V\subset E : \forall x\in V\ \exists U\in\mathcal{U}_0 :\ x+U\subset V\},$$

is finitely intersection stable and stable by taking (arbitrary) unions. Also $\emptyset, E\in\mathcal{T}$. Thus $\mathcal{T}$ is a topology. Also note that for $x\in E$,

$$\mathcal{U}_x=\{x+U : U\in\mathcal{U}_0\}$$

is a neighborhood basis of x.

We need to show that addition and multiplication by scalars is continuous. Assume $(x_i : i\in I)$ and $(y_i : i\in I)$ converge in E to $x\in E$ and $y\in E$, respectively, and let $U\in\mathcal{U}_0$. By (b) there is a $V\in\mathcal{U}_0$ with $V+V\subset U$. We can therefore choose i_0 so that $x_i\in x+V$ and $y_i\in x+V$, for $i\ge i_0$, and, thus, $x_i+y_i\in x+y+V+V\subset x+y+U$, for $i\ge i_0$. This proves the continuity of the addition in E.

Assume $(x_i : i\in I)$ converges in E to x, $(\lambda_i : i\in I)$ converges in $\mathbb{K}$ to λ and let $U\in\mathcal{U}_0$. Then choose first (using property (b)) $V\in\mathcal{U}_0$ so that $V+V\subset U$. Then, by property (c) choose $W\in\mathcal{U}_0$, so that for all $\rho\in\mathbb{K}$, $|\rho|\le R:=|\lambda|+1$ it follows that $\rho W\subset V$ and, using (d) choose $\varepsilon\in(0,1)$ so that $\rho x\in W$, for all $\rho\in\mathbb{K}$, with $|\rho|\le\varepsilon$. Finally choose $i_0\in I$ so that $x_i\in x+W$ and $|\lambda-\lambda_i|<\varepsilon$ (and thus $|\lambda_i|<R$ for $i\ge i_0$), for all $i\ge i_0$ in I (and thus also $|\lambda_i|<R$ for $i\ge i_0$).

$$\lambda_i x_i=\lambda_i(x_i-x)+(\lambda_i-\lambda)x+\lambda x\in\lambda x+\lambda_i W+V\subset\lambda x+V+V\subset\lambda x+U.$$

If $\mathcal{U}_0$ satisfies (e) and if $x \neq y$ are in E, then we can choose $U \in \mathcal{U}_0$ so that $y - x \notin U$ and then, using the already proven fact that addition and multiplication by scalars is continuous, there is V so that $V - V \subset U$. It follows that $x + V$ and $y + V$ are disjoint. Indeed, if $x + v_1 = y + v_2$, for some $v_1, v_2 \in V$ it would follows that $y - x = v_2 - v_1 \in U$, which is a contradiction.

If (f) is satisfied then E is locally convex since we observed before that $\mathcal{U}_x = \{x + U : U \in \mathcal{U}_0\}$ is a neighborhood basis of x, for each $x \in E$. $\square$

Let E be a vector space over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let F be a subspace of

$$E^{\#} = \{f : E \to \mathbb{K} \text{ linear}\}.$$

Assume that for each $x \in E$ there is an $x^* \in F$ so that $x^*(x) \neq 0$, we say in that case that F *is separating the elements of* E *from* 0. Consider

$$\mathcal{U}_0 = \Big\{ \bigcap_{j=1}^{n} \{x \in E : |x_i^*(x)| < \varepsilon_i\} : n \in \mathbb{N}, x_i^*, \in F, \text{ and } \varepsilon_i > 0,\ i = 1, \ldots, n \Big\}.$$

$\mathcal{U}_0$ is finitely intersection stable and it is easily checked that $\mathcal{U}_0$ satisfies that assumptions (b)-(f). It follows therefore that $\mathcal{U}_0$ is the neighborhood basis of a topology which turns E into locally convex Hausdorff space.

Definition 2.1.5. If E is a topological vector space over $\mathbb{K}$, we call

$$E^* = \{f : E \to \mathbb{K} : f \text{ linear and continuous}\}.$$

Definition 2.1.6. [The Topology $\sigma(E, F)$]
Let E be a vector space and let F be a separating subspace of $E^{\#}$.

Then we denote the locally convex Hausdorff topology generated by

$$\mathcal{U}_0 = \Big\{ \bigcap_{j=1}^{n} \{x \in E : |x_i^*(x)| < \varepsilon_i\} : n \in \mathbb{N}, x_i^* \in F, \text{ and } \varepsilon_i > 0\,,\ i = 1, \ldots, n \Big\},$$

by $\sigma(E, F)$.

If E is a locally convex space we call $\sigma(E, E^*)$, as in the case of Banach spaces, the *Weak Topology on* E and denote it also by w. If E, say $E = F^*$, for some locally convex space F, we call $\sigma(F^*, F)$ the weak* topology and denote it by w^* (if no confusion can happen).

From the Hahn Banach Theorem for Banach spaces it follows that the weak topology turns a Banach space X into a Hausdorff space, and we can see $\big(X, \sigma(X, X^*)\big)$ as a locally convex space. Similarly $\big(X^*, \sigma(X^*, X)\big)$ is a locally convex space which is Hausdorff.

Proposition 2.1.7. *Assume that X is a Banach space and that X^* denotes its dual with respect to the norm. Then*

$$\big(X,\sigma(X,X^*)\big)^* = X^* \text{ and } \big(X^*,\sigma(X^*,X)\big)^* = X.$$

Proposition 2.1.7 follows from a more general principle.

Proposition 2.1.8. *Let E be a locally convex space and E^* its dual space. Equip E^* with the topology $\sigma(E^*,E)$. Then $\big(E^*,\sigma(E^*,E)\big)$, and $\big(E,\sigma(E,E^*)\big)$ are locally convex spaces whose duals are E and E^*, respectively (where we identify $e\in E$ in the canonical way with a map defined on E^*).*

Remark. Proposition 2.1.8 means the following: Start with an arbitrary locally convex space E, and let E^* be its dual. Then for the topology $\sigma(E^*,E)$, *i.e.* the coarsest topology on E^* for which all elements of E are continuous, you have "reflexivity" in the sense that the dual of the locally convex space $(E^*,\sigma(E^*,E))$ is E again.

Proof of Proposition 2.1.8. We will only show that $\big(E^*,\sigma(E^*,E)\big)^* = E$ and leave the second part as an exercise. It is clear that E belongs to $\big(E^*,\sigma(E^*,E)\big)^*$ in the following sense: If $e\in E$ and if $\chi(e)$ is the function on E^* which assigns to $f\in E^*$ the scalar $\langle f,e\rangle$, then $\chi(e)$ is in $\big(E^*,\sigma(E^*,E)\big)^*$. From now on we identify e with $\chi(e)$ and simply write e instead of $\chi(e)$.

Assume $\phi: E^*\to\mathbb{K}$ is linear and $\sigma(E^*,E)$-continuous. We need to show that $\phi=\chi(e)=e$ for some $e\in E$.

$$U=\{f\in E^*:|\langle\phi,f\rangle|<1\}=\phi^{-1}(-1,1)$$

is then an $\sigma(E,E^*)$-open neighborhood and thus there are $e_1,e_2,\ldots,e_n\in E$ and $\varepsilon>0$ so that

$$\bigcap_{j=1}^n\{f\in E^*|\ |\langle e_j,f\rangle|<\varepsilon\}\subset U.$$

It follows from this that

$$\bigcap_{j=1}^n \ker(e_j)\subset\ker(\phi).$$

Indeed,

$$\bigcap_{j=1}^n \ker(e_j)=\bigcap_{\delta>0}\bigcap_{j=1}^n\{f\in E^*|\ |\langle e_j,f\rangle|<\delta\varepsilon\}$$

$$
\begin{aligned}
&= \bigcap_{\delta>0} \delta \cdot \bigcap_{j=1}^{n} \{f \in E^*| \, |\langle e_j, f\rangle| < \varepsilon\} \\
&\subset \bigcap_{\delta>0} \delta \cdot U \\
&= \bigcap_{\delta>0} \delta \cdot \{f \in E^* : |\langle \phi, f\rangle| < 1\} \\
&= \bigcap_{\delta>0} \{f \in E^* : |\langle \phi, f\rangle| < \delta\} = \ker(\phi).
\end{aligned}
$$

Now an easy linear algebra argument implies that ϕ is a linear combination of $e_1, e_2, \ldots, e_n$ which yields that $\phi \in E$. $\square$

Proposition 2.1.9. *Let E be a vector space and let F be a separating subspace of $E^{\#}$.*

For a net $(x_i)_{i\in I} \subset E$ and $x \in E$

$$\lim_{i\in I} x_i = x \text{ in } \sigma(E,F) \iff \forall x^* \in F \quad \lim_{i\in I} \langle x^*, x_i\rangle = \langle x^*, x\rangle.$$

2.2 Geometric Version of the Hahn-Banach Theorem for locally convex spaces

We want to formulate a *geometric version of the Hahn-Banach Theorem.*

Definition 2.2.1. A subset A of a vector space V over $\mathbb{K}$ is called *convex* if for all $a, b \in A$ and all $\lambda \in [0,1]$ also $\lambda a + (1-\lambda)b \in A$.
If $A \subset V$ we define the *convex hull of A* by

$$\begin{aligned}\mathrm{conv}(A) &= \bigcap \{C : A \subset C \subset V, C \text{ convex }\}\\ &= \Big\{\sum_{j=1}^n \lambda_j a_j : n\in\mathbb{N}, \lambda_j\in[0,1], a_i\in A, \text{ for } i=1,\dots,n, \text{ and } \sum_{j=1}^n \lambda_j=1\Big\}.\end{aligned}$$

A subset $A \subset V$ is called *absorbing* if for all $x \in V$ there is an $0 < r < \infty$ so that $x/r \in A$. For an absorbing set A we define the *Minkowski functional* by

$$\mu_A : V \to [0,\infty), x \mapsto \inf\{\lambda > 0 : x/\lambda \in A\}.$$

A is called *symmetric* if for all $\lambda \in \mathbb{K}$, $|\lambda| = 1$, and all $x \in A$, it follows that $\lambda x \in A$.

Lemma 2.2.2. *Assume C is a convex and absorbing subset of a vector space V. Then μ_C is a sublinear functional on V, and*

$$\{v \in V : \mu_C(v) < 1\} \subset C \subset \{v \in V : \mu_C(v) \le 1\}. \tag{2.1}$$

If V is a locally convex space space and if 0 is in the open kernel of C, then μ_C is continuous at 0.

Proof. Since C is absorbing $0 \in C$ and $\mu_C(0) = 0$. If $u, v \in V$ and $\varepsilon > 0$ is arbitrary, we find $0 < \lambda_u < \mu_C(u) + \varepsilon$ and $0 < \lambda_v < \mu_C(v) + \varepsilon$, so that $u/\lambda_u \in C$ and $v/\lambda_v \in C$ and thus

$$\frac{u+v}{\lambda_u+\lambda_v} = \frac{\lambda_u}{\lambda_u+\lambda_v}\frac{u}{\lambda_u} + \frac{\lambda_v}{\lambda_u+\lambda_v}\frac{v}{\lambda_v} \in C,$$

which implies that $\mu_C(u+v) \le \lambda_u + \lambda_v \le \mu_C(u) + \mu_C(v) + 2\varepsilon$, and, since, $\varepsilon > 0$ is arbitrary, $\mu_C(u+v) \le \mu_C(u) + \mu_C(v)$.

Finally for $\lambda > 0$ and $v \in V$

$$\mu_C(\lambda v) = \inf\Big\{r > 0 : \frac{\lambda v}{r} \in C\Big\} = \lambda \inf\Big\{\frac{r}{\lambda} : \frac{\lambda v}{r} \in C\Big\} = \lambda\mu_C(v).$$

To show the first inclusion in (2.1) assume $v \in V$ with $\mu_C(v) < 1$, there is a $0 < \lambda < 1$ so that $v/\lambda \in C$, and, thus,

$$v = \lambda \frac{v}{\lambda} + (1-\lambda)0 \in C.$$

The second inclusion is clear since for $v \in C$ it follows that $v = \frac{v}{1} \in C$.

If V is a locally convex space and $0 \in C^0$, then there is a an open convex neighborhood U of 0, so that $0 \in U \subset C$. Now let (x_i) be a net which converges in V to 0. Since with U also εU is a neighborhood of 0, for $\varepsilon > 0$, we obtain for any $\varepsilon > 0$ an $i_0 \in I$, so that for all $i \geq i_0$ in I it follows that $x_i \in \varepsilon U$.

$$\mu_C(x_i) \leq \mu_U(x_i) \leq \varepsilon \mu_{\varepsilon U}(x_i) \leq \varepsilon.$$

□

Theorem 2.2.3. (The Geometric Hahn-Banach Theorem for locally convex spaces) *Let C be a non empty, closed convex subset of a locally convex and Hausdorff space E and let $x_0 \in E \setminus C$.*

Then there is an $x^* \in E^*$ so that

$$\sup_{x \in C} \Re(\langle x^*, x\rangle) < \Re(\langle x^*, x_0\rangle).$$

Proof. We first assume that $\mathbb{K} = \mathbb{R}$ and we also assume w.l.o.g. that $0 \in C$ (otherwise pass to $C - x$ and $x_0 - x$ for some $x \in C$). Let U be convex open neighborhood of 0 so that $C \cap (x_0 + U) = \emptyset$, then let V be an open neighborhood of 0 so that $V - V \subset U$ and let $D = C + V$. It follows that also $(x_0 + V) \cap D = \emptyset$. Therefore $\mu_D(z) \geq 1$, for all $z \in x_0 + V$. Since V is open there is a $0 < \delta < 1$ so that $(1-\delta)x_0 \in x_0 + V$ and thus $\mu_D(x_0) = \frac{1}{1-\delta}\mu_D((1-\delta)x_0) > 1$.

From Lemma 2.2.2 it follows that μ_D is a sublinear functional on E, which is continuous at 0.

On the one dimensional space $Y = \text{span}(x_0)$ define

$$f : Y \to \mathbb{R}, \quad \alpha x_0 \mapsto \alpha \mu_D(x_0).$$

Then $f(y) \leq \mu_D(y)$ for all $y \in Y$ (if $y = \alpha x_0$, with $\alpha > 0$ this follows from the positive homogeneity of μ_D, and if $\alpha < 0$ this is clear). By Theorem 1.4.2 we can extend f to a linear function F, defined on all of E, with $F(x) \leq \mu_D(x)$ for all $x \in E$. Since μ_D is continuous at 0 it follows F is continuous at 0 and thus in E^*.

Moreover, if $x \in C$ it follows that $F(x_0) > 1 \geq \sup_{x \in C} \mu_D(x)) \geq 1$. If $\mathbb{K} = \mathbb{C}$ we first choose F, by considering E to be a real locally convex space, and then put $f(x) = F(x) - iF(ix)$. It is then easily checked that F is a complex linear bounded functional on E. $\square$

Corollary 2.2.4. *Assume that A and B are two convex closed subsets of a locally convex space E, with for which there is an open neighborhood U of 0 with $(A+U) \cap (B+U) = \emptyset$.*

Then there is an $x^* \in E^*$ and $\alpha \in \mathbb{R}$ so that

$$\Re(\langle x^*, x\rangle) \le \alpha \le \Re(\langle x^*, y\rangle), \text{ for all } x \in A \text{ and } y \in B.$$

Proof. Consider

$$C = \overline{A - B} = \overline{\{x - y : x \in A \text{ and } y \in B\}}.$$

we note that $0 \notin C$ is convex and that, applying Theorem 2.2.3, we obtain an $x^* \in X^*$ so that

$$\sup_{x \in C} \Re(\langle x^*, x\rangle) < \Re(\langle x^*, 0\rangle) = 0.$$

But this means that for all $x \in A$ and all $y \in B$ $\Re(\langle x^*, x - y\rangle) < 0$ and thus

$$\Re(\langle x^*, x\rangle) < \Re(\langle x^*, y\rangle).$$

$\square$

An easy consequence of the geometrical version of the Hahn-Banach Theorem 2.2.3 is the following two observation.

Proposition 2.2.5. *If A is a convex subset of a Banach space X then*

$$\overline{A}^w = \overline{A}^{\|\cdot\|}.$$

If a representation of the dual space of a Banach space X is not known, it might be hard to verify weak convergence of a sequence directly. The following Corollary of Proposition 2.2.5 states an equivalent criterium for a sequence to be weakly null without using the dual space of X.

Corollary 2.2.6. *For a bounded sequence (x_n) in Banach space X it follows that (x_n) is weakly null if and only if for all subsequences (z_n), all $\varepsilon > 0$ there is a convex combination $z = \sum_{j=1}^k \lambda_j z_j$ of (z_j) (i.e. $\lambda_i \ge 0$, for $i = 1, 2, \ldots, k$, and $\sum_{j=1}^l \lambda_j = 1$) so that $\|z\| \le \varepsilon$.*

2.3 Reflexivity and Weak Topology

Proposition 2.3.1. *If X is a Banach space and Y is a closed subspace of X, then $\sigma(Y, Y^*) = \sigma(X, X^*) \cap Y$, i.e. the weak topology on Y is the weak topology on X restricted to Y.*

Theorem 2.3.2. (Theorem of Alaoglu, c.f. [Fol, Theorem 5.18])
B_{X^*} is w^* compact for any Banach space X.

Sketch of a proof. Consider the map

$$\Phi : B_X^* \to \prod_{x\in X} \{\lambda \in \mathbb{K} : |\lambda| \le \|x\|\}, \qquad x^* \mapsto (x^*(x) : x \in X).$$

Then we check that Φ is continuous with respect to w^* topology on B_{X^*} and the product topology on $\prod_{x\in X}\{\lambda \in \mathbb{K} : |\lambda| \le \|x\|\}$, has a closed image, and is a homeorphism from B_X^* onto its image.

Since by the Theorem of Tychanoff $\prod_{x\in X}\{\lambda \in \mathbb{K} : |\lambda| \le \|x\|\}$ is compact, $\Phi(B_{X^*})$ is a compact subset, which yields (via the homeomorphism Φ^{-1}) that B_{X^*} is compact in the w^* topology. $\square$

Theorem 2.3.3. (Theorem of Goldstein)
B_X is (via the canonical embedding) w^* dense in $B_{X^{**}}$.

Proof. We need to show that $\overline{\chi(B_X)}^{\sigma(X^{**},X^*)} = B_{X^{**}}$.

Now $\overline{\chi(B_X)}^{\sigma(X^{**},X^*)}$ is closed in the locally convex space $\big(X^{**}, \sigma(X^{**}, X^*)\big)$ whose dual is by Proposition 2.1.7 X^*. So assume that $x_0^{**} \in B_{X^{**}} \setminus \overline{\chi(B_X)}^{\sigma(X^{**},X^*)}$. Then by the Geometrical Hahn Banach Theorem 2.2.3 we can find $x^* \in X^*$ so that

$$\sup_{x^{**}\in\overline{\chi(B_X)}^{\sigma(X^{**},X^*)}} \Re(x^{**}(x^*)) < \Re(x_0^{**}(x^*))$$

But

$$\sup_{x^{**}\in\overline{\chi(B_X)}^{\sigma(X^{**},X^*)}} \Re(x^{**}(x^*)) \ge \sup_{x\in B_X} \Re(x^*(x)) = \|x^*\| \text{ and } \Re(x^*(x)) \le \|x^*\|$$

which is a contradiction. $\square$

Theorem 2.3.4. *Let X be a Banach space. Then X is reflexive if and only if B_X is compact in the weak topology.*

Proof. Let $\chi : X \hookrightarrow X^{**}$ be the canonical embedding.
"$\Rightarrow$" If X is reflexive and thus χ is onto it follows that χ is an homeomorphism between $(B_X, \sigma(X, X^*))$ and $(B_{X^{**}}, \sigma(X^{**}, X^*))$. But by the Theorem of Alaoglu 2.3.2 $(B_{X^{**}}, \sigma(X^{**}, X^*))$ is compact.
"$\Leftarrow$" Assume $x^{**} \in B_{X^{**}}$. By Goldstein's Theorem 2.3.3 there is a net $(x_i)_{i\in I} \subset B_X$, for which $(\chi(x_i) : i \in I)$ converges in $\sigma(X^{**}, X^*)$ to x^{**}. Since B_X is assumed to be $\sigma(X, X^*)$ compact there is a subnet $(x_j : j \in J)$ which converges in $\sigma(X, X^*)$ to some $x \in B_X$ thus it follows for all $x^* \in X^*$ that

$$x^{**}(x^*) = \lim_{i\in I} x^*(x_i) = \lim_{j\in J} x^*(x_j) = x^*(x) = \chi(x)(x^*)$$

which implies that $x^{**} = \chi(x)$.

□

Theorem 2.3.5. *For a Banach space X the following are equivalent.*

a) X is reflexive,

b) X^* is reflexive,

c) every closed subspace of X is reflexive.

Proof. "(a)$\Rightarrow$(c)" Assume $Y \subset X$ is a closed subspace. Proposition 2.2.5 yields that $B_Y = B_X \cap Y$ is a $\sigma(X, X^*)$-closed and, thus, $\sigma(X, X^*)$-compact subset of B_X. Since, by the Theorem of Hahn-Banach (Corollary 1.4.4), every $y^* \in Y^*$ can be extended to an element in X^*, it follows that $\sigma(Y, Y^*)$ is the restriction of $\sigma(X, X^*)$ to the subspace Y. Thus, B_Y is $\sigma(Y, Y^*)$-compact, which implies, by Theorem 2.3.4 that Y is reflexive.
"(a)$\Rightarrow$(b)" If X is reflexive then $\sigma(X^*, X^{**}) = \sigma(X^*, X)$. Since by the Theorem of Alaoglu 2.3.2 B_{X^*} is $\sigma(X^*, X)$-compact the claim follows from Theorem 2.3.4.
"(c)$\Rightarrow$(a)" clear.
"(b)$\Rightarrow$(a)" If X^* is reflexive, then, by "(a)$\Rightarrow$(b)" , applied to X^*, X^{**} is also reflexive and thus, the implication "(a)$\Rightarrow$(c)" yields that X is reflexive. □

Similar ideas as in the proof of Theorem 2.3.3 are used to show the following result which characterizes when a Banach space X is a dual space of another space.

Theorem 2.3.6. *Assume that X is a Banach space and Z is a closed subspace of X^*, so that B_X is compact in the topology $\sigma(X, Z)$, and so that $\|x\| = \sup_{z\in B_Z} |z(x)|$.*

Then Z^* is isometrically isomorphic to X and the map

$$T : X \to Z^*, \quad x \mapsto f_x, \text{ with } f_x(z) = \langle z, x\rangle, \text{ for } x \in X \text{ and } z \in Z,$$

is an isometrical isomorphism onto Z^*.

Proof. We first note that $T(B_X)$ is $\sigma(Z^*, Z)$ dense in B_{Z^*}. Indeed, if this is not true we can apply the Geometric Hahn Banach Theorem for locally convex spaces (Theorem 2.2.3) applied to the locally convex space $\big(Z^*, \sigma(Z^*, Z)\big)$ whose dual is by Proposition 2.1.7 $\big(Z, \sigma(Z, Z^*)\big)$, and obtain elements $z^* \in S_{Z^*}$ and $z \in S_Z$ so that

$$1 = \|z\| = \sup_{x \in B_X} \langle x, z\rangle < \langle z^*, z\rangle = 1,$$

which is a contradiction.

Secondly, our assumption says that $T(B_X)$ is $\sigma(Z^*, Z)$-compact. To see that note that if $(x_i)_{i \in I}$ is a net in X and $z^* \in Z^*$, then

$$\begin{aligned} & (f_{x_i})_{i\in I} \text{ converges to } z^* \text{ with respect to } \sigma(Z^*, Z) \\ \iff & \lim_{i\in I} \langle x_i, z\rangle = \langle z^*, z\rangle \text{ for all } z \in Z \\ \iff & z^* \in T(B_X) \text{ and } \sigma(X, Z^*) - \lim_{i \in I} \langle x_i, z\rangle = z^* \text{ (By assumption).} \end{aligned}$$

$\square$

2.4 Annihilators, Complemented Subspaces

Definition 2.4.1. (Annihilators, Pre-Annihilators)
Assume X is a Banach space. Let $M \subset X$ and $N \subset X^*$. We call

$$M^\perp = \{x^* \in X^* : \forall x \in M \ \langle x^*, x\rangle = 0\} \subset X^*,$$

the *annihilator of* M and

$$N_\perp = \{x \in X : \forall x^* \in N \ \langle x^*, x\rangle = 0\} \subset X,$$

the *pre-annihilator of* N.

Proposition 2.4.2. *Let X be a Banach space, and assume $M \subset X$ and $N \subset X^*$.*

a) *$M^\perp$ is a closed subspace of X^*, $M^\perp = (\overline{\text{span}(M)})^\perp$, and $(M^\perp)_\perp = \overline{\text{span}(M)}$,*

b) *$N_\perp$ is a closed subspace of X, $N_\perp = (\text{span}(N))_\perp$, and $\overline{\text{span}(N)} \subset (N_\perp)^\perp$.*

c) $\overline{\text{span}(M)} = X \iff M^\perp = \{0\}$.

Proof. We only show (a), (b) can be shown similarly and (c) is clear, and we only show the third claim of (a). If $x \in \overline{\text{span}(M)}$ and $x^* \in M^\perp$ then $x^*(x) = 0$, and thus $x \in (M^\perp)_\perp$.

Assume $x_0 \in (M^\perp)_\perp$ but $x_0 \notin \overline{\text{span}(M)}$, then by the Corollary 1.4.5 of Hahn Banach Theorem there is an $x^* \in X^*$ for which $x^*(x_0) > 0$ and $x^*|_{\overline{\text{span}(M)}} \equiv 0$, and thus $x^* \in M^\perp$ which implies that $x^*(x_0) = 0$ which is a contradiction. □

Proposition 2.4.3. *If X is Banach space and $Y \subset X$ is a closed subspace then $(X/Y)^*$ is isometrically isomorphic to $Y^\perp$ via the operator*

$$\Phi : (X/Y)^* \to Y^\perp, \text{ with } \Phi(z^*)(x) = z^*(\overline{x}).$$

(recall $\overline{x} := x + Y \in X/Y$ for $x \in X$).

Proof. Let $Q : X \to X/Y$ be the quotient map.

For $z^* \in (X/Y)^*$, $\Phi(z^*)$, as defined above, can be written as $\Phi(z^*) = z^* \circ Q$. Thus $\Phi(z^*) \in X^*$. Since $Q(Y) = \{0\}$ it follows that $\Phi(z^*) \in Y^\perp$. For $z^* \in (X/Y)^*$ we have

$$\|\Phi(z^*)\| = \sup_{x \in B_X} \langle z^*, Q(x)\rangle = \sup_{\overline{x} \in B_{X/Y}} \langle z^*, \overline{x}\rangle = \|z^*\|_{(X/Y)^*},$$

where the second equality follows on the one hand from the fact that $\|Q(x)\| \leq \|x\|$, for $x \in X$, and on the other hand, from the fact that for any $\overline{x} = x+Y \in X/Y$ there is a sequence $(y_n) \subset Y$ so that $\limsup_{n\to\infty} \|x + y_n\| = \|\overline{x}\|$.

Thus Φ is an isometric embedding. In order to show that Φ is onto let $x^* \in Y^\perp \subset X^*$. We define

$$z^* : X/Y \to \mathbb{K}, \quad x + Y \mapsto \langle x^*, x\rangle.$$

First note that this map is well defined (since $\langle x^*, x + y_1\rangle = \langle x^*, x + y_2\rangle$ for $y_1, y_2 \in Y$). Since x^* is linear, z^* is also linear, and $|\langle z^*, \overline{x}\rangle| = |\langle x^*, x\rangle|$, for all $x \in X$, and thus $\|z^*\|_{(X/Y)^*} = \|x^*\|$. Finally, since

$$\langle \Phi(z^*), x\rangle = \langle z^*, Q(x)\rangle = \langle x^*, x\rangle,$$

it follows that $\Phi(z^*) = x^*$, and thus that Φ is surjective. $\square$

Proposition 2.4.4. *Assume X and Y are Banach spaces and $T \in L(X, Y)$. Then*

$$T(X)^\perp = \mathcal{N}(T^*) \text{ and } \overline{T^*(Y^*)} \subset \mathcal{N}(T)^\perp \tag{2.2}$$

$$\overline{T(X)} = \mathcal{N}(T^*)_\perp \text{ and } T^*(Y^*)_\perp = \mathcal{N}(T). \tag{2.3}$$

Proof. We only prove (2.2). The verification of (2.3) is similar. For $y^* \in Y^*$

$$\begin{aligned} y^* \in T(X)^\perp &\iff \forall x \in X \quad \langle y^*, T(x)\rangle = 0 \\ &\iff \forall x \in X \quad \langle T^*(y^*), x\rangle = 0 \\ &\iff T^*(y^*) = 0 \iff y^* \in \mathcal{N}(T^*), \end{aligned}$$

which proves the first part of (2.2), and for $y^* \in Y^*$ and all $x \in \mathcal{N}(T)$, it follows that $\langle T^*(y^*), x\rangle = \langle y^*, T(x)\rangle = 0$, which implies that $T^*(Y^*) \subset \mathcal{N}(T)^\perp$, and, thus, $\overline{T^*(X^*)} \subset \mathcal{N}(T)^\perp$ since , $\mathcal{N}(T)^\perp$ is closed. $\square$

Definition 2.4.5. Let X be a Banach space and let U and V be two closed subspaces of X. We say that X is *the complemented sum of U and V* and we write $X = U \oplus V$, if for every $x \in X$ there are $u \in U$ and $v \in V$, so that $x = u + v$ and so that this representation of x as sum of an element of U and an element of V is unique.

We say that a closed subspace Y of X *is complemented in* X if there is a closed subspace Z of X so that $X = Y \oplus Z$.

Remark. Assume that the Banach space X is the complemented sum of the two closed subspaces U and V. We note that this implies that $U \cap V = \{0\}$.

We can define two maps

$$P : X \to U \text{ and } Q : X \to V$$

where we define $P(x) \in U$ and $Q(x) \in V$ by the equation $x = P(x) + Q(y)$, with $P(x) \in U$ and $Q(x) \in V$ (which, by assumption, has a unique solution). Note that P and Q are linear. Indeed, if $P(x_1) = u_1$, $P(x_2) = u_2$, $Q(x_1) = v_1$, $Q(x_2) = v_2$, then for $\lambda, \mu \in \mathbb{K}$ we have $\lambda x_1 + \mu x_2 = \lambda u_1 + \mu u_2 + \lambda v_1 + \mu v_2$, and thus, by uniqueness $P(\lambda x_1 + \mu x_2) = \lambda u_1 + \mu u_2$, and $Q(\lambda x_1 + \mu x_2) = \lambda v_1 + \mu v_2$.

Secondly it follows that $P \circ P = P$, and $Q \circ Q = Q$. Indeed, for any $x \in X$ we we write $P(x) = P(x) + 0 \in U + V$, and since this representation of $P(x)$ is unique it follows that $P(P(x)) = P(x)$. The argument for Q is the same.

Finally it follows that, again using the uniqueness argument, that P is the identity on U and Q is the identity on V.

We therefore proved that

a) P is linear,

b) the image of P is U

c) P is idempotent, i.e. $P^2 = P$

We say in that case that P is a *linear projection onto* U. Similarly Q is a a linear projection onto V, and P and Q *are complementary to each other*, meaning that $P(X) \cap Q(X) = \{0\}$ and $P + Q = Id$. A linear map $P : X \to X$ with the properties (a) and (c) is called *projection.*

The next Proposition will show that P and Q as defined in above remark are actually bounded.

Lemma 2.4.6. *Assume that X is the complemented sum of two closed subspaces U and V. Then the projections P and Q as defined in above remark are bounded.*

Proof. Consider the norm $|||\cdot|||$ on X defined by

$$|||x||| = \|P(x)\| + \|Q(x)\|, \text{ for } x \in X.$$

We claim that $(X, \|\!|\cdot|\!\|)$ is also a Banach space. Indeed if $(x_n) \subset X$ with

$$\sum_{n=1}^{\infty} \|\!|x_n|\!\| = \sum_{n=1}^{\infty} \|P(x_n)\| + \sum_{n=1}^{\infty} \|Q(x_n)\| < \infty.$$

Then $u = \sum_{n=1}^{\infty} P(x_n) \in U$, $v = \sum_{n=1}^{\infty} Q(x_n) \in V$ (U and V are assumed to be closed) converge in U and V with respect to $\|\cdot\|$, respectively. Since $\|\cdot\| \le \|\!|\cdot|\!\|$ also $x = \sum_{n=1}^{\infty} x_n$ converges with respect to $\|\cdot\|$ and

$$x = \sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} \sum_{j=1}^{n} \big(P(x_j)+Q(x_j)\big) = \lim_{n\to\infty} \sum_{j=1}^{n} P(x_j) + \lim_{n\to\infty} \sum_{j=1}^{n} Q(x_j) = u+v,$$

and

$$\begin{aligned} \Big\|\!\Big| x - \sum_{j=1}^{n} x_n \Big|\!\Big\| &= \Big\|\!\Big| u - \sum_{j=1}^{n} P(x_n) + v - \sum_{j=1}^{n} Q(x_n) \Big|\!\Big\| \\ &= \Big\| u - \sum_{j=1}^{n} P(x_n) \Big\| + \Big\| v - \sum_{j=1}^{n} Q(x_n) \Big\| \to_{n\to\infty} 0, \end{aligned}$$

(here all series are meant to converge with respect to $\|\cdot\|$) which proves that $(X, \|\!|\cdot|\!\|)$ is complete.

Since the identity is a bijective linear bounded operator from $(X, \|\!|\cdot|\!\|)$ to $(X, \|\cdot\|)$ it has by Corollary 1.3.6 of the Closed Graph Theorem a continuous inverse and is thus an isomorphy. Since $\|P(x)\| \le \|\!|x|\!\|$ and $\|Q(x)\| \le \|\!|x|\!\|$ we deduce our claim. □

Proposition 2.4.7. *Assume that X is a Banach space and that $P : X \to X$, is a bounded projection onto a closed subspace of X.*

Then $X = P(X) \oplus \mathcal{N}(P)$.

Theorem 2.4.8. *There is no linear bounded operator $T : \ell_\infty \to \ell_\infty$ so that the kernel of T equals to c_0.*

Corollary 2.4.9. *c_0 is not complemented in ℓ_∞.*

Proof of Theorem 2.4.8. For $n \in \mathbb{N}$ we let e_n^* be the *n-th coordinate functional on ℓ_∞*, i.e.

$$e_n^* : \ell_\infty \to \mathbb{K}, \qquad x = (x_j) \mapsto x_n.$$

Step 1. If $T : \ell_\infty \to \ell_\infty$ is bounded and linear, then

$$\mathcal{N}(T) = \bigcap_{n=1}^{\infty} \mathcal{N}(e_n^* \circ T).$$

Indeed, note that

$$x \in \mathcal{N}(T) \iff \forall n \in \mathbb{N} \quad e_n^*(T(x)) = \langle e_n^*, T(x)\rangle = 0.$$

In order to prove our claim we will show that c_0 cannot be the intersection of the kernel of countably many functionals in ℓ_∞^*.

Step 2. There is an uncountable family $(N_\alpha : \alpha \in I)$ of infinite subsets of $\mathbb{N}$ for which $N_\alpha \cap N_\beta$ is finite whenever $\alpha \neq \beta$ are in I.

Write the rational numbers $\mathbb{Q}$ as a sequence $(q_j : j \in \mathbb{N})$, and choose for each $r \in \mathbb{R}$ a sequence $(n_k(r) : k \in \mathbb{N})$, so that $(q_{n_k(r)} : k \in \mathbb{N})$ converges to r. Then, for $r \in \mathbb{R}$ let $N_r = \{n_k(r) : k \in \mathbb{N}\}$. The family $(N_r : r \in \mathbb{R})$ then satisfies the claim in Step 2.

For $\alpha \in I$, put $x_\alpha = 1_{N_\alpha} \in \ell_\infty$, i.e.

$$x_\alpha = (\xi_k^{(\alpha)} : k \in \mathbb{N}) \text{ with } \xi_k^{(\alpha)} = \begin{cases} 1 & \text{if } k \in N_\alpha \\ 0 & \text{if } k \notin N_\alpha. \end{cases}$$

Step 3. If $f \in \ell_\infty^*$ and $c_0 \subset \mathcal{N}(f)$ then $\{\alpha \in I : f(x_\alpha) \neq 0\}$ is countable.

In order to verify Step 3 let $A_n = \{\alpha : |f(x_\alpha)| \geq 1/n\}$, for $n \in \mathbb{N}$. It is enough to show that for $n \in \mathbb{N}$ the set A_n is finite. To do so, let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be distinct elements of A_n and put $x = \sum_{j=1}^k \mathrm{sign}\big(\overline{f(x_{\alpha_j})}\big) x_{\alpha_j}$ (for $a \in \mathbb{C}$ we put $\mathrm{sign}(a) = a/|a|$) and deduce that $f(x) \geq k/n$. Now consider $M_j = N_{\alpha_j} \setminus \bigcup_{i \neq j} N_{\alpha_i}$. Then $N_{\alpha_j} \setminus M_j$ is infinite, and thus it follows for

$$\tilde{x} = \sum_{j=1}^k \mathrm{sign}(f(x_{\alpha_j})) 1_{M_j}$$

that $f(x) = f(\tilde{x})$ (since $x - \tilde{x} \in c_0$). Since the M_j, $j = 1, 2, \ldots, k$ are pairwise disjoint, it follows that $\|\tilde{x}\|_\infty = 1$, and thus

$$\frac{k}{n} \leq f(x) = f(\tilde{x}) \leq \|f\|.$$

Which implies that A_n can have at most $n\|f\|$ elements.

Step 4. If $c_0 \subset \bigcap_{n=1}^\infty \mathcal{N}(f_n)$, for a sequence $(f_n) \subset \ell_\infty^*$, then there is an $\alpha \in I$ so that $x_\alpha \in \bigcap_{n=1}^\infty \mathcal{N}(f_n)$. In particular this implies that $c_0 \neq \bigcap_{n \in \mathbb{N}} \mathcal{N}(f_n)$.

