Limitierte Mengen in Banachräumen

Dissertation

zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften
an der Fakultät für Mathematik
der Ludwig-Maximilians-Universität
München

vorgelegt

von Thomas Schlumprecht

München, den 22. Dezember 1987

Erster Berichterstatter: Prof. Dr. J. Batt

Zweiter Berichterstatter: Prof. Dr. V. Eberhardt

Tag der mündlichen Prüfung: 20.Juli 1988

Contents

Foreword

It began with an error: I. Gelfand [21, II §2, Satz 1] and S. Mazur (cited from [38, p.22]) independently stated the following assertion:

A subset A of a Banach space X is relatively compact if and only if each w*-zero sequence of the dual X' converges uniformly on A.

(We have formulated the assertion using modern terminology).

It is easy to see that on a relatively compact set $C \subset X$, every weak*-zero sequence converges uniformly. Indeed, let $\varepsilon > 0$ be arbitrary. Then there are finitely many $x_1, x_2, ..., x_n \in X$ such that $C \subset \bigcup_{i=1}^{n} x_i + B_\varepsilon(X)$. This implies that $\lim_{n\to\infty} \sup_{x\in C} |\langle x'_n, x\rangle| \leq \varepsilon \sup_{n\in\mathbb{N}} \| x'_n \|$ for each weak*-zero sequence $(x'_n : n \in \mathbb{N}) \subset X'$, and thus, the assertion, since a weak*-zero sequence in the dual of a Banach space is bounded.

However, it was observed by R. S. Phillips [46, p.525, line 6] that the converse is not true. He showed [46, p.539, 7.5 and preceding remarks] that on the unit-basis of c_0, viewed as subset of ℓ_∞, all weak*-zero sequences of ℓ'_∞ converge uniformly. Nowadays, this observation can easily be deduced from two results, both proven by A. Grothendieck [22, p.139, Theorem 1 and p.168, Theorem 9], which state that $C(K)$-spaces (and ℓ_∞ is representable as a $C(K)$) enjoy the Dunford-Pettis property, which means that all $\sigma(\ell'_\infty, \ell''_\infty)$-zero sequences converge uniformly on $\sigma(\ell_\infty, \ell'_\infty)$-compact sets, and also that $\sigma(\ell'_\infty, \ell_\infty)$-convergence of sequences implies $\sigma(\ell'_\infty, \ell''_\infty)$-convergence.

G. Köthe [37, p.196, Definition] introduced the notion "begrenzt" for subsets A of locally convex spaces E having the property that each $\sigma(E', E)$-zero sequence converges uniformly on A. Grothendieck [24] translated this notion into "limited". Thus, we will call a subset A of a Banach space X *limited in X* or *X-limited* if each $\sigma(X', X)$-zero sequence converges uniformly on A, i.e. if

$$\lim_{n\to\infty} \sup_{x\in A} |\langle x'_n, x\rangle| = 0 \qquad \text{whenever } x'_n \underset{n\to\infty}{\longrightarrow} 0 \text{ in } \sigma(X', X).$$

Following J. Diestel [8] we will call a Banach space X a *Gelfand-Phillips space* or say that X has the *Gelfand-Phillips property* , if all X-limited sets are relatively norm-compact.

Before we begin the discussion of these ideas, we want to give a short survey of I. Gelfand's arguments because they lead us to the important difference between limitedness and various types of compactness, namely norm, weak and conditional weak compactness. He first observed that limited sets of Banach spaces are bounded. Indeed, if $A \subset X$ is unbounded, we find a norm-zero sequence $(x'_n : n \in \mathbb{N}) \subset X'$ such that $\limsup_{n\to\infty} \sup_{x\in A} \langle x'_n, x\rangle \neq 0$. He also observed, and this was still correct, that each limited set in a separable space is relatively compact. He deduced this by using the fact that a separable Banach space can be isometrically embedded in a C(K)-space, where K is compact and metrizable, and then applying the theorem of Arzelà and Ascoli. His final step was to make the following conclusion (we translate literally):

Since any countable set $x_1, x_2 \ldots$ can be embedded in a separable space E, this part of the theorem (that limitedness implies compactness) is proven.