Indeed, Step 3 yields that

$$C = \{\alpha \in I : f_n(x_\alpha) \neq 0 \text{ for some } n \in \mathbb{N}\} = \bigcup_{n \in \mathbb{N}} \{\alpha \in I : f_n(x_\alpha) \neq 0\},$$

is countable, and thus $I \setminus C$ is not empty. $\square$

Remark. Assume that Z is any subspace of ℓ_∞ which is isomorphic to c_0, then Z is not complemented. The proof of that statement is a bit harder.

Theorem 2.4.10. [So] *Assume Y is a subspace of a separable Banach space X and $T : Y \to c_0$ is linear and bounded. Then T can be extended to a linear and bounded operator $\tilde{T} : X \to c_0$. Moreover, $\tilde{T}$ can be chosen so that $\|\tilde{T}\| \le 2\|T\|$.*

Corollary 2.4.11. *Assume that X is a separable Banach space which contains a subspace Y which is isomorphic to c_0. Then Y is complemented in X.*

Proof. Let $T : Y \to c_0$ be an isomorphism. Then extend T to $\tilde{T} : X \to c_0$ and put $P = T^{-1} \circ \tilde{T}$. □

Proof of Theorem 2.4.10. Note that an operator $T : Y \to c_0$ is defined by a $\sigma(Y^*, Y)$ null sequence $(y_n^*) \subset Y^*$, i.e.

$$T : Y \to c_0, \qquad y \mapsto \big(\langle y_n^*, y\rangle : n \in \mathbb{N}\big).$$

We would like to use the Hahn Banach Theorem and extend each y_n^* to an element $x_n^* \in X_n^*$, with $\|y_n^*\| = \|x_n^*\|$, and define

$$\tilde{T}(x) := (\langle x_n^*, x\rangle : n \in \mathbb{N}), \qquad x \in X.$$

But the problem is that (x_n^*) might not be $\sigma(X^*, X)$ convergent to 0, and thus we can only say that $(\langle x_n^*, x\rangle : n \in \mathbb{N}) \in \ell_\infty$, but not necessarily in c_0. Thus we will need to *change the x_n^* somehow* so that they are still extensions of the y_n^* but also $\sigma(X^*, X)$ null.

Let $B = \|T\| B_{X^*}$. B is $\sigma(X^*, X)$-compact and metrizable (since X is separable). Denote the metric which generates the $\sigma(X^*, X)$-topology by $d(\cdot, \cdot)$. Put $K = B \cap Y^\perp$. Since $Y^\perp \subset X^*$ is $\sigma(X^*, X)$-closed, K is $\sigma(X^*, X)$-compact . Also note that every $\sigma(X^*, X)$-accumulation point of (x_n^*) lies in K. Indeed, this follows from the fact that $x_n^*(y) = y_n^*(y) \to_{n\to\infty} 0$, for all $y \in Y$. This implies that $\lim_{n\to\infty} d(x_n^*, K) = 0$, thus we can choose $(z_n^*) \subset K$ so that $\lim_{n\to\infty} d(x_n^*, z_n^*) = 0$, and thus $(x_n^* - z_n^*)$ is $\sigma(X^*, X)$-null and for $y \in Y$ it follows that $\langle x_n^* - z_n^*, y\rangle = \langle x_n^*, y\rangle$, $n \in \mathbb{N}$. Choosing therefore

$$\tilde{T} : X \to c_0, \qquad x \mapsto (\langle x_n^* - z_n^*, x\rangle : n \in \mathbb{N}),$$

yields our claim. □

Remark. Zippin [Zi] proved the converse of Theorem: if Z is an infinite-dimensional separable Banach space admitting a projection from any separable Banach space X containing it, then Z is isomorphic to c_0.

2.5 The Theorem of Eberlein Smulian

For infinite dimensional Banach spaces the weak topology is not metrizable (see Exercise in Homework). Nevertheless compactness in the weak topology can be characterized by sequences.

Theorem 2.5.1. (The Theorem of Eberlein- Smulian)
Let X be a Banach space. For subset K the following are equivalent.

a) K is relatively $\sigma(X, X^*)$ compact, i.e. $\overline{K}^{\sigma(X,X^*)}$ is compact.

b) Every sequence in K contains a $\sigma(X, X^*)$-convergent subsequence.

c) Every sequence in K has a $\sigma(X, X^*)$-accumulation point.

We will need the following Lemma.

Lemma 2.5.2. *Let X be a Banach space and assume that there is a countable set $C = \{x_n^* : n \in \mathbb{N}\} \subset B_{X^*}$, so that $C_\perp = \{0\}$. In that case we say that C* is total for X.

Consider for x, y

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} |\langle x_n^*, x - y \rangle|.$$

Then d is a metric on X, and for any $\sigma(X, X^*)$-compact set K, $\sigma(X, X^*)$ coincides on K with the metric generated by d.

The proof of Lemma 2.5.2 goes along the lines of an Exercise in this section.

Lemma 2.5.3. *Assume that X is separable. Then there is a countable total set $C \subset X^*$.*

Proof. Let $D \subset X$ be dense, and choose by the Corollary 1.4.6 of the Theorem of Hahn Banach for each element $x \in D$, an element $y_x^* \in S_{X^*}$ so that $\langle y_x^*, x \rangle = \|x\|$. Put $C = \{y_x^* : x \in D\}$. If $x \in X$, $x \neq 0$, is arbitrary then there is a sequence $(x_k) \subset D$, so that $\lim_{k\to\infty} x_k = x$, and thus $\lim_{k\to\infty} \langle y_{x_k}^*, x \rangle = \|x\| > 0$. Thus there is an $x^* \in C$ so that $\langle x^*, x \rangle \neq 0$, which implies that C is total.

□

Proof of Theorem 2.5.1. "(a)⇒(b)" Assume that K is $\sigma(X, X^*)$-compact (if necessary, pass to the closure) and let $(x_n) \subset K$ be a sequence, and put $X_0 = \overline{\text{span}(x_n : n \in \mathbb{N})}$. X_0 is a separable Banach space. By Proposition 2.2.5 the topology $\sigma(X_0, X_0^*)$ coincides with the restriction of $\sigma(X, X^*)$ to X_0. Thus, $K_0 = K \cap X_0$ is $\sigma(X_0, X_0^*)$-compact. Since X_0 is separable, by Lemma 2.5.3 there exists a countable set $C \subset B_{X_0^*}$, so that $C_\perp = \{0\}$.

It follows therefore from Lemma 2.5.2 that $(K_0, \sigma(X_0, X_0^*) \cap K_0)$ is metrizable and thus (x_n) has a convergent subsequence in K_0. Again, using the fact that on X_0 the weak topology coincides with the weak topology on X, we deduce our claim.

"(b)⇒(c)" clear.

"(c)⇒(a)" Assume $K \subset X$ satisfies (c). We first observe that K is (norm) bounded. Indeed, for $x^* \in X^*$, the set $A_{x^*} = \{\langle x^*, x\rangle : x \in K\} \subset \mathbb{K}$ is the continuous image of A (under x^*) and thus has the property that every sequence has an accumulation point in $\mathbb{K}$. This implies that A_{x^*} is bounded in $\mathbb{K}$ for all $x^* \in X^*$, but this implies by the Banach Steinhaus Theorem 1.3.8 that $A \subset X$ must be bounded.

Let $\chi : X \hookrightarrow X^{**}$ be the canonical embedding. By the Theorem of Alaoglu 2.3.2, it follows that $\overline{\chi(K)}^{\sigma(X^{**},X^*)}$ is $\sigma(X^{**}, X^*)$-compact. Therefore it will be enough to show that $\overline{\chi(K)}^{\sigma(X^{**},X^*)} \subset \chi(X)$ (because this would imply that every net $(\chi(x_i) : i \in I) \subset \chi(K)$ has a subnet which $\sigma(\chi(X), X^*))$ converges to some element $\chi(x) \in \chi(X)$).

So, fix $x_0^{**} \in \overline{\chi(K)}^{\sigma(X^{**},X^*)}$. Recursively we will choose for each $k \in \mathbb{N}$, $x_k \in K$, and for each $k \in \mathbb{N}$ a finite set $A_k^* \subset S_{X^*}$, so that

$$\text{(2.4)} \quad \big|\langle x_0^{**} - \chi(x_k), x^*\rangle\big| < \frac{1}{k} \text{ for all } x^* \in \bigcup_{0\le j<k} A_j^*, \text{ if } k \ge 1,$$

$$\text{(2.5)} \quad \forall x^{**} \in \text{span}(x_0^{**}, \chi(x_j), 0 \le j \le k)\ \|x^{**}\| \ge \max_{x^* \in A_k^*} |\langle x^{**}, x^*\rangle| \ge \frac{\|x^{**}\|}{2}.$$

For $k = 0$ choose $A_0^* = \{x^*\}$, $x^* \in S_X^*$, with $|x^*(x_0^{**})| \ge \|x_0^{**}\|/2$, then condition (2.5) is satisfied, while condition (2.4) is vacuous.

Assuming that $x_1, x_2, \ldots, x_{k-1}$ and $A_0^*, A_1^*, \ldots, A_{k-1}^*$ have been chosen for some $k > 1$, we can first choose $x_k \in K$ so that (2.4) is satisfied (since A_j^* is finite for $j = 1, 2, \ldots, k-1$), and then, since $\text{span}(x_0^{**}, \chi(x_j), j \le k)$ is a finite dimensional space we can choose $A_k^* \subset S_{X^*}$ so that (2.5) holds.

By our assumption (c) the sequence (x_k) has an $\sigma(X, X^*)$- accumulation point x_0. By Proposition 2.3.1 it follows that

$$x_0 \in Y = \overline{\text{span}(x_k : k \in \mathbb{N})}^{\|\cdot\|} = \overline{\text{span}(x_k : k \in \mathbb{N})}^{\sigma(X,X^*)}.$$

We will show that $x_0^{**} = \chi(x_0)$ (which will finish the proof). First note that for any $x^* \in \bigcup_{j\in\mathbb{N}} A_j^*$

$$\big|\langle x_0^{**} - \chi(x_0), x^*\rangle\big| \le \liminf_{k\to\infty} \big(\big|\langle x_0^{**} - \chi(x_k), x^*\rangle\big| + \big|\langle x^*, x_k - x_0\rangle\big|\big) = 0.$$

Secondly consider the space $Z = \overline{\text{span}(x_0^{**}, \chi(x_k), k\in\mathbb{N})}^{\|\cdot\|} \subset X^{**}$ it follows from (2.5) that the set of restrictions of elements of $\bigcup_{k=1}^\infty A_k^*$ to Z is total in Z and thus that

$$x_0^{**} - \chi(x_0) \in Z \cap \Big(\bigcup_{k=1}^\infty A_k^*\Big)_\perp = \{0\},$$

which implies our claim. $\square$

2.6 Characterizations of Reflexivity by Pták

We present several characterization of the reflexivity of a Banach space, due to Pták [Ptak]. We assume in this section that our Banach spaces are defined over the real field $\mathbb{R}$.

Theorem 2.6.1. *The following conditions for a Banach space X are equivalent*

1. *X is not reflexive.*
2. *For each $\theta \in (0,1)$ there are sequences $(x_i)_{i=1}^\infty \subset B_X$ and $(x_i^*)_{i=1}^\infty \subset B_{X^*}$, so that*

$$x_j^*(x_i) = \begin{cases} \theta & \text{if } j \le i, \text{ and} \\ 0 & \text{if } j > i. \end{cases} \tag{2.6}$$

3. *For some $\theta > 0$ there are sequences $(x_i)_{i=1}^\infty \subset B_X$ and $(x_i^*)_{i=1}^\infty \subset B_{X^*}$, for which (2.6) holds.*
4. *For each $\theta \in (0,1)$ there is a sequence $(x_i)_{i=1}^\infty \subset B_X$, so that*

$$\operatorname{dist}\big(\operatorname{conv}(x_1, x_2, \ldots, x_n), \operatorname{conv}(x_{n+1}, x_{n+2}, \ldots)\big) \ge \theta. \tag{2.7}$$

5. *For some $\theta > 0$ there is a sequence $(x_i)_{i=1}^\infty \subset B_X$, so that (2.7) holds.*

For the proof we will need Helly's Lemma.

Lemma 2.6.2. *Let Y be an infinite-dimensional normed linear space y_1^*, $y_2^*, \ldots, y_n^* \in Y^*$, $M > 0$ and let $c_1, c_2, \ldots, c_n$ be scalars.*

The following are equivalent

(M) The Moment Condition
For all $\varepsilon > 0$ there exists $y \in Y$ with

$$\|y\| = M + \varepsilon \text{ and } y_k^*(y) = c_k \text{ for } k = 1, 2, \ldots, n.$$

(H) Helly's Condition

$$\Big|\sum_{j=1}^n a_j c_j\Big| \le M \Big\|\sum_{j=1}^n a_j y_j^*\Big\| \text{ for any sequence } (a_j)_{j=1}^n \text{ of scalars.}$$

Proof. "$(M) \Rightarrow (H)$". Let $\varepsilon > 0$ and assume $y \in Y$ satisfies the condition in (M). Then

$$\Big|\sum_{j=1}^n a_j c_j\Big| = \Big|\sum_{j=1}^n a_j y_j^*(y)\Big| \le \|y\| \cdot \Big\|\sum_{j=1}^n a_j y_j^*\Big\| = (M+\varepsilon)\Big\|\sum_{j=1}^n a_j y_j^*\Big\|,$$

which implies (H), since $\varepsilon > 0$ was arbitrary.

"$(H) \Rightarrow (M)$" We will first need a Lemma

Lemma 2.6.3. *Let X be a Banach space and assume that $x_1^*, x^*2, \ldots, x_n^*$ are linear independent in X^*. Then there exists $x_1, x_2, \ldots, x_n \in X$ so that*

$$x_j^*(x_i) = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases}$$

Proof. By the Theorem of Hahn Banach there are $x_1^{**}, x_2^{**}, \ldots, x_n^{**} \in X^{**}$ so that $x_j^{**}(x_i^*) = \delta_{i,j}$ for $1 \le i, j \le n$. Let $\varepsilon > 0$ (to be chosen later small enough). By Goldstein's Theorem 2.3.3 there are $z_1, z_2, \ldots z_n$ in X, so that $|x_j^*(z_i)| < \varepsilon$, if $i \ne j$ and $|x_i^*(z_i) - 1| < \varepsilon$. Let $A = (a_{i,j})_{i,j=1}^n$ be the n by matrix defined by $a_{i,j} = x_j^*(x_i)$. Assuming that ε has been chosen small enough, we deduce that A is invertible and let $B = (b_{i,j})_{i,j=1}^n$ be its inverse Defining now $x_i = \sum_{s=1}^n b_{i,s} z_s$, it follows that

$$x_j^*(x_i) = \sum_{s=1}^n b_{j,s} x_j^*(x_s) = \sum_{s=1}^n b_{j,s} a_{s,i} = \delta_{j,i}.$$

□

Assume (H) and let $\varepsilon > 0$. We can assume that not all the y_k^* are vanishing (otherwise also all the c_k have to be equal to 0, and any $y \in \{y_1^*, y_2^*, \ldots y_n^*\}^\perp$ with $\|y\| = M + \varepsilon$ will satisfy the conditions in (M)). We can also, for the same reason assume that not all c_j's vanish. Secondly, we can assume, after reordering the y_j^*, that for some $k \in \{1, 2, \ldots, n\}$ the sequence $(y_j^*)_{j=1}^k$ is linear independent and $y_{k+1}^*, y_{k+2}^*, \ldots, y_n^* \in \text{span}(y_j^* : j = 1, 2, \ldots, k)$. This implies that if we have a $y \in Y$, with $\|y\| = M + \varepsilon$ and $y_j^*(y) = c_j$, for $j = 1, 2, \ldots, k$, then it also follows that $y_j^*(y) = c_j$, for $j = k+1, k+2, \ldots, n$. Indeed, for $j = k+1, k+2, \ldots, n$, choose scalars $(a_i^{(j)} : i = 1, 2, \ldots, k)$ so that $y_j^* = \sum_{i=1}^k a_i^{(j)} y_i^*$, for $j = k+1, k+2, \ldots, n$. Now, the inequality in (H) implies that $c_j = \sum_{i=1}^k a_i^{(j)} c_i$, for $j = k+1, k+2, \ldots, n$. Indeed, choose $a_j = -1$, $a_i = 0$, if $i \in \{k+1, k+2, \ldots, n\} \setminus \{j\}$ and $a_i = a_i^{(j)}$, if $i \in \{1, 2, \ldots, k\}$, which implies that the right hand of the equation in (M) vanishes. This yields that $y_j^*(y) = c_j$, for $j = k+1, k+2, \ldots, n$.

We can therefore restrict ourselves to satisfy the second condition in (H) for all $j = 1, 2, \ldots, k$. Define for $j = 1, 2, \ldots, k$ the affine subspace $H_j = \{y \in Y : y_j^*(y) = c_j\}$. Then

$$G = \bigcap_{j=1}^{k} H_j = \{y \in Y : y_j^*(y) = c_j, \text{ for } j = 1, 2 \ldots n\}$$

is not empty, by Lemma 2.6.3, and if we pick $y \in G$, then $G = y + G_0$, where G_0 is the closed subspace

$$G_0 = \bigcap_{j=1}^{k} \{y \in Y : y_j^*(y) = 0\}.$$

We need to show that

$$N := \inf \big\{ \|y\| : y \in G \big\} \leq M. \tag{2.8}$$

Then our claim would follow, since the Intermediate Value implies that there must be some y in G for which $N < M + \varepsilon \leq \|y\| < \infty$. Without loss of generality we can assume that $N > 0$. We define $\tilde{G} = \overline{\text{span}(G)}$, and note that if $y \in G$

$$\tilde{G} = \text{span}(y_0, G_0) = \{ry : r \in \mathbb{K}, y \in G\} \text{ where } y_0 \in G. \tag{2.9}$$

We choose a functional g^* in the dual of the span of G so that $g^*(y) = N$, for all $y \in G$ This can be done by picking a fixed point $y_0 \in G$, and choosing by Hahn Banach $g^* \in \tilde{G}^*$, with $g^*(y_0) = N$ and which vanishes on the linear closed subspace G_0.

We note that $\|g^*\| \geq 1$. Indeed, otherwise choose a sequence $(y_n) \subset G$, with $\lim_{n\to\infty} \|y_n\| = N$, and note that

$$N = g^*(y_n) \leq \|g^*\| \cdot \|y_n\| \to_{n\to\infty} \|g^*\| N < N$$

which is a contradiction.

Secondly, we note that $\|g^*\| \leq 1$, we use (2.9) and find $r \in \mathbb{R}$ and $y \in G$ so that $g^*(ry) > \|ry\| \geq |r|N$, which is a contradiction since $g^*(ry) = rN$.

Thus $\|y^*\| = 1$.

We let y^* be a Hahn Banach extension of g^* to a functional defined on all of Y.

For all $y \in Y$, we have that if $y_j^*(y) = c_j$, for $j = 1, 2, \ldots, k$ (and thus $y \in G$) it follows that $y^*(y) = N$. Thus, we have for all $y \in Y$ if $y_j^*(y) = 0$, for $j = 1, 2, \ldots, k$, then $y^*(y) = 0$, in other words, the intersection of the null spaces of the y_j^*, $j = 1, 2, \ldots, k$, is a subset of the null space of y^*. This means that y^* is a linear combination of the y_j^*, $j = 1, 2, \ldots, k$, say $y^* = \sum_{j=1}^k a_j y_j^*$. This also implies that $N = y^*(y) = \sum_{j=1}^k a_j y_j^*(y) = \sum_{j=1}^k a_j c_j$, for $y \in G$.

Thus, by our assumption (H)

$$N = \frac{N}{\|y^*\|} = \frac{\sum_{j=1}^k a_j c_j}{\left\| \sum_{j=1}^k a_j y_j^* \right\|} \le M,$$

which proves our claim (2.8) and finishes the proof of the Lemma.

□

Proof of Theorem 2.6.1. "(i)⇒(ii)"
Claim:Assume that X is not reflexive and that $\theta \in (0,1)$. Thn there is a functional $x^{***} \in X^{***}$ that $\|x^{***}\| = 1$, $x^{***}|_X \equiv 0$ and $x^{***}(x^{**}) > \theta$ for some $x^{**} \in X^{**}$, with $\|x^{**}\| < 1$

Indeed, by Proposition 2.4.3

$$(X^{**}/\xi(X))^* = \chi(X)^\perp = \{x^{***} : x^{***}|_{\chi(X)} \equiv 0\}.$$

Since X is not reflexive, we pick $z^{**} \in X^{**}$ so that

$$\|z^{**} + \chi(X)\|_{X^{**}/\chi(X)} = \inf_{y^{**} \in z^* * + \chi(X)} \|y^* * \|_{X^* *} = 1.$$

Using Hahn Banach, we find $x^{***} \in S_{\chi(X)^\perp}$ with $x^{***}(z^{**}) = 1$. Choose $\varepsilon > 0$ so that $\frac{1}{(1+\varepsilon)^2} > \theta$, then choose $y^{**} \in z^{**} + \chi(X)$ with $\|y^{**}\| < 1+\varepsilon$ and finally let $x^{**} = y^{**}/(1+\varepsilon)^2$. It follows $\|x^{**}\| < 1$ and $x^{***}(x^{**}) = (1+\varepsilon)^2 > \theta$.

Now we will choose inductively $x_n \in B_X$ and $x_n^* \in B_{X^*}$, $n \in \mathbb{N}$, at each step assuming that the condition (2.6) holds up to n, and additionally, that $x^{**}(x_n^*) = \theta$.

For $n = 1$ we simply choose $x_1^* \in S_{X^*}$ so that $x^{**}(x_1^*) = \theta$ and then we choose $x_1 \in B_X$ so that $x_1^*(x_1) = \theta$. Assuming we have chosen $x_1, x_2, \ldots, x_n$ and $x_1^*, x_2^*, \ldots, x_n^*$ so that

$$x_j^*(x_i) = \begin{cases} \theta & \text{if } j \le i \le n, \text{ and} \\ 0 & \text{if } i < j \le n. \end{cases} \tag{2.10}$$

Since $x^{***}(x_j) = 0$ for $j = 1, 2, \ldots, n$ and $x^{***}(x^{**}) > \theta$, the elements $x_1, x_2, \ldots, x_n, x^{**}$, seen as functionals on X^*, together with the numbers

$0, 0, \ldots, 0, \theta$ and $M = \frac{\theta}{x^{***}(x^{**})} < 1$ satisfy Helly's condition (H). Indeed, for scalars $a_1, \ldots, a_{n+1}$ we have

$$\begin{aligned} |a_{n+1}|\theta &= M|a_{n+1}x^{***}(x^{**})| \\ &= M\Big|x^{***}\Big(\sum_{j=1}^{n} a_j x_j + a_{n+1}x^{**}\Big)\Big| \le M\Big\|\sum_{j=1}^{n} a_j x^{***}(x_j) + a_{n+1}x^{**}\Big\|. \end{aligned}$$

We can therefore choose an $x^*_{n+1} \in X^*$, $\|x^*_{n+1}\| \le 1$ so that $x^*_{n+1}(x_j) = 0$ for all $j = 1, 2, \ldots, n$ and $x^{**}(x^*_{n+1}) = \theta$.

Secondly, we note that the functionals $x^*_1, x^*_2, \ldots, x^*_{n+1}$, the numbers $\theta, \theta, \ldots, \theta$, and the number $M = \|x^{**}\| < 1$ satisfy Helly's condition. Indeed, for scalars $a_1, \ldots, a_{n+1}$ we have

$$\Big|\sum_{j=1}^{n+1} a_j\theta\Big| = \Big|\sum_{j=1}^{n+1} a_j x^{**}(x^*_j)\Big| \le M\Big\|\sum_{j=1}^{n+1} a_j x^*_j\Big\|.$$

We can therefore find $x_{n+1} \in B_X$, so that $x^*_j(x_{n+1}) = \theta$, for all $j = 1, 2, \ldots, n$.

"(ii)⇒(iv)" and "(iii)⇒(v)" Fix a $\theta \in (0, 1)$ for which there are sequences $(x_j) \subset B_X$ and $(x^*_j) \subset B_{X^*}$ for which (2.6) holds. Let $x = \sum_{j=1}^{n} a_j x_j \in \mathrm{conv}(x_1, x_2, \ldots, x_n)$ and $z = \sum_{j=n+1}^{\infty} b_j x_j \in \mathrm{conv}(x_{n+1}, x_{n+2}, \ldots)$ then

$$\|z - x\| \ge x^*_{n+1}(z - x) = x^*_{n+1}(y) = \theta,$$

which implies our claim.

"(iv)⇒(v)" obvious.

"(v)⇒(i)" Assume that for $\theta > 0$ and the sequence $(x_j) \subset B_X$ satisfies (2.7). Now assume that our claim is false and X is reflexive.

Define $C_n = \overline{\mathrm{conv}(x_j : j \ge n+1)}$, for $n \in \mathbb{N}$, then the sets C_n, $n \in \mathbb{N}$, are weakly compact, $C_1 \supset C_2 \supset \ldots$. Thus there is an element $v \in \bigcap_{n\in\mathbb{N}} C_n$. We can approximate v by some $u \in \mathrm{conv}(x_j : j \in \mathbb{N})$, with $\|u - v\| < \theta/2$. There is some n so that $v \in \mathrm{conv}(x_1, \ldots, x_n)$. But now it follows, since $u \in C_{n+1}$, that $\mathrm{dist}\big(\mathrm{conv}(x_1, \ldots, x_n), \mathrm{conv}(x_{n+1}, x_{n+2}, \ldots)\big) \le \|v - u\| < \theta/2$, which is a contradiction and finishes the proof.

□

2.7 The Principle of Local Reflexivity

In this section we proof a result by J. Lindenstrauss and H. Rosenthal [LR] which states that for a Banach space X the finite dimensional subspaces of the bidual X^{**} are *in a certain sense have "similar" finite dimensional subspaces of X.*

Theorem 2.7.1. [LR] [The Principle of Local Reflexivity]
Let X be a Banach space and let $F \subset X^{**}$ and $G \subset X^*$ be finite dimensional subspaces of X^{**} and X^* respectively.

Then, given $\varepsilon > 0$, there is a subspace E of X containing $F \cap X$ (we identify X with its image under the canonical embedding) with $\dim E = \dim F$ and an isomorphism $T : F \to E$ with $\|T\| \cdot \|T^{-1}\| \leq 1 + \varepsilon$ such that

$$T(x) = x \text{ if } x \in F \cap X \text{ and} \tag{2.11}$$

$$\langle x^*, T(x^{**})\rangle = \langle x^{**}, x^*\rangle \text{ if } x^* \in G, x^{**} \in F. \tag{2.12}$$

We need several Lemmas before we can prove Theorem 2.7.1. The first one is a corollary of the Geometric Hahn-Banach Theorem

Proposition 2.7.2. (Variation of the Geometric Version of the Theorem of Hahn Banach)
Assume that X is a Banach space and $C \subset X$ is convex with $C^\circ \neq \emptyset$ and let $x \in X \setminus C$ (so x could be in the boundary of C). Then there exists an $x^* \in X^*$ so that

$$\Re\langle x^*, z\rangle < \langle x^*, x\rangle \text{ for all } z \in C^0,$$

and, if moreover C is absolutely convex (i.e. if $\rho x \in C$ for all $x \in C$ and $\rho \in \mathbb{K}$, with $|\rho| \leq 1$), then

$$|\langle x^*, z\rangle| < 1 = \langle x^*, x\rangle \text{ for all } z \in C^0.$$

Lemma 2.7.3. *Assume $T : X \to Y$ is a bounded linear operator between the Banach spaces X and Y and assume that $T(X)$ is closed.*

Suppose that for some $y \in Y$ there is an $x^{**} \in X^{**}$ with $\|x^{**}\| < 1$, so that $T^{**}(x^{**}) = y$. Then there is an $x \in X$, with $\|x\| < 1$ so that $T(x) = y$.

Proof. We first show that there is an $x \in X$ so that $T(x) = y$. Assume this where not true, then we could find by the Hahn-Banach Theorem (Corollary 1.4.5) an element $y^* \in Y^*$, so that $y^*(z) = 0$, for all $z \in T(X)$ and $\langle y^*, y\rangle = 1$ ($T(X)$ is closed). But this yields $\langle T^*(y^*), x\rangle = \langle y^*, T(x)\rangle = 0$, for all $x \in X$, and, thus, $T^*(y^*) = 0$. Thus

$$0 = \langle x^{**}, T^*(y^*)\rangle = \langle T^{**}(x^{**}), y^*\rangle = \langle y, y^*\rangle = 1,$$

which is a contradiction.

Secondly, assume that $y \in T(X) \setminus T(B_X^\circ)$. Since T is surjective onto its (closed) image $Z = T(X)$ it follows from the Open Mapping Theorem that $T(B_X^\circ)$ is open in Z, and we can use variation of the geometric version of the Hahn-Banach Theorem, Proposition (2.7.2), and chose $z^* \in Z^*$, so that $\langle z^*, T(x)\rangle < 1 = \langle z^*, y\rangle$ for all $x \in B_X^\circ$. Again by the Theorem of Hahn-Banach (Corollary 1.4.4) we can extend z^* to an element y^* in Y^*. It follows that

$$\|T^*(y^*)\| = \sup_{x\in B_X^\circ} \langle T^*(y^*), x\rangle = \sup_{x\in B_X^\circ} \langle z^*, T(x)\rangle \le 1,$$

and thus, since $\|x^{**}\| < 1$, it follows that

$$|\langle y^*, y\rangle| = |\langle y^*, T^{**}(x^{**})\rangle| = |\langle x^{**}, T^*(y^*)\rangle| < 1,$$

which is a contradiction. □

Lemma 2.7.4. *Let $T : X \to Y$ be a bounded linear operator between two Banach spaces X and Y with closed range, and assume that $F : X \to Y$ has finite rank.*

Then $T + F$ also has closed range.

Proof. Assume the claim is not true. Put $S = T + K$ and consider the map

$$\overline{S} : X/\mathcal{N}(S) \to Y, \quad x + \mathcal{N}(S) \to S(x)$$

which is a well defined linear bounded Operator, and which by Proposition 1.3.11 cannot be an isomorphism onto its image.

Therefore we can choose sequence $(\overline{z_n})$ in $X/\mathcal{N}(S)$, with $\|\overline{z}_n\| = 1$ and $x_n \in \overline{z_n}$, with $1 \le \|x_n\| \le 2$, for $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} \overline{S}(\overline{z_n}) = \lim_{n\to\infty} S(x_n) = 0 \text{ and } \operatorname{dist}(x_n, \mathcal{N}(S)) \ge 1.$$

Since the sequence $(F(x_n) : n \in \mathbb{N})$ is a bounded sequence in a finite dimensional space, we can, after passing to a subsequence, assume that $(F(x_n) : n\in\mathbb{N})$ converges to some $y \in Y$ and, hence,

$$\lim_{n\to\infty} T(x_n) = -y.$$

Since T has closed range there is an $x \in X$, so that $T(x) = -y$. Using again the equivalences in Proposition 1.3.11 and the fact that $T(x_n) \to -y = T(x)$, if $n \nearrow \infty$, it follows for some constant $C > 0$ that

$$\lim_{n\to\infty} \operatorname{dist}\big(x - x_n, \mathcal{N}(T)\big) \le \lim_{n\to\infty} C\big\|T(x - x_n)\big\| = 0,$$

and, thus,

$$y - F(x) = \lim_{n\to\infty} F(x_n) - F(x) \in F(\mathcal{N}(T)),$$

so we can write $y - F(x)$ as

$$y - F(x) = F(u), \text{ where } u \in \mathcal{N}(T).$$

Thus

$$\lim_{n\to\infty} \operatorname{dist}(x_n - x - u, \mathcal{N}(T)) = 0 \text{ and } \lim_{n\to\infty} \|F(x_n) - F(x) - F(u)\| = 0.$$

$F|_{\mathcal{N}(T)}$ has also closed range, Proposition 1.3.11 yields (C being some positive constant)

$$\limsup_{n\to\infty} \operatorname{dist}(x_n - x - u, \mathcal{N}(F) \cap \mathcal{N}(T)) \le \limsup_{n\to\infty} C\|F(x_n) - F(x) - F(u)\| = 0.$$

Since $T(x+u) = -y = -F(x+u)$ (by choice of u), and thus $(T+F)(x+u) = 0$ which means that $x + u \in \mathcal{N}(T + F)$. Therefore

$$\begin{aligned}\limsup_{n\to\infty} \operatorname{dist}(x_n, \mathcal{N}(T + F)) &= \limsup_{n\to\infty} \operatorname{dist}(x_n - x - u, \mathcal{N}(T + F)) \\ &\le \limsup_{n\to\infty} \operatorname{dist}(x_n - x - u, \mathcal{N}(T) \cap \mathcal{N}(F)) = 0.\end{aligned}$$

But this contradicts our assumption on the sequence (x_n). □

Lemma 2.7.5. *Let X be a Banach space, $A = (a_{i,j})_{i\le m, j\le n}$ an m be n matrix and $B = (b_{i,j})_{i\le p, j\le n}$ a p by n matrix, and assume that B has only real entries (even if $\mathbb{K} = \mathbb{C}$).*

Suppose that $y_1, \ldots, y_m \in X$, $y_1^*, \ldots, y_p^* \in X^*$, $\xi_1, \ldots, \xi_p \in \mathbb{R}$, and $x_1^{**}, \ldots, x_n^{**} \in B^\circ_{X^{**}}$ satisfy the following equations:

$$\sum_{j=1}^n a_{i,j} x_j^{**} = y_i, \text{ for all } i = 1, 2, \ldots, m, \text{ and} \tag{2.13}$$

$$\Big\langle y_i^*, \sum_{j=1}^n b_{i,j} x_j^{**} \Big\rangle = \xi_i, \text{ for all } i = 1, 2, \ldots, p. \tag{2.14}$$

Then there are vectors $x_1, \ldots, x_n \in B^\circ_X$ satisfying:

$$\sum_{j=1}^n a_{i,j} x_j = y_i, \text{ for all } i = 1, 2, \ldots, m, \text{ and} \tag{2.15}$$

$$\Big\langle y_i^*, \sum_{j=1}^n b_{i,j} x_j \Big\rangle = \xi_i, \text{ for all } i = 1, 2, \ldots, p. \tag{2.16}$$

Proof. Recall from Linear Algebra that we can write the matrix A as a product $A = U \circ P \circ V$, where U and V are invertible and P is of the form

$$P = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where r is the rank of A and I_r the identity on $\mathbb{K}^r$.

For a general s by t matrix $C = (c_{i,j})_{i\le s, j\le t}$ consider the operator

$$T_C : \ell_\infty^t(X) \to \ell_\infty^s(X), \quad (x_1, x_2, \ldots, x_t) \mapsto \Big(\sum_{j=1}^t c_{i,j} x_j : i = 1, 2, \ldots, m\Big).$$

If $s = t$ and if C is invertible then T_C is an isomorphism. Also if $C^{(1)}$ and $C^{(2)}$ are two matrices so that the number of columns of $C^{(1)}$ is equal to the number of rows of $C^{(2)}$ one easily computes that $T_{C^{(1)}\circ C^{(2)}} = T_{C^{(1)}} \circ T_{C^{(2)}}$. Secondly it is clear that T_P is a closed operator (P defined as above), since T_P is simply the projection onto the first r coordinates in $\ell_\infty^n(X)$.

It follows therefore that $T_A = T_U \circ T_P \circ T_V$ is an operator with closed range. Secondly define the operator

$$\begin{aligned} &S_A : \ell_\infty^n(X) \to \ell_\infty^m(X) \oplus \ell_\infty^p, \\ &(x_1, \ldots x_n) \mapsto \left(T_A(x_1, \ldots x_n), \Big(\Big\langle y_i^*, \sum_{j=1}^n b_{i,j} x_j \Big\rangle\Big)_{i=1}^p \right). \end{aligned}$$

S_A can be written as the sum of T_A and a finite rank operator and has therefore also closed range by Lemma 2.7.4.

Since the second adjoint of S_A^{**} is the operator

$$\begin{aligned} &S_A^{**} : \ell_\infty^n(X^{**}) \to \ell_\infty^m(X^{**}) \oplus \ell_\infty^p, \\ &(x_1^{**}, \ldots x_n^{**}) \mapsto \left(T_A^{**}(x_1^{**}, \ldots x_n^{**}), \Big(\Big\langle y_i^*, \sum_{j=1}^n b_{i,j} x_j^{**} \Big\rangle\Big)_{i=1}^p \right) \end{aligned}$$

with

$$T_A^{**} : \ell_\infty^n(X^{**}) \to \ell_\infty^m(X^{**}),\ (x_1^{**}, x_2^{**}, \ldots x_n^{**}) \mapsto \Big(\sum_{j=1}^t a_{i,j} x_j^{**} : i = 1, 2, \ldots, m\Big),$$

our claim follows from Lemma 2.7.3. □

Lemma 2.7.6. *Let E be a finite dimensional space and $(x_i)_{i=1}^N$ is an ε-net of S_E for some $0<\varepsilon<1/3$. If $T:E\to E$ is a linear map so that*

$$(1-\varepsilon)\le \|T(x_j)\|\le (1+\varepsilon),\ \text{for all } j=1,2,\dots N.$$

Then

$$\frac{1-3\varepsilon}{1-\varepsilon}\|x\|\le \|T(x)\|\le \frac{1+\varepsilon}{1-\varepsilon}\|x\|,\ \text{for all } x\in E,$$

and thus

$$\|T\|\cdot\|T^{-1}\|\le \frac{(1+\varepsilon)^2}{(1-\varepsilon)(1-3\varepsilon)}.$$

We are now ready to proof Theorem 2.7.1.

Proof of Theorem 2.7.1. Let $F\subset X^{**}$ and $G\subset X^*$ be finite dimensional subspaces, and let $0<\varepsilon<1$. Choose $\delta>0$, so that $\frac{(1+\delta)^2}{(1-\delta)(1-3\delta)}<\varepsilon$, and a δ-net $(x_j^{**})_{j=1}^N$ of S_F. It can be shown that $(x_j^{**})_{j=1}^N$ span all of F, but we can also simply assume without loss of generality, that it does, since we can add a basis of F.

Let

$$S:\mathbb{R}^N\to F,\qquad (\xi_1,\xi_2,\dots,\xi_N)\mapsto \sum_{j=1}^N \xi_j x_j^{**},$$

and note that S is surjective.

Put $H=S^{-1}(F\cap X)$, and let $(a^{(i)}:i=1,2,\dots,m)$ be a basis of H, write $a^{(i)}$ as $a^{(i)}=(a_{i,1},a_{i,2},\dots a_{i,N})$, and define A to be the m by N matrix $A=(a_{i,j})_{i\le m,j\le N}$. For $i=1,2,\dots,m$ put

$$y_i=S(a^{(i)})=\sum_{j=1}^N a_{i,j}x_j^{**}\in F\cap X,$$

choose $x_1^*,x_2^*,\dots,x_N^*\in S_{X^*}$ so that $\langle x_j^{**},x_j^*\rangle>1-\delta$, and pick a basis $\{g_1^*,g_2^*,\dots g_\ell^*\}$ of G.

Consider the following system of equations in N unknowns z_1^{**}, z_2^{**}, $\dots,z_N^{**}$ in X^{**}:

$$\begin{aligned}
&\sum_{j=1}^N a_{i,j}z_j^{**}=y_i \text{ for } i=1,2,\dots,m\\
&\langle z_j^{**},x_j^*\rangle=\langle x_j^{**},x_j^*\rangle \text{ for } j=1,2,\dots,N \text{ and}\\
&\langle z_j^{**},g_k\rangle=\langle x_j^{**},g_k^*\rangle \text{ for } j=1,2,\dots,N \text{ and } k=1,2,\dots,\ell.
\end{aligned}$$

By construction $z_j^{**} = x_j^{**}$, $j = 1, 2, \dots, N$, is a solution to these equations. Since $\|x_j^{**}\| = 1 < 1 + \delta$, for $j = 1, 2, \dots, N$, we can use Lemma 2.7.5 and find $x_1, x_2, \dots x_N \in X$, with $\|x_j\| = 1 < 1 + \delta$, for $j = 1, 2, \dots, N$, which solve above equations.

Define

$$S_1 : \mathbb{R}^N \to X, \qquad (\xi_1, \xi_2, \dots, \xi_N) \mapsto \sum_{j=1}^N \xi_j x_j.$$

We claim that the null space of S is contained in the null space of S_1. Indeed if we assumed that $\xi \in \mathbb{K}^N$, and $\sum_{j=1}^N \xi_j x_j = 0$, but $\sum_{j=1}^N x_j^{**} \neq 0$, then, Lemma 2.7.6 (consider the operator $F \to \mathbb{R}^N$, $x^{**} \mapsto \langle x^{**}, x_j^* \rangle$) there is an $i \in \{1, 2, \dots N\}$ so that

$$\Big\langle x_i^*, \sum_{j=1}^N x_j^{**} \Big\rangle \neq 0,$$

but since $\langle x_j^{**}, x_i^* \rangle = \langle x_j, x_i^* \rangle$ this is a contradiction.

It follows therefore that we can find a linear map $T : F \to X$ so that $S_1 = TS$. Denoting the standard basis of $\mathbb{R}^N$ by $(e_i)_{i \le N}$ we deduce that $x_i = S_1(e_i) = T \circ S(e_i) = T(x_i^{**})$, and thus

$$1 + \delta > \|x_i\| = \|T(x_i^{**})\| \ge |\langle x_i^*, x_i \rangle| = \langle x_i^{**}, x_i^* \rangle| > 1 - \delta.$$

By Lemma 2.7.6 and the choice of δ it follows therefore that $\|T\| \cdot \|T^{-1}\| \le 1 + \varepsilon$.

Note that for $\xi \in H = S^{-1}(F \cap X)$, say $\xi = \sum_{i=1}^m \beta_i a^{(i)}$, we compute

$$\begin{aligned} S_1(\xi) &= \sum_{i=1}^m \beta_i S_1(a^{(i)}) = \sum_{i=1}^m \beta_i \sum_{j=1}^N a_{i,j} x_j \\ &= \sum_{i=1}^m \beta_i \sum_{j=1}^N a_{i,j} x_j^{**} = \sum_{i=1}^m \beta_i S(a^{(i)}) = S(\xi). \end{aligned}$$

We deduce therefore for $x \in F \cap X$, that $T(x) = x$.

Finally from the third part of the system of equations it follows, that

$$\langle x^*, T(x_j^{**}) \rangle = \langle x^*, x_j \rangle = \langle x^*, x_j \rangle, \text{ for all } j = 1, 2, \dots, N \text{ and } x^* \in G\ ,$$

and, thus (since the x_j^{**} span all of F), that

$$\langle x^*, T(x^{**}) \rangle = \langle x^{**}, x^* \rangle, \text{ for all } x^{**} \in F \text{ and } x^* \in G.$$

$\square$

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Chapter 3

Bases in Banach Spaces

Like every vectorspace a Banach space X has an *algebraic* or *Hamel basis*, i.e. a subset $B \subset X$, so that every $x \in X$ is in a unique way the (finite) linear combination of elements in B. This definition does not take into account that we can take infinite sums in Banach spaces and that we might want to represent elements $x \in X$ as converging series (with possibly infinite non zero elements). Hamel bases are also not very useful for Banach spaces, since the coordinate functionals might not be continuous.

3.1 Schauder Bases

Definition 3.1.1. (Schauder bases of Banach Spaces)
Let X be an infinite dimensional Banach space. A sequence $(e_n) \subset X$ is called *Schauder basis of* X, or simply a *basis of* X, if for every $x \in X$, there is a unique sequence of scalars $(a_n) \subset \mathbb{K}$ so that

$$x = \sum_{n=1}^{\infty} a_n e_n.$$

Examples 3.1.2. For $n \in \mathbb{N}$ let

$$e_n = (\underbrace{0, \ldots 0}_{n-1 \text{ times}}, 1, 0, \ldots) \in \mathbb{K}^{\mathbb{N}}$$

Then (e_n) is a basis of ℓ_p, $1 \leq p < \infty$ and c_0. We call (e_n) *the unit vector basis of* ℓ_p and c_0, respectively.

Remarks. Assume that X is a Banach space and (e_n) a basis of X.

a) (e_n) is linear independent.

b) $\text{span}(e_n : n \in \mathbb{N})$ is dense in X, in particular X is separable.

c) Every element x is uniquely determined by the sequence (a_n) so that $x = \sum_{j=1}^{\infty} a_n e_n$. So we can identify X with a space of sequences in $\mathbb{K}^{\mathbb{N}}$, for which $\sum a_n e_n$ converges in X.

Proposition 3.1.3. *Let X be a normed linear space and assume that $(e_n) \subset X$ has the property that each $x \in X$ can be uniquely represented as a series*

$$x = \sum_{n=1}^{\infty} a_n e_n, \text{ with } (a_n) \subset \mathbb{K}$$

(we could call (e_n) Schauder basis of X *but we want to reserve this term only if X is a Banach space).*

For $n \in \mathbb{N}$ and $x \in X$ define $e_n^*(x) \in \mathbb{K}$ to be the unique element in $\mathbb{K}$, so that

$$x = \sum_{n=1}^{\infty} e_n^*(x) e_n.$$

Then $e_n^* : X \to \mathbb{K}$ is linear.

For $n \in \mathbb{N}$ let

$$P_n : X \to \text{span}(e_j : j \leq n), \quad x \mapsto \sum_{j=1}^{n} e_n^*(x) e_n.$$

Then $P_n : X \to X$ are linear projections onto $\text{span}(e_j : j \leq n)$ *and the following properties hold:*

a) $\dim(P_n(X)) = n$,

b) $P_n \circ P_m = P_m \circ P_n = P_{\min(m,n)}$, *for $m, n \in \mathbb{N}$,*

c) $\lim_{n\to\infty} P_n(x) = x$, *for every $x \in X$.*

Conversely if $(P_n : n \in \mathbb{N})$ is a sequence of linear projections satisfying (a), (b), and (c), and moreover are bounded, and if $e_1 \in P_1(X) \setminus \{0\}$ and $e_n \in P_n(X) \cap \mathcal{N}(P_{n-1})$, with $e_n \neq 0$, if $n > 1$, then each $x \in X$ can be uniquely represented as a series

$$x = \sum_{n=1}^{\infty} a_n e_n, \text{ with } (a_n) \subset \mathbb{K},$$

so in particular (e_n) is a Schauder basis of X in the case that X is a Banach space.

Proof. The linearity of e_n^* follows from the unique representation of every $x \in X$ as $x = \sum_{j=1}^{\infty} e_n^*(x)e_n$, which implies that for x and y in X and $\alpha, \beta \in \mathbb{K}$,

$$\begin{aligned}\alpha x + \beta y &= \lim_{n\to\infty} \alpha \sum_{j=1}^{n} e_j^*(x)e_j + \beta \sum_{j=1}^{n} e_j^*(y)e_j \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} (\alpha e_j^*(x) + \beta e_j^*(y))e_j = \sum_{j=1}^{\infty} (\alpha e_j^*(x) + \beta e_j^*(y))e_j,\end{aligned}$$

and, on the other hand

$$\alpha x + \beta y = \sum_{j=1}^{\infty} e_j^*(\alpha x + \beta y)e_j,$$

thus, by uniqueness, $e_j^*(\alpha x + \beta y) = \alpha e_j^*(x) + \beta e_j^*(y)$, for all $j \in \mathbb{N}$. The conditions (a), (b) and (c) are clear.

Conversely, assume that (P_n) is a sequence of bounded and linear projections satisfying (a), (b), and (c). By (b) $P_{n-1}(X) = P_n \circ P_{n-1}(X) \subset P_n(X)$, for $n \in \mathbb{N}$ (put $P_0 = 0$)and, thus, by (a), the codimension of $P_{n-1}(X)$ inside $P_n(X)$ is 1. So if $e_1 \in P_1(X) \setminus \{0\}$ and $e_n \in P_n(X) \cap \mathcal{N}(P_{n-1})$, if $n > 1$, then for $x \in X$, by (b)

$$\begin{aligned}P_{n-1}(P_n(x) - P_{n-1}(x)) &= P_{n-1}(x) - P_{n-1}(x) = 0, \\ &\text{and thus } P_n(x) - P_{n-1}(x) \in \mathcal{N}(P_{n-1}) \text{ and} \\ P_n(x) - P_{n-1}(x) &= P_n(P_n(x) - P_{n-1}(x)) \in P_n(X),\end{aligned}$$

and therefore $P_n(x) - P_{n-1}(x) \in P_n(X) \cap \mathcal{N}(P_{n-1})$. Thus, we can write $P_n(x) - P_{n-1}(x) = a_n e_n$, for $n \in \mathbb{N}$, and it follows from (c) that (letting $P_0 = 0$)

$$x = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} \sum_{j=1}^{n} P_j(x) - P_{j-1}(x) = \lim_{n\to\infty} \sum_{j=1}^{n} a_j e_j = \sum_{j=1}^{\infty} a_j e_j.$$

In order to show uniqueness of this representation of x assume $x = \sum_{j=1}^{\infty} b_j e_j$. From the continuity of $P_m - P_{m-1}$, for $m \in \mathbb{N}$ it follows that

$$a_m e_m = (P_m - P_{m-1})(x) = \lim_{n\to\infty} (P_m - P_{m-1})\Big(\sum_{j=1}^{n} b_j e_j\Big) = b_m e_m,$$

and thus $a_m = b_m$. $\square$

Definition 3.1.4. (Canonical Projections and Coordinate functionals)
Let X be a normed linear space and assume that (e_i) satisfies the assumptions of Proposition 3.1.3. The linear functionals (e_n^*) as defined in Proposition 3.1.3 are called the *Coordinate Functionals for* (e_n) and the projections P_n, $n \in \mathbb{N}$, are called the *Canonical Projections for* (e_n).

Proposition 3.1.5. *Suppose X is a normed linear space and assume that $(e_n) \subset X$ has the property that each $x \in X$ can be uniquely represented as a series*

$$x = \sum_{n=1}^{\infty} a_n e_n, \text{ with } (a_n) \subset \mathbb{K}.$$

If the canonical projections are bounded, and, moreover, $\sup_{n\in\mathbb{N}} \|P_n\| < \infty$ *(i.e. uniformly the P_n are bounded), then (e_i) is a Schauder basis of its completion $\tilde{X}$.*

Proof. Let $\tilde{P}_n : \tilde{X} \to \tilde{X}$, $n \in \mathbb{N}$, be the unique extensions bounded of P_n. Since P_n has finite dimensional range it follows that $\tilde{P}_n(\tilde{X}) = P_n(X) = \text{span}(e_j : j \le n)$ is finite dimensional and, thus, closed. $(\tilde{P}_n)$ satisfies therefore (a) of Proposition 3.1.3. Since the P_n are continuous, and satisfy the equalities in (b) of Proposition 3.1.3 on a dense subset of $\tilde{X}$, (b) is satisfied on all of $\tilde{X}$. Finally, (c) of Proposition 3.1.3 is satisfied on a dense subset of $\tilde{X}$, and we deduce for $\tilde{x} \in \tilde{X}$, $\tilde{x} = \lim_{k\to\infty} x_k$, with $x_k \in X$, for $k \in \mathbb{N}$, that

$$\|\tilde{x} - \tilde{P}_n(\tilde{x})\| \le \|\tilde{x} - x_k\| + \sup_{j\in\mathbb{N}} \|P_j\| \|\tilde{x} - x_k\| + \|x_k - P_n(x_k)\|$$

and, since (P_n) is uniformly bounded, we can find for given $\varepsilon > 0$, k large enough so that the first two summands do not exceed ε, and then we choose $n \in \mathbb{N}$ large enough so that the third summand is smaller than ε. It follows therefore that also (c) is satisfied on all of $\tilde{X}$. Thus, our claim follows from the second part of Proposition 3.1.3 applied to $\tilde{X}$. $\square$

Our goal is now to show the converse of Proposition 3.1.3, and prove that if (e_n) is a Schauder basis, then the canonical projections are uniformly bounded, and thus that the coordinate functionals are bounded.

Theorem 3.1.6. *Let X be a Banach space with a basis (e_n) and let (e_n^*) be the corresponding coordinate functionals and (P_n) the canonical projections. Then P_n is bounded for every $n \in \mathbb{N}$ and*

$$b = \sup_{n\in\mathbb{N}} ||P_n||_{L(X,X)} < \infty,$$

and thus $e_n^* \in X^*$ *and*

$$\|e_n^*\|_{X^*} = \frac{\|P_n - P_{n-1}\|}{\|e_n\|} \le \frac{2b}{\|e_n\|}.$$

We call b the basis constant of (e_j). *If* $b = 1$ *we say that* (e_i) *is a monotone basis.*

Furthermore there is an equivalent renorming $|\!|\!|\cdot|\!|\!|$ *of* $(X, \|\cdot\|)$ *for which* (e_n) *is a monotone basis for* $(X, |\!|\!|\cdot|\!|\!|)$.

Proof. For $x \in X$ we define

$$|\!|\!|x|\!|\!| = \sup_{n\in\mathbb{N}} \|P_n(x)\|,$$

since $\|x\| = \lim_{n\to\infty} \|P_n(x)\|$, it follows that $\|x\| \le |\!|\!|x|\!|\!| < \infty$ for $x \in X$.

It is clear that $|\!|\!|\cdot|\!|\!|$ is a norm on X. Note that for $n \in \mathbb{N}$

$$\begin{aligned} |\!|\!|P_n|\!|\!| &= \sup_{x\in X, |\!|\!|x|\!|\!|\le 1} |\!|\!|P_n(x)|\!|\!| \\ &= \sup_{x\in X, |\!|\!|x|\!|\!|\le 1} \sup_{m\in\mathbb{N}} \|P_m \circ P_n(x)\| \\ &= \sup_{x\in X, |\!|\!|x|\!|\!|\le 1} \sup_{m\in\mathbb{N}} \|P_{\min(m,n)}(x)\| \le 1. \end{aligned}$$

Thus the projections P_n are uniformly bounded on $(X, |\!|\!|\cdot|\!|\!|)$. Let $\tilde{X}$ be the completion of X with respect to $|\!|\!|\cdot|\!|\!|$, $\tilde{P}_n$, for $n \in \mathbb{N}$, the (unique) extension of P_n to an operator on $\tilde{X}$. We note that the $\tilde{P}_n$ also satisfy the conditions (a), (b) and (c) of Proposition 3.1.3. Indeed (a) and (b) are purely algebraic properties which are satisfied by the first part of Proposition 3.1.3. Moreover for $x \in X$ then

$$\begin{aligned} (3.1) \qquad |\!|\!|x - P_n(x)|\!|\!| &= \sup_{m\in\mathbb{N}} \|P_m(x) - P_{\min(m,n)}(x)\| \\ &= \sup_{m\ge n} \|P_m(x) - P_n(x)\| \to 0 \text{ if } n \to \infty, \end{aligned}$$

which verifies condition (c). Thus, it follows therefore from the second part of Proposition 3.1.3, the above proven fact that $|\!|\!|P_n|\!|\!| \le 1$, for $n \in \mathbb{N}$, and Proposition 3.1.5, that (e_n) is a Schauder basis of the completion of $(X, |\!|\!|\cdot|\!|\!|)$ which we denote by $(\tilde{X}, |\!|\!|\cdot|\!|\!|)$.

We will now show that actually $\tilde{X} = X$, and thus that, $(X, |\!|\!|\cdot|\!|\!|)$ is already complete. Then it would follow from Corollary 1.3.6 of the Closed Graph Theorem that $|\!|\!|\cdot|\!|\!|$ is an equivalent norm, and thus that

$$C = \sup_{n\in\mathbb{N}} \sup_{x\in B_X} \|P_n(x)\| = \sup_{x\in B_X} |\!|\!|x|\!|\!| < \infty.$$

So, let $\tilde{x} \in \tilde{X}$ and write it (uniquely) as $\tilde{x} = \sum_{j=1}^{\infty} a_j e_j$, where this convergence happens in $\|\!|\cdot|\!\|$. Since $\|\cdot\| \leq \|\!|\cdot|\!\|$, and since X is complete the series $\sum_{j=1}^{\infty} a_j e_j$ also converges with respect to $\|\cdot\|$ in X to say $x \in X$.
Important Side Note: This means that the sequence of partial sums $\big(\sum_{j=1}^{n} a_j e_j\big)$ converges in $(X, \|\cdot\|)$ to x, which means that (a_n) is *the unique sequence in $\mathbb{K}$, for which $x = \sum_{j=1}^{\infty} a_j e_j$.* In particular this means that

$$P_n(x) = \sum_{j=1}^{n} a_j e_j = \tilde{P}_n(\tilde{x}), \text{ for all } n \in \mathbb{N}.$$

But now (3.1) yields that $P_n(x)$ also converges in $\|\!|\cdot|\!\|$ to x.

This means (since $(P_n(x)$ cannot converge to two different elements) that $x = \tilde{x}$, which finishes our proof. □

After reading the proof of Theorem 3.1.6 one might ask whether the last part couldn't be generalized and whether the following could be true: If $\|\cdot\|$ and $\|\!|\cdot|\!\|$ are two norms on the same linear space X, so that $\|\cdot\| \leq \|\!|\cdot|\!\|$, and so that $(\|\cdot\|, X)$ is complete, does it then follow that $(X, \|\!|\cdot|\!\|)$ is also complete (and thus $\|\cdot\|$ and $\|\!|\cdot|\!\|$ are equivalent norms). The answer is negative, as the following example shows.

Example 3.1.7. Let $X = \ell_2$ with its usual norm $\|\cdot\|_2$ and let $(b_\gamma : \gamma \in \Gamma) \subset S_{\ell_2}$ be a Hamel basis of Γ (Γ is necessarily uncountable). For $x \in \ell_2$ define $\|\!|x|\!\|$,

$$\|\!|x|\!\| = \sum_{\gamma \in \Gamma} |x_\gamma|,$$

where $x = \sum_{\gamma \in \Gamma} x_\gamma b_\gamma$ is the unique representation of x as a finite linear combination of elements of $(b_\gamma : \gamma \in \Gamma)$. Since $\|b_\gamma\|_2$, for $\gamma \in \Gamma$, it follows for $x = \sum_{\gamma \in \Gamma} x_\gamma b_\gamma \in \ell_2$ from the triangle inequality that

$$\|\!|x|\!\| = \sum_{\gamma \in \Gamma} |x_\gamma| = \sum_{\gamma \in \Gamma} \|x_\gamma b_\gamma\|_2 \geq \Big\| \sum_{\gamma \in \Gamma} x_\gamma b_\gamma \Big\|_2 = \|x\|_2.$$

Finally both norms $\|\cdot\|$ and $\|\!|\cdot|\!\|$, cannot be equivalent. Indeed, for arbitrary $\varepsilon > 0$, there is an uncountable set $\Gamma' \subset \Gamma$, so that $\|b_\gamma - b_{\gamma'}\|_2 < \varepsilon$, $\gamma, \gamma' \in \Gamma'$, ($\Gamma$ is uncountable but S_{ℓ_2} is in the $\|\cdot\|_2$-norm separable). For any two different elements $\gamma, \gamma' \in \Gamma'$ it follows that

$$\|b_\gamma - b_{\gamma'}\| < \varepsilon < 2 = \|\!|b_\gamma - b_{\gamma'}|\!\|.$$

Since $\varepsilon > 0$ was arbitrary this proves that $\|\cdot\|$ and $\|\!|\cdot|\!\|$ cannot be equivalent.

Definition 3.1.8. (Basic Sequences)
Let X be a Banach space. A sequence $(x_n) \subset X\setminus\{0\}$ is called *basic sequence* if it is a basis for $\overline{\text{span}(x_n : n \in \mathbb{N})}$.

If (e_j) and (f_j) are two basic sequences (in possibly two different Banach spaces X and Y). We say that (e_j) and (f_j) are *isomorphically equivalent* if the map

$$T : \text{span}(e_j : j \in \mathbb{N}) \to \text{span}(f_j : j \in \mathbb{N}), \quad \sum_{j=1}^{n} a_j e_j \mapsto \sum_{j=1}^{n} a_j f_j,$$

extends to an isomorphism between the Banach spaces between $\overline{\text{span}(e_j : j \in \mathbb{N})}$ and $\overline{\text{span}(f_j : j \in \mathbb{N})}$.

Note that this is equivalent with saying that there are constants $0 < c \leq C$ so that for any $n \in \mathbb{N}$ and any sequence of scalars $(\lambda_j)_{j=1}^n$ it follows that

$$c\Big\|\sum_{j=1}^{n} \lambda_j e_j\Big\| \leq \Big\|\sum_{j=1}^{n} \lambda_j f_j\Big\| \leq C\Big\|\sum_{j=1}^{n} \lambda_j e_j\Big\|.$$

Proposition 3.1.9. *Let X be Banach space and $(x_n : n \in \mathbb{N}) \subset X \setminus \{0\}$. The (x_n) is a basic sequence if and only if there is a constant $K \geq 1$, so that for all $m < n$ and all scalars $(a_j)_{j=1}^n \subset \mathbb{K}$ we have*

$$\Big\|\sum_{i=1}^{m} a_i x_i\Big\| \leq K\Big\|\sum_{i=1}^{n} a_i x_i\Big\|. \tag{3.2}$$

In that case the basis constant is the smallest of all $K \geq 1$ so that (3.2) *holds.*

Proof. "$\Rightarrow$" Follows from Theorem 3.1.6, since $K := \sup_{n\in\mathbb{N}} \|P_n\| < \infty$ and $P_m\big(\sum_{i=1}^n a_i x_i\big) = \sum_{i=1}^m a_i x_m$, if $m \leq n$ and $(a_i)_{i=1}^n \subset \mathbb{K}$.
"$\Leftarrow$" Assume that there is a constant $K \geq 1$ so that for all $m < n$ and all scalars $(a_j)_{j=1}^n \subset \mathbb{K}$ we have

$$\Big\|\sum_{i=1}^{m} a_i x_i\Big\| \leq K\Big\|\sum_{i=1}^{n} a_i x_i\Big\|.$$

We first note that this implies that (x_n) is linear independent. Indeed, if we assume that $\sum_{j=1}^n a_j x_j = 0$, for some choice of $n \in \mathbb{N}$ and $(a_j)_{j=1}^n \subset \mathbb{K}$, and not all of the a_j are vanishing, we first observe that at least two of $a_j's$ cannot be equal to 0 (since $x_j \neq 0$, for $j \in \mathbb{N}$), thus if we let $m := \min\{j : a_j \neq 0\}$, it follows that $\sum_{j=1}^m a_j x_j \neq 0$, but $\sum_{j=1}^n a_j x_j = 0$, which contradicts our assumption.

It follows therefore that (x_n) is a Hamel basis for (the vector space) $\text{span}(x_j : j \in \mathbb{N})$, which implies that the projections P_n are well defined on $\text{span}(x_j : j \in \mathbb{N})$, and satisfy (a), (b), and (c) of Proposition 3.1.3. Moreover, it follows from our assumption that

$$\|P_m\| = \sup\left\{\Big\|\sum_{j=1}^m a_j x_j\Big\| : n \in \mathbb{N}, (a_j)_{j=1}^n \subset \mathbb{K},\ \Big\|\sum_{j=1}^n a_j x_j\Big\| \leq 1\right\} \leq K.$$

Thus, our claim follows from Proposition 3.1.5.

Also note that the proof of "$\Rightarrow$" implies that the smallest constant so that 3.2 is at most as big as the basis constant, and the proof of "$\Leftarrow$" yielded that it is at least as large as the basis constant. $\square$

Remark. It was for a long time an open problem whether or not every separable Banach space admits a Schauder basis. 1973 this was solved by Enflo [En] in the negative. He constructed the first separable Banach space which does not admit a Schauder basis.

Every separable Hilbert space has a basis (for example an orthogonal basis). Thus, every subspace of a Hilbert space has also a basis. It was shown [Jo] that only Banach space which in some sense are "very close" to a Hilbert space, have the property that each of their subspaces have bases.

3.2 Bases of $C[0,1]$ and $L_p[0,1]$

In the previous section we introduced the unit vector bases of ℓ_p and c_0. Less obvious is it to find bases of function spaces like $C[0,1]$ and $L_p[0,1]$.

3.2.1 The Schauder or Spline basis on $C[01]$

Let $(t_n) \subset [0,1]$ be a dense sequence in $[0,1]$, and assume that $t_1 = 0$, $t_2 = 1$. It follows that

$$\text{(3.3)} \qquad \text{mesh}(t_1, t_2, \ldots t_n) \to 0, \text{ if } n \to \infty, \text{ where}$$
$$\text{mesh}(t_1, t_2, \ldots t_n) = \max_{i=1,2,\ldots n} \big\{|t_i - t_j| : t_j \text{ is neighbor of } t_i\big\}.$$

For $f \in C[0,1]$ we let $P_1(f)$ to be the constant function taking the value $f(0)$, and for $n \geq 2$ we let $P_n(f)$ be the piecewise linear function which interpolates the f at the points $t_1, t_2, \ldots t_n$. More precisely, let $0 = s_1 < s_2 < \ldots s_n = 1$ be the increasing reordering of $\{t_1, t_2, \ldots t_n\}$, then define $P_n(f)$ by

$$P_n(f) : [0,1] \to \mathbb{K}, \text{ with}$$
$$P_n(f)(s) = \frac{s_j - s}{s_j - s_{j-1}} f(s_{j-1}) + \frac{s - s_{j-1}}{s_j - s_{j-1}} f(s_j), \text{ for } s \in [s_{j-1}, s_j].$$

We note that $P_n : C[0,1] \to C[0,1]$ is a linear projection and that $\|P_n\| = 1$, and that (a), (b), (c) of Proposition 3.1.3 are satisfied. Indeed, the image of $P_n(C[0,1]]$ is generated by the functions $f_1 \equiv 1$, $f_2(s) = s$, for $s \in [0,1]$, and for $n \geq 2$, $f_n(s)$ is the functions with the property $f(t_n) = 1$, $f(t_j) = 0$, $j \in \{1, 2, \ldots\} \setminus \{t_n\}$, and is linear between any t_j and the next bigger t_i. Thus $\dim(P_n(C[0,1])) = n$. Property (b) is clear, and property (c) follows from the fact that elements of $C[0,1]$ are uniformly continuous, and condition (3.3).

Also note that for $n > 1$ it follows that $f_n \in P_n(C[0,1]) \cap \mathcal{N}(P_{n-1}) \setminus \{0\}$ and thus it follows from Proposition 3.1.3 that (f_n) is a monotone basis of $C[0,1]$.

3.2.2 The Haar basis of $L_p[0,1]$

Now we define a basis of $L_p[0,1]$, *the Haar basis of* $L_p[0,1]$. Let

$$T = \{(n,j) : n \in \mathbb{N}_0, j = 1, 2, \ldots, 2^n\} \cup \{0\}.$$

We partially order the elements of T as follows

$$(n_1, j_1) < (n_2, j_2) \iff [(j_2 - 1)2^{-n_2}, j_2 2^{-n_2}] \subsetneq [(j_1 - 1)2^{-n_1}, j_1 2^{-n_1}]$$

$$\iff (j_1-1)2^{-n_1} \le (j_2-1)2^{-n_2} < j_2 2^{-n_2} \le j_1 2^{-n_1}, \text{ and } n_1 < n_2$$
$$\text{whenever } (n_1,j_1),(n_2,j_2) \in T$$

and

$$0 < (n,j), \quad \text{whenever } (n,j) \in T \setminus \{0\}$$

Let $1 \le p < \infty$ be fixed. We define *the Haar basis* $(h_t)_{t\in T}$ and the *in L_p normalized Haar basis* $(h_t^{(p)})_{t\in T}$ as follows.

$h_0 = h_0^{(p)} \equiv 1$ on $[0,1]$ and for $n \in \mathbb{N}_0$ and $j = 1,2,\ldots,2^n$ we put

$$h_{(n,j)} = 1_{[(j-1)2^{-n},(j-\frac{1}{2})2^{-n})} - 1_{[(j-\frac{1}{2})2^{-n},j2^{-n})}.$$

and we let

$$\begin{aligned}
\Delta_{(n,j)} &= \mathrm{supp}(h_{(n,j)}) = \big[(j-1)2^{-n}, j2^{-n}\big),\\
\Delta^+_{(n,j)} &= \big[(j-1)2^{-n}, (j-\frac{1}{2})2^{-n}\big)\\
\Delta^-_{(n,j)} &= \big[(j-\frac{1}{2})2^{-n}, j2^{-n}\big).
\end{aligned}$$

We let $h^{(\infty)}_{(n,j)} = h_{(n,j)}$. And for $1 \le p < \infty$

$$h^{(p)}_{(n,j)} = \frac{h_{(n,j)}}{\|h_{(n,j)}\|_p} = 2^{n/p}\big(1_{[(j-1)2^{-n},(j-\frac{1}{2})2^{-n}} - 1_{[(j-\frac{1}{2})2^{-n}),j2^{-n})}\big).$$

Note that $\|h_t\|_p = 1$ for all $t \in T$ and that $\mathrm{supp}(h_t) \subset \mathrm{supp}(h_s)$ if and only if $s \le t$.

Theorem 3.2.1. *If one orders $(h_t^{(p)})_{t\in T}$ linearly in any order compatible with the order on T then $(h_t^{(p)})$ is a monotone basis of $L_p[0,1]$ for all $1 \le p < \infty$.*

Remark. a linear order compatible with the order on T is for example the *lexicographical order*

$$h_0, h_{(0,1)}, h_{(1,1)}, h_{(1,2)}, h_{(2,1)}, h_{(2,2)}, \ldots.$$

Important observation: if $(h_t : t \in T)$ is linearly ordered into $h_0, h_1, \ldots,$ which is compatible with the partial order of T, then the following is true:

If j, n are in $\mathbb{N}$ and $j < n$ then h_j is constant on the support of h_n, thus we obtain:

If $n \in \mathbb{N}$ an if

$$h = \sum_{j=1}^{n-1} a_j h_j,$$

is any linear combination of the first $n-1$ elements, then h is constant on the support of h_n. Moreover, h can be written as a step function

$$h = \sum_{j=1}^{N} b_j 1_{[s_{j-1}, s_j)},$$

with $0 = s_0 < s_1 < \dots s_N$, so that

$$\int_{s_{j-1}}^{s_j} h_n(t)dt = 0.$$

As we will see later, if $1 < p < \infty$, any linear ordering of $(h_t : t \in T)$ is a basis of $L_p[0,1]$, but not necessarily a monotone one.

Proof of Theorem 3.2.1. First note that the indicator functions on all dyadic intervals are in $\text{span}(h_t : t\in T)$. Indeed:

$$1_{[0,1/2)} = \frac{h_0 + h_{(0,1)}}{2},$$

$$1_{(1/2,1]} = \frac{h_0 - h_{(0,1)}}{2},$$

$$1_{[0,1/4)} = \frac{1_{[0,1/2)} - h_{(1,1)}}{2}.$$

Since the indicator functions on all dyadic intervals are dense in $L_p[0,1]$ it follows that $\overline{\text{span}(h_t : t\in T)} = L_p[0,1]$.

Let (h_n) be a linear ordering of $(h_t^{(p)})_{t\in T}$ which is compatible with the ordering of T.

Let $n \in \mathbb{N}$ and let $(a_i)_{i=1}^n$ be a scalar sequence. We need to show that

$$\Big\| \sum_{i=1}^{n-1} a_i h_i \Big\| \le \Big\| \sum_{i=1}^{n} a_i h_i \Big\|.$$

As noted above, on the set $A = \text{supp}(h_n)$ the function $f = \sum_{i=1}^{n-1} a_i h_i$ is constant, say $f(x) = a$, for $x \in A$. Therefore we can write

$$1_A(f + a_n h_n) = 1_{A^+}(a + a_n) + 1_{A^-}(a - a_n),$$

where A^+ is the first half of the interval A and A^- the second half. From the convexity of $[0, \infty) \ni r \mapsto r^p$, we deduce that

$$\frac{1}{2}\big[|a + a_n|^p + |a - a_n|^p\big] \geq |a|^p,$$

and thus

$$\begin{aligned}\int |f + a_n h_n|^p dx &= \int_{A^c} |f|^p dx + \int_A |a + a_n|^p 1_{A^+} + |a - a_n|^p 1_{A^-} dx \\ &= \int_{A^c} |f|^p dx + \frac{1}{2} m(A)\big[|a + a_n|^p + |a - a_n|^p\big] \\ &\geq \int_{A^c} |f|^p dx + m(A)|a|^p = \int |f|^p dx\end{aligned}$$

which implies our claim. □

Proposition 3.2.2. *Since for $1 \leq p < \infty$, and $1 < q \leq \infty$, with $\frac{1}{p} + \frac{1}{q}$ it is easy to see that for $s, t \in T$*

$$\langle h_s^{(p)}, h_t^{(q)} \rangle = \delta(s, t), \tag{3.4}$$

we deduce that $(h_t^{(q)})_{t\in T}$ are the coordinate functionals of $(h_t^{(p)})_{t\in T}$.

3.3 Shrinking, and boundedly complete bases

Proposition 3.3.1. *Let (e_n) be a Schauder basis of a Banach space X, and let (e_n^*) be the coordinate functionals and (P_n) the canonical projections for (e_n).*

Then

a) $P_n^*(x^*) = \sum_{j=1}^{n} \langle x^*, e_j\rangle e_j^* = \sum_{j=1}^{n} \langle \chi(e_j), x^*\rangle e_j^*$, *for* $n \in \mathbb{N}$ *and* $x^* \in X^*$.

b) $x^* = \sigma(X^*, X) - \lim_{n\to\infty} P_n^*(x^*)$, *for* $x^* \in X^*$.

c) (e_n^*) *is a Schauder basis of* $\overline{\mathrm{span}(e_n^* : n \in \mathbb{N})}$ *whose coordinate functionals are* (e_n).

Proof. (a) For $n \in \mathbb{N}$, $x^* \in X^*$ and $x = \sum_{j=1}^{\infty} \langle e_j^*, x\rangle e_j \in X$ it follows that

$$\langle P_n^*(x^*), x\rangle = \langle x^*, P_n(x)\rangle = \Big\langle x^*, \sum_{j=1}^{n} \langle e_j^*, x\rangle e_j \Big\rangle = \Big\langle \sum_{j=1}^{n} \langle x^*, e_j\rangle e_j^*, x \Big\rangle$$

and thus

$$P_n^*(x^*) = \sum_{j=1}^{n} \langle x^*, e_j\rangle e_j^*.$$

(b) For $x \in X$ and $x^* \in X^*$

$$\langle x^*, x\rangle = \lim_{n\to\infty} \langle x^*, P_n x\rangle = \lim_{n\to\infty} \langle P_n^*(x^*), x\rangle.$$

(c) It follows for $m \le n$ and $(a_i)_{i=1}^n \subset \mathbb{K}$, that

$$\begin{aligned} \Big\| \sum_{i=1}^{m} a_i e_i^* \Big\| &= \sup_{x \in B_X} \Big| \sum_{i=1}^{m} a_i \langle e_i^*, x\rangle \Big| \\ &= \sup_{x \in B_X} \Big| \sum_{i=1}^{n} a_i \langle e_i^*, P_m(x)\rangle \Big| \\ &\le \Big\| \sum_{i=1}^{n} a_i e_i^* \Big\| \|P_m\| \le \sup_{j \in \mathbb{N}} \|P_j\| \cdot \Big\| \sum_{i=1}^{n} a_i e_i^* \Big\| \end{aligned}$$

It follows therefore from Proposition 3.1.9 that (e_n^*) is a basic sequence, thus, a basis of $\overline{\mathrm{span}(e_n^*)}$, Since $\langle \chi(e_j), e_i^*\rangle = \langle e_i^*, e_j\rangle = \delta_{i,j}$, it follows that $(\chi(e_n))$ are the coordinate functionals for (e_n^*). $\square$

Remark. If X is a space with basis (e_n) one can identify X with a vector space of sequences $x = (\xi_n) \subset \mathbb{K}$. If (e_n^*) are coordinate functionals for (e_n) we can also identify the subspace $\overline{\mathrm{span}(e_n^* : n\in\mathbb{N})}$ with a vector space of sequences $x^* = (\eta_n) \subset \mathbb{K}$. The way such a sequence $x^* = (\eta_n) \in X^*$ acts on elements in X is via the *infinite scalar product*:

$$\langle x^*, x\rangle = \Big\langle \sum_{n\in\mathbb{N}} \eta_n e_n^*, \sum_{n\in\mathbb{N}} \xi_n e_n \Big\rangle = \sum_{n\in\mathbb{N}} \eta_n \xi_n.$$

We want to address two questions for a basis (e_n) of a Banach space X and its coordinate functionals (e_j^*):

1. Under which conditions does it follow that $X^* = \overline{\mathrm{span}(e_n^*)}$?

2. Under which condition does it follow that the map $J : X \to \overline{\mathrm{span}(e_n^*)}^*$, with
$$J(x)(z^*) = \langle z^*, x\rangle, \text{ for } x\in X \text{ and } z^* \in \overline{\mathrm{span}(e_n^*)},$$
an isomorphy or even an isometry?