This is the crucial point: it is, in general, not true that if Y is a subspace of X, then each $\sigma(Y', Y)$-zero sequence converges uniformly on a subset $A \subset Y$ if each $\sigma(X', X)$-zero sequence converges uniformly on A. Here we arrive at the important difference between limited sets and the above mentioned types of compactness. Norm, weak and conditional weak compactness are properties of the appropriate topology restricted to A. Limitedness for an $A \subset X$ depends not only on the set itself and its topology, but also on the space in which we consider A. Therefore, it is always necessary to specify the space in which we are seeking the limitedness of a given set.

Our investigations on limited sets in Banach spaces can be divided into the following four categories:

1) Limitedness and compactness.
2) Limitedness and geometric properties.
3) Limitedness in C(K)-spaces.
4) Limitedness in combinations of Banach spaces.

Limitedness and compactness

In section (1.1) (Lemma (1.1.5)), we recall a result due to J. Bourgain and J. Diestel [4], who deduced from Rosenthal's ℓ_1 theorem (see (0.2.2)) that the limited sets in any Banach space are weakly conditionally compact. We observe (Proposition (1.1.7)) that Grothendieck spaces X which enjoy the Dunford-Pettis property have the property that, conversely, every conditionally weakly compact subset of X is X-limited.

In (1.2), we will formulate sufficient conditions of a Banach space to have the Gelfand-Phillips property. The most general condition is the following one (Proposition (1.2.2)((c) $\Rightarrow$(d))):

If the dual unit ball $B_1(X')$ of a Banach space X contains a weak*-sequential pre-compact subset D which norms the elements of X up to a constant $c > 0$ (i.e. if $\| x \| \leq c \sup_{x' \in D} |\langle x', x\rangle|$ for each $x \in X$), then X has the Gelfand-Phillips property.

This easily proven statement leads us to our first examples of Gelfand-Phillips spaces (Examples (1.2.4) and (1.2.5)). We observe, for example, that weakly compactly generated Banach spaces, Banach lattices not containing c_o, and $C(K)$-spaces, with K containing a dense and sequentially pre-compact subset, are all Gelfand-Phillips spaces. In section (5.3), we show that the above condition is not necessary for the Gelfand-Phillips property of a Banach space (Theorem (5.3.3)).

J. Bourgain and J. Diestel [4] discovered that limited sets are relatively weakly compact in any Banach space which does not contain ℓ_1. We will generalize this result by showing that limited sets are relatively weakly compact in any Banach space X whose dual does not contain a copy of $L_1(\{0,1\}^{\omega_1})$ (Corollary (2.3.3)). This result will be discussed in more detail in the next category.

Limitedness and geometric properties

Already the definition of limited sets indicates a close relationship between limitedness in a Banach space X and sequential convergence in the weak* topology of its dual X'. Thus, it is natural that our investigations are related to those of [25, 26, 27, 31, 34, 35] which treat the relationship between weak*-convergence of sequences in X' and geometric properties of X and X', like the property of X containing $\ell_1(\Gamma)$ and of X' containing $L_1(\{0,1\}^\Gamma)$.

On the one hand, R. Haydon [31] found an example of a Banach space which shows that the failure of the w*-sequential compactness of $B_1(X')$ does not imply that X contains $\ell_1(\Gamma)$ for an uncountable set Γ. By modifying this example, J. Hagler and E. Odell [25] showed that the failure of the w*-sequential compactness of $B_1(X')$ does not even imply that X contains ℓ_1. By sharpening the construction of R. Haydon [31] and applying the factorization method of W. J. Davis, T. Figiel, W.B. Johnson, and A. Pelczinsky [7], we will give an example of a Banach space which does not contain ℓ_1 and does not have the Gelfand-Phillips property (Theorem (5.2.4)).