We need first the following definition and some observations.

Definition 3.3.2. [Block Bases]
Assume (x_n) is a basic sequence in Banach space X, *a block basis of* (x_n) is a sequence $(z_n) \subset X \setminus \{0\}$, with

$$z_n = \sum_{j=k_{n-1}+1}^{k_n} a_j x_j, \text{ for } n \in \mathbb{N}, \text{ where } 0 = k_0 < k_1 < k_2 < \ldots \text{ and } (a_j) \subset \mathbb{K}.$$

We call (z_n) a *convex block of* (x_n) if the a_j are non negative and $\sum_{j=k_{n-1}+1}^{k_n} a_j = 1$.

Proposition 3.3.3. *The block basis (z_n) of a basic sequence (x_n) is also a basic sequence, and the basis constant of (z_n) is smaller or equal to the basis constant of (x_n).*

Proof. Let K be the basis constant of (x_n), let $m \le n$ in $\mathbb{N}$, and $(b_i)_{i=1}^n \subset \mathbb{K}$. Then

$$\Big\| \sum_{i=1}^m b_i z_i \Big\| = \Big\| \sum_{i=1}^m \sum_{j=k_{i-1}+1}^{k_i} b_i a_j x_j \Big\|$$

$$\le K\Big\|\sum_{i=1}^{n}\sum_{j=k_{i-1}+1}^{k_i} b_i a_j x_j\Big\| = K\Big\|\sum_{i=1}^{n} b_i z_i\Big\|$$

□

Theorem 3.3.4. *For a Banach space with a basis (e_n) and its coordinate functionals (e_n^*) the following are equivalent.*

a) $X^* = \overline{\mathrm{span}(e_n^* : n\in\mathbb{N})}$ *(and, thus, by Proposition 3.3.1, (e_n^*) is a basis of X^* whose canonical projections are P_n^*).*

b) *For every $x^*\in X^*$,*

$$\lim_{n\to\infty}\|x^*|_{\mathrm{span}(e_j:j>n)}\| = \lim_{n\to\infty}\sup_{x\in\mathrm{span}(e_j:j>n),\|x\|\le 1}|\langle x^*,x\rangle| = 0.$$

c) *Every bounded block basis of (e_n) is weakly convergent to* 0.

We call the basis (e_n) shrinking *if these conditions hold .*

Remark. Recall that by Corollary 2.2.6 the condition (c) is equivalent with

c') Every bounded block basis of (e_n) has a further convex block which converges to 0 in norm.

Proof of Theorem 3.3.4. "(a)⇒(b)" Let $x^*\in X^*$ and, using (a), write it as $x^* = \sum_{j=1}^\infty a_j e_j^*$. Then

$$\begin{aligned}\lim_{n\to\infty}\sup_{x\in\mathrm{span}(e_j:j>n)\|x\|\le 1}|\langle x^*,x\rangle| &= \lim_{n\to\infty}\sup_{x\in\mathrm{span}(e_j:j>n),\|x\|\le 1}|\langle x^*,(I-P_n)(x)\rangle|\\ &= \lim_{n\to\infty}\sup_{x\in\mathrm{span}(e_j:j>n),\|x\|\le 1}|\langle (I-P_n^*)(x^*),x\rangle|\\ &\le \lim_{n\to\infty}\|(I-P_n^*)(x^*)\| = 0.\end{aligned}$$

"(b)⇒(c)" Let (z_n) be a bounded block basis of (x_n), say

$$z_n = \sum_{j=k_{n-1}+1}^{k_n} a_j x_j, \text{ for } n\in\mathbb{N}, \text{ with } 0=k_0<k_1<k_2<\ldots \text{ and } (a_j)\subset\mathbb{K}.$$

and $x^*\in X^*$. Then, letting $C=\sup_{j\in\mathbb{N}}\|z_j\|$,

$$|\langle x^*,z_n\rangle| \le \sup_{z\in\mathrm{span}(e_j:j\ge k_{n-1}),\|z\|\le C}|\langle x^*,z\rangle| \to_{n\to\infty} 0, \text{ by condition (b)},$$

thus, (z_n) is weakly null.
"$\neg$(a)$\Rightarrow \neg$(c)" Assume there is an $x^* \in S_{X^*}$, with $x^* \notin \overline{\text{span}(e_j^* : j \in \mathbb{N})}$. It follows for some $0 < \varepsilon \le 1$

$$\varepsilon = \limsup_{n\to\infty} \|x^* - P_n^*(x^*)\| > 0. \tag{3.5}$$

By induction we choose $z_1, z_2, \ldots$ in B_X and $0 = k_0 < k_1 < \ldots$, so that $z_n = \sum_{j=k_{n-1}+1}^{k_n} a_j e_j$, for some choice of $(a_j)_{j=k_{n-1}+1}^{k_n}$ and $|\langle x^*, z_n\rangle| \ge \varepsilon/2(1+K)$, where $K = \sup_{j\in\mathbb{N}} \|P_j\|$. Indeed, let $z_1 \in B_X \cap \text{span}(e_j)$, so that $|\langle x^*, z_1\rangle| \ge \varepsilon/2(1+K)$ and let $k_1 = \min\{k : z_1 \in \text{span}(e_j : j \le k)$. Assuming $z_1, z_2, \ldots z_n$ and $k_1 < k_2 < \ldots k_n$ has been chosen. Using (3.5) we can choose $m > k_n$ so that $\|x^* - P_m^*(x^*)\| > \varepsilon/2$ and then we let $\tilde{z}_{n+1} \in B_X \cap \text{span}(e_i : i \in \mathbb{N})$ with

$$|\langle x^* - P_m^*(x^*), \tilde{z}_{n+1}\rangle| = |\langle x^*, \tilde{z}_{n+1} - P_m(\tilde{z}_{n+1})\rangle| > \varepsilon/2.$$

Finally choose

$$z_{n+1} = \frac{\tilde{z}_{n+1} - P_m(\tilde{z}_{n+1})}{1+K} \in B_X$$

and

$$k_{n+1} = \min\big\{k : z_{n+1} \in \text{span}(e_j : j \le k)\big\}.$$

It follows that (z_n) is a bounded block basis of (e_n) which is not weakly null. $\square$

Examples 3.3.5. Note that the unit vector bases of ℓ_p, $1 < p < \infty$, and c_0 are shrinking. But the unit vector basis of ℓ_1 is not shrinking (consider $(1,1,1,1,1,1\ldots) \in \ell_1^* = \ell_\infty$).

Proposition 3.3.6. *Let (e_j) be a shrinking basis for a Banach space X and (e_j^*) its coordinate functionals. Put*

$$Y = \Big\{(a_i) \subset \mathbb{K} : \sup_n \Big\| \sum_{j=1}^n a_j e_j \Big\| < \infty\Big\}.$$

Then Y with the norm

$$|||(a_i)||| = \sup_{n\in\mathbb{N}} \Big\| \sum_{j=1}^n a_j e_j \Big\|,$$

is a Banach space and

$$T : X^{**} \to Y, \quad x^{**} \mapsto (\langle x^{**}, e_j^*\rangle)_{j\in\mathbb{N}},$$

is well defined and an isomorphism between X^{**} and Y.

If (e_n) is monotone then T is an isometry.

Remark. Note that if $a_j = 1$, for $j \in \mathbb{N}$, then in c_0

$$\sup_{n\in\mathbb{N}} \Big\| \sum_{j=1}^{n} a_j e_j \Big\|_{c_0} = 1,$$

but the series $\sum_{j\in\mathbb{N}} a_j e_j$ does not converge in c_0.

Considering X as a subspace of X^{**} (via the canonical embedding) the image of X under T is the space of sequences

$$Z := \Big\{ (a_i) \in Y : \sum_{j=1}^{\infty} a_j e_j \text{ converges in } X \Big\}.$$

Proof of Proposition 3.3.6. Let K denote the basis constant of (e_n), (e_n^*) the coordinate functionals, and (P_n) the canonical projections. It is straightforward to check that Y is a vector space and that $|||\cdot|||$ is a norm on Y.

For $x^* \in X^*$ and $x^{**} \in X^{**}$ we have by Proposition 3.3.1

$$P_n^*(x^*) = \sum_{j=1}^{n} \langle x^*, e_j \rangle e_j^* \text{ and}$$

$$\langle P_n^{**}(x^{**}), x^* \rangle = \Big\langle x^{**}, \sum_{j=1}^{n} \langle x^*, e_j \rangle e_j^* \Big\rangle = \sum_{j=1}^{n} \langle x^{**}, e_j^* \rangle \langle x^*, e_j \rangle = \Big\langle x^*, \sum_{j=1}^{n} \langle x^{**}, e_j^* \rangle e_j \Big\rangle,$$

which implies that

$$(3.6) \qquad |||T(x^{**})||| = \sup_{n\in\mathbb{N}} \Big\| \sum_{j=1}^{n} \langle x^{**}, e_j^* \rangle e_j \Big\| = \sup_{n\in\mathbb{N}} \|P_n^{**}(x^{**})\| \le K \|x^{**}\|.$$

Thus T is bounded and $\|T\| \le K$.

Assume that $(a_n) \in Y$. We want to find $x^{**} \in X^{**}$, so that $T(x^{**}) = (a_n)$. Put

$$x_n^{**} = \sum_{j=1}^{n} a_j e_j, \text{ for } n \in \mathbb{N}.$$

(where we identify X with its canonical image in X^{**} and, thus, e_j with $\chi(e_j) \in X^{**}$) Since

$$\|x_n^{**}\|_{X^{**}} = \Big\| \sum_{j=1}^{n} a_j e_j \Big\|_X \le |||(a_i)|||, \text{ for all } n \in \mathbb{N},$$

and since X^* is separable (and thus $(B_{X^{**}}, \sigma(X^{**}, X^*))$ is metrizable) $(x_n^{**}$ has a w^*-converging subsequence $x_{n_j}^{**}$ to an element x^{**} with

$$\|x^{**}\| \leq \limsup_{n\to\infty} \|x_n^{**}\| \leq |||(a_j)|||.$$

It follows for $m \in \mathbb{N}$ that

$$\langle x^{**}, e_m^* \rangle = \lim_{j\to\infty} \langle x_{n_j}^{**}, e_m^* \rangle = a_m,$$

and thus it follows that $T(x^{**}) = (a_j)$, and thus that T is surjective.

Finally, since (e_n^*) is a basis for X^* it follows for any x^{**}

$$\begin{aligned}
|||T(x^{**})||| &= \sup_{n\in\mathbb{N}} \Big\| \sum_{j=1}^n \langle x^{**}, e_j^* \rangle e_j \Big\| \\
&= \sup_{n\in\mathbb{N}, x^*\in B_{X^*}} \Big| \sum_{j=1}^n \langle x^{**}, e_j^* \rangle \langle x^*, e_j \rangle \Big| \\
&= \sup_{x^*\in B_{X^*}} \sup_{n\in\mathbb{N}} \big\langle x^{**}, P_n^*(x^*) \big\rangle \geq \|x^{**}\| \text{ (since } P_n^*(x^*) \to x^* \text{ if } n \to \infty),
\end{aligned}$$

which proves that T is an isomorphism, and, that $|||T(x^*)||| \geq \|x^{**}\|$, for $x^{**} \in X^{**}$. Together with (3.6) that shows T is an isometry if $K = 1$. $\square$

Now we want to discuss the"dual problem". Let (e_j) be the basis of a Banach space X and $(e_j^*)_{j=1}^\infty$ its coordinate functionals. Let $Z = \overline{\text{span}(e_j^* : j \in \mathbb{N})} \subset X^*$. Consider the Operator:

$$S : X \to Z^*, \quad x \mapsto \chi(x)|_Z \quad (\textit{ i.e. } T(x)(z) = z(x), \text{ for } z \in Z.$$

Question: Under which conditions is S an onto isomorphism?

We first show that it is always an isomorphic embedding:

Lemma 3.3.7. *Let X be a Banach space with a basis (e_n), with basis constant K and let (e_n^*) be its coordinate functionals. Let $Z = \overline{\text{span}(e_n^* : n\in\mathbb{N})} \subset X^*$ and define the operator*

$$S : X \to Z^*, \quad x \mapsto \chi(x)|_Z \quad \textit{i.e. } S(x)(z) = \langle z, x \rangle, \textit{ for } z \in Z \textit{ and } x \in X.$$

Then S is an isomorphic embedding of X into Z^* and for all $x \in X$.

$$\frac{1}{K}\|x\| \leq \|S(x)\| \leq \|x\|.$$

Moreover, the sequence $(S(e_n)) \subset Z^*$ are the coordinate functionals of (e_n^*) (which by Proposition 3.3.1 is a basis of Z).

Proof. For $x \in X$ note that

$$\|S(x)\| = \sup_{z\in Z,\|z\|_{X^*}\le 1} |\langle z, x\rangle| \le \sup_{x^*\in B_{X^*}} |\langle x^*, x\rangle| = \|x\|,$$

By Corollary 1.4.6 of the Hahn Banach Theorem.

On the other hand, again by using that Corollary of the Hahn Banach Theorem, we deduce that

$$\begin{aligned}
\|x\| &= \sup_{w^*\in B_{X^*}} |\langle w^*, x\rangle| \\
&= \sup_{w^*\in B_{X^*}} \lim_{n\to\infty} |\langle w^*, P_n(x)\rangle| \\
&= \sup_{w^*\in B_{X^*}} \lim_{n\to\infty} |\langle P_n^*(w^*), x\rangle| \\
&\le \sup_{n\in\mathbb{N}} \sup_{w^*\in B_{X^*}} |\langle P_n^*(w^*), x\rangle \\
&\le \sup_{n\in\mathbb{N}} \sup_{z\in \mathrm{span}(e_j^*:j\le n),\|z\|\le K} |\langle z, x\rangle = K\|S(x)\|.
\end{aligned}$$

□

Theorem 3.3.8. *Let X be a Banach space with a basis (e_n), and let (e_n^*) be its coordinate functionals. Let $Z = \overline{\mathrm{span}(e_n^* : n\in\mathbb{N})} \subset X^*$. Then the following are equivalent*

a) X is isomorphic to Z^*, via the map S as defined in Lemma 3.3.7

b) (e_n^*) is a shrinking basis of Z.

c) If $(a_j) \subset \mathbb{K}$, with the property that

$$\sup_{n\in\mathbb{N}} \Big\| \sum_{j=1}^{n} a_j e_j \Big\| < \infty,$$

then $\sum_{j=1}^{\infty} a_j e_j$ converges.

In that case we call (e_n) boundedly complete.

Proof. "(a)⇒(b)" Assuming condition (a) we will verify condition (b) of Theorem 3.3.4 for Z and its basis (e_n^*). So let $z^* \in Z^*$. By (a) we can write $z^* = S(x)$ for some $x \in X$. Since $x = \lim_{n\to\infty} P_n(x)$, where (P_n) are the canonical projection for (e_n), we deduce that

$$\sup_{w\in\mathrm{span}(e_j^*:j>n),\|w\|\le 1} \langle z^*, w\rangle = \sup_{w\in\mathrm{span}(e_j^*:j>n),\|w\|\le 1} \langle S(x), w\rangle$$

$$\begin{aligned}
&= \sup_{w\in\mathrm{span}(e_j^*:j>n),\|w\|\le 1} \langle w, x\rangle \\
&= \sup_{w\in\mathrm{span}(e_j^*:j>n),\|w\|\le 1} \langle w, (I-P_n)(x)\rangle \\
&\le \|(I-P_n)(x)\| \to_{n\to\infty} 0.
\end{aligned}$$

It follows now from Theorem 3.3.4 that (e_j^*) is a shrinking basis of Z.
"(b)⇒(c)" Assume (b) and let $(a_j)\subset\mathbb{K}$ so that

$$|||(a_j)||| = \sup_{n\in\mathbb{N}}\Big\|\sum_{j=1}^n a_j e_j\Big\| = \sup_{n\in\mathbb{N}}\Big\|\sum_{j=1}^n a_j\chi(e_j)\Big\| < \infty.$$

The sequence $(x_n^{**})\subset X^{**}$, with $x_n^{**}=\sum_{j=1}^n a_j\chi(e_j)$, is bounded in X^{**} and must therefore have an $\sigma(X^{**},X^*)$-converging subnet whose limit we denote by x^{**}. It follows that $a_j=\langle x^{**},e_j^*\rangle$, for all $j\in\mathbb{N}$.

Let z^* be the restriction of x^{**} to the space Z (which is a subspace of X^*). Since by assumption (e_j^*) is a shrinking basis of Z and since by Lemma 3.3.7 $(S(e_j))_{j\in\mathbb{N}}$ are the coordinate functionals we can write z^* in a unique way as

$$z^* = \sum_{j=1}^\infty b_j S(e_j).$$

But this means that $a_j=\langle x^{**},e_j^*\rangle=\langle z^*,e_j^*\rangle=b_j$, for all $j\in\mathbb{N}$ and since S is an isomorphism between X and its image it follows that $\sum_{j=1}^\infty a_je_j$ converges in norm in X.
"(c)⇒ (a)" By Lemma 3.3.7 it is left to show that the operator S is surjective. Thus, let $z^*\in Z^*$. Since (e_n^*) is a basis of Z and $(S(e_n))\subset Z^*$ are the coordinate functionals of (e_n^*), it follows from Proposition 3.3.1 that z^* is the w^* limit of (z_n^*) where

$$z_n^* = \sum_{j=1}^n \langle z^*, e_j^*\rangle S(e_j).$$

Since w^*-converging sequences are bounded it follows that

$$\sup_{n\in\mathbb{N}}\Big\|\sum_{j=1}^n \langle z^*, e_j^*\rangle S(e_j)\Big\| < \infty$$

and, thus, by Lemma 3.3.7

$$\sup_{n\in\mathbb{N}}\Big\|\sum_{j=1}^n \langle z^*, e_j^*\rangle e_j\Big\| < \infty.$$

By our assumption (c) it follows therefore that $x = \sum_{j=1}^{\infty} \langle z^*, e_j^* \rangle e_j$ converges in X, and moreover

$$S(x) = \lim_{n\to\infty} \sum_{j=1}^{n} \langle z^*, e_j^* \rangle S(e_j) = z^*,$$

which proves our claim. $\square$

Theorem 3.3.9. *Let X be a Banach space with a basis (e_n). Then X is reflexive if and only if (e_j) is shrinking and boundedly complete, or equivalently if (e_j) and (e_j^*) are shrinking.*

Proof. Let (e_n^*) be the coordinate functionals of (e_n) and (P_n) be the canonical projections for (e_n).
"$\Rightarrow$" Assume that X is reflexive. By Proposition 3.3.1 it follows for every $x^* \in X^*$

$$x^* = w^* - \lim_{n\to\infty} P_n^*(x^*) = w - \lim_{n\to\infty} P_n^*(x^*),$$

which implies that $x^* \in \overline{\text{span}(e_n^* : n \in \mathbb{N})}^w$, and thus, by Proposition 2.2.5 $x^* \in \overline{\text{span}(e_n^* : n \in \mathbb{N})}^{\|\cdot\|}$. It follows therefore that $x^* = \overline{\text{span}(e_n^* : n \in \mathbb{N})}^{\|\cdot\|}$ and thus that (e_j) is shrinking (by Proposition 3.3.1).

Thus X^* is a Banach space with a basis (e_j^*) which is also reflexive. We can therefore apply to X^* what we just proved for X and deduce that (e_n^*) is a shrinking basis for X^*. But, by Theorem 3.3.8 (in this case $Z = X^*$) this means that (e_n) is boundedly complete.
"$\Leftarrow$" Assume that (e_n) is shrinking and boundedly complete, and let $x^{**} \in X^{**}$. Then

$$\begin{aligned}
x^{**} &= \sigma(X^{**}, X^*) - \lim_{n\to\infty} \sum_{j=1}^{n} \langle x^{**}, e_j^* \rangle \chi(e_j) \\
&\quad \left[\begin{array}{l} \text{By Proposition 3.3.1 and the fact that } X^* = \overline{\text{span}(e_j^* : j \in \mathbb{N})} \\ \text{has } (e_j^*) \text{ as a basis, since } (e_j) \text{ is shrinking} \end{array}\right] \\
&= \|\cdot\| - \lim_{n\to\infty} \sum_{j=1}^{n} \langle x^{**}, e_j^* \rangle \chi(e_j) \in \chi(X) \\
&\quad \left[\begin{array}{c} \text{Since } \sup_{n\in\mathbb{N}} \|\sum_{j=1}^{n} \langle P^{**}(x^{**}), e_j^* \rangle e_j\| < \infty, \text{ and} \\ \text{since } (e_j) \text{ is boundedly complete} \end{array}\right]
\end{aligned}$$

which proves our claim. $\square$

The last Theorem in this section describes by *how much one can perturb a basis of a Banach space* X and still have a basis of X.

Theorem 3.3.10. (The small Perturbation Lemma)
Let (x_n) *be a basic sequence in a Banach space* X*, and let* (x^*_n) *be the coordinate functionals (they are elements of* $\overline{\text{span}(x_j : j\in\mathbb{N})}^*$*) and assume that* (y_n) *is a sequence in* X *such that*

$$c = \sum_{n=1}^{\infty} \|x_n - y_n\| \cdot \|x^*_n\| < 1. \tag{3.7}$$

Then there exists an onto isomorphism $S : X \to X$*, with*

$$(1-c)\|x\| \le \|S(x)\| \le (1+c)\|x\|$$

and $s(x_j) = y_j$*, for all* $j \in \mathbb{N}$*.*
Moreover:

a) (y_n) *is also basic in* X *and isomorphically equivalent to* (x_n)*, more precisely*

$$(1-c)\Big\|\sum_{n=1}^{\infty} a_n x_n\Big\| \le \Big\|\sum_{n=1}^{\infty} a_n y_n\Big\| \le (1+c)\Big\|\sum_{n=1}^{\infty} a_n x_n\Big\|,$$

for all in X *converging series* $x = \sum_{n\in\mathbb{N}} a_n x_n$*.*

b) *If* $\overline{\text{span}(x_j : j\in\mathbb{N})}$ *is complemented in* X*, then so is* $\overline{\text{span}(y_j : j\in\mathbb{N})}$*.*

c) *If* (x_n) *is a Schauder basis of all of* X*, then* (y_n) *is also a Schauder basis of* X *and it follows for the coordinate functionals* (y^*_n) *of* (y_n)*, that* $y^*_n \in \overline{\text{span}(x^*_j : j\in\mathbb{N})}$*, for* $n\in\mathbb{N}$*.*

Proof. By Corollary 1.4.4 of the Hahn Banach Theorem we extend the functionals x^*_n to functionals $\tilde{x}^*_n \in X^*$, with $\|\tilde{x}^*_n\| = \|x^*_n\|$, for all $n\in\mathbb{N}$.

Consider the operator:

$$T : X \to X, \quad x \mapsto \sum_{n=1}^{\infty} \langle \tilde{x}^*_n, x\rangle (x_n - y_n).$$

Since $\sum_{n=1}^{\infty} \|x_n - y_n\| \cdot \|x^*_n\| < 1$, T is well defined, linear and bounded and $\|T\| \le c < 1$. It follows $S = Id - T$ is an isomorphism between X and it self. Indeed, for $x \in X$ we have, $\|S(x)\| \geq \|x\| - \|T\| \cdot \|x\| \geq (1-c)\|x\|$ and if $y \in X$, define $x = \sum_{n=0}^{\infty} T^n(y)$ ($T^0 = Id$) then

$$(Id - T)(x) = \sum_{n=0}^{\infty} T^n(y) - T\Big(\sum_{n=0}^{\infty} T^n(y)\Big) = \sum_{n=0}^{\infty} T^n(y) - \sum_{n=1}^{\infty} T^n(y) = y.$$

Thus $Id - T$ is surjective, and, it follows from Corollary 1.3.6 that $Id - T$ is an isomorphism between X and itself.
(a) We have $(I - T)(x_n) = y_n$, for $n \in \mathbb{N}$, this means in particular that (y_n) is basic and (x_n) and (y_n) are isomorphically equivalent.
(b) Let $P : X \to \overline{\text{span}(x_n : n \in \mathbb{N})}$ be a bounded linear projection onto $\overline{\text{span}(x_n : n \in \mathbb{N})}$. Then it is easily checked that

$$Q : X \to \overline{\text{span}(y_n : n \in \mathbb{N})}, \quad x \mapsto (Id - T) \circ P \circ (Id - T)^{-1}(x),$$

is a linear projection onto $\overline{\text{span}(y_n : n \in \mathbb{N})}$.
(c) If $X = \overline{\text{span}(x_n : n \in \mathbb{N})}$, then, since $I - T$ is an isomorphism, $(y_n) = ((I - T)(x_n))$ is also a Schauder basis of X.

Moreover define for k and i in $\mathbb{N}$,

$$y^*_{(i,k)} = \sum_{j=1}^{k} \langle y_i^*, x_j \rangle x_j^* = \sum_{j=1}^{k} \langle \chi(x_j), y_i^* \rangle x_j^* \in \overline{\text{span}(x_j^* : j \in \mathbb{N})}.$$

It follows from Proposition 3.3.1, part (b), that $w^* - \lim_{k\to\infty} y^*_{(i,k)} = y_i^*$, which implies that $y_i^*(x) = \sum_{j=1}^{\infty} \langle y_i^*, x_j \rangle \langle x_j^*, x \rangle$, for all $x \in X$, and thus for $k \geq i$

$$\begin{aligned}
\|y_i^* - y^*_{(i,k)}\| &= \sup_{x \in B_X} |\langle y_i^* - y^*_{(i,k)}, x \rangle| \\
&= \sup_{x \in B_X} \Big| \sum_{j=k+1}^{\infty} \langle y_i^*, x_j \rangle \langle x_j^*, x \rangle \Big| \\
&= \sup_{x \in B_X} \Big| \sum_{j=k+1}^{\infty} \langle y_i^*, x_j - y_j \rangle \langle x_j^*, x \rangle \Big| \\
&\leq \|y_i^*\| \sum_{j=k+1}^{\infty} \|x_j - y_j\| \cdot \|x_j^*\| \to 0, \text{ if } k \to \infty.
\end{aligned}$$

so it follows that $y_i^* = \|\cdot\| - \lim_{k\to\infty} y^*_{(k,i)} \in \overline{\text{span}(x_j^* : j \in \mathbb{N})}$ for every $i \in \mathbb{N}$, which finishes the proof of our claim (c). $\square$

3.4 Unconditional Bases

As shown in the Homework there are basic sequences which are no longer basic sequences if one reorders them (like the Haarbasis in $L_[0,1]$ or the summing basis in c_0). *Unconditional bases* are defined to be bases which are bases no matter how one reorders them.

We will first observe the following result on *unconditionally converging series*

Theorem 3.4.1. *For a sequence (x_n) in Banach space X the following statements are equivalent.*

a) *For any reordering (also called* permutation*) σ of $\mathbb{N}$ (i.e. $\sigma : \mathbb{N} \to \mathbb{N}$ is bijective) the series $\sum_{n\in\mathbb{N}} x_{\sigma(n)}$ converges.*

b) *For any $\varepsilon > 0$ there is an $n \in \mathbb{N}$ so that whenever $M \subset \mathbb{N}$ is finite with $\min(M) > n$, then $\|\sum_{n\in M} x_n\| < \varepsilon$.*

c) *For any subsequence (n_j) the series $\sum_{j\in\mathbb{N}} x_{n_j}$ converges.*

d) *For sequence $(\varepsilon_j) \subset \{\pm 1\}$ the series $\sum_{j=1}^{\infty} \varepsilon_j x_{n_j}$ converges.*

In the case that above conditions hold we say that the series $\sum x_n$ converges unconditionally.

Proof. "(a)⇒(b)" Assume that (b) is false. Then there is an $\varepsilon > 0$ and for every $n \in \mathbb{N}$ there is a finite set $M \subset \mathbb{N}$, $n < \min M$, so that $\|\sum_{j\in M} x_j\| \geq \varepsilon$. We can therefore, recursively choose finite subsets of $\mathbb{N}$, M_1, M_2, M_3 etc. so that $\min M_{n+1} > \max M_n$ and $\|\sum_{j\in M_n} x_j\| \geq \varepsilon$, for $n \in \mathbb{N}$. Now consider a bijection $\sigma : \mathbb{N} \to \mathbb{N}$, which on each interval of the form $[\max M_{n-1} + 1, \max M_n]$ (with $M_0 = 0$) is as follows: The interval $[\max M_{n-1}+1, \max M_{n-1} + \#M_n]$ will be mapped to M_n, and $[\max M_{n-1} + \#M_n, \max M_n]$ will be mapped to $[\max M_{n-1} + 1, \max M_n] \setminus M_n$. It follows then for each $n \in \mathbb{N}$ that

$$\Big\| \sum_{j=\max M_{n-1}+1}^{\max M_{n-1}+\#M_n} x_{\sigma(j)} \Big\| = \Big\| \sum_{j\in M_n} x_j \Big\| \geq \varepsilon,$$

and, thus, the series $\sum x_{\sigma(n)}$ cannot be convergent, which is a contradiction.

"(b)⇒(c)" Let (n_j) be a subsequence of $\mathbb{N}$. For a given $\varepsilon > 0$, use condition (b) and choose $n \in \mathbb{N}$, so that $\|\sum_{j\in M} x_j\| < \varepsilon$, whenever $M \subset \mathbb{N}$ is finite and $\min M > n$. This implies that for all $i_0 \leq i < j$, with $i_0 = \min\{s : n_s > n\}$, it follows that $\|\sum_{s=i}^{j} x_{n_s}\| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary this means that the sequence $(\sum_{s=1}^{j} x_{n_s})_{j\in\mathbb{N}}$ is Cauchy and thus convergent.

"(c)⇒(d)" If (ε_n) is a sequence of ± 1's, let $N^+ = \{n \in \mathbb{N} : \varepsilon_n = 1\}$ and $N^- = \{n \in \mathbb{N} : \varepsilon_n = -1\}$. Since

$$\sum_{j=1}^n \varepsilon_j x_j = \sum_{j\in N^+, j\le n} x_j - \sum_{j\in N^-, j\le n} x_j, \text{ for } n \in \mathbb{N},$$

and since $\sum_{j\in N^+,j\le n} x_j$ and $\sum_{j\in N^-,j\le n} x_j$ converge by (c), it follows that $\sum_{j=1}^n \varepsilon_j x_j$ converges.

"(d)⇒ (b)" Assume that (b) is false. Then there is an $\varepsilon > 0$ and for every $n \in \mathbb{N}$ there is a finite set $M \subset \mathbb{N}$, $n < \min M$, so that $\|\sum_{j\in M} x_j\| \ge \varepsilon$. As above choose finite subsets of $\mathbb{N}$, M_1, M_2, M_3 etc. so that $\min M_{n+1} > \max M_n$ and $\|\sum_{j\in M_n} x_j\| \ge \varepsilon$, for $n \in \mathbb{N}$. Assign $\varepsilon_n = 1$ if $n \in \bigcup_{k\in\mathbb{N}} M_k$ and $\varepsilon_n = -1$, otherwise.

Note that the series $\sum_{n=1}^\infty (1+\varepsilon_n)x_n$ cannot converge because

$$\sum_{j=1}^k \sum_{i\in M_j} x_i = \frac{1}{2} \sum_{n=1}^{\max M_k} (1+\varepsilon_n)x_n, \text{ for } k \in \mathbb{N}.$$

Thus at least one of the series $\sum_{n=1}^\infty x_n$ and $\sum_{n=1}^\infty \varepsilon_n x_n$ cannot converge.

"¬(b)⇒ ¬(a)" Assume that $\sigma : \mathbb{N} \to \mathbb{N}$ is a permutation for which $\sum x_{\sigma(j)}$ is not convergent. Then we can find an $\varepsilon > 0$ and $0 = k_0 < k_1 < k_2 < \ldots$ so that

$$\Big\| \sum_{j=k_{n-1}+1}^{k_n} x_{\sigma(j)} \Big\| \ge \varepsilon.$$

Then choose $M_1 = \{\sigma(1), \ldots \sigma(k_1)\}$ and if $M_1 < M_2 < \ldots M_n$ have been chosen with $\min M_{j+1} > \max M_j$ and $\|\sum_{i\in M_j} x_i\| \ge \varepsilon$, if $i = 1, 2, \ldots, n$, choose $m \in \mathbb{N}$ so that $\sigma(j) > \max M_n$ for all $j > k_m$ (we are using the fact that for any permutaion σ, $\lim_{j\to\infty} \sigma(j) = \infty$) and let

$$M_{n+1} = \{\sigma(k_m + 1), \sigma(k_m + 2), \ldots \sigma(k_{m+1})\},$$

then $\min(M_{n+1}) > \max M_n$ and $\|\sum_{i\in M_j} x_i\| \ge \varepsilon$. It follows that (b) is not satisfied. □

Proposition 3.4.2. *In case that the series $\sum x_n$ is unconditionally converging, then $\sum x_{\sigma(j)} = \sum x_j$ for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$.*

Definition 3.4.3. A basis (e_j) for a Banach space X is called *unconditional*, if for every $x \in X$ the expansion $x = \sum \langle e_j^*, x\rangle e_j$ converges unconditionally, where (e_j^*) are coordinate functionals of (e_j).

A sequence $(x_n) \subset X$ is called *unconditional basic sequence* if (x_n) is an unconditional basis of $\overline{\text{span}(x_j : j \in \mathbb{N})}$.

Proposition 3.4.4. *For a sequence of non zero elements (x_j) in a Banach space X the following are equivalent.*

a) (x_j) *is an unconditional basic sequence,*

b) There is a constant C, so that for all $n \in \mathbb{N}$, all $A \subset \{1, 2, \dots, n\}$ and all scalars $(a_j)_{j=1}^n \subset \mathbb{K}$,

$$\Big\| \sum_{j \in A} a_j x_j \Big\| \le C \Big\| \sum_{j=1}^n a_j x_j \Big\|. \tag{3.8}$$

c) There is a constant C', so that for all $n \in \mathbb{N}$, all $(\varepsilon_j)_{j=1}^n \subset \{\pm 1\}$ and all scalars $(a_j)_{j=1}^n \subset \mathbb{K}$,

$$\Big\| \sum_{j=1}^n \varepsilon_j a_j x_j \Big\| \le C' \Big\| \sum_{j=1}^n a_j x_j \Big\|. \tag{3.9}$$

In that case we call the smallest constant $C = K_s$ which satisfies (3.8) *the* supression-unconditional constant of (x_n) *for all n, $A \subset \{1, 2, \dots, n\}$ and all scalars $(a_j)_{j=1}^n \subset \mathbb{K}$ and we call the smallest constant $C' = K_u$ so that* (3.9) *holds for all n, $(\varepsilon_j)_{j=1}^n \subset \{\pm 1\}$ and all scalars $(a_j)_{j=1}^n \subset \mathbb{K}$ the* unconditional constant of (x_n).

Moreover, it follows

$$K_s \le K_u \le 2K_s. \tag{3.10}$$

Proof. "(a)⇒(b)" Assume that (b) does not hold. We can assume that (x_n) is a basic sequence with constant b. Then (Exercise) we choose recursively $k_0 < k_1 < k_2, \dots$, $A_n \subset \{k_{n-1}+1, k_{n-1}+1, \dots k_n\}$, and scalars $(a_j)_{j=k_{n-1}+1}^{k_n}$ so that

$$\Big\| \sum_{j \in A_n} a_j x_j \Big\| \ge 1 \text{ and } \Big\| \sum_{j=k_{n-1}+1}^{k_n} a_j x_j \Big\| \le \frac{1}{n^2} \text{ for all } n \in \mathbb{N}.$$

For any $l < m$, we can choose $i \le j$ so that $k_{i-1} < l \le k_i$ and $k_{j-1} < m \le k_j$, and thus

$$\Big\| \sum_{s=l}^m a_s x_s \Big\| \le \Big\| \sum_{s=l}^{k_i} a_s x_s \Big\| + \sum_{t=i+1}^{j-1} \Big\| \sum_{s=k_{t-1}+1}^{k_t} a_s x_s \Big\| + \Big\| \sum_{s=k_{j-1}+1}^{m} a_s x_s \Big\|$$

(where the second term is defined to be 0, if $i \ge j-1$)

$$\leq \frac{2b}{(i-1)^2} + \sum_{t=i+1}^{j-1} \frac{1}{t^2} + \frac{2b}{(j-1)^2}$$

It follows therefore that $x = \sum_{j=1}^{\infty} a_j x_j$ converges, but by Theorem 3.4.1 (b) it is not unconditionally.
"(b) $\iff$ (c)" and (3.10) follows from the following estimates for $n \in \mathbb{N}$, $(a_j)_{j=1}^n \subset \mathbb{K}$, $A \subset \{1, 2, \ldots, n\}$ and $(\varepsilon_j)_{j=1}^n \subset \{\pm 1\}$

$$\Big\| \sum_{j=1}^n \varepsilon_j a_j x_j \Big\| \leq \Big\| \sum_{j=1, \varepsilon_j = 1}^n a_j x_j \Big\| + \Big\| \sum_{j=1, \varepsilon_j = -1}^n a_j x_j \Big\| \text{ and}$$

$$\Big\| \sum_{j \in A} a_j x_j \Big\| \leq \frac{1}{2} \left[\Big\| \sum_{j \in A} a_j x_j + \sum_{j \in \{1,2,\ldots\} \setminus A} a_j x_j \Big\| + \Big\| \sum_{j \in A} a_j x_j - \sum_{j \in \{1,2,\ldots\} \setminus A} a_j x_j \Big\| \right].$$

"(b)$\Rightarrow$"(a) First, note that (b) implies by Proposition 3.1.9 that (x_n) is a basic sequence. Now assume that for some $x = \sum_{j=1}^{\infty} a_j x_j \in \overline{\operatorname{span}(x_j : j \in \mathbb{N})}$ is converging but not unconditionally converging. It follows from the equivalences in Theorem 3.4.1 that there is some $\varepsilon > 0$ and of $\mathbb{N}$, M_1, M_2, M_3 etc. so that $\min M_{n+1} > \max M_n$ and $\| \sum_{j \in M_n} a_j x_j \| \geq \varepsilon$, for $n \in \mathbb{N}$. On the other hand it follows from the convergence of the series $\sum_{j=1}^{\infty} a_j x_j$ that

$$\limsup_{n \to \infty} \Big\| \sum_{j=1+\max(M_{n-1})}^{\max(M_n)} a_j x_j \Big\| = 0,$$

and thus

$$\sup_{n \to \infty} \frac{\Big\| \sum_{j \in M_n} a_j x_j \Big\|}{\Big\| \sum_{j=1+\max(M_{n-1})}^{\max(M_n)} a_j x_j \Big\|} = \infty,$$

which is a contradiction to condition (b). $\square$

Proposition 3.4.5. *Assume that X is a Banach space over the field $\mathbb{C}$ with an unconditional basis (e_n), then it follows if $\sum_{j=1}^{\infty} \alpha_n e_n$ is convergent and $(\beta_n) \subset \{\beta \in \mathbb{C} : |\beta| = 1\}$ that $\sum_{j=1}^{\infty} \beta_n \alpha_n e_n$ is also converging and*

$$\Big\| \sum_{n \in \mathbb{N}} \beta_n \alpha_n e_n \Big\| \leq 2K_u \Big\| \sum_{n \in \mathbb{N}} \alpha_n e_n \Big\|.$$

Proof. Exercise . $\square$

Proposition 3.4.6. *If X is a Banach space with an unconditional basis, then the coordinate functionals (e_n^*) are also a unconditional basic sequence, with the same unconditional constant and the same suppression-uncondtional constant.*

Proof. Let K_u and K_s be the unconditional and suppression unconditional constant of X.

Let $x^* = \sum_{n\in\mathbb{N}} \eta_n e_n^*$ and $(\varepsilon_n) \subset \{\pm 1\}$ then

$$\begin{aligned}
\Big\| \sum_{n\in\mathbb{N}} \varepsilon_n \eta_n e_n^* \Big\|_{X^*} &= \sup_{x=\sum_{n=1}^\infty \xi_n e_n \in B_X} \Big\langle \sum_{n\in\mathbb{N}} \varepsilon_n \eta_n e_n^*, \sum_{n=1}^\infty \xi_n e_n \Big\rangle \\
&= \sup_{x=\sum_{n=1}^\infty \xi_n e_n \in B_X} \sum_{n\in\mathbb{N}} \varepsilon_n \eta_n \xi_n \\
&= \sup_{x=\sum_{n=1}^\infty \xi_n e_n \in B_X} \Big\| \sum_{n\in\mathbb{N}} \eta_n e_n^* \Big\| \cdot \Big\| \sum_{n\in\mathbb{N}} \varepsilon_n \xi_n e_n \Big\| \\
&\le K_u \Big\| \sum_{n\in\mathbb{N}} \eta_n e_n^* \Big\|.
\end{aligned}$$

Using the Hahn Banach Theorem we can similarly show that if K_u^* is the unconditional constant of (e_n^*) then

$$\Big\| \sum_{n\in\mathbb{N}} \xi_n \varepsilon_n e_n \Big\|_X \le K_u^* \le \Big\| \sum_{n\in\mathbb{N}} \xi_n \varepsilon_n \Big\|_X$$

Thus $K_u = K_u^*$. A similar argument works to show that K_s is equal to the suppression unconditional constant of (e_n^*). $\square$

The following Theorem about spaces with unconditional basic sequences was shown By James [Ja]

Theorem 3.4.7. *Let X be a Banach space with an unconditional basis (e_j). Then either X contains a copy of c_0, or a copy of ℓ_1 or X is reflexive.*

We will need first the following Lemma (Exercise)

Lemma 3.4.8. *Let X be a Banach space with an unconditional basis (e_n) and let K_u its constant of unconditionality. Then it follows for any converging series $\sum_{n\in\mathbb{N}} a_n e_n$ and a bounded sequence of scalars $(b_n) \subset \mathbb{K}$, that $\sum_{n\in\mathbb{N}} a_n b_n e_n$ is also converging and*

$$\Big\| \sum_{n\in\mathbb{N}} a_n b_n e_n \Big\| \le K \sup_{n\in\mathbb{N}} |b_n| \Big\| \sum_{n=1}^\infty a_n e_n \Big\|,$$

where $K = K_u$, if $\mathbb{K} = \mathbb{R}$, and $K = 2K_u$, if $\mathbb{K} = \mathbb{C}$.

Proof of Theorem 3.4.7. We will prove the following two statements for a space X with unconditional basis (e_n).
Claim 1: If (e_n) is not boundedly complete then X contains a copy of c_0.
Claim 2: If (e_n) is not shrinking then X contains a copy of ℓ_1.

Together with Theorem 3.3.9, this yields the statement of Theorem 3.4.7.

Let K_u be the constant of unconditionality of (e_n) and let $K'_u = K_u$, if $\mathbb{K} = \mathbb{R}$, and $K'_U = 2K_u$, if $\mathbb{K} = \mathbb{C}$.
Proof of Claim 1: If (e_n) is not boundedly complete there is, by Theorem 3.3.8, a sequence of scalars (a_n) such that

$$\sup_{n\in\mathbb{N}} \Big\| \sum_{j=1}^{n} a_j e_j \Big\| = C_1 < \infty, \text{ but } \sum_{j=1}^{\infty} a_j e_j \text{ does not converge.}$$

This implies that there is an $\varepsilon > 0$ and sequences (m_j) and (n_j) with $1 \le m_1 \le n_1 < m_2 \le n_2 < \ldots$ in $\mathbb{N}$ so that if we put $y_k = \sum_{j=m_k}^{n_k} a_j e_j$, for $k \in \mathbb{N}$, it follows that $\|y_k\| \ge \varepsilon$, and also

$$\|y_k\| \le \Big\| \sum_{j=1}^{n_k} a_j e_j \Big\| + \Big\| \sum_{j=1}^{m_k-1} a_j e_j \Big\| \le 2C_1.$$

For any $k \in \mathbb{N}$ and any sequence of scalars $(\lambda_j)_{j=1}^{k}$ it follows therefore from Lemma 3.4.8, that

$$\Big\| \sum_{j=1}^{k} \lambda_j y_j \Big\| \le 2K_u \max_{j\le k} |\lambda_j| \Big\| \sum_{j=1}^{k} y_j \Big\| \le 2K_u K_s \sup_{j\le k} |\lambda_j| \Big\| \sum_{i=1}^{n_k} a_i e_j \Big\| \le 2K_u K_s C_1 \sup_{j\le k} |\lambda_j|.$$

On the other hand for every $j_0 \le n$ that

$$\Big\| \sum_{j=1}^{n} \lambda_j y_j \Big\| \ge \frac{1}{K_s} \|\lambda_{j_0} y_{j_0}\| \ge \frac{\varepsilon}{K_s} \max_{j\le n} |\lambda_j|.$$

Letting $c = \varepsilon/K_u$ and $C = 2K_u K_s C_1$, it follows therefore for any $n \in \mathbb{N}$ and any sequence of scalars $(\lambda_j)_{j=1}^{n}$ that

$$c\|(\lambda)_{j=1}^{n}\|_{c_0} \le \Big\| \sum_{j=1}^{n} \lambda_j y_j \Big\| \le C\|(\lambda)_{j=1}^{n}\|_{c_0},$$

which means that (y_j) and the unit vector basis of c_0 are isomorphically equivalent.

Proof of Claim 2. (e_n) is not shrinking then there is by Theorem 3.3.4 a bounded block basis (y_n) of (e_n) which is not weakly null. After passing to a subsequence we can assume that there is a $x^* \in X^*$, $\|x^*\| = 1$, so that

$$\varepsilon = \inf_{n\in\mathbb{N}} |\langle x^*, y_n\rangle| > 0.$$

We also can assume that $\|y_n\| = 1$, for $n \in \mathbb{N}$ (otherwise replace y_n by $y_n/\|y_n\|$ and change ε accordingly).

We claim that (y_n) is isomorphically equivalent to the unit vector basis of ℓ_1. Let $n \in \mathbb{N}$ and $(a_j)_{j=1}^n \subset \mathbb{K}$. By the triangle inequality we have

$$\Big\|\sum_{j=1}^n a_j y_j\Big\| \le \sum_{j=1}^n |a_j|,$$

On the other hand we can choose for $j = 1, 2, \ldots, n$ $\varepsilon_j = \mathrm{sign}(a_j\langle x^*, y_j\rangle)$ if $\mathbb{K} = \mathbb{R}$ and $\varepsilon_j = \overline{a_j\langle x^*, y_j\rangle}/|a_j\langle x^*, y_j\rangle|$, if $\mathbb{K} = \mathbb{C}$ (if $a_j = 0$, simply let $\varepsilon = 1$) and deduce from Lemma 3.4.8

$$\Big\|\sum_{j=1}^n a_j y_j\Big\| \ge \frac{1}{K'_u}\Big\|\sum_{j=1}^n \varepsilon_j a_j y_j\Big\| \ge \Big|\sum_{j=1}^n \varepsilon_j a_j\langle x^*, y_j\rangle\Big| \ge \varepsilon\sum_{j=1}^n |a_j|,$$

which implies that (y_n) is isomorphically equivalent to the unit vector basis of ℓ_1. □

Remark. It was for long time an open problem whether or not every infinite dimensional Banach space contains an unconditional basis sequence. If this were so, every infinite dimensional Banach space would contain a copy of c_0 or a copy of ℓ_1, or has an infinite dimensional reflexive subspace space. In [GM], Gowers and Maurey proved the existence of a Banach space which does not contain any unconditional basic sequences. Later then Gowers [Go] constructed a space which does not contain any copy of c_0 or ℓ_1, and has no infinite dimensional reflexive subspace.

3.5 James' Space

The following space J was constructed by R. C. James [Ja1]. It is a space which is not reflexive and does not contain a subspace isomorphic to c_0 or ℓ_1. By Theorem 3.4.7 it does not have an unconditional basis. Moreover we will prove that $J^{**}/\chi(J)$ is one dimensional and that J is isomorphically isometric to J^{**} (but of course not via the canonical mapping).

We will define the space J over the real numbers $\mathbb{R}$.

For a sequence $(\xi_n) \subset \mathbb{R}$ we define the *quadratic variation to be*

$$\begin{aligned}\|(\xi_n)\|_{qv} &= \sup\Big\{\Big(\sum_{j=1}^{l}|\xi_{n_j}-\xi_{n_{j-1}}|^2\Big)^{1/2} : l\in\mathbb{N} \text{ and } 1\le n_0<n_1<\ldots n_l\Big\}\\ &= \sup\big\{\|(\xi_{n_j}-\xi_{n_{j-1}} : j=1,2,\ldots l)\|_{\ell_2} : l\in\mathbb{N} \text{ and } 1\le n_0<n_1<\ldots n_l\big\}\end{aligned}$$

and the *cyclic quadratic variation norm to be*

$$\|(\xi_n)\|_{cqv} = \sup\Big\{\frac{1}{\sqrt{2}}\Big(|\xi_{n_0}-\xi_{n_l}|^2+\sum_{j=1}^{l}|\xi_{n_j}-\xi_{n_{j-1}}|^2\Big)^{1/2} : l\in\mathbb{N} \text{ and } 1\le n_0<n_1<\ldots n_l\Big\}.$$

Remark. Let $(\xi_n) \subset \mathbb{R}$ with $\|(\xi_n)\|_{qv}$ and assume that $n_0 < n_1 < n_2 < \ldots$ are such that

$$\|(\xi_n)\|_{qv} = \Big(\sum_{j=1}^{l}|\xi_{n_j}-\xi_{n_{j-1}}|^2\Big)^{1/2} \tag{3.11}$$

(for example if (ξ_n) has only finitely many non zero coefficients the supremum is achieved). We note the following:

1. We can assume that $n_0 = 1$ (otherwise we add it)
2. If $n_{j-1} < n < n_j$ then x_j lies between $x_{n_{j-1}}$ and x_{n_j}, for $1 < j \le l$ Other wise we could add n to the n_i's, and make the sum in (3.11) larger
3. If $x_{j-1} < x_{n_j}$ then $x_{n_{j+1}} \le x_{n_j}$ (zig-zag condition), for $1 < j < l$.
4. x_j is a local extreme point in the sequence $(x_{n_j-1}, x_{n_j}, x_{n_j+1})$, but this does not mean that every local extreme point must be among the n_j's.

Note that for a bounded sequences $(\xi_n), (\eta_n) \subset \mathbb{R}$

$$\|(\xi_n+\eta_n)\|_{qv} = \sup\big\{\|(\xi_{n_i}+\eta_{n_i}-\xi_{n_{i-1}}-\eta_{n_{i-1}})_{i=1}^{l}\|_2 : l\in\mathbb{N}, n_0<n_1<\ldots n_l\big\}$$

$$
\begin{aligned}
&\leq \sup\big\{\|(\xi_{n_i}-\xi_{n_{i-1}})_{i=1}^l\|_2 + \|(\eta_{n_i}-\eta_{n_{i-1}})_{i=1}^l\|_2 : l\in\mathbb{N}, n_0<\ldots n_l\big\}\\
&\leq \sup\big\{\|(\xi_{n_i}-\xi_{n_{i-1}})_{i=1}^l\|_2 : l\in\mathbb{N}, n_0<n_1<\ldots n_l\big\}\\
&\qquad + \sup\big\{\|(\eta_{n_i}-\eta_{n_{i-1}})_{i=1}^l\|_2 : l\in\mathbb{N}, n_0<n_1<\ldots n_l\big\}\\
&= \|(\xi_n)\|_{qv} + \|(\eta_n)\|_{qv}
\end{aligned}
$$

and similarly

$$\|(\xi_n+\eta_n)\|_{cqv} \leq \|(\xi_n)\|_{cqv} + \|(\eta_n)\|_{cqv}.$$

and we note that

$$\frac{1}{\sqrt{2}}\|(\xi_n)\|_{qv} \leq \|(\xi_n)\|_{cqv} \leq \sqrt{2}\|(\xi_n)\|_{qv}.$$

Thus $\|\cdot\|_{qv}$ and $\|\cdot\|_{cqv}$ are two equivalent semi norms on the vector space

$$\tilde{J} = \big\{(\xi_n)\subset\mathbb{R} : \|(\xi_n)\|_{qv}<\infty\big\}$$

and since

$$\|(\xi_n)\|_{qv}=0 \iff \|(\xi_n)\|_{cqv}=0 \iff (\xi_n) \text{ is constant}$$

$\|\cdot\|_{qv}$ and $\|\cdot\|_{cqv}$ are two equivalent norms on the vector space

$$J = \big\{(\xi_n)\subset\mathbb{R} : \lim_{n\to\infty}\xi_n = 0 \text{ and } \|(\xi_n)\|_{qv}<\infty\big\}.$$

Proposition 3.5.1. *The space J with the norms $\|\cdot\|_{qv}$ and $\|\cdot\|_{cqv}$ is complete and, thus, a Banach space.*

Proof. The proof is similar to the proof of showing that ℓ_p is complete. Let (x_k) be a sequence in J with $\sum_{k\in\mathbb{N}}\|x_k\|_{qv}<\infty$ and write $x_k=(\xi_{(k,j)})_{j\in\mathbb{N}}$, for $k\in\mathbb{N}$. Since for $j,k\in\mathbb{N}$ it follows that

$$|\xi_{(k,j)}| = \lim_{n\to\infty}|\xi_{(k,j)}-\xi_{(k,n)}| \leq \|x_k\|_{qv}$$

it follows that

$$\xi_j = \sum_{k\in\mathbb{N}}\xi_{(k,j)}$$

exists and for $x=(\xi_j)$ it follows that $x\in c_0$ (c_0 is complete) and

$$\|x\|_{qv} = \sup\Big\{\Big(\sum_{j=1}^l|\xi_{n_j}-\xi_{n_{j-1}}|^2\Big)^{1/2} : l\in\mathbb{N} \text{ and } 1\leq n_0<n_1<\ldots n_l\Big\}$$

$$\le \sup\Big\{\sum_{k\in\mathbb{N}}\Big(\sum_{j=1}^{l}|\xi_{(k,n_j)}-\xi_{(k,n_{j-1})}|^2\Big)^{1/2} : l\in\mathbb{N} \text{ and } 1\le n_0<\dots n_l\Big\}$$
$$\le \sum_{k\in\mathbb{N}}\|x_k\|_{qv}<\infty$$

and for $m\in\mathbb{N}$

$$\Big\|x-\sum_{k=1}^{m}x_k\Big\|_{qv}$$
$$=\sup\Big\{\Big(\sum_{j=1}^{\infty}\Big|\sum_{k=m+1}^{\infty}\xi_{(k,n_j)}-\xi_{(k,n_{j-1})}\Big|^2\Big)^{1/2} : l\in\mathbb{N} \text{ and } 1\le n_0\le\dots n_l\Big\}$$
$$\le \sup\Big\{\sum_{k=m+1}^{\infty}\Big(\sum_{j=1}^{l}|\xi_{(k,n_j)}-\xi_{(k,n_{j-1})}|^2\Big)^{1/2} : l\in\mathbb{N} \text{ and } 1\le n_0\le\dots n_l\Big\}$$

(By the triangle inequality in ℓ_2)

$$\le \sum_{k=m+1}^{\infty}\|x_k\|_{qv}\to 0 \text{ for } m\to\infty.$$

□

Proposition 3.5.2. *The unit vector basis (e_i) is a monotone basis of J for both norms, $\|\cdot\|_{qv}$ and $\|\cdot\|_{cqv}$.*

Proof. First we claim that $\overline{\operatorname{span}(e_j : j\in\mathbb{N})}=J$. Indeed, if $x=(\xi_n)\in J$, and $\varepsilon>0$ we choose l and $1\le n_0<n_1<\dots n_l$ in $\mathbb{N}$ so that

$$\sum_{j=1}^{l}|\xi_{n_j}-\xi_{n_{j-1}}|^2>\|x\|_{qv}^2-\varepsilon.$$

But this implies that

$$\|x-\sum_{j=1}^{n_l+1}\xi_je_j\|=\|(\underbrace{0,0,\dots 0}_{(n_l+1)\text{ times}},\xi_{n_l+2},\xi_{n_l+3},\dots)\|<\varepsilon.$$

In order to show monotonicity, assume $m<n$ are in $\mathbb{N}$ and $(a_i)_{i=1}^n\subset\mathbb{R}$. For $i\in\mathbb{N}$ let

$$\xi_i=\begin{cases}a_i & \text{if } i\le m\\ 0 & \text{otherwise}\end{cases} \quad\text{and } \eta_i=\begin{cases}a_i & \text{if } i\le n\\ 0 & \text{otherwise.}\end{cases}$$

For $x = \sum_{i=1}^\infty \xi_i e_i$ and $y = \sum_{i=1}^\infty \eta_i e_i$ we need to show that $\|x\|_{qv} \le \|y\|_{qv}$ and $\|x\|_{cqv} \le \|y\|_{cqv}$. So choose l and $n_0 < n_1 < \dots n_l$ in $\mathbb{N}$ so that

$$\|x\|_{qv}^2 = \sum_{j=1}^{l} |\xi_{n_j} - \xi_{n_{j-1}}|^2.$$

Then we can assume that $n_l > n$ (otherwise replace l by $l+1$ and add $n_{l+1} = n+1$) and we can assume that $n_{l-1} \le m$ (otherwise we drop all the n_j's in $(m, n]$), and thus

$$\|x\|_{qv}^2 = \sum_{j=1}^{l} |\xi_{n_j} - \xi_{n_{j-1}}|^2 = \sum_{j=1}^{l} |\eta_{n_j} - \eta_{n_{j-1}}|^2 \le \|y\|_{qv}.$$

The argument for the cyclic variation norm is similar. □

Our next goal is to show that (e_n) is a shrinking basis of J. We need the following lemma

Lemma 3.5.3. *For any normalized block basis (u_i) of e_i in J, and $m \in \mathbb{N}$ and any scalars $(a_i)_{i=1}^m$ it follows that*

$$\Big\| \sum_{i=1}^{m} a_i u_i \Big\| \le \sqrt{5} \|(a_i)_{i=1}^n\|_2. \tag{3.12}$$

Proof. Let $(\eta_j) \subset \mathbb{R}$ and $k_0 = 0 < k_1 < k_2 < \dots$ in $\mathbb{N}$ so that for $i \in \mathbb{N}$

$$u_i = \sum_{j=k_{i-1}+1}^{k_i} \eta_j e_j.$$

Let for $i = 1, 2, 3 \dots m$ and $j = k_{i-1}+1, k_{i-1}+2, \dots k_i$ put $\xi_j = a_i \cdot \eta_j$, and

$$x = \sum_{i=1}^{n} a_i u_i = \sum_{j=1}^{k_n} \xi_j e_j.$$

For given $l \in \mathbb{N}$ and $1 \le n_0 < n_1 < \dots < n_l$ we need to show that

$$\sum_{j=1}^{l} |\xi_{n_j} - \xi_{n_{j-1}}|^2 \le 5 \sum_{i=1}^{m} a_i^2. \tag{3.13}$$

We put $\xi_j = \eta_j = 0$, whenever $j > k_m$.

For $i = 1, 2, \ldots, m$ define $A_i = \{1 \le j \le l : k_{i-1} < n_{j-1} < n_j \le k_i\}$ and $A_m = \{1 \le j \le l : k_{m-1} < n_j\}$. It follows that

$$\sum_{j \in A_i} |\xi_j - \xi_{j-1}|^2 = a_i^2 \sum_{j \in A_i} |\eta_j - \eta_{j-1}|^2 \le a_i^2 \|u_i\|_{qv},$$

and thus

$$\sum_{j \in \bigcup_{i=1}^n A_i} |\xi_j - \xi_{j-1}|^2 \le \sum_{i=1}^n a_i^2.$$

Now let $A = \bigcup_{i=1}^n A_i$ and $B = \{j \le l : j \notin A\}$. For each $j \in B$ there must exist $l(j)$ and $m(j)$ in $\{1, 2, \ldots, m-1\}$ so that

$$k_{l(j)-1} < n_{j-1} \le k_{l(j)} \le k_{m(j)} < n_j \le k_{m(j)+1}$$

and thus

$$\begin{aligned} |\xi_{n_j} - \xi_{n_{j-1}}|^2 &= |a_{m(j)+1}\eta_{n_j} - a_{l(j)}\eta_{n_{j-1}}|^2 \\ &\le 2a_{m(j)+1}^2 \eta_{n_j}^2 + 2a_{l(j)}^2 \eta_{n_{j-1}}^2 \le 2a_{m(j)+1}^2 + 2a_{l(j)}^2 \end{aligned}$$

(for the last inequality note that $|\eta_i| \le 1$ since $\|u_j\| = 1$).

For $j, j' \in B$ it follows that $l(j) \ne l(j')$ and $m(j) \ne m(j')$, $j \ne j'$ and thus

$$\begin{aligned} \sum_{j=1}^l |\xi_{n_j} - \xi_{n_{j-1}}|^2 &= \sum_{j \in A} |\xi_{n_j} - \xi_{n_{j-1}}|^2 + \sum_{j \in B} |\xi_{n_j} - \xi_{n_{j-1}}|^2 \\ &\le \sum_{i=1}^n a_i^2 + 2\sum_{j \in B} a_{l(j)}^2 + 2\sum_{j \in B} a_{m(j)+1}^2 \le 5\sum_{i=1}^n a_i^2, \end{aligned}$$

which finishes the proof of our claim. □

Corollary 3.5.4. *The unit vector basis (e_n) is shrinking in J.*

Proof. Let (u_n) be any block basis of (e_n), which is w.l.o.g. normalized. Then by Lemma 3.5.3

$$\frac{1}{n}\Big\| \sum_{j=1}^n u_j \Big\|_{qv} \le \sqrt{5}/\sqrt{n} \to 0 \text{if } n \to \infty.$$

By Corollary 2.2.6 (u_n) is therefore weakly null. Since (u_n) was an arbitrary block basis of (e_n) this yields by Theorem 3.3.8 that (e_n) is shrinking. □

Definition 3.5.5. (Skipped Block Bases)
Assume X is a Banach space with basis (e_n). A *Skipped Block Basis of* (e_n) is a sequence (u_n) for which there are $0 = k_0 < k_1 < k_2 < \dots$ in $\mathbb{N}$, and $(a_j) \subset \mathbb{K}$ so that

$$u_n = \sum_{j=k_{n-1}+1}^{k_n-1} a_j e_j, \text{ for } n \in \mathbb{N}.$$

(i.e. the k_n's *are skipped*).

Proposition 3.5.6. *Every normalized skipped block sequence of the unit vector basis in J is isomorphically equivalent to the unit vector basis in ℓ_2. Moreover the constant of equivalence is $\sqrt{5}$.*

Proof. Assume that

$$u_n = \sum_{j=k_{n-1}+1}^{k_n-1} a_j e_j, \text{ for } n \in \mathbb{N}.$$

with $0 = k_0 < k_1 < k_2 < \dots$ in $\mathbb{N}$, and $(a_j) \subset \mathbb{K}$, and $a_{k_n} = 0$, for $n \in \mathbb{N}$. For $n \in \mathbb{N}$ we can find l_n and $k_{n-1} = p_0^{(n)} < p_1^{(n)} < \dots p_{l_n} = k_n$ in $\mathbb{N}$ so that

$$\|u_n\|_{qv}^2 = \sum_{j=1}^{l_n} \big(a_{p_j^{(n)}} - a_{p_{j-1}^{(n)}}\big)^2 = 1.$$

Now let $m \in \mathbb{N}$ and $(b_i)_{i=1}^m \subset \mathbb{R}$ we can *string the $p_j^{(n)}$'s together* and deduce:

$$\Big\|\sum_{n=1}^m b_n u_n\Big\|_{qv}^2 \geq \sum_{i=1}^m b_i^2 \sum_{j=1}^{l_n} \big(a_{p_{j-1}^{(n)}} - a_{p_{j-1}^{(n)}}\big)^2 = \sum_{i=1}^m b_i^2.$$

On the other hand it follows from Lemma 3.5.3 that

$$\Big\|\sum_{n=1}^m b_n u_n\Big\|_{qv}^2 \leq 5 \sum_{i=1}^m b_i^2.$$

□

Corollary 3.5.7. *J is* hereditarily ℓ_2*, meaning every infinite dimensional subspace of J has a further subspace which is isomorphic to ℓ_2.*

Proof. Let Z be an infinite dimensional subspace of J. By induction we choose for each $n \in \mathbb{N}$, $z_n \in Z$, $u_n \in J$ and $k_n \in \mathbb{N}$, so that

$$\|z_n\|_{qv} = \|u_n\|_{qv} = 1 \text{ and } \|z_n - u_n\|_{qv} < 2^{-4-n}, \tag{3.14}$$

$$u_n \in \operatorname{span}(e_j : k_{n-1} < j < k_n) \tag{3.15}$$

Having accomplished that, (u_n) is a skipped block basis of (e_n) and by Proposition 3.5.6 isomorphically equivalent to the unit vector basis of ℓ_2. Letting (u_n^*) be the coordinate functionals of (u_n) it follows that $\|u_n^*\| \le \sqrt{5}$, for $n \in \mathbb{N}$, and thus, by the third condition in (3.14),

$$\sum_{n=1}^{\infty} \|u_n^*\| \|u_n - z_n\| \le \sqrt{5} 2^{-4} < 1,$$

which implies by the Small Perturbation Lemma, Theorem 3.3.10, that (z_n) is also isomorphically equivalent to unti vector bais in ℓ_2.

We choose $z_1 \in S_Z$ arbitrarily, and then let $u_1 \in \operatorname{span}(e_j : j \in \mathbb{N})$, with $\|u_1\|_{qv} = 1$ and $\|u_1 - z_1\|_{qv} < 2^{-4}$. Then let $k_1 \in \mathbb{N}$ so that $u_1 \in \operatorname{span}(e_j : j < k_1)$. If we assume that $z_1, z_2, \ldots, \ldots z_n$, $u_1, u_2, \ldots, u_n$, and $k_1 < k_2 < \ldots k_n$ have been chosen we choose $z_{n+1} \in Z \cap \{e_1^*, \ldots e_{k_n}^*\}_\perp$ (note that this space is infinite dimensional and a subspace of $\overline{\operatorname{span}(e_j : j > k_{n+1})}$) and then choose $u_{n+1} \in \operatorname{span}(e_j : j > k_{n+1})$, $\|u_{n+1}\|_{qv} = 1$, with $\|u_{n+1} - z_{n+1}\|_{qv} < 2^4 2^{-n-1}$ and let $k_{n+1} \in \mathbb{N}$ so that $u_{n+1} \in \operatorname{span}(e_j : j < k_{n+1})$. □

Using the fact that (e_n) is a monotone and shrinking basis of J (see Proposition 3.5.2 and Corollary 3.5.4) we can use Proposition 3.3.6 to represent the bidual J^{**} of J. We will now use the cyclic variation norm.

$$J^{**} = \Big\{(\xi_n) \subset \mathbb{R} : \sup_{n\in\mathbb{N}} \Big\| \sum_{j=1}^{n} \xi_i e_i \Big\|_{cqv} < \infty \Big\} \tag{3.16}$$

and for $x^{**} = (\xi_n) \in J^{**}$

$$\begin{aligned} \|x^{**}\|_{J^{**}} &= \sup_{n\in\mathbb{N}} \|(\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots)\|_{cqv} \\ &= \sup_{l\in\mathbb{N}, k_0<k_1<\ldots k_l} \max \Bigg(\Big((\xi_{k_0} - \xi_{k_l})^2 + \sum_{j=1}^{l} (\xi_{k_{j-1}} - \xi_{k_j})^2 \Big)^{1/2}, \\ &\qquad \Big(\xi_{k_0}^2 + \xi_{k_l}^2 + \sum_{j=1}^{l} (\xi_{k_{j-1}} - \xi_{k_j})^2 \Big)^{1/2} \Bigg). \end{aligned} \tag{3.17}$$

The second equality in (3.17) can be seen as follows: Fix an $n \in \mathbb{N}$ and consider

$$x^{(n)} = (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots), \text{thus } x^{(n)} = (\xi_j^{(n)}), \text{ with } \xi_j^{(n)} = \begin{cases} \xi_j & \text{if } j \leq n \\ 0 & \text{else} \end{cases}.$$

Now we let l and $1 \leq k_1 < k_2 < \ldots < k_l$ in $\mathbb{N}$ be chosen so that

$$\|x^{(n)}\|_{cqv}^2 = \frac{1}{2}\Big((\xi_{k_0}^{(n)} - \xi_{k_l}^{(n)})^2 + \sum_{j=1}^{l}(\xi_{k_j}^{(n)} - \xi_{k_{j-1}}^{(n)})^2\Big).$$

There are two cases: Either $k_l \leq n$. In this case $\xi_{k_j}^{(n)} = \xi_{k_j}$, for all $j \leq l$, and thus

$$\|x^{(n)}\|_{cqv}^2 = \Big((\xi_{k_0} - \xi_{k_l})^2 + \sum_{j=1}^{l}(\xi_{k_{j-1}} - \xi_{k_j})^2\Big)^{1/2},$$

which leads to the first term in above "max". Or $k_l > n$. Then we can assume without loss of generality that $k_{l-1} \leq n$ (otherwise we can drop k_{l-1}) and we note that $\xi_{k_l}^{(n)} = 0$, while $\xi_{k_j}^{(n)} = \xi_{k_j}$ for all $j \leq l-1$, and thus

$$\|x^{(n)}\|_{cqv}^2 = \frac{1}{2}\Big(\xi_{k_0}^2 + \sum_{j=1}^{l}(\xi_{k_{j-1}} - \xi_{k_j})^2\Big)^{1/2} = \Big(\xi_{k_0}^2 + \xi_{k_{l-1}}^2 + \sum_{j=1}^{l-1}(\xi_{k_{j-1}} - \xi_{k_j})^2\Big)^{1/2},$$

which, after renaming $l-1$ to be l, leads to the second term above "max".

Remark. Note that there is a difference between

$$\|(\xi_1, \xi_2, \ldots)\|_{cqv}$$

and

$$\sup_{n \in \mathbb{N}} \|(\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots)\|_{cqv}$$

and there is only equality if $\lim_{n\to\infty} \xi_n = 0$.

It follows that for all $x^{**} = (\xi_n) \in J^{**}$, that $e_\infty^*(x) = \lim_{n\to\infty} \xi_n$ exists, that $(1, 1, 1, 1, \ldots) \in J^{**} \setminus J$, and that

$$x^{**} - e_\infty^*(x)(1, 1, 1, \ldots) \in J.$$

Theorem 3.5.8. *J is not reflexive, does not contain an isomorphic copy of c_0 or ℓ_1 and the codimension of J in J^{**} is 1.*

Proof. We only need to observe that it follows from the above that

$$\begin{aligned} J^{**} &= \big\{(\xi_j) \subset \mathbb{R} : \|(\xi_j)\|_{cqv} < \infty\big\} \\ &= \big\{(\xi_j) + \xi_\infty(1,1,1\ldots) : \|(\xi_j)\|_{cqv} < \infty,\ \lim_{j\to\infty} \xi_j = 0 \text{ and } \xi_\infty \in \mathbb{R}\big\}, \end{aligned}$$

where the second equality follows from the fact that if (ξ_n) has finite quadratic variation then $\lim_{j\to\infty} \xi_j$ exists. □

It follows therefore from Theorem 3.4.7

Corollary 3.5.9. *J does not have an unconditional basis.*

Theorem 3.5.10. *The operator*

$$T : J^{**} \to J, \quad x^{**} = (\xi_j) \mapsto (\eta_j) = (-e^*_\infty(x^{**}), \xi_1 - e^*_\infty(x^{**}), \xi_2 - e^*_\infty(x^{**}), \ldots)$$

is an isometry between J^{**} and J with respect to the cyclic quadratic variation.

Proof. Let $x^{**} = (\xi_j) \in J^{**}$ and

$$z = (\eta_j) = (-e^*_\infty(x^{**}), \xi_1 - e^*_\infty(x^{**}), \xi_2 - e^*_\infty(x^{**}), \ldots.$$

By (3.17)

$$\begin{aligned} &\sqrt{2}\|x^{**}\| \\ &= \sup_{l\in\mathbb{N}, k_0<k_1<\ldots k_l} \max\Bigg(\Big((\xi_{k_0} - \xi_{k_l})^2 + \sum_{j=1}^{l} (\xi_{k_j} - \xi_{k_{j-1}})^2\Big)^{1/2}, \\ &\qquad \Big(\xi_{k_0}^2 + \xi_{k_l}^2 + \sum_{j=1}^{l} (\xi_{k_j} - \xi_{k_{j-1}})^2\Big)^{1/2}\Bigg) \\ &= \sup_{l\in\mathbb{N}, k_0<k_1<\ldots k_l} \max\Bigg(\Big((\eta_{k_0+1} - \eta_{k_l+1})^2 + \sum_{j=1}^{l} (\eta_{k_j+1} - \eta_{k_{j-1}+1})^2\Big)^{1/2}, \\ &\qquad \Big((\eta_{k_0+1} + e^*_\infty(x^{**}))^2 + (\eta_{k_l+1} + e^*_\infty(x^{**}))^2 + \sum_{j=1}^{l} (\eta_{k_j+1} - \eta_{k_{j-1}+1})^2\Big)^{1/2}\Bigg) \\ &= \sup_{l\in\mathbb{N}, k_0<k_1<\ldots k_l} \max\Bigg(\Big((\eta_{k_0+1} - \eta_{k_l+1})^2 + \sum_{j=1}^{l} (\eta_{k_j+1} - \eta_{k_{j-1}+1})^2\Big)^{1/2}, \end{aligned}$$

$$
\Big((\eta_{k_0+1}-\eta_1)^2+(\eta_1-\eta_{k_l+1})^2+\sum_{j=1}^{l}(\eta_{k_j+1}-\eta_{k_{j-1}+1})^2\Big)^{1/2}\Bigg)
$$

$$
=\max\Big(\sup_{l\in\mathbb{N},1<k_0<k_1<\dots k_l}\Bigg(\Big((\eta_{k_0}-\eta_{k_l})^2+\sum_{j=1}^{l}(\eta_{k_j}-\eta_{k_{j-1}})^2\Big)^{1/2},
$$

$$
\sup_{l\in\mathbb{N},1=k_0<k_1<\dots k_l}\Big((\eta_{k_0}-\eta_{k_l})^2+\sum_{j=1}^{l}(\eta_{k_j}-\eta_{k_{j-1}})^2\Big)^{1/2}\Bigg)
$$

(For the first part we rename k_j+1 to to be k_j, for the second part, we rename 1 to be k_0, k_0+1 to be k_1,....., and k_l+1 to be k_{l+1}, and then we rename $l+1$ to be l)

$$
=\sup_{l\in\mathbb{N},k_0<k_1<\dots k_l}\Bigg(\Big((\eta_{k_0}-\eta_{k_l})^2+\sum_{j=1}^{l}(\eta_{k_j}-\eta_{k_{j-1}})^2\Big)^{1/2},
$$

$$
=\sqrt{2}\|z\|_{cqv}.
$$

Since T is surjective this implies the claim. $\square$

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Chapter 4

Convexity and Smoothness

4.1 Strict Convexity, Smoothness, and Gateaux Differentiablity

Definition 4.1.1. Let X be a Banach space with a norm denoted by $\|\cdot\|$. A map

$$f : X \setminus \{0\} \to X^* \setminus \{0\}, \quad f \mapsto f_x$$

is called a *support mapping* whenever:

a) $f(\lambda x) = \lambda f_x$, for $\lambda > 0$ and

b) If $x \in S_X$, then $\|f_x\| = 1$ and $f_x(x) = 1$ (and thus $f_x(x) = \|x\|^2$ for all $x \in X$).

Often we only define f_x for $x \in S_X$ and then assume that $f_x = \|x\| f_{x/\|x\|}$, for all $x \in X \setminus \{0\}$.

For $x \in X$ a *support functional of* x is an element $x^* \in X^*$, with $\|x^*\| = \|x\|$ and $\langle x^*, x \rangle = \|x\|^2$. Thus a support map is a map $f_{(\cdot)} : X \to X^*$, which assigns to each $x \in X$ a support functional of x.

We say that X is *smooth at* $x_0 \in S_X$ if there exists a unique $f_x \in S_{X^*}$, for which $f_x(x) = 1$, and we say that X is *smooth* if it is smooth at each point of S_X.

The Banach space X is said to have *Gateaux differentiable norm at* $x_0 \in S_X$, if for all $y \in S_X$

$$\rho(x_0, y) = \lim_{h \to 0} \frac{\|x_0 + hy\| - \|x_0\|}{h}$$

exists, and we say that $\|\cdot\|$ is *Gateaux differentiable* if it is Gateaux differentiable norm at each $x_0 \in S_X$.

Example 4.1.2. For $X = L_p[0,1]$, $1 < p < \infty$ the function

$$f : L_p[0,1] \to L_q[0,1], \quad f_x(t) = \operatorname{sign}(x(t))\left|\frac{x(t)}{\|x\|_p}\right|^{p/q}\|x\|_p = \|x\|_p^{1-\frac{p}{q}}|x(t)|^{\frac{p}{q}}$$

is a (and the only) support function for $L_p[0,1]$.

In $L_1[0,1]$, not every element has a unique support functional!

In order to establish a relation between Gateaux differentiability and smoothness we observe the following equalities and inequalities for any $x \in X$, $y \in S_X$, and $h > 0$:

$$\begin{aligned}
\frac{f_x(y)}{\|x\|} &= \frac{f_x(hy)}{h\|x\|} \\
&= \frac{\overbrace{f_x(x) - \|x\|^2}^{=0} + f_x(hy)}{h\|x\|} \\
&= \frac{f_x(x+hy) - \|x\|^2}{h\|x\|} \\
&\leq \frac{|f_x(x+hy)| - \|x\|^2}{h\|x\|} \\
&\leq \frac{\|f_x\|\|x+hy\| - \|x\|^2}{h\|x\|} \\
&= \frac{\|x+hy\| - \|x\|}{h} \\
&= \frac{\|x+hy\|^2 - \|x+hy\|\|x\|}{h\|x+hy\|} \\
&\leq \frac{\|x+hy\|^2 - |f_{x+hy}(x)|}{h\|x+hy\|} \\
&= \frac{f_{x+hy}(x+hy) - |f_{x+hy}(x)|}{h\|x+hy\|} \\
&= \frac{hf_{x+hy}(y) + f_{x+hy}(x) - |f_{x+hy}(x)|}{h\|x+hy\|} \\
&\leq \frac{hf_{x+hy}(y)}{h\|x+hy\|} = \frac{f_{x+hy}(y)}{\|x+hy\|}
\end{aligned}$$

and thus for any $x \in X$, $y \in S_X$, and $h > 0$:

$$\frac{f_x(y)}{\|x\|} \leq \frac{|f_x(x+hy)| - \|x\|}{h\|x\|} \leq \frac{\|x+hy\| - \|x\|}{h} \leq \frac{f_{x+hy}(y)}{\|x+hy\|}. \tag{4.1}$$

Theorem 4.1.3. *Assume X is a Banach space and $x_0 \in S_X$.*
The following statements are equivalent:

a) X is smooth at x_0.

b) Every support mapping $f : x \mapsto f_x$ is norm to w^* continuous from S_X to S_{X^*} at the point x_0.

c) There exists a support mapping $f_{(\cdot)} : x \mapsto f_x$ which is norm to w^* continuous from S_X to S_{X^*} at the point x_0.

d) The norm is Gateaux differentiable at x_0.

In that case

$$f_{x_0}(y) = \rho(x_0, y) = \lim_{h\to 0} \frac{\|x_0 + hy\| - \|x_0\|}{h} \text{ for all } y \in S_X.$$

Proof. $\neg$(b) $\Rightarrow$ $\neg$(a). Assume that $(x_i) \subset S_X$ is a net, which converges in norm to x_0, but for which f_{x_i} does not converge in w^* to f_{x_0}, where $f_{(\cdot)} : X \to X^*$ is a support map. After passing to a subnet we can assume by Alaoglu's Theorem 2.3.2 that (f_{x_i}) converges in w^* to some $x^* \in B_{X^*}$ (which is not f_{x_0}).

As

$$\begin{aligned}
&|x^*(x_0) - 1| \\
&\quad = |x^*(x_0) - f_{x_i}(x_i)| \\
&\quad \le |x^*(x_0) - f_{x_i}(x_0)| + |f_{x_i}(x_0 - x_i)| \\
&\quad \le |x^*(x_0) - f_{x_i}(x_0)| + \|x_0 - x_i\| \to_{i\in I} 0,
\end{aligned}$$

it follows that $x^*(x_0) = 1$, and since $\|x^*\| \le 1$ we must have $\|x^*\| = 1$. Since $x^* \ne f_{x_0}$, X cannot be smooth at x_0.
(b) $\Rightarrow$ (c) is clear (since by The Theorem of Hahn Banach there is always at least one support map).
(c) $\Rightarrow$ (d) Follows from (4.1), and from applying (4.1) to $-y$ instead of y which gives

$$\frac{\|x_0 - hy\| - \|x_0\|}{-h\|x_0\|} = -\frac{\|x_0 + h(-y)\| - \|x_0\|}{h\|x_0\|} \le -\frac{f_{x_0}(-y)}{\|x_0\|} = f_{x_0}(y)$$

and

$$\frac{\|x_0 - hy\| - \|x_0\|}{-h\|x_0\|} = -\frac{\|x_0 + h(-y)\| - \|x_0\|}{h\|x_0\|} \ge -\frac{f_{x_0+h(-y)}(-y)}{\|x_0 + h(-y)\|} = \frac{f_{x_0+h(-y)}(y)}{\|x_0 + h(-y)\|}.$$

(d) $\Rightarrow$ (a) Let $f \in S_{x^*}$ be such that $f(x_0) = \|x_0\| = 1$. Since (4.1) is true for any support function it follows that

$$f(y) \leq \frac{\|x_0 + hy\| - \|x_0\|}{h}, \text{ for all } y \in S_X \text{ and } h > 0,$$

and

$$\frac{\|x_0 - hy\| - \|x_0\|}{-h} = -\frac{\|x_0 + (-y)\| - \|x_0\|}{h} \leq -f(-y) = f(y)$$
$$\text{for all } y \in S_X \text{ and } h > 0.$$

Thus, by assumption (d), $\rho(x_0, y) = f(y)$, which proves the uniqueness of $f \in S_{X^*}$ with $f(x_0) = 1$. $\square$

Definition 4.1.4. A Banach space X with norm $\|\cdot\|$ is called *strictly convex* whenever $S(X)$ contains no non-trivial line segement, i.e. if for all $x, y \in S_X$, $x \neq y$ it follows that $\|x + y\| < 2$.

Theorem 4.1.5. *If X^* is strictly convex then X is smooth, and if X^* is smooth the X is strictly convex.*

Proof. If X is not smooth then there exists an $x_0 \in S_X$, and two functionals $x^* \neq y^*$ in S_{X^*} with $x^*(x_0) = y^*(x_0) = 1$ but this means that

$$\|x^* + y^*\| \geq (x^* + y^*)(x_0) = 2,$$

which implies that X^* is not strictly convex. If X is not strictly convex then there exist $x \neq y$ in S_X so that $\|\lambda x + (1 - \lambda)y\| = 1$, for all $0 \leq \lambda \leq 1$. So let $x^* \in S_{X^*}$ such that

$$x^*\Big(\frac{x + y}{2}\Big) = 1.$$

But this implies that

$$1 = x^*\Big(\frac{x + y}{2}\Big) = \frac{1}{2}x^*(x) + \frac{1}{2}x^*(y) \leq \frac{1}{2} + \frac{1}{2} = 1,$$

which implies that $x^*(x) = x^*(y) = 1$, which by viewing x and y to be elements in X^{**}, implies that X^* is not smooth. $\square$

4.2 Uniform Convexity and Uniform Smoothness

Definition 4.2.1. Let X be a Banach space with norm $\|\cdot\|$. We say that *the norm of X is Fréchet differentiable at $x_0 \in S_X$* if

$$\lim_{h\to 0} \frac{\|x_0 + hy\| - \|x_0\|}{h}$$

exists uniformly in $y \in S_X$.

We say that *the norm of X is Fréchet differentiable* if the norm of X is Fréchet differentiable at each $x_0 \in S_X$.

Remark. By Theorem 4.1.3 it follows from the Frechét differentiability of the norm at x_0 that there a unique support functional $f_{x_0} \in S_X^*$ and

$$\lim_{h\to 0} \left| \frac{\|x_0 + hy\| - \|x_0\|}{h} - f_{x_0}(y) \right| = 0,$$

uniformly in y and thus that (put $z = hy$)

$$\lim_{z\to 0} \frac{\|x_0 + z\| - \|x_0\| - f_{x_0}(z)}{\|z\|} = 0.$$

In particular, if X has a Fréchet differentiable norm it follows from Theorem 4.1.3 that there is a unique support map $x \to f_x$.

Proposition 4.2.2. *Let X be a Banach space with norm $\|\cdot\|$. Then the norm is Fréchet differentiable if and only if the support map is norm-norm continuous.*

Proof. (We assume that $\mathbb{K} = \mathbb{R}$) "$\Rightarrow$" Assume that $(x_n) \subset S_X$ converges to x_0 and put $x_n^* = f_{x_n}$, $n \in \mathbb{N}$, and $x_0^* = f_{x_0}$. It follows from Theorem 4.1.3 that $x_n^*(x_0) \to 1$, for $n \to \infty$. Assume that our claim were not true, and we can assume that for some $\varepsilon > 0$ we have $\|x_n^* - x_0^*\| > 2\varepsilon$, and therefore we can choose vectors $z_n \in S_X$, for each $n \in \mathbb{N}$ so that $(x_n^* - x_0^*)(z_n) > 2\varepsilon$. But then

$$\begin{aligned}
x_0^*(x_0) - x_n^*(x_0) &\le \big(x_0^*(x_0) - x_n^*(x_0)\big)\Big(\frac{1}{\varepsilon}\big(\underbrace{x_n^*(z_n) - x_0^*(z_n)}_{>2\varepsilon}\big) - 1\Big) \\
&= \big(x_n^*(x_0) - x_0^*(x_0)\big) + \frac{1}{\varepsilon}\big(x_0^*(z_n) - x_n^*(z_n)\big)\big(x_n^*(x_0) - x_0^*(x_0)\big) \\
&= \big(x_n^* - x_0^*\big)\Big(x_0 + z_n\frac{1}{\varepsilon}\big(x_0^*(x_0) - x_n^*(x_0)\big)\Big)
\end{aligned}$$

$$\begin{aligned}
&\leq \Big|x_n^*\Big(x_0 + z_n\frac{1}{\varepsilon}\big(x_0^*(x_0) - x_n^*(x_0)\big)\Big)\Big| - \\
&\qquad - x_0^*\Big(x_0 + z_n\frac{1}{\varepsilon}\big(x_0^*(x_0) - x_n^*(x_0)\big)\Big) \\
&\leq \Big\|x_0 + z_n\frac{1}{\varepsilon}\big(x_0^*(x_0) - x_n^*(x_0)\big)\Big\| \\
&\qquad - \|x_0\| - x_0^*\Big(z_n\frac{1}{\varepsilon}\big(x_0^*(x_0) - x_n^*(x_0)\big)\Big).
\end{aligned}$$

Thus if we put

$$y_n = z_n\frac{1}{\varepsilon}\big(x_0^*(x_0) - x_n^*(x_0)\big),$$

it follows that $\|y_n\| \to 0$, if $n \to \infty$, and, using the Fréchet differentiability of the norm that (note that $\big(x_0^*(x_0) - x_n^*(x_0)\big)/\|y_n\| = \varepsilon$) we deduce that

$$0 < \varepsilon = \frac{x_0^*(x_0) - x_n^*(x_0)}{\|y_n\|} \leq \frac{\|x_0 + y_n\| - \|x_0\| - x_0^*(y_n)}{\|y_n\|} \to_{n\to\infty} 0,$$

which is a contradiction.
"$\Leftarrow$" From (4.1) it follows that for $x, y \in S_X$, and $h \in \mathbb{R}$

$$\begin{aligned}
&\left|\frac{\|x+hy\| - \|x\|}{h} - f_x(y)\right| \\
&\qquad \leq \left|\frac{f_{x+hy}(y)}{\|x+hy\|} - f_x(y)\right| \\
&\qquad \leq \big|f_{x+hy}(y) - f_x(y)\big| + \left|\frac{f_{x+hy}(y)}{\|x+hy\|} - f_{x+hy}(y)\right| \\
&\qquad \leq \|f_{x+hy} - f_x\| + \left|\frac{1}{1+|h|} - 1\right|\|f_{x+hy}\|,
\end{aligned}$$

which converges uniformly in y to 0 and proves our claim. □

Definition 4.2.3. Let X be a Banach space with norm $\|\cdot\|$.
We say that the norm is *uniformly Fréchet differentiable on* S_X if

$$\lim_{h\to 0}\Big|\frac{\|x+hy\| - \|x\|}{h} - f_x(y)\Big|,$$

uniformly in $x \in S_X$ and $y \in S_X$. In other words if for all $\varepsilon > 0$ there is a $\delta > 0$ so that for all $x, y \in S_X$ and all $h \in \mathbb{R}$, $0 < |h| < \delta$

$$\Big|\frac{\|x+hy\| - \|x\|}{h} - f_x(y)\Big| < \varepsilon.$$

X is *uniformly convex* if for all $\varepsilon > 0$ there is a $\delta > 0$ so that for all $x, y \in S_X$ with $\|x - y\| \geq \varepsilon$ it follows that $\|(x + y)/2\| < 1 - \delta$.

We call

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, \|x - y\| \geq \varepsilon\right\}, \text{ for } \varepsilon \in [0, 2]$$

the *modulus of uniform convexity of X*.

X is called *uniform smooth* if for all $\varepsilon > 0$ there exists a $\delta > 0$ so that for all $x, y \in S_X$ and all $h \in (0, \delta]$

$$\|x + hy\| + \|x - hy\| < 2 + \varepsilon h.$$

The *modulus of uniform smoothness of X* is the map $\rho : [0, \infty) \to [0, \infty)$

$$\rho_X(\tau) = \sup\left\{\frac{\|x+z\|}{2} + \frac{\|x-z\|}{2} - 1 : x, z \in X, \|x\| = 1, \|z\| \leq \tau\right\}.$$

Remark. X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for all $\varepsilon > 0$. X is uniformly smooth if and only if $\lim_{\tau \to 0} \rho_X(\tau)/\tau = 0$.

Theorem 4.2.4. *For a Banach space X the following statements are equivalent.*

a) There exists a support map $x \to f_x$ which uniformly continuous on S_X with respect to the norms.

b) The norm on X is uniformly Fréchet differentiable on S_X.

c) X is uniformly smooth.

d) X^* is uniformly convex.

e) Every support map $x \to f_x$ is uniformly continuous on S_X with respect to the norms.

Proof. "(a)⇒(b)" We proceed as in the proof of Proposition 4.2.2. From (4.1) it follows that for $x, y \in S_X$, and $h \in \mathbb{R}$

$$\left|\frac{\|x + hy\| - \|x\|}{h} - f_x(y)\right|$$
$$\leq \left|\frac{f_{x+hy}(y)}{\|x + hy\|} - f_x(y)\right|$$

$$\leq \big|f_{x+hy}(y) - f_x(y)\big| + \left|\frac{f_{x+hy}(y)}{\|x+hy\|} - f_{x+hy}(y)\right|$$
$$\leq \|f_{x+hy} - f_x\| + \left|\frac{1}{1+|h|} - 1\right|\|f_{x+hy}\|$$

which converges by (a) uniformly in x and y, to 0.
"(b)⇒(c)". Assuming (b) we can choose for $\varepsilon > 0$ a $\delta > 0$ so that for all $h \in (0,\delta)$ and all $x, y \in S_X$

$$\left|\frac{\|x+hy\| - \|x\|}{h} - f_x(y)\right| < \varepsilon/2.$$

But this implies that for all $h \in (0,\delta)$ and all $x, y \in S_X$ we have

$$\begin{aligned}&\|x+hy\| + \|x-hy\| \\ &\quad = 2 + h\left(\frac{\|x+hy\| - \|x\|}{h} - f_x(y) + \Big(\frac{\|x+h(-y)\| - \|x\|}{h} - f_x(-y)\Big)\right) \\ &\quad \leq 2 + \varepsilon h,\end{aligned}$$

which implies our claim.
"(c)⇒(d)". Let $\varepsilon > 0$. By (c) we can find $\delta > 0$ such that for all $x \in S_X$ and $z \in X$, with $\|z\| \leq \delta$, we have $\|x+z\| + \|x-z\| \leq 2 + \varepsilon\|z\|/4$.

Let $x^*, y^* \in S_{X^*}$ with $\|x^* - y^*\| \geq \varepsilon$. There is a $z \in X$, $\|z\| \leq \delta/2$ so that $(x^* - y^*)(z) \geq \varepsilon\delta/2$. This implies

$$\begin{aligned}\|x^* + y^*\| &= \sup_{x\in S_X} \big|(x^*+y^*)(x)\big| \\ &= \sup_{x\in S_X} \big|x^*(x+z) + y^*(x-z) - (x^*-y^*)(z)\big| \\ &\leq \sup_{x\in S_X} \|x+z\| + \|x-z\| - \varepsilon\delta/2 \\ &\leq 2 + \varepsilon\|z\|/4 - \varepsilon\delta/2 < 2 - \varepsilon\delta/4.\end{aligned}$$

"(d)⇒(e)". Let $x \mapsto f_x$ be a support functional. By (d) we can choose for $\varepsilon >$ a δ so that for all $x^*, y^* \in S_{X^*}$ we have $\|x^* - y^*\| < \varepsilon$, whenever $\|x^* + y^*\| > 2 - \delta$.

Assume now that $x, y \in S_X$ with $\|x - y\| < \delta$. Then

$$\|f_x + f_y\| \geq \frac{1}{2}(f_x + f_y)(x+y)$$

$$= f_x(x) + f_y(y) + \frac{1}{2} f_x(y-x) + \frac{1}{2} f_y(x-y)$$
$$\geq 2 - \|x-y\| \geq 2 - \delta,$$

which implies that $\|f_x - f_y\| < \varepsilon$, which proves our claim.
"(e)⇒(a)". Clear. □

Theorem 4.2.5. *Every uniformly convex and every uniformly smooth Banach space is reflexive.*

Proof. Assume that X is uniformly convex, and let $x^{**} \in S_{X^{**}}$. Since B_X is w^*-dense in $B_{X^{**}}$ we can find a net $(x_i)_{i\in I}$ which converges with respect to w^* to x^{**}. Since for every $\eta > 0$ there is a $x^* \in S_{X^*}$ with $\lim_{i\in I} x^*(x_i) = x^{**}(x^*) > 1-\eta$, it follows that $\lim_{i\in I} \|x_i\| = 1$ and we can therefore assume that $\|x_i\| = 1$, $i \in I$. We claim that $\chi(x_i)$ is a Cauchy net with respect to the norm to x^{**}, which would finish our proof.

So let $\varepsilon > 0$ and choose δ so that $\|x+y\| > 2-\delta$ implies that $\|x-y\| < \varepsilon$, for any $x, y \in S_X$. Then choose $x^* \in S_{X^*}$, so that $x^{**}(x^*) > 1 - \delta/4$, and finally let $i_0 \in I$ so that $x^{**}(x_i) \geq 1 - \delta/2$, for all $i \geq i_0$. It follows that

$$\|x_i + x_j\| \geq x^*(x_i + x_j) \geq 2 - \delta \text{ whenever } i, j \geq i_0,$$

and thus $\|x_i - x_j\| < \varepsilon$, which verifies our claim.

If X is uniformly smooth it follows from Theorem 4.2.4 that X^* is uniformly convex. The first part yields that X^* is reflexive, which implies that X is reflexive. □

Chapter 5

L_p-spaces

5.1 Reduction to the Case ℓ_p and L_p

The main (and only) result of this section is the following Theorem.

Theorem 5.1.1. *Let $1 \le p < \infty$ and let (Ω, Σ, μ) be a separable measure space, i.e. Σ is generated by a countable set of subsets of Ω.*

Then there is a countable set I so that $L_p(\Omega, \Sigma, \mu)$ is isometrically isomorphic to $L_p[0,1] \oplus_p \ell_p(I)$ or to $\ell_p(I)$.

Moreover, if (Ω, Σ, μ) has no atoms, and is not 0, we can choose I to be empty and, thus, $L_p(\Omega, \Sigma, \mu)$ is isometrically isomorphic to $L_p[0,1]$.

Proof. First note that the assumption that Σ is generated by a countable set say $\mathcal{D} \subset \mathcal{P}(\Omega)$ implies that $L_p(\mu)$ is separable. Indeed, the algebra generated by $\mathcal{D}$ is $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$, where $\mathcal{A}_n$ is defined recursively for every $n \in \mathbb{N}$ as follows: $\mathcal{A}_1 = \mathcal{D}$, and, assuming, $\mathcal{A}_n$ is defined we let first

$$\mathcal{A}'_{n+1} = \Big\{ \bigcup_{j=1}^{k} B_j : k \in \mathbb{N}, B_j \in \mathcal{A}_n \text{ or } B_j^c \in \mathcal{A}_n \Big\}$$

and then

$$\mathcal{A}_{n+1} = \Big\{ \bigcap_{j=1}^{k} B_j : k \in \mathbb{N}, B_j \in \mathcal{A}_n \text{ or } B_j^c \in \mathcal{A}_n \Big\}.$$

This proves that $\mathcal{A}$ is countable. Then we observe that $\text{span}(1_A : A \in \mathcal{A})$ is dense in $L_p(\mu)$

We first reduce to the σ-finite case.

Step 1: $L_p(\Omega,\Sigma,\mu)$ is isometrically isomorphic to a space $L_p(\Omega',\Sigma',\mu')$ where (Ω',Σ',μ') is a σ-finite measure space.

Let $(f_n) \subset L_p(\Omega,\Sigma,\mu)$ be a dense sequence in $L_p(\Omega,\Sigma,\mu)$ and define

$$\Omega' = \bigcup_{n\in\mathbb{N}} \big\{|f_n| > 0\big\}.$$

Since $\{|f_n| > 0\}$ is a countable union of sets of finite measure, namely

$$\{|f_n| > 0\} = \bigcup_{m\in\mathbb{N}} \{|f_n| > 1/m\}$$

Ω' is also σ-finite. Moreover, for any $f \in L_p(\Omega,\Sigma,\mu)$ it follows that $\{|f| > 0\} \subset \Omega'$ μ a.e. Therefore we can choose $\Sigma' = \Sigma|_{\Omega'} = \{A \in \Sigma : A \subset \Omega'\}$ and $\mu' = \mu|_{\Sigma'}$.

Step 2: Assume (Ω,Σ,μ) is a σ-finite measure space. Let I be the set of all atoms of (Ω,Σ,μ). Recall that $A \in \Sigma$ is called an *atom*, if $\mu(A) > 0$ and if for every measurable $B \subset A$, either $\mu(B) = \mu(A)$, or $\mu(B) = 0$. Since μ is σ-finite, I is countable, and $\mu(A) < \infty$ for all $A \in I$. We put $\Omega' = \Omega \setminus \bigcup_{A\in I} A$, $\Sigma' = \Sigma|_{\Omega'}$ and $\mu' = \mu|_{\Sigma'}$. Then

$$\begin{aligned} &T : L_p(\Omega,\Sigma,\mu) \to \ell_p(I) \oplus_p L_p(\Omega',\Sigma',\mu'),\\ &f \mapsto \left(\Big(\frac{1}{\mu^{1/p}(A)}\int_A f d\mu : A \in I\Big), f|_{\Omega'}\right), \end{aligned}$$

is an isometry onto $\ell_p(I) \oplus_p L_p(\Omega',\Sigma',\mu')$.

Now either $\mu' = 0$ or it is an atomless σ-finite measure.
In the next step we reduce to the case of μ being an atomless probability measure.

Step 3: Assume that (Ω,Σ,μ) is σ-finite, atomless and not 0. Then there is an atomless probability μ' on (Ω,Σ) so that $L_p(\Omega,\Sigma,\mu)$ is isometrically isomorphic to the space $L_p(\Omega,\Sigma,\mu')$.

Since (Ω,Σ,μ) is σ-finite there is an $f \in L_1(\Omega,\Sigma,\mu)$, with $f(\omega) > 0$ for all $\omega \in \Omega$ and $\|f\|_1 = 1$. Let μ' be the measure whose Radon Nikodym derivative with respect to μ is f (thus μ' is a probability measure) and consider the operator

$$T : L_p(\Omega,\Sigma,\mu) \to L_p(\Omega,\Sigma,\mu'), \quad g \mapsto g \cdot f^{-1/p},$$

which is an isometry onto $L_p(\Omega,\Sigma,\mu')$. Using an operator like T is often called "change of density" -argument.

Step 4: Reduction to $[0,1]$. Assume (Ω,Σ,μ) is an atomless countably generated probability space. Let $(B_n)\subset\Sigma$ be a sequence which generates Σ. By induction we choose for each $n\in\mathbb{N}_0$ a finite Σ-partition $\mathcal{P}_n = (P_1^{(n)},P_2^{(n)},\dots P_{k_n}^{(n)})$ of Ω with the following properties:

(5.1) $\{B_1,B_2,\dots B_n\}\subset\sigma(\mathcal{P}_n)$ (the σ-algebra generated by $\mathcal{P}_n$),

(5.2) $\mu(P_i^{(n)})\le 2^{-n}$, for $i=1,2,\dots,k_n$,

(5.3) $\mathcal{P}_n$ is a subpartition of $\mathcal{P}_{n-1}$ if $n>1$, i.e. for each $i\in\{1,\dots k_{n-1}\}$ there are $s_n(i)\le t_n(i)$ in $\{1,\dots k_n\}$, so that

$$P_i^{(n-1)}=\bigcup_{j=s_n(i)}^{t_n(i)} P_j^{(n)}.$$

Put for $n\in\mathbb{N}$ and $1\le i\le k_n$

$$\tilde{P}_i^{(n)}=\Big[\sum_{j\le i-1}\mu(P_j^{(n)}),\sum_{j\le i}\mu(P_j^{(n)})\Big),\ \text{if } j<k_n \text{ and}$$

$$\tilde{P}_{k_n}^{(n)}=\Big[\sum_{j\le k_n-1}\mu(P_j^{(n)}),\sum_{j\le k_n}\mu(P_j^{(n)})\Big]$$

and $\tilde{\mathcal{P}}^{(n)}=(\tilde{P}_1^{(n)},\tilde{P}_2^{(n)},\dots,\tilde{P}_{k_n}^{(n)})$. Then $\tilde{\mathcal{P}}^{(n)}$ is a Borel partition of $[0,1]$ into intervals, with $\lambda(\tilde{P}_i^{(n)})=\mu(P_i^{(n)})$, for each $i\le k_n$, and $\bigcup_{n\in\mathbb{N}}\tilde{P}^{(n)}$ generate the Borel σ-algebra on $[0,1]$.

For $n\in\mathbb{N}$ put

$$V_n=\Big\{\sum_{i=1}^{k_n}a_i\chi_{P_i^{(n)}} : a_i \text{ scalars}\Big\},$$

Then V_n is a vector space and $V=\bigcup_n V_n$ is a dense subspace of $L_P(\mu)$. Similarly $\tilde{V}$, with

$$\tilde{V}_n=\Big\{\sum_{i=1}^{k_n}a_i\chi_{\tilde{P}_i^{(n)}} : a_i \text{ scalars}\Big\},$$

is a dense subspace of $L_p[0,1]$, and

$$T:V\to\tilde{V},\quad \sum_{i=1}^{k_n}a_i\chi_{P_i^{(n)}}\mapsto\sum_{i=1}^{k_n}a_i\chi_{\tilde{P}_i^{(n)}},$$

is an isometry whose image is dense in $L_p[0,1]$. Thus T extends to an isometry from $L_p(\mu)$ onto $L_p[0,1]$. $\square$

5.2 Uniform Convexity and Uniform Smoothness of L_p, $1 < p < \infty$

Let (Ω, Σ, μ) be a measure space. The first goal of this section is to prove the following

Theorem 5.2.1. *Let $1 < p < \infty$ and denote the modulus of uniform convexity of $L_p(\mu)$ by δ_p. Then for any $1 < p < \infty$ there is a $c_p > 0$ so that.*

$$\delta_p(\varepsilon) \geq \begin{cases} c_p \varepsilon^2 & \text{if } 1 < p < 2 \\ c_p \varepsilon^p & \text{if } 2 < p < \infty. \end{cases}$$

Lemma 5.2.2. *Assume $\xi, \eta \in \mathbb{R}$*

a) If $2 \leq p < \infty$, then

$$|\xi + \eta|^p + |\xi - \eta|^p \geq 2(|\xi|^p + |\eta|^p).$$

b) If $0 < p \leq 2$

$$|\xi + \eta|^p + |\xi - \eta|^p \leq 2(|\xi|^p + |\eta|^p).$$

If $p \neq 2$ equality in (a) and (b) only holds if either ξ or η is zero.

Proof. If $p = 2$ we have equality by the binomial formula.
If $2 < p < \infty$ and $\alpha, \beta \in \mathbb{R}$, we apply Hölder's inequality to the function

$$\{1, 2\} \to \{\alpha^2, \beta^2\}, 1 \mapsto \alpha^2,\ 2 \mapsto \beta^2,$$

the counting measure on $\{1, 2\}$, and the exponents $p/2$ and $p/(p-2)$.

$$\alpha^2 + \beta^2 \leq (|\alpha|^p + |\beta|^p)^{2/p} 2^{(p-2)/p}, \text{ and, thus,}$$
$$|\alpha|^p + |\beta|^p \geq (\alpha^2 + \beta^2)^{p/2} 2^{(2-p)/2}. \tag{5.4}$$

If $0 < p < 2$ we can replace p by $4/p$ and obtain

$$|\alpha|^{4/p} + |\beta|^{4/p} \geq (\alpha^2 + \beta^2)^{2/p} 2^{(p-2)/p},$$

and if we replace $|\alpha|$ and $|\beta|$ by $|\alpha|^{p/2}$ and $|\beta|^{p/2}$ respectively, we obtain

$$|\alpha|^2 + |\beta|^2 \geq (|\alpha|^p + |\beta|^p)^{2/p} 2^{(p-2)/p}, \text{ or}$$
$$|\alpha|^p + |\beta|^p \leq 2^{(2-p)/2} (|\alpha|^2 + |\beta|^2)^{p/2}. \tag{5.5}$$

Since

$$0 \le \frac{\xi^2}{\eta^2+\xi^2} \le 1$$

we derive that

$$\frac{|\xi|^p}{(|\eta|^2+|\xi|^2)^{p/2}} \begin{cases} \le \frac{\xi^2}{\eta^2+\xi^2} & \text{if } 2<p \\ \ge \frac{\xi^2}{\eta^2+\xi^2} & \text{if } 2>p. \end{cases} \tag{5.6}$$

Forming similar inequalities by exchanging the roles of η and ξ and adding them we get

$$|\eta|^p+|\xi|^p \begin{cases} \le (|\eta|^2+|\xi|^2)^{p/2} & \text{if } 2<p \\ \ge (|\eta|^2+|\xi|^2)^{p/2} & \text{if } 2>p. \end{cases} \tag{5.7}$$

Note that equality in (5.7) can only hold if $\eta=0$ or $\xi=0$.

Letting now $\alpha=|\xi+\eta|$ and $\beta=|\xi-\eta|$ we deduce from (5.4) and (5.7) if $p>2$

$$\begin{aligned} |\xi+\eta|^p+|\xi-\eta|^p &\ge \big(|\xi+\eta|^2+|\xi-\eta|^2\big)^{p/2}2^{(2-p)/2} \\ &= 2\big(\xi^2+\eta^2\big)^{p/2} \ge 2\big(|\eta|^p+|\xi|^p\big), \end{aligned}$$

which finishes the proof of part (a), while part (b) follows from applying (5.5) and (5.7). □

Corollary 5.2.3. *Let $0<p<\infty$ and $f,g\in L_p(\mu)$. Then*

$$\|f+g\|_p^p+\|f-g\|_p^p \begin{cases} \ge 2(\|f\|_p^p+\|g\|_p^p) & \textit{if } p\ge 2 \\ \le 2(\|f\|_p^p+\|g\|_p^p) & \textit{if } p\le 2. \end{cases}$$

If $p\ne 2$ equality only holds if $f\cdot g=0$ μ-almost everywhere.

Lemma 5.2.4. *Let $1<p<2$. Then there is a positive constant $C=C(p)$ so that*

$$\Big(\Big|\frac{s-t}{C}\Big|^2+\Big|\frac{s+t}{2}\Big|^2\Big)^{1/2} \le \Big(\frac{|s|^p+|t|^p}{2}\Big)^{1/p}. \tag{5.8}$$

Proof. We can assume without loss of generality that $s=1>|t|$ and need therefore to show that for some $C>0$ and all $t\in[-1,1]$ we have

$$\Big(\frac{1-t}{C}\Big)^2 \le \phi(t) = \Big(\frac{1+|t|^p}{2}\Big)^{2/p} - \Big(\frac{1+t}{2}\Big)^2. \tag{5.9}$$

Since ϕ is strictly positive on $[-1,0]$ we only need to find C so that (5.8) holds for all $t \in [0,1]$. Since $\xi \mapsto \xi^{1/p}$ is strictly concave it follows for all $0 < t < 1$

$$\Big(\frac{1}{2} + \frac{t^p}{2}\Big)^{2/p} > \Big(\frac{1}{2} + \frac{t}{2}\Big)^2,$$

we only need to show that

$$\lim_{t \to 1^-} \frac{\phi(t)}{(1-t)^2} > 0. \tag{5.10}$$

We compute

$$\begin{aligned}\frac{d^2}{dt^2}\phi(t) &= \frac{d}{dt}\Big[2^{-(2/p)+1}(1+t^p)^{(2/p)-1}t^{p-1} - \frac{1}{2}(1+t)\Big]\\ &= 2^{-(2/p)+1}(2-p)(1+t^p)^{(2/p)-2}t^{2p-2}\\ &\qquad + 2^{-(2/p)+1}(p-1)(1+t^p)^{(2/p)-1}t^{p-2} - \frac{1}{2}\end{aligned}$$

and thus

$$\frac{d}{dt}\phi(t)\big|_{t=1} = 0, \text{ and}$$

$$\frac{d^2}{dt^2}\phi(t)\big|_{t=1} = (2-p)(1/2) + (p-1) - (1/2) = (p-1)/2 > 0$$

Applying now twice the L'Hospital rule, we deduce our wanted inequality (5.10) $\square$

Via integrating, Lemma 5.2.4 yields

Corollary 5.2.5. *If $1 < p \le 2$ and $f, g \in L_p(\mu)$ and if $C = C(p)$ is as in Lemma 5.2.4, it follows*

$$\begin{aligned}\bigg\|\Big(\Big|\frac{f-g}{C}\Big|^2 + \Big|\frac{f+g}{2}\Big|^2\Big)^{1/2}\bigg\|_p &\le \bigg\|\Big(\frac{|f|^p + |g|^p}{2}\Big)^{1/p}\bigg\|_p\\ &= 2^{-1/p}\Big(\|f\|^p + \|g\|^p\Big)^{1/p}.\end{aligned} \tag{5.11}$$

Proposition 5.2.6. *If $1 \le p < q < \infty$ and $f_j \in L_p$, $j = 1, 2, \ldots$ then*

$$\bigg(\sum_{j=1}^n \|f_i\|_p^q\bigg)^{1/q} \le \bigg\|\Big(\sum_{j=1}^n |f_j|^q\Big)^{1/q}\bigg\|_p.$$

Proof. We can assume without loss of generality that

$$\sum_{j=1}^n \|f_i\|_p^q = 1.$$

We estimate

$$\begin{aligned}
\Big\| \Big(\sum_{j=1}^n |f_j|^q \Big)^{1/q} \Big\|_p &= \Big\| \Big(\sum_{j=1}^n \|f_j\|_p^q \Big(\frac{|f_j|}{\|f_j\|_p} \Big)^q \Big)^{1/q} \Big\|_p \\
&= \Big\| \Big(\Big(\sum_{j=1}^n \|f_j\|_p^q \Big(\frac{|f_j|}{\|f_j\|_p} \Big)^q \Big)^{p/q} \Big)^{1/p} \Big\|_p \\
&\geq \Big\| \Big(\sum_{j=1}^n \|f_j\|_p^q \Big(\frac{|f_j|}{\|f_j\|_p} \Big)^p \Big)^{1/p} \Big\|_p \\
&\quad \text{(We use the concavity of the function } \xi \mapsto \xi^{p/q}.) \\
&\geq \Big\| \sum_{j=1}^n \|f_j\|_p^q \Big(\frac{|f_j|}{\|f_j\|_p} \Big)^p \Big\|_1^{1/p} = 1,
\end{aligned}$$

which proves our claim. □

Proof of Theorem 5.2.1. For $2 \leq p < \infty$ we will deduce our claim from Corollary 5.2.3. For $f, g \in L_p(\mu)$, with $\|f\| = \|g\| = 1$, we deduce from the first inequality in Corollary 5.2.3

$$2^p = \frac{1}{2}\big[\|(f+g)-(f-g)\|^p + \|(f+g)+(f-g)\|^p\big] \geq \|f+g\|^p + \|f-g\|^p$$

and thus, using the approximation $(2^p+\xi)^{1/p} = 2 + \frac{1}{p}2^{1-p}\xi + o(\xi)$, we deduce that

$$\|f+g\| \leq \big(2^p - \|f-g\|^p\big)^{1/p} = 2 - \frac{1}{p}2^{1-p}\|f-g\|^p + o(\|f-g\|^p),$$

which implies our claim.

Now assume that $1 < p < 2$. Let $f, g \in S_{L_p(\mu)}$ with $\varepsilon = \|f-g\|_p > 0$. Let $C = C(p)$ be the constant in Corollary 5.2.5.

We deduce from Proposition 5.2.6 and Corollary 5.2.5 that

$$\Big(\Big\| \frac{f-g}{C} \Big\|_p^2 + \Big\| \frac{f+g}{2} \Big\|_p^2 \Big)^{1/2} \leq \Big\| \Big(\Big| \frac{f-g}{C} \Big|^2 + \Big| \frac{f+g}{2} \Big|^2 \Big)^{1/2} \Big\|_p$$

$$\leq \left\| \left(\frac{|f|^p + |g|^p}{2} \right)^{1/p} \right\|_p$$
$$= 2^{-1/p} \Big(\|f\|^p + \|g\|^p \Big)^{1/p} = 1.$$

Solving for $\|(f+g)/2\|_p$ leads to

$$\left\| \frac{f+g}{2} \right\|_p \leq \sqrt{1 - \left\| \frac{f-g}{C} \right\|_p^2} = 1 - \frac{\varepsilon^2}{2C} + o(\varepsilon^2)$$

which implies our claim. □

5.3 On "Small Subspaces" of L_p

By *small subspaces* of $L_p[0,1]$ we usually mean subspaces which are not isomorphic to the whole space. Khintchine's theorem, says that $L_p[0,1]$, $1 \le p \le \infty$ contains isomorphic copies of ℓ_2, which are complemented if $1 < p < \infty$.

Note that all the arguments below can be made in a general probability space $(\Omega, \Sigma, \mathbb{P})$ on which a *Rademacher sequence* (r_i) *exists*, i.e. (r_i) is an independent sequence of random variables for which $\mathbb{P}(r_i = 1) = \mathbb{P}(r_i = -1) = 1/2$.

Theorem 5.3.1. [Khintchine's Theorem]
$L_p[0,1]$, $1 \le p \le \infty$ contains a subspaces isomorphic to ℓ_2, if $1 < p < \infty$ $L_p[0,1]$, contains a complemented subspaces isomorphic to ℓ_2.

Remark. By Theorem 5.1.1 the conclusion of Theorem 5.3.1 holds for all spaces $L_p(\mu)$, as long as μ is a measure on some measurable space (Ω, Σ) for which there is in $\Omega' \subset \Omega$, $\Omega' \in \Sigma$ so that $\mu|_{\Omega'}$ is a non zero atomless measure.

Definition 5.3.2. *The Rademacher functions* are the functions:

$$r_n : [0,1] \to \mathbb{R}, \qquad t \mapsto \operatorname{sign}(\sin 2^n \pi t), \text{ whenever } n \in \mathbb{N}.$$

Lemma 5.3.3. [Khintchine inequality]
For every $p \in [1, \infty)$ there are numbers $0 < A_p \le 1 \le B_p$ so that for any $m \in \mathbb{N}$ and any scalars $(a_i)_{i=1}^m$.

$$A_p\Big(\sum_{i=1}^m |a_i|^2\Big)^{1/2} \le \Big\| \sum_{i=1}^m a_i r_i \Big\|_{L_p} \le B_p\Big(\sum_{i=1}^m |a_i|^2\Big)^{1/2}. \tag{5.12}$$

Proof. We prove the claim for Banach spaces over the reals. The complex case can be easily deduced (using some worse constants).

Since for $p > r \ge 1$

$$\Big\| \sum_{i=1}^m a_i r_i \Big\|_{L_p} \ge \Big\| \sum_{i=1}^m a_i r_i \Big\|_{L_r},$$

it is enough to prove the right hand inequality for all even integers, and then choose $B_p = B_{p'}$ with $p' = 2\lceil \frac{p}{2} \rceil$, for $1 \le p < \infty$ and the left hand inequality for $p = 1$, and take $A_p = A_1$.

We first show the existence of B_{2k} for any $k \in \mathbb{N}$. For scalars $(a_i)_{i=1}^m$ we deduce

$$\int_0^1 \Big(\sum_{i=1}^m a_i r_i(t)\Big)^{2k} d$$

$$= \sum_{\substack{(\alpha_1,\alpha_2,\dots\alpha_m)\in\mathbb{N}_0^m \\ \sum\alpha_i=2k}} A(\alpha_1,\alpha_2,\dots\alpha_m) a_1^{\alpha_1} a_2^{\alpha_2}\dots a_m^{\alpha_m} \int_0^1 r_1^{\alpha_1}(t) r_2^{\alpha_2}(t)\dots r_m^{\alpha_m}(t)\,dt$$

where $A(\alpha_1,\alpha_2,\dots\alpha_m) = \dfrac{\Big(\sum_{i=1}^m \alpha_i\Big)!}{\prod_{i=1}^m \alpha_i!}$

$$= \sum_{\substack{(\beta_1,\beta_2,\dots\beta_m)\in\mathbb{N}_0^m \\ \sum\beta_i=k}} A(2\beta_1,2\beta_2,\dots,2\beta_m) a_1^{2\beta_1} a_2^{2\beta_2}\dots a_m^{2\beta_m}$$

[Note that above integral vanishes if one of the exponents is odd, and that it equals otherwise to 1].

On the other hand

$$\Big(\sum |a_i|^2\Big)^k$$

$$= \Big(\sum_{\substack{(\beta_1,\beta_2,\dots\beta_m)\in\mathbb{N}_0^m \\ \sum\beta_i=k}} A(\beta_1,\beta_2,\dots,\beta_m) a_1^{2\beta_1} a_2^{2\beta_2}\dots a_m^{2\beta_m}\Big)$$

$$= \sum_{\substack{(\beta_1,\beta_2,\dots\beta_m)\in\mathbb{N}_0^m \\ \sum\beta_i=k}} \frac{A(\beta_1,\beta_2,\dots,\beta_m)}{A(2\beta_1,2\beta_2,\dots,2\beta_m)} A(2\beta_1,2\beta_2,\dots,2\beta_m) a_1^{2\beta_1} a_2^{2\beta_2}\dots a_m^{2\beta_m}$$

$$\geq \min_{\substack{(\beta_1,\beta_2,\dots\beta_m)\in\mathbb{N}_0^m \\ \sum\beta_i=k}} \frac{A(\beta_1,\beta_2,\dots,\beta_m)}{A(2\beta_1,2\beta_2,\dots,2\beta_m)} \sum_{\substack{(\beta_1,\beta_2,\dots\beta_m)\in\mathbb{N}_0^m \\ \sum\beta_i=k}} A(2\beta_1,2\beta_2,\dots,2\beta_m) a_1^{2\beta_1} a_2^{2\beta_2}\dots a_m^{2\beta_m}$$

$$= \min_{\substack{(\beta_1,\beta_2,\dots\beta_m)\in\mathbb{N}_0^m \\ \sum\beta_i=k}} \frac{A(\beta_1,\beta_2,\dots,\beta_m)}{A(2\beta_1,2\beta_2,\dots,2\beta_m)} \int_0^1 \Big(\sum_{i=1}^m a_i r_i\Big)^{2k} dt$$

which implies our claim if put

$$B_{2k}^{-2k} = \min_{\substack{(\beta_1,\beta_2,\dots\beta_m)\in\mathbb{N}_0^m \\ \sum\beta_i=k}} \frac{A(\beta_1,\beta_2,\dots,\beta_m)}{A(2\beta_1,2\beta_2,\dots,2\beta_m)} = \min_{m\leq k} \min_{\substack{(\beta_1,\beta_2,\dots\beta_m)\in\mathbb{N}_0^m \\ \sum\beta_i=k}} \frac{A(\beta_1,\beta_2,\dots,\beta_m)}{A(2\beta_1,2\beta_2,\dots,2\beta_m)}.$$

In order to show that we can choose $A_1 > 0$, to satisfy (5.12) we observe that for $f(t) = \sum_{i=1}^m a_i r_i(t)$

$$\begin{aligned}
\int_0^1 |f(t)|^2\,dt &= \int_0^1 |f(t)|^{2/3}|f(t)|^{4/3}\,dt \\
&\leq \Big[\int_0^1 |f(t)|\,dt\Big]^{2/3}\Big[\int_0^1 |f(t)|^4\,dt\Big]^{1/3} \\
&\ \text{[By Hölders inequality for } p = 3/2 \text{ and } q = 3] \\
&\leq \Big[\int_0^1 |f(t)|\,dt\Big]^{2/3} B_4^{4/3}\Big[\sum_{i=1}^m a_i^2\Big]^{2/3}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^1 |f(t)|\,dt &\geq \left[B_4^{-4/3}\int_0^1 |f(t)|^2\,dt\Big(\sum_{i=1}^m a_i^2\Big)^{-2/3}\right]^{3/2} \\
&= \Big[B_4^{-4/3}\sum_{i=1}^m |a_i|^2\Big(\sum_{i=1}^m a_i^2\Big)^{-2/3}\Big]^{3/2} = B_4^{-2}\Big(\sum_{i=1}^m a_i^2\Big)^{1/2}
\end{aligned}$$

which proves our claim if we let $A_1 = B_4^{-2}$. □

Proof of Theorem 5.3.1. Since the Rademacher functions are an orthonormal basis inside $L_2[0,1]$ it follows from Lemma 5.3.3 that ℓ_2 is isomorphically embeddable in $L_p[0,1]$, for $1 \leq p < \infty$. Secondly, for $2 \leq p < \infty$ the formal identity $I : L_p[0,1] \to L_2[0,1]$ is bounded and the restriction of I to $\overline{\mathrm{span}(r_i : i \in \mathbb{N})}$ is an isomorphism onto $\overline{\mathrm{span}(r_i : i \in \mathbb{N})}$. We conclude that the map:

$$P : L_p[0,1] \to \overline{\mathrm{span}(r_i : i \in \mathbb{N})}, \qquad f \mapsto \sum_{n=1}^\infty \Big(\int_0^1 f(s) r_n(s) ds\Big) r_n,$$

is a projection onto $\overline{\mathrm{span}(r_i : i \in \mathbb{N})}$, which proves that ℓ_2 is isomorphic to a complemented subspace of $L_p[0,1]$, if $2 \leq p < \infty$. The same conclusion follows also for $1 < p < 2$ by duality. □

Remark. The constants A_p and B_p $1 \leq p < \infty$, exhibited in the proof of Khintchine's inequality in Lemma 5.3.3 are far from being optimal. These optimal constants where determined by Uffe Haagerup [Ha]. He proved the following:

Theorem 5.3.4. [Ha] *For $0 < p < \infty$ the inequality 5.12 in Lemma 5.3.3 holds for all finite sequences $(a_j)_{j=1}^m$ of scalars and the following numbers A_p and B_p:*

$$A_p = \begin{cases} 2^{1/2-1/p} & \text{if } 0 < p \le p_0, \\ 2^{1/2}\left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}}\right)^{1/p} & \text{if } p_0 < p < 2, \\ 1 & \text{if } 2 \le p < \infty \end{cases}$$

and

$$B_p = \begin{cases} 1 & \text{if } 0 < p \le 2 \\ 2^{1/2}\left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}}\right)^{1/p} & \text{if } 2 < p < \infty. \end{cases}$$

Here $\Gamma(\cdot)$ is the "Gamma-function":

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt,$$

and $p_0 \in (1, 2)$ is the solution to the equation

$$\Gamma((p+1)/2) = \frac{\sqrt{\pi}}{2}$$

($p_0 \approx 1.84742$). Moreover A_p and B_p are optimal in the following sense. If $A > A_p$ or $B < B_p$ then there is a choice of $m \in \mathbb{N}$ and scalars $(a_j)_{j=1}^m$, for which 5.12 in Lemma 5.3.3 is violated, if one replaces A_p by A, or B_p by B, respectively.

The next Theorem on subspaces of L_p is due to Kadets and Pełczyński. We first state the *Extrapolation Principle.*

Theorem 5.3.5. [The Extrapolation Principle]
Let $X \subset L_p[0, 1]$, be a linear subspace on which $\|\cdot\|_{p_1}$ and $\|\cdot\|_{p_2}$, where $p_1 < p_2$, are finite and equivalent. Thus, there is a $C \ge 1$ so that

$$\|f\|_{p_1} \le \|f\|_{p_2} \le C\|f\|_{p_1} \qquad \text{whenever } f \in X.$$

(first inequality holds always by Hölder inequality).

Then for all $0 < p \le p_1$ and all $x \in X$

$$C^{(p_2/p)(1-(1/\lambda))}\|x\|_{p_1} \le \|x\|_p \le \|x\|_{p_1},$$

where $\lambda \in (0, 1)$ is defined by $p_1 = \lambda p + (1 - \lambda)p_2$.

Proof. Let $0 < p \le p_1$ and choose $0 < \lambda < 1$ so that $p_1 = \lambda p + (1-\lambda)p_2$. For $x \in X$ it follows

$$\begin{aligned}
\|x\|_{p_1} &= \left[\int |x(t)|^{\lambda p} \cdot |x(t)|^{(1-\lambda)p_2}\, dt\right]^{1/p_1} \\
&\le \left[\int |x(t)|^{p}\, dt\right]^{\lambda/p_1} \cdot \left[\int |x(t)|^{p_2}\, dt\right]^{(1-\lambda)/p_1} dt \\
&\quad \text{[Hölder inequality for exponents } 1/\lambda \text{ and } 1/(1-\lambda)] \\
&= \|x\|_p^{\frac{p\lambda}{p_1}} \|x\|_{p_2}^{\frac{p_2(1-\lambda)}{p_1}} \le C^{\frac{p_2}{p_1}(1-\lambda)} \|x\|_{p_1}^{\frac{p_2}{p_1}(1-\lambda)} \|x\|_p^{\frac{p\lambda}{p_1}}
\end{aligned}$$

thus (since $1 - \frac{p_2}{p_1}(1-\lambda) = \frac{\lambda p}{p_1}$)

$$\begin{aligned}
&\|x\|_{p_1}^{\frac{\lambda p}{p_1}} \le C^{\frac{p_2}{p_1}(1-\lambda)} \|x\|_p^{\frac{p\lambda}{p_1}} \text{ and thus} \\
&\|x\|_{p_1} \le C^{\frac{p_2}{p}(\frac{1}{\lambda}-1)} \|x\|_p
\end{aligned}$$

which yields that

$$C^{(p_2/p)(1-(1/\lambda))} \|x\|_{p_1} \le \|x\|_p.$$

□

Remark. The *Interpolation* is obvious, and follows from applying Hölder's Theorem twice:

Assume as in the previous Theorem that $X \subset L_p[0,1]$,is a linear subspace on which $\|\cdot\|_{p_1}$ and $\|\cdot\|_{p_2}$, where $p_1 < p_2$, are C-equivalent. Thus, there is a $C \ge 1$ so that

$$\|f\|_{p_1} \le \|f\|_{p_2} \le C\|f\|_{p_1} \qquad \text{whenever } f \in X.$$

Then for all $p \in (p_1, p_2)$

$$\|f\|_{p_1} \le \|f\|_p \le \|f\|_{p_2} \le C\|f\|_{p_1}.$$

Theorem 5.3.6 (Kadets and Pełczyński)**.** *Assume $2 < p < \infty$ and assume that X is a closed subspace of $L_p[0,1]$. Then:*

Either there is an $0 < r < p$ so that $\|\cdot\|_r$ and $\|\cdot\|_p$ are equivalent norms on X. In that case it follows that X is isomorphic to a Hilbert space, X is complemented in $L_p[0,1]$ and the constant of isomorphism as well as the constant of complementation only depend on r, p and the equivalence constant between $\|\cdot\|_r$ and $\|\cdot\|_p$ on X.

Or $\|\cdot\|_r$ and $\|\cdot\|_p$ are not equivalent on X for some $r < p$. Then X contains for any $\varepsilon > 0$ a sequence which is $(1+\varepsilon)$-equivalent to the ℓ_p-unit vector basis.

Proof. Let X be (w.l.o.g) an infinite dimensional subspace of $L_p[0,1]$. If for some $r < p$ the norms $\|\cdot\|_p$ and $\|\cdot\|_r$ are equivalent on X it follows from Theorem 5.3.5 and the following remark that $\|\cdot\|_2$ and $\|\cdot\|_p$ are equivalent norms on X and the constant of equivalence only depends on r, p and the equivalence constant of $\|\cdot\|_r$ and $\|\cdot\|_p$. Thus, X is isomorphic to a separable Hilbert space. Moreover X, seen as a linear subspace of $L_2[0,1]$, is closed and thus complemented. Let $P : L_2[0,1] \to X$ be the orthogonal projection from $L_2[0,1]$ onto X. Then $Q = P \circ I$, where $I : L_p[0,1] \to L_2[0,1]$ is the formal identity, is a projection from $L_p[0,1]$ onto X.

Assume for all $r < p$ the norms $\|\cdot\|_p$ and $\|\cdot\|_r$ are not equivalent on X and let $\varepsilon > 0$. For $n \in \mathbb{N}$, choose inductively $r_n < p$, $M_n > 1$ and $f_n \in X$ so that

$$M_n \ge 2^n \text{ and, } \int_{\{|f|>M_n\}} |f_i(t)|^p \, dt < 2^{-n-1}\varepsilon, \tag{5.13}$$
$$\text{whenever } 1 \le i < n \text{ and } f \in B_{L_p[0,1]}$$

$$M_n^{p-r_n} = 2 \tag{5.14}$$

$$\|f_n\|_{r_n} < 2^{-n-1}\varepsilon, \text{ and } \|f_n\|_p = 1. \tag{5.15}$$

Indeed, for $n = 1$ let $M_1 = 2$ (which satisfies (5.13), since the second condition is vacuous). Then choose $r_1 < p$ close enough to p so that (5.14) holds. Since $\|\cdot\|_{r_1}$ and $\|\cdot\|_p$ are not equivalent on X, and we can choose $f_1 \in S_X$ so that (5.15) holds.

Assuming $f_1, f_2, \dots f_{n-1}, r_1, r_2, \dots, r_{n-1}$, and $M_1, M_2, \dots M_{n-1}$ have been chosen, we first choose $\eta > 0$ so that for all $i = 1, 2, \dots, n$, and all measurable $A \subset [0,1]$ with $m(A) < \eta$ and all $i = 1, 2, \dots, n-1$, it follows that

$$\int_A |f_i(t)|^p dt < 2^{-n-1}\varepsilon.$$

Now for any $f \in B_{L_p[0,1]}$ and any $M > 0$ we have

$$m(\{|f| > M\}) \le \frac{1}{M^p} \int |f(t)|^p \, dt \le \frac{1}{M^p},$$

So choosing $M_n = \max(2^n, \frac{1}{\eta^{1/p}})$, we deduce (5.13). We can then choose $r_n \in (0,p)$ close enough to p, so that (5.14), and since by assumption $\|\cdot\|_{r_n}$ and $\|\cdot\|_p$ are not equivalent on X, we can choose $f_n \in X$ so that (5.15) holds. This finishes the recursion.

Then

$$\int_{|f_n|<M_n} |f_n(t)|^p\,dt \le \int M_n^{p-r_n}|f(t)|^{r_n}\,dt \le 2\|f_n\|_{r_n} < 2^{-n}\varepsilon.$$

For $n\in$ put $A_n = \{f_n \ge M_n\}\setminus\bigcup_{j>n}\{|f_j|\ge M_j\}$ and $g_n = f_n 1_{A_n}$. Then the g_n's have disjoint support and

$$\begin{aligned}\|f_n - g_n\|_p^p &\le \int_{|f_n|<M_n} |f_n(t)|^p\,dt + \sum_{j>n}\int_{|f_j|>M_j}|f_n(t)|^p\,dt\\ &\le 2^{-n}\varepsilon + \sum_{j>n}\int_{|f_j|>M_j}|f_n(t)|^p\,dt < 2^{-n}\varepsilon + \sum_{j>n}2^{j-1}\varepsilon = 2^{1-n}\varepsilon,\end{aligned}$$

Fix $\delta>0$. For ε small enough (depending on δ), it follows that (g_n) is $(1+\delta)$-equivalent to the ℓ_p-unit vector basis (since the g_n have disjoint support . By choosing δ small enough we can secondly ensure that

$$\sum_{n\in\mathbb{N}} \|g_n - f_n\|_p \|g_n^*\|_q < 1,$$

where the (g_n^*) are the coordinate functionals of (g_n). Applying now the Small Perturbation Lemma yields that (f_n) is also equivalent to the ℓ_p unit basis. □

Remark. The Theorem of Kadets and Pełczyński started the investigation of complemented subspaces of $L_p[0,1]$, $2<p<\infty$. Here are some results:

Johnson-Odell 1974: Every complemented subspace of $L_p[0,1]$ which does not contain ℓ_2, must be a subspace of ℓ_p. In other words if X is an infinite dimensional complemented subspace of $L_p[0,1]$ it must be either ℓ_2 or ℓ_p or contain $\ell_p\oplus\ell_2$ (we are using here also that ℓ_p is *prime*, i.e that every infinite dimensional complemented subspace of ℓ_p is isomorphic to ℓ_p).

Bourgain-Rosenthal-Schechtman 1981: There are uncountable many non isomorphic complemented subspaces of $L_p[0,1]$.

Haydon-Odell-Schlumprecht 2011: If X is a complemented subspace of $L_p[0,1]$ which does not isomorphically embed into $\ell_2\oplus\ell_p$ then it must contain $\ell_p(\ell_2)$.

Next Question: Assume that X is a complemented subspace of $L_p[0,1]$ which is not contained in an isomorphic copy of $\ell_p(\ell_2)$. What can we say about X?

5.4 The spaces ℓ_p, $1 \le p < \infty$, and c_0 are prime spaces

The main goal of this section is show that the spaces ℓ_p, $1 \le p < \infty$, and c_0 are *prime spaces*.

Definition 5.4.1. A Banach space X is said to be *prime* if every complemented subspace of X is isomorphic to X.

The following Theorem is due to Pełczyński.

Theorem 5.4.2. *The spaces ℓ_p, $1 \le p < \infty$, and c_0 are prime.*

We will prove this theorem using the *Pełczynski Decomposition Method*, an argument which is important in its own right and also very pretty. Before doing that we need some lemmas. The first one was, up to the "moreover part" a homework problem and can be easily deduced from the Small Perturbation Lemma.

Lemma 5.4.3. (The Gliding Hump Argument)

Let X be a Banach space with a basis (e_i) and Y an infinite dimensional closed subspace of X. Let $\varepsilon > 0$. Then Y contains a normalized sequence (y_n) which is basic and $(1-\varepsilon)^{-1}$-equivalent to some normalized block basis (u_n).

Moreover, if the span of (u_n) is complemented in X, so is the span of (y_n).

Proof. Without loss of generality we can assume that $\|e_n\| = 1$, for $n \in \mathbb{N}$.

Let b be the basis constant, and (e_j^*) the coordinated functionals of (e_n). Let $\delta_n \subset (0,1)$ a null sequence, with $\sum_{n=1}^{\infty} \delta_n \le \varepsilon/2b$. By induction we choose for every $n \in \mathbb{N}$ $y_n, u_n \in S_X$ and $k_n \in \mathbb{N}$, so that:

a) $0 = k_0 < k_1 < k_2 < \ldots$,

b) $u_n \in \operatorname{span}(e_j : k_{n-1} + 1 \le j \le k_n)$, and

c) $y_n \in Y$, and $\|u_n - y_n\| < \delta_n$.

For $n = 1$ we simply choose any $y_1 \in S_Y$, and then by density of $\operatorname{span}(e_j : j \in \mathbb{N})$ in X an element $x_1 \in \operatorname{span}(e_j : j \in \mathbb{N})$, with $\|x_1\| = 1$ and choose $k_1 \in \mathbb{N}$ so that $x_1 \in \operatorname{span}(e_j \in \mathbb{N})$.

Assuming k_n has been chosen we can choose $y_{n+1} \in \bigcap_{i \le k_n} \mathcal{N}(e_i^*) \cap S_X$. Since $\operatorname{span}(e_j : j \in \mathbb{N}, j > k_n)$ is dense in $\bigcap_{i \le k_n} \mathcal{N}(e_i^*)$, we can choose

$u_{n+1} \in \mathrm{span}(e_j : j \in \mathbb{N}, j > k_n) \cap S_X$ so that $\|x_{n+1} - u_{n+1}\| < \delta_{n+1}$, and finally choose k_{n+1}, so that $u_{n+1} \in \mathrm{span}(e_j : j \in \mathbb{N}, k_n < j \le k_{n+1})$.

Since the basis constant of (u_n) does not exceed b (Proposition 3.3.3) we deduce for the coordinate functionals (u_n^*) of (u_n) that

$$\sup_{n\in\mathbb{N}} \|u_n^*\| \ \le \sup_{n\in\mathbb{N}} \frac{2b}{\|u_n\|} = 2b,$$

and thus

$$\sum_{j=1}^{n} \|y_n - u_n\| \cdot \|u_n^*\| \le 2b \sum_{j=1}^{\infty} \delta_n \le \varepsilon,$$

and we conclude therefore our claim form the Small Perturbation Lemma 3.3.10. $\square$

Proposition 5.4.4. *Block bases in ℓ_p and c_0 are isometrically equivalent to the unit vector basis and their closed linear span is 1-complemented in ℓ_p, or c_0.*

Proof. We only present the proof for ℓ_p, $1 \le p < \infty$, the c_0-case works in the same way. Let (u_n) be a normalized block basis, and write u_n, $n\in\mathbb{N}$, as

$$u_n = \sum_{j=k_{n-1}+1}^{k_n} a_j e_j, \text{with } 0 = k_0 < k_1 < k_2 < \ldots \text{ and } (a_n) \subset \mathbb{K}.$$

It follows for $m\in\mathbb{N}$ and $(b_n)_{n=1}^m \subset \mathbb{K}$, that

$$\Big\| \sum_{n=1}^{m} b_n u_n \Big\|_p^p = \sum_{n=1}^{m} \sum_{j=k_{n-1}+1}^{k_n} |b_n|^p |a_j|^p = \sum_{j=1}^{m} |b_j|^p,$$

and thus (u_n) is isometrically equivalent to (e_n).

For $n \in \mathbb{N}$ choose $u_n^* \in \ell_q$, $u_n^* \in \mathrm{span}(e_j^* : k_{n-1} < j \le k_n)$, $\|u_n^*\|_q = 1$, so that $\langle u_n^*, u_n\rangle = 1$, and define

$$P : \ell_p \to \overline{\mathrm{span}(u_n : j\in\mathbb{N})}, x \mapsto \sum \langle x, u_n^*\rangle u_n.$$

For $x = \sum_{j=1}^\infty x_j e_j \in \ell_p$ it follows that

$$|\langle u_n^*, x\rangle|^p = \Big|\Big\langle u_n^*, \sum_{j=k_{n-1}+1}^{k_n} x_j e_j \Big\rangle\Big| \le \sum_{j=k_{n-1}+1}^{k_n} |x_j|^p,$$

and, thus, that

$$\|P(x)\|_p^p = \sum_{n=1}^{\infty} \sum_{j=k_{n-1}+1}^{k_n} |a_j|^p |\langle u_n^*, x\rangle|^p \le \sum_{n=1}^{\infty} \langle u_n^*, x\rangle|^p \le \sum_{n=1}^{\infty} \sum_{j=k_{n-1}+1}^{k_n} |x_j|^p = \|x\|_p^p.$$

This shows that $\|P\| \le 1$, and, since moreover $P(u_n) = u_n$, and $P(X) \subset \overline{\text{span}(u_j : j\in\mathbb{N})}$, it follows that P is a projection onto $\overline{\text{span}(u_j : j\in\mathbb{N})}$ of norm 1. $\square$

Remark. It follows from Lemma 5.4.3 and Proposition 5.4.4 for $X = \ell_p$ or c_0 that every subspace Y of X has a further subspace Z which is complemented in X and isomorphic to X. We call a space X which has this property *complementably minimal*, a notion introduced by Casazza. In particular if Y is any complemented subspace of X the pair (Y, X) has the *Schröder Bernstein property*, which means that X is isomorphic to a subspace Y, and Y is isomorphic to a complemented subsapce of X.

It was for long time an open question whether a complementably minimal space is prime, and an even longer open problem was the question whether or not ℓ_p and c_0 are the only separable prime spaces. The first question would have a positive answer if all Banach spaces X and Y for which (X, Y) has the Schröder Bernstein property then it follows that X and Y are isomorphic. It is also open if complementably minimal spaces have to be prime.

Then Gowers and Maurey [GM2] constructed a space X(this is a variation of the space cited in [GM] and also does not contain any unconditional basis sequence) which only has trivial complemented subspaces, namely the finite and cofinite dimensional subspaces which has the property that all the cofinite dimensional subspaces are isomorphic to X. Thus, this space is prime, but not ℓ_p or c_0.

Then Gowers [Go2] also found a counterexamples to the Schröder Bernstein problem, which also does not contain any unconditional basic sequence.

Both questions are still open for spaces with unconditional basic sequences, and thus spaces with *lots of complemented subspaces.* In [Sch] a space with a 1-unconditional space was constructed which is complementably minimal (shown in [AS]) but does not contain ℓ_p or c_0. This space together with some complemented subspace Y must either be a counterexample to the Schröder Bernstein Problem, or it is new prime space.

The *Pełczyński Decomposition Method* now proves that a complementably minimal space is prime, if you assume some additional assumptions which are all satisfied by ℓ_p or c_0.

Let's start with a very easy and general observation.

Proposition 5.4.5. *If X and Y are Banach spaces, with the property that X is isomorphic to a complemented subspace of Y and if X is isomorphic to its square, i.e. $X \sim X \oplus X$, then Y is isomorphic to $X \oplus Y$.*

In particular if X and Y are ismorphic to their squares, isomorphic to complemented subspaces of each other, then it follows that $X \sim X \oplus Y \sim Y$.

Proof. Let Z be a complemented subspace of Y so that $Y \sim X \oplus Z$. Then

$$Y \sim X \oplus Z \sim (X \oplus X) \oplus Z \sim X \oplus (X \oplus Z) \sim X \oplus Y.$$

$\square$

Remark. It is easy to see that $\ell_p \sim \ell_p \oplus \ell_p$, $1 \le p < \infty$ and $c_0 \sim c_0 \oplus c_0$, but it is not clear how to show directly that any complemented subspace of ℓ_p or c_0 is isomorphic to its square. So we will need an additional property of ℓ_p and c_o. Nevertheless we can easily deduce the following Corollary from Proposition 5.4.5 and Khintchine's Theorem 5.2.1.

Corollary 5.4.6. *For $1 < p < \infty$ it follows that $L_p[0,1]$ is isomorphic to $L_p[0,1] \oplus L_2[0,1]$.*

Proof of Theorem 5.4.2. Let $X = \ell_p$ or c_0. From now on we consider on all complemented sums the ℓ_p-sum, respectively c_0-sum. Note that $X \sim (\oplus_{j\in\mathbb{N}} X)_X$ (actually isometrically)

Let Y be a complemented subspace of X, by Proposition 5.4.5 we only need to show that $X \sim X \oplus Y$, and that can be seen as follows: we let Z be a subspace of X so that $X \sim Y \oplus Z$, then

$$\begin{aligned}
Y \oplus X &\sim Y \oplus (\oplus_{n\in\mathbb{N}} X)_X \\
&\sim Y \oplus (\oplus_{n\in\mathbb{N}} (Z \oplus Y))_X \\
&\sim Y \oplus Z \oplus (\oplus_{n\in\mathbb{N}} (Y \oplus Z))_X \\
&\text{(consider } \big(y_1, (z_1, x_1, z_2, x_2, \ldots)\big) \mapsto \big((y_1, z_1), (x_1, z_2, x_2, \ldots)\big) \\
&\sim (\oplus_{n\in\mathbb{N}} (Y \oplus Z))_X \\
&\sim (\oplus_{n\in\mathbb{N}} X)_X \sim X.
\end{aligned}$$

$\square$

One more open question:

Remark. $L_1[0,1]$ cannot be prime since ℓ_1 is isomorphic to a complemented subspaces of $L_1[0,1]$, but it is a famous open problem whether or not this is the only *other* complemented subspace? Are all the complemented subspaces of $L_1[0,1]$ either isomorphic to ℓ_1 or to $L_1[0,1]$?

5.5 The Haar basis is Unconditional in $L_p[0,1]$, $1 < p < \infty$

Theorem 5.5.1. (Unconditionality of the Haar basis in L_p)
Let $1 < p < \infty$. Then $(h_t^{(p)})$ is an unconditional basis of $L_p[0,1]$. More precisely, for any two families $(a_t)_{t\in T}$ and $(b_t)_{t\in T}$ in $c_{00}(T)$ with $|a_t| \le |b_t|$, for all $t \in T$, it follows that

$$\Big\| \sum_{t\in T} a_t h_t^{(p)} \Big\| \le (p^* - 1) \Big\| \sum_{t\in T} b_t h_t^{(p)} \Big\|, \tag{5.16}$$

where

$$p^* = \max\big(p, \frac{p}{p-1}\big) = \begin{cases} p & \text{if } p \ge 2 \\ p/(p-1) & \text{if } p \le 2 \end{cases}$$

We will prove the theorem for $2 < p < \infty$. For $p = 2$ it is clear since $(h_t^{(2)})$ is orthonormal and for $1 < p < 2$ it follows from Propostion 3.4.5 by duality (note that $p^* = q^*$ if $\frac{1}{p} + \frac{1}{q} = 1$).

We first need the following technical Lemma which presents the "heart of the proof of Theorem 5.5.1"

Lemma 5.5.2. *Let $2 < p < \infty$ and define*

$$v : \mathbb{C} \times \mathbb{C} \to [0, \infty), \qquad (x, y) \mapsto |y|^p - (p-1)^p |x|^p, \text{ and} \tag{5.17}$$

$$u : \mathbb{C} \times \mathbb{C} \to [0, \infty), \qquad (x, y) \mapsto \alpha_p \big(|x| + |y|\big)^{p-1} \big(|y| - (p-1)|x|\big) \tag{5.18}$$

$$\text{with } \alpha_p = p\Big(1 - \frac{1}{p}\Big)^{p-1}.$$

Then it follows for $x, y, a, b \in \mathbb{C}$, with $|a| \le |b|$

$$v(x, y) \le u(x, y), \tag{5.19}$$

$$u(-x, -y) = u(x, y), \tag{5.20}$$

$$u(0, 0) = 0, \text{ and} \tag{5.21}$$

$$u(x + a, y + b) + u(x - a, y - b) \le 2u(x, y). \tag{5.22}$$

Proof. Let $x, y, a, b \in \mathbb{C}$, $|a| \le |b|$ be given. (5.20) and (5.21) are trivially satisfied. Since u and v are both p-homogeneous (i.e. $u(\alpha x, \alpha y) = |\alpha|^p u(x, y)$ for $\alpha, x, y \in \mathbb{C}$) we can assume that $|x| + |y| = 1$ in order to show (5.19). Thus, the inequality (put $s = |x|$ and, thus, $1 - s = |y|$) reduces to show

$$F(s) = \alpha_p(1 - ps) - (1 - s)^p + (p-1)^p s^p \ge 0 \text{ for } 0 \le s \le 1 \text{ and } 2 \le p. \tag{5.23}$$

In order to verify (5.23), first show that $F(0) > 0$. Indeed, by concavity of $\ln x$ it follows that

$$\ln p = \ln\big((p-1)+1\big) < \ln(p-1) + \frac{1}{p-1},$$

and, thus,

$$\ln(p-1)+1 = \ln(p-1) + \frac{1}{p-1} + \frac{p-2}{p-1} > \ln p + \frac{p-2}{p-1} > \ln p + \frac{p-2}{p}.$$

Integrating both sides of the inequality

$$\ln(x-1)+1 > \ln x + \frac{x-2}{x} = \ln x + 1 - \frac{2}{x}$$

from $x = 2$ to $p > 2$, implies that

$$\ln\big((p-1)^{p-1}\big)(p-1) = \ln(p-1) > (p-2)\ln p = \ln\big(p^{p-2}\big)$$

and, thus,

$$(p-1)^{p-1} > p^{p-2},$$

which yields

$$\alpha_p = p\Big(1-\frac{1}{p}\Big)^{p-1} = \frac{(p-1)^{p-1}}{p^{p-2}} > 1$$

and thus the claim that $F(0) > 0$.

Secondly, we claim that $F(1) > 0$. Indeed,

$$\begin{aligned} F(1) &= \alpha_p(1-p) + (p-1)^p \\ &= -\frac{(p-1)^p}{p^{p-2}} + (p-1)^p = (p-1)^p\left[1 - \frac{1}{p^{p-2}}\right] > 0. \end{aligned}$$

Thirdly, we compute the first and second derivative of F and get

$$\begin{aligned} F'(s) &= -\alpha_p p + p(1-s)^{p-1} + (p-1)^p p s^{p-1}, \text{ and} \\ F''(s) &= -p(p-1)(1-s)^{p-2} + (p-1)^{p+1} p s^{p-2} \end{aligned}$$

and deduce that $F(\frac{1}{p}) = F'(\frac{1}{p}) = 0$, $F''(\frac{1}{p}) > 0$ and that $F''(s)$ vanishes for exactly one value of s (because it is the difference of an increasing and a decreasing function). Thus, $F(s)$ cannot have more points at which it vanishes and it follows that $F(s) \geq 0$ for all $s \in [0,1]$ and we deduce (5.19).

Finally we need to show (5.22). We can (by density argument) assume that x and a as well as y and b are linear independent as two-dimensional vectors over $\mathbb{R}$. This implies that $|x+ta|$ and $|y+tb|$ can never vanish, and, thus, that the function

$$G:\mathbb{R}\to\mathbb{R},\quad t\mapsto t=u(x+ta,y+tb),$$

is infinitely often differentiable.

We compute the second derivative of G at 0, getting

$$\begin{aligned}G''(0)=\alpha_p\Bigg[&-p(p-1)\big(|a|^2-|b|^2\big)\big(|x|+|y|\big)^{p-2}\\&-p(p-2)\big(|b|^2-\Re(\langle\frac{y}{|y|},b\rangle^2)|y|^{-1}\big(|x|+|y|\big)^{p-1}\\&-p(p-1)(p-2)\Big(\Re(\langle\frac{x}{|x|},a\rangle)+\Re(\langle\frac{y}{|y|},b\rangle)\Big)^2|x|\big(|x|+|y|\big)^{p-3}\Bigg].\end{aligned}$$

A detailed computation of $G''(0)$ will be given in the appendix of this section.

Inspecting each term we deduce (recall that $|a|\ge|b|$) from the Cauchy inequality that $G''(0)<0$. Since for $t\ne0$ it follows that $G''(t)=\tilde G''(0)$ where

$$\tilde G(s):\mathbb{R}\to\mathbb{R},\quad s\mapsto u(\underbrace{x+ta}_{\tilde x}+sa,\underbrace{y+tb}_{\tilde y}+sb),$$

we deduce that $G''(t)\le0$ for all $t\in\mathbb{R}$. Thus, G is a concave function which yields

$$\frac12[u(x+a,y+b)+u(x-a,y-b)]=\frac12[G(1)+G(-1)]\le G(0)=u(x,y),$$

which proves (5.22). $\square$

Now we are ready to deduce Theorem 5.5.1:

Proof of Theorem 5.5.1. Assume that $\tilde h_n$ is normalized in L_∞ so that $h_n=\tilde h_n/\|\tilde h_n\|_p$ is a linear reordering of $(h_t^{(p)})_t\in T$ which is compatible with the order on T. For $n\in\mathbb{N}$ let $f_n=\sum_{i=1}^n a_i\tilde h_i$ and $g_n=\sum_{i=1}^n b_i\tilde h_i$, where $(a_i)_{i=1}^n$, $(b_i)_{i=1}^n$ in $\mathbb{R}$, with $|a_j|\ge|b_j|$, for $j=1,2,\dots,n$, we need to show that $\|g_n\|_p\le(1-p^*)\|f_n\|$. The fact that we are considering the normalization in $L_\infty[0,1]$ instead of the normalization in $L_p[0,1]$ (i.e. $\tilde h_n$ instead of h_n) will not effect the outcome. We deduce from (5.19) that

$$\|g_n\|^p-(p-1)^p\|f_n\|^p=\int_0^1 v(f_n(t),g_n(t))\,dt\le\int_0^1 u(f_n(t),g_n(t))\,dt.$$

Let $A = \text{supp}(\tilde{h}_n)$, $A^+ = A \cap \{\tilde{h}_n > 0\}$ and $A^- = A \cap \{\tilde{h}_n < 0\}$. Since f_{n-1} and g_{n-1} are constant on A we deduce

$$\begin{aligned}
&\int_0^1 u(f_n(t), g_n(t))\,dt \\
&\qquad = \int_{[0,1]\setminus A} u(f_{n-1}(t), g_{n-1}(t))\,dt \\
&\qquad\qquad + \int_{A^+} u(f_{n-1}(t) + a_n, g_{n-1}(t) + b_n)\,dt \\
&\qquad\qquad + \int_{A^-} u(f_{n-1}(t) - a_n, g_{n-1}(t) - b_n)\,dt \\
&\qquad = \int_{[0,1]\setminus A} u(f_{n-1}(t), g_{n-1}(t))\,dt \\
&\qquad\quad + \frac{1}{2}\int_A u(f_{n-1}(t) + a_n, g_{n-1}(t) + b_n) + u(f_{n-1}(t) - a_n, g_{n-1}(t) - b_n)\,dt \\
&\qquad \le \int_{[0,1]\setminus A} u(f_{n-1}(t), g_{n-1}(t))\,dt + \int_A u(f_{n-1}(t), g_{n-1}(t))\,dt \\
&\qquad \text{[By (5.22)]} \\
&\qquad = \int_0^1 u(f_{n-1}(t), g_{n-1}(t))\,dt
\end{aligned}$$

Iterating this argument yields

$$\begin{aligned}
\int_0^1 u(f_n(t), g_n(t))\,dt &\le \int_0^1 u(f_1(t), g_1(t))\,dt \\
&= u(a_1, b_1) \\
&= \frac{1}{2}\big(u(a_1, b_1) + u(-a_1, -b_1)\big) \text{ [By (5.20)]} \\
&\le u(0,0) = 0 \text{ [By (5.21) and (5.22)]},
\end{aligned}$$

which implies our claim that $\|g_n\| \le (p-1)\|f_n\|$. $\square$

From the unconditionality of the Haar basis and Khintchine's Theorem we now can deduce the following equivalent representation of the norm on L_p.

Theorem 5.5.3 (The square-function norm)**.** *Let $1 < p < \infty$ and let (f_n) be an unconditional basic sequence in $L_p[0,1]$. For example (f_n) could be a linear ordering of the Haar basis. Then there is a constant $C \ge 1$, only depending on the unconditionality constant of (f_i) and the constants A_p and*

B_p in Khintchine's Inequality (Lemma 5.3.3) so that for any $g = \sum_{i=1}^\infty a_i f_i \in \overline{\mathrm{span}(f_i : i \in \mathbb{N})}$ it follows that

$$\frac{1}{C}\Big\|\sum_{i=1}^\infty \Big(|a_i|^2|f_i|^2\Big)^{1/2}\Big\|_p \le \|g\|_p \le C\Big\|\sum_{i=1}^\infty \Big(|a_i|^2|f_i|^2\Big)^{1/2}\Big\|_p,$$

which means that $\|\cdot\|_p$ is on $\overline{\mathrm{span}(f_i : i \in \mathbb{N})}$ equivalent to the norm

$$|||f||| = \Big\|\sum_{i=1}^\infty \Big(|a_i|^2|f_i|^2\Big)^{1/2}\Big\|_p = \Big\|\sum_{i=1}^\infty |a_i|^2|f_i|^2\Big\|_{p/2}^{1/2}.$$

Proof. For two positive numbers A and B and $c > 0$ we write: $A \sim_c B$ if $\frac{1}{c}A \le B \le cA$. Let K_p be the *Khintchine constant for L_p*, i.e the smallest number so that for the Rademacher sequence (r_n)

$$\Big\|\sum_{i=1}^\infty a_i r_i\Big\|_p \sim_{K_p} \Big(\sum_{i=1}^\infty |a_i|^2\Big)^{1/2} \text{ for } (a_i) \subset \mathbb{K},$$

and let b_u be the unconditionality constant of (f_i), i.e.

$$\Big\|\sum_{i=1}^\infty \sigma_i a_i f_i\Big\|_p \sim_{b_u} \Big\|\sum_{i=1}^\infty a_i f_i\Big\|_p \text{ for } (a_i) \subset \mathbb{K} \text{ and } (\sigma_i) \subset \{\pm 1\}.$$

We consider $L_p[0,1]$ in a natural way as subspace of $L_p[0,1]^2$, with $\tilde{f}(s,t) := f(s)$ for $f \in L_p[0,1]$. Then let $r_n(t) = r_n(s,t)$ be the n-th Rademacher function action on the second coordinate, i.e

$$r_n(s,t) = \mathrm{sign}(\sin(2^n \pi t)),\ (s,t) \in [0,1]^2.$$

It follows from the b_u-unconditionality for any $(a_j)_{j=1}^m \subset \mathbb{K}$, that

$$\begin{aligned}\Big\|\sum_{j=1}^m a_j f_j(\cdot)\Big\|_p^p &\sim_{b_u^p} \Big\|\sum_{j=1}^m a_j f_j(\cdot) r_j(t)\Big\|_p^p \\ &= \int_0^1 \Big(\sum_{j=1}^m a_j f_j(s) r_j(t)\Big)^p ds \text{ for all } t \in [0,1],\end{aligned}$$

and integrating over all $t \in [0,1]$ implies

$$\Big\|\sum_{j=1}^m a_j f_j(\cdot)\Big\|_p^p \sim_{b_u^p} \int_0^1 \int_0^1 \left(\sum_{j=1}^m a_j f_j(s) r_j(t)\right)^p ds\, dt$$

$$
\begin{aligned}
&= \int_0^1 \int_0^1 \left(\sum_{j=1}^m a_j f_j(s) r_j(t) \right)^p dt\, ds \text{(By Theorem of Fubini)} \\
&= \int_0^1 \Big\| \sum_{j=1}^m a_j f_j(s) r_j(\cdot) \Big\|_p^p ds \\
&\sim_{K_p^p} \int_0^1 \Big(\sum_{j=1}^m |a_j f_j(s)|^2 \Big)^{p/2} ds = \Big\| \Big(\sum_{j=1}^m |a_j f_j|^2 \Big)^{1/2} \Big\|_p,
\end{aligned}
$$

which proves our claim using $C = K_p b_u$. □

Let (h_j) a compatible ordering of the Haar basis. For $1 \leq p < \infty$ and a measurable function $f = \sum_{j=1}^\infty a_j h_j \in L_p$ we define

$$
\|f\|_{H_p} = \big| \big(\sum_{j=1}^m |a_j h_j|^2 \big\|_p = \left(\int_0^1 \Big(\sum_{j=1}^\infty |a_j|^2 1_{\mathrm{supp}(h_j)} \Big)^{p/2} \right)^{1/p},
$$

is called

Corollary 5.5.4. *For $1 < p < \infty$ the norms $\|\cdot\|_{H_p}$ and the usual L_p- norm are equivalent. But $\|\cdot\|_{H_1}$ and the L_1 are not equivalent (otherwise would the Haar basis be uncondition al in L_1.*

5.6 Appendix: Detailed computation of $G''(0)$, as defined in the proof of Lemma 5.5.2:

We write $x = x_1 + ix_2$, $y = y_1 + iy_2$, $a = a_1 + ia_2$, and $b = b_1 + ib_2$ be in $\mathbb{C}$, with $|a| \leq |b|$. We define:

$$
\begin{aligned}
&f : \mathbb{R} \times \mathbb{R}, \quad (s,t) \mapsto (s+t)^{p-1}\big(t - (p-1)s\big) = \\
&s : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto |x + a\xi| = \sqrt{(x_1 + a_1\xi)^2 + (x_2 + a_2\xi)^2} \\
&t : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto |y + b\xi| = \sqrt{(y_1 + b_1\xi)^2 + (y_2 + b_2\xi)^2} \\
&G : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto f\big(s(\xi), t(\xi)\big) = \frac{1}{\alpha_p} u(|x + a\xi|, |y + b\xi|).
\end{aligned}
$$

We will compute the second derivative of G with respect to ξ.

First we compute the partial first and second derivatives of $f(s,t)$:

$$
f_s(s,t) = (p-1)(s+t)^{p-2}(t - (p-1)s) - (p-1)(s+t)^{p-1} \tag{5.24}
$$

$$
\begin{aligned}
&= (p-1)(s+t)^{p-2}\big(t-(p-1)s-s-t\big)\\
&= -p(p-1)(s+t)^{p-2}s\\
f_t(s,t) &= (p-1)\big(s+t\big)^{p-2}\big(t-(p-1)s\big)+(s+t)^{p-1}\\
&= \big(s+t\big)^{p-2}\big((p-1)t-(p-1)^2s+s+t\big)\\
&= \big(s+t\big)^{p-2}\big(pt-p(p-2)s\big)=p\big(s+t\big)^{p-2}\big(t-(p-2)s\big)
\end{aligned}
$$

$$
\begin{aligned}
f_{s,s}(s,t) &= -p(p-1)(p-2)(s+t)^{p-3}s-p(p-1)(s+t)^{p-2} \qquad (5.25)\\
&= -p(p-1)(s+t)^{p-3}\big((p-2)s+s+t\big)
\end{aligned}
$$

$$
f_{s,t}(s,t) = -p(p-1)(p-2)(s+t)^{p-3}s \qquad (5.26)
$$

$$
\begin{aligned}
f_{t,t}(s,t) &= p(p-2)(s+t)^{p-3}\big(t-(p-2)s\big)+p(s+t)^{p-2} \qquad (5.27)\\
&= p(s+t)^{p-3}\big((p-2)t-(p-2)^2s+s+t\big)\\
&= p(s+t)^{p-3}\big((p-1)t-((p-2)^2-1)s\big)
\end{aligned}
$$

Secondly we compute the first and second derivatives of $s(\xi)$ and $t(\xi)$.

$$
\begin{aligned}
\frac{ds}{d\xi} &= \frac{(x_1+\xi a_1)a_1+(x_2+\xi a_2)a_2}{\sqrt{(x_1+a_1\xi)^2+(x_2+a_2\xi)^2}} \qquad (5.28)\\
&= \frac{(x_1+\xi a_1)a_1+(x_2+\xi a_2)a_2}{s}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dt}{d\xi} &= \frac{(y_1+\xi b_1)b_1+(y_2+\xi b_2)b_2}{\sqrt{(y_1+b_1\xi)^2+(y_2+b_2\xi)^2}} \qquad (5.29)\\
&= \frac{(y_1+\xi b_1)b_1+(y_2+\xi b_2)b_2}{t}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2s}{\xi^2} &= \frac{(a_1^2+a_2^2)s-\frac{((x_1+\xi a_1)a_1+(x_2+\xi a_2)a_2)^2}{s}}{s^2} \qquad (5.30)\\
&= \frac{|a|^2}{s}-\frac{\big((x_1+\xi a_1)a_1+(x_2+\xi a_2)a_2\big)^2}{s^3}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2t}{\xi^2} &= \frac{(b_1^2+b_2^2)t-\frac{((y_1+\xi b_1)b_1+(y_2+\xi b_2)b_2)^2}{s}}{s^2} \qquad (5.31)\\
&= \frac{|b|^2}{t}-\frac{\big((y_1+\xi b_1)b_1+(y_2+\xi b_2)b_2\big)^2}{t^3}
\end{aligned}
$$

and thus

$$
\frac{ds}{\xi}\Big|_{\xi=0} = \frac{\langle x,a\rangle}{|x|}, \quad \frac{dt}{\xi}\Big|_{\xi=0} = \frac{\langle y,b\rangle}{|y|}, \qquad (5.32)
$$

$$
\frac{d^2s}{\xi^2}\Big|_{\xi=0} = \frac{|a|^2}{|x|}-\frac{\langle x,a\rangle^2}{|x|^3}, \quad \frac{d^2t}{\xi^2}\Big|_{\xi=0} = \frac{|b|^2}{|y|}-\frac{\langle y,b\rangle^2}{|y|^3}, \qquad (5.33)
$$

(here we mean by $\langle x,a\rangle$ and $\langle y,b\rangle$ the scalar product in $\mathbb{R}^2$, where x,y,a,b are seen as vectors in $\mathbb{R}^2$). Thus

$$
\begin{aligned}
G'(0) &= f_s(|x|,|y|)s'(0) + f_t(|x|,|y|)t'(0) \\
G''(0) &= f_{s,s}(|x|,|y|)\big(s'(0)\big)^2 + f_s(|x|,|y|)s''(0) \\
&\qquad + 2f_{s,t}(|x|,|y|)s'(0)t'(0) + f_{t,t}(|x|,|y|)\big(t'(0)\big)^2 + f_t(|x|,|y|)t''(0) \\
&= -p(p-1)(|x|+|y|)^{p-3}\big((p-2)|x|+|x|+|y|\big)\frac{\langle x,a\rangle^2}{|x|^2} \\
&\quad - p(p-1)(|x|+|y|)^{p-2}|x|\left[\frac{|a|^2}{|x|} - \frac{\langle x,a\rangle^2}{|x|^3}\right] \\
&\quad - 2p(p-1)(p-2)\big(|x|+|y|\big)^{p-3}|x|\frac{\langle x,a\rangle}{|x|}\frac{\langle y,b\rangle}{|y|} \\
&\quad + p(|x|+|y|)^{p-3}\big((p-1)|y| - ((p-2)^2-1)|x|\big)\frac{\langle y,b\rangle^2}{|y|^2} \\
&\quad + p\big(|x|+|y|\big)^{p-2}\big(|y|-(p-2)|x|\big)\left[\frac{|b|^2}{|y|} - \frac{\langle y,b\rangle^2}{|y|^3}\right] \\
&= |a|^2\big(-p(p-1)(|x|+|y|)^{p-2}\big) \\
&\quad - |b|^2\left(p(|x|+|y|)^{p-2} - p(|x|+|y|)^{p-2}(p-2)\frac{|x|}{|y|}\right) \\
&\quad - \left\langle\frac{x}{|x|},a\right\rangle^2\big(p(p+1)(|x|+|y|)^{p-3}\big((p-2)|x|+|x|+|y|\big)+p(p+1)(|x|+|y|)^{p-2}\big) \\
&\quad - 2\frac{\langle x,a\rangle}{|x|}\frac{\langle y,b\rangle}{|y|}p(p-1)(p-2)\big(|x|+|y|\big)^{p-3}|x| \\
&\quad + \left\langle\frac{y}{|y|},b\right\rangle^2\Bigg(p(|x|+|y|)^{p-3}\big((p-1)|y|-((p-2)^2-1)|x|\big) \\
&\qquad\qquad - p\big(|x|+|y|\big)^{p-2}\Big(1-(p-2)\frac{|x|}{|y|}\Big)\Bigg) \\
&= \big(|b|^2-|a|^2\big)\big(p(p-1)(|x|+|y|)^{p-2}\big) \\
&\quad - |b|^2p(p-2)(|x|+|y|)^{p-1}|y|^{-1} \\
&\quad - \left\langle\frac{x}{|x|},a\right\rangle^2p(p-1)(p-2)|x|\big(|x|+|y|\big)^{p-3} \\
&\quad - 2\frac{\langle x,a\rangle}{|x|}\frac{\langle y,b\rangle}{|y|}p(p-1)(p-2)\big(|x|+|y|\big)^{p-3}|x|
\end{aligned}
$$

$$
\begin{aligned}
&- \left\langle \frac{y}{|y|}, b \right\rangle^2 p(p-1)(p-2)|x|\big(|x|+|y|\big)^{p-3} \\
&+ \left\langle \frac{y}{|y|}, b \right\rangle^2 p(p-2)(|x|+|y|)^{p-1}|y|^{-1}
\end{aligned}
$$

$$
\left[
\begin{array}{c}
\text{We note that the factor of } |b|^2 \text{ is:} \\
p(p-1)(|x|+|y|)^{p-2} + \big(p - p(p-1) - p(p-2)\tfrac{|x|}{|y|}\big)(|x|+|y|)^{p-2} \\
= p(p-1)(|x|+|y|)^{p-2} - p(p-2)\big(1 - \tfrac{|x|}{|y|}\big)(|x|+|y|)^{p-2} \\
= p(p-1)(|x|+|y|)^{p-2} - p(p-2)(|x|+|y|)^{p-1}|y|^{-1} \\
\text{and the factor of } \big\langle \tfrac{y}{|y|}, b \big\rangle^2 \text{ equals:} \\
(p{+}1)p(|x|+|y|)^{p-3}|y| - p(p{+}1)(p{+}3)(|x|+|y|)^{p-3}|x| \\
\qquad -p(|x|+|y|)^{p-2}\Big(1 - (p{+}2)\tfrac{|x|}{|y|}\Big) \\
= -p(p-1)(p-2)|x|\big(|x|+|y|\big)^{p-3} + (p-1)p(|x|+|y|)^{p-2} \\
\qquad -p(|x|+|y|)^{p-2}\Big(1 - (p{+}2)\tfrac{|x|}{|y|}\Big) \\
= -p(p-1)(p-2)|x|\big(|x|+|y|\big)^{p-3} \\
\qquad +(p-2)p(|x|+|y|)^{p-2} + (p-2)p\tfrac{|x|}{|y|} \\
= -p(p-1)(p-2)|x|\big(|x|+|y|\big)^{p-3} + p(p-2)(|x|+|y|)^{p-1}|y|^{-1}
\end{array}
\right]
$$

$$
\begin{aligned}
&= \big(|b|^2 - |a|^2\big)p(p-1)(|x|+|y|)^{p-2} \\
&- |b|^2 p(p-2)(|x|+|y|)^{p-1}|y|^{-1} + \left\langle \frac{y}{|y|}, b \right\rangle^2 (|x|+|y|)^{p-1}|y|^{-1} \\
&\quad - \left(\left\langle \frac{x}{|x|}, a \right\rangle + \left\langle \frac{y}{|y|}, b \right\rangle \right)^2 p(p-1)(p-2)|x|\big(|x|+|y|\big)^{p-3}.
\end{aligned}
$$

Chapter 6

Greedy bases

6.1 Characterization of Greedy bases, by Temlyakov and Konyagin

We start with the *Threshold Algorithm*:

Definition 6.1.1. Let X be a separable Banach space with a normalized basis (e_n), and let (e_n^*) be the coordinate functionals. For $n \in \mathbb{N}$ and $x \in X$ let $\Lambda_n \subset \mathbb{N}$,, with $\#\Lambda_n = n$ so that

$$\min_{i\in\Lambda_n} |e_i^*(x)| \geq \max_{i\in\mathbb{N}\setminus\Lambda_n} |e_i^*(x)|,$$

i.e. we are reordering $(e_i^*(x))$ into $(e_{\lambda(i)}^*(x))$, so that

$$|e_{\lambda_1}^*(x)| \geq |e_{\lambda_2}^*(x)| \geq |e_{\lambda_3}^*(x)| \geq \dots,$$

and for $n \in \mathbb{N}$ we put

$$\Lambda_n = \{\lambda_1, \lambda_2, \dots \lambda_n\}.$$

Then define for $n \in \mathbb{N}$

$$G_n^T(x) = \sum_{i\in\Lambda_n} e_i^*(x)e_i.$$

(G_n^T) is called the *Threshold Algorithm*.

Definition 6.1.2. A normalized basis (e_i) is called *Quasi-Greedy*, if for all x

$$x = \lim_{n\to\infty} G_n^T(x). \tag{QG}$$

A basis is called greedy if there is a constant C so that

$$\text{(G)} \qquad \big\|x - G_T(x)\big\| \le C\sigma_n(x),$$

where we define

$$\sigma_n(x) = \sigma_n\big(x, (e_j)\big) = \inf_{\Lambda\subset\mathbb{N}, \#\Lambda=n} \inf_{z\in\text{span}(e_j : j\in\Lambda)} \|z - x\|.$$

In that case we say that (e_i) is C-greedy. We call the smallest constant C for which (G) holds *the greedy constant of* (e_n) and denote it by C_g.

Recall the definition of the unconditional constant and suppression unconditional constant of a basis (e_i):

$$C_u = \sup\Big\{\Big\|\sum_{i=1}^\infty a_i b_i e_i\Big\| : x = \sum_{i=1}^\infty a_i e_i \in B_X \text{ and } |b_i| \le 1\Big\}$$
$$C_s = \sup\Big\{\Big\|\sum_{i\in A} a_i e_i\Big\| : x = \sum_{i=1}^\infty a_i e_i \in B_X \text{ and } A \subset \mathbb{N}\Big\}.$$

Recall that a basis (e_n) of a Banach space X is unconditional if and only if for all $x = \sum_{n=1}^\infty x_n e_n \in X$ and any permutation $\pi : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n=1}^\infty x_{\pi(n)} e_{\pi(n)}$ also converges to x. This implies in particular that every unconditional basis must be quasi greedy.

Example 6.1.3. The shrinking basis (e_n) in James space J is not quasi greedy.

Recall

$$\Big\|\sum_{j=1}^n x_n e_n\Big\|_{qv} = \sup\Big\{\Big(\sum_{j=1}^l |\xi_{n_j} - \xi_{n_{j-1}}|^2\Big)^{1/2} : l\in\mathbb{N} \text{ and } 1 \le n_0 < n_1 < \dots n_l\Big\}$$

For $n \in \mathbb{N}$, let

$$z_n = \Big(\underbrace{1, 1-\frac{1}{n}, 1, 1-\frac{1}{n}, \dots, 1, 1-\frac{1}{n}}_{2n \text{ coordinates}}, 0, 0, \dots\Big),$$

then $\|z_n\|_{qv} \le c$, where c does not depend on n. But

$$\|G_n^T(z_n)\| \ge 2n$$

Now we can *concatinate infinitely many small enough multiples of the z_n's*, *i.e.*, let $n_1 < n_2 < n_3 < \dots$ fast increasing (faster than k^2), say $n_k = 2^k$, $k \in \mathbb{N}$,

$$y_k = \Big(\underbrace{0,0,\dots,0}_{\sum_{j<k} n_j}, \frac{1}{k^2}\underbrace{\Big(1, 1-\frac{1}{n_k}, 1, 1-\frac{1}{n_k}, \dots, 1, 1-\frac{1}{n_k}\Big)}_{2n_k}, 0, 0, \dots\Big).$$

Then

$$x = \sum_{k=1}^{\infty} y_k$$

converges in J, but, if we let $N_k = \sum_{j=1}^{k-1} 2n_j + n_k$ we deduce that

$$\lim_{k\to\infty} \|G^T_{N_k}(x)\|_{qv} \geq \lim_{k\to\infty} \frac{2^k}{k^2} = \infty.$$

Definition 6.1.4. We call a normalized basic sequence *democratic* if there is a constant C so that for all finite $E, F \subset \mathbb{N}$, with $\#E = \#F$ it follows that

$$\Big\|\sum_{j\in E} e_j\Big\| \leq C \Big\|\sum_{j\in F} e_j\Big\| \tag{6.1}$$

In that case we call the smallest constant, so that (6.1) holds, the *Constant of Democracy of* (e_i) and denote it by C_d.

The following characterization of greedy bases is due to Konyagin and Temlyakov:

Theorem 6.1.5. [KT1] *A normalized basis (e_n) is greedy if and only it is unconditional and democratic. In this case*

$$\max(C_s, C_d) \leq C_g \leq C_d C_s C_u^2 + C_u, \tag{6.2}$$

where C_u is the unconditional constant and C_s is the suppression constant.

Remark. The proof will show that the first inequality is sharp. Recently it was shown in [DOSZ1] that the second inequality is also sharp.

Proof of Theorem 6.1.5. "$\Leftarrow$" Assume that (e_i) is unconditional and democratic. Let $x = \sum e_i^*(x)e_i \in X$, $n \in \mathbb{N}$ and let $\eta > 0$. Choose $\tilde{x} = \sum_{i\in\Lambda_n^*} a_i e_i$ so that $\#\Lambda_n^* = n$ which is up to η the best n term approximation to x (since we allow a_i to be 0, we can assume that $\#\Lambda$ is exactly n), i.e.

$$\|x - \tilde{x}\| \leq \sigma_n(x) + \eta. \tag{6.3}$$

Let Λ_n be a set of n coordinates for which

$$b := \min_{i\in\Lambda_n} |e_i^*(x)| \geq \max_{i\in\mathbb{N}\setminus\Lambda_n} |e_i^*(x)| \text{ and } G_n^T(x) = \sum_{i\in\Lambda_n} e_i^*(x)e_i.$$

We need to show that

$$\|x - G_n^T(x)\| \leq (C_d C_s C_u^2 + C_u)(\sigma_n(x) + \eta).$$

Then

$$x - G_n^T(x) = \sum_{i\in\mathbb{N}\setminus\Lambda_n} e_i^*(x)e_i = \sum_{i\in\Lambda_n^*\setminus\Lambda_n} e_i^*(x)e_i + \sum_{i\in\mathbb{N}\setminus(\Lambda_n^*\cup\Lambda_n)} e_i^*(x)e_i.$$

But we also have

$$
\begin{aligned}
(6.4)\qquad \Big\|\sum_{i\in\Lambda_n^*\setminus\Lambda_n} e_i^*(x)e_i\Big\| &\leq bC_u\Big\|\sum_{i\in\Lambda_n^*\setminus\Lambda_n} e_i\Big\| \\
&\leq bC_uC_d\Big\|\sum_{i\in\Lambda_n\setminus\Lambda_n^*} e_i\Big\| \\
&\text{[Note that } \#(\Lambda_n\setminus\Lambda_n^*) = \#(\Lambda_n^*\setminus\Lambda_n)] \\
&\leq C_u^2C_d\Big\|\sum_{i\in\Lambda_n\setminus\Lambda_n^*} e_i^*(x)e_i\Big\| \\
&\text{[Note that } |e_i^*(x)| \geq b \text{ if } i\in\Lambda_n\setminus\Lambda_n^*\ ] \\
&\leq C_sC_u^2C_d\Big\|\sum_{i\in\Lambda_n^*}(e_i^*(x)-a_i)e_i + \sum_{i\in\mathbb{N}\setminus\Lambda_n^*} e_i^*(x)e_i\Big\| \\
&\text{[On } \mathbb{N}\setminus\Lambda_n^* \text{ take all coefficients } e_i^*(x) \\
&\qquad \text{and on } \Lambda_n^* \text{ the coefficients } e_i^*(x)-a_i] \\
&= C_sC_u^2C_d\|x-\tilde{x}\| \leq C_sC_u^2C_d(\sigma_n(x)+\eta)
\end{aligned}
$$

and

$$
\begin{aligned}
(6.5)\qquad \Big\|\sum_{i\in\mathbb{N}\setminus(\Lambda_n^*\cup\Lambda_n)} e_i^*(x)e_i\Big\| &\leq C_s\Big\|\sum_{i\in\Lambda_n^*}(e_i^*(x)-a_i)e_i + \sum_{i\in\mathbb{N}\setminus\Lambda_n^*} e_i^*(x)e_i\Big\| \\
&= C_s\|x-\tilde{x}\| \leq C_s(\sigma_n(x)+\eta).
\end{aligned}
$$

This shows that (e_i) is greedy and, since $\eta > 0$ is arbitrary, we deduce that $C_g \leq C_sC_u^2C_d + C_s$.

"$\Rightarrow$" Assume that (e_i) is greedy. In order to show that (e_i) is democratic let $\Lambda_1, \Lambda_2 \subset \mathbb{N}$ with $\#\Lambda_1 = \#\Lambda_2$. Let $\eta > 0$ and put $m = \#(\Lambda_2 \setminus \Lambda_1)$ and

$$x = \sum_{i\in\Lambda_1} e_i + (1+\eta) \sum_{i\in\Lambda_2\setminus\Lambda_1} e_i.$$

Then it follows

$$\begin{aligned}\Big\|\sum_{i\in\Lambda_1} e_i\Big\| &= \|x - G_m^T(x)\| \\ &\le C_g\sigma_m(x) \text{ (since } (e_i) \text{ is } C_g\text{-greedy)} \\ &\le C_g\Big\|x - \sum_{i\in\Lambda_1\setminus\Lambda_2} e_i\Big\| = C_g\Big\|\sum_{i\in\Lambda_1\cap\Lambda_2} e_i + (1+\eta)\sum_{i\in\Lambda_2\setminus\Lambda_1} e_i\Big\|.\end{aligned}$$

Since $\eta > 0$ can be taken arbitrary, we deduce that

$$\Big\|\sum_{i\in\Lambda_1} e_i\Big\| \le C_g\Big\|\sum_{i\in\Lambda_2} e_i\Big\|.$$

Thus, it follows that (e_i) is democratic and $C_d \le C_g$.

In order to show that (e_i) is unconditional let $x = \sum e_i^*(x)e_i \in X$ have finite support S. Let $\Lambda \subset S$ and put

$$y = \sum_{i\in\Lambda} e_i^*(x)e_i + b\sum_{i\in S\setminus\Lambda} e_i,$$

with $b > \max_{i\in S} |e_i^*(x)|$. For $n = \#(S \setminus \Lambda)$ it follows that

$$G_n^T(y) = b\sum_{i\in S\setminus\Lambda} e_i,$$

and since (e_i) is greedy we deduce that (note that $\#\text{supp}(y-x) = n$)

$$\Big\|\sum_{i\in\Lambda} e_i^*(x)e_i\Big\| = \|y - G_n^T(y)\| \le C_g\sigma_n(y) \le C_g\|y-(y-x)\| = C_g\|x\|,$$

which implies that (e_i) is unconditional with $C_s \le C_g$. $\square$

6.2 The Haar basis is greedy in $L_p[0,1]$ and $L_p(\mathbb{R})$

Recall $(h_t)_{t\in T}$, with

$$T = \{(n,j) : n \in \mathbb{N}_0, j = 0,1,2,\ldots,2^n-1\} \cup \{0\},$$

and $h_0 = 1_{[0,1]}$ and for $n = 0, 1, 2, \ldots$, and $j = 0, 1, 2, \ldots, 2^n - 1$

$$h_{(n,j)} = 1_{[j2^{-n}, j2^{-n}+2^{-n-1})} - 1_{[j2^{-n}+2^{-n-1}, (j+1)2^{-n})}.$$

$$h^{(p)}_{(n,j)} = 2^{n/p} h_{(n,j)}.$$

Theorem 6.2.1. *For $1 < p < \infty$ there are two constants $c_p \le C_p$, depending only on p, so that for all $n \in \mathbb{N}$ and all $A \subset T$ with $\#A = n$*

$$c_p n^{1/p} \le \Big\| \sum_{t \in A} h^{(p)}_t \Big\| \le C_p n^{1/p}.$$

In particular $(h^{(p)}_t)_{t \in T}$ is democratic in $L_p[0,1]$.

With Theorem 6.1.5 and Theorem 5.5.1 we deduce that

Corollary 6.2.2. *The Haar Basis of $L_p[0,1]$, $1 < p < \infty$ is greedy.*

The proof will follow from the following three Lemmas.

Lemma 6.2.3. *For any $0 < q < \infty$ there is a $d_q > 0$ so that the following holds.*

Let $n_1 < n_2 < \ldots n_k$ be integers and let $E_j \subset [0,1]$ be measurable for $j = 1, \ldots k$. Then we have

$$\int_0^1 \Big(\sum_{j=1}^k 2^{n_j/q} 1_{E_j}(x) \Big)^q dx \le d_q \sum_{j=1}^k 2^{n_j} m(E_j).$$

Proof. Define

$$f(x) = \sum_{j=1}^k 2^{n_j/q} 1_{E_j}(x).$$

For $j = 1, \ldots k$ write $E'_j = E_j \setminus \bigcup_{i=j+1}^k E_i$. It follows that for $x \in E'_j$

$$f(x) \le \sum_{i=1}^j 2^{n_i/q} \le \sum_{i=1}^{n_j} 2^{i/q} = \frac{2^{(n_j+1)/q} - 1}{2^{1/q} - 1} \le \underbrace{\frac{2^{1/q}}{2^{1/q} - 1}}_{d_q^{1/q}} 2^{n_j/q}.$$

Thus

$$\int_0^1 f(x)^q dx \le d_q \sum_{j=1}^k 2^{n_j} m(E'_j) \le d_q \sum_{j=1}^k 2^{n_j} m(E_j),$$

which finishes the proof. $\square$

Lemma 6.2.4. *For $1 < p < \infty$ there is a $C_p > 0$ so that for all $n \in \mathbb{N}$, $A \subset T$ with $\#A = n$, and $(\varepsilon_t) \subset \{-1, 1\}$ it follows that*

$$\Big\| \sum_{t\in A} \varepsilon_t h_t^{(p)} \Big\|_p \le C_p n^{1/p}.$$

Proof. Let $n_1 < n_2 < \ldots < n_k$ be all the integers n_i for which there is a $t \in A$ so that $m(\text{supp}(h^{(p)}t)) = 2^{-n_i}$. For $j = 1, \ldots k$ put

$$E_j = \bigcup_{i\in\{0,1,\ldots 2^{n_j}-1\},(n_j,i)\in A} \text{supp}(h^{(p)}(i, n_j)).$$

Since

$$m(E_j) = 2^{-n_j}\#\{i \in \{0, 1, \ldots 2^{n_j} - 1\}, (n_j, i) \in A\}$$

and thus

$$\#\{i \in \{0, 1, \ldots 2^{n_j} - 1\}, (n_j, i) \in A\} = 2^{n_j} m(E_j).$$

It follows therefore that

$$n = \begin{cases} \sum_{j=1}^k \#\{i \in \{0, 1, \ldots 2^{n_j} - 1\}, (n_j, i) \in A\} = \sum_{j=1}^k 2^{n_j} m(E_j) & \text{if } 0 \notin A \\ 1 + \sum_{j=1}^k 2^{n_j} m(E_j) & \text{if } 0 \in A. \end{cases}$$

Assume without loss of generality that $0 \notin A$. It follows that

$$\Big\| \sum_{t\in A} \varepsilon_t h_t^{(p)} \Big\|_p \le \left[\int_0^1 \Big[\sum_{j=1}^k 2^{n_j/p} 1_{E_j} \Big]^p dx \right]^{1/p} \le d_p^{1/p} \left[\sum_{j=1}^k 2^{n_j} m(E_j) \right]^{1/p} = d_p^{1/p} n^{1/p}.$$

[d_p as in Lemma 6.2.3]

□

Lemma 6.2.5. *For $1 < p < \infty$ there is a $c_p > 0$ so that for all $n \in \mathbb{N}$, $A \subset T$ with $\#A = n$, and $(\varepsilon_t) \subset \{-1, 1\}$ it follows that*

$$\Big\| \sum_{t\in A} \varepsilon_t h_t^{(p)} \Big\|_p \ge c_p n^{1/p}.$$

Proof. Note that for $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q}$ and $s, t \in T$ it follows that

$$\langle h_t^{(p)}, h_s^{(q)} \rangle = \delta_{(t,s)},$$

thus the claim follows from the fact that the $h_t^{(p)}$'s are normalized in $L_p[0,1]$ and by Lemma 6.2.4 using the duality between $L_p[0,1]$ and $L_q[0,1]$. Indeed,

$$\begin{aligned}\Big\| \sum_{t\in A} \varepsilon_t h_t^{(p)} \Big\| &\geq \left\langle \sum_{t\in A} \varepsilon_t h_t^{(p)}, \frac{\sum_{t\in A} \varepsilon_t h_t^{(q)}}{\left\| \sum_{t\in A} \varepsilon_t h_t^{(q)} \right\|} \right\rangle \\ &= \frac{n}{\left\| \sum_{t\in A} \varepsilon_t h_t^{(q)} \right\|} \geq \frac{n^{1/p}}{c_q},\end{aligned}$$

where c_q is chosen like in Lemma 6.2.5. Our claim follows therefore by letting $C_p = 1/c_q$. □

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