On the other hand, J. Hagler and W. B. Johnson [26] showed that a Banach space X, admitting in its dual a bounded sequence which has no w*-convergent absolutely convex block basis (in Definition (2.1.1) we will consider this property as a property of X and denote it by (ACBH)), contains a copy of ℓ_1. R. Haydon

[34] sharpened this result by showing that (ACBH) implies that X' contains a copy of $L_1(\{0,1\}^{\omega_1})$. In [26] and [34], it was observed that non-reflexive Grothendieck spaces enjoy the property (ACBH). In [4], J. Bourgain and J. Diestel observed a similar condition for Banach spaces having limited sets which are not relatively weakly compact: to prove their result ($\ell_1 \not\subset X \Rightarrow$ all limited sets of X are relatively weakly compact), they showed first that any space X which contains limited sets that are not relatively weakly compact must admit bounded sequences in X' which do not have w*-convergent convex blocks (we will denote this property by (CBH)). In chapter 2 (Theorem (2.1.3)), we will prove the following generalization of R. Haydon's result, which leads to a generalization of J. Bourgain's and J. Diestel's result:

A dual space which contains a bounded sequence without a weak convergent convex block contains a copy of* $L_1(\{0,1\}^{\omega_1})$.

<u>Limitedness in C(K)–spaces</u>

Since each Banach space can be isometrically embedded in a C(K)-space (where K is compact), the investigation of limited sets in C(K)-spaces is of special interest.

In (3.1) (Theorem (3.3)), we will show how to construct from a given limited and not relatively compact set $A \subset C(K)$ a normed limited sequence $(f_n : n \in \mathbb{N}) \subset C(K)$ which consists of positive elements with pairwise disjoint supports. Together with the result in (1.3) (Theorem (1.3.2)), where we will show that for a sequence $(x_n : n \in \mathbb{N})$ in X which is equivalent to the unit-basis of c_0, limitedness is equivalent to the condition that no subspace generated by a subsequence of $(x_n : n \in \mathbb{N})$ is complemented in X, we deduce a characterization of the Gelfand-Phillips property of C(K)-spaces. Moreover, we will have reduced the problem of limitedness in C(K)-spaces to the limitedness of positive sequences in C(K) with pairwise disjoint supports.

In the second part of chapter 3, we provide some auxiliary results which will be needed in the sequel. They deal with the following question: Suppose that $(f_n : n \in \mathbb{N}) \subset C(K)$ is weakly conditionally compact but not limited. Which additional properties can a weak*-zero sequence $(\mu_n : n \in \mathbb{N})$ have, for which

$$(+) \qquad \limsup_{n \in \mathbb{N}} |\langle \mu_n, f_n \rangle| > 0\,?$$

We arrive at the following result (Corollary (3.2.5)):

Suppose that $A \subset C(K)$ *is weakly conditionally compact and not limited. Consider, moreover, a sequence* $(F_n : n \in \mathbb{N})$ *of closed and pairwise disjoint subsets of* K *with the following property: For any disjoint infinite* $N_1, N_2 \subset \mathbb{N}$ *there are infinite* $\tilde{N}_1 \subset N_1$ *and* $\tilde{N}_2 \subset N_2$ *such that* $\overline{\bigcup_{n \in \tilde{N}_1} F_n}$ *and* $\overline{\bigcup_{n \in \tilde{N}_2} F_n}$ *are disjoint.*
Then a $\sigma(C(K), C(K)')$*-zero sequence* $(\mu_n : n \in \mathbb{N})$ *and a subsequence* $(k_n : n \in \mathbb{N})$ *of* $\mathbb{N}$ *can be chosen such that* $\inf_{n \in \mathbb{N}} \langle \mu_n, f_{k_n} \rangle > 0$ *and, moreover, such that the support of each* μ_n *has an open neighborhood* O_n *for which* $(O_n : n \in \mathbb{N})$ *is pairwise disjoint and* $O_n \cap \overline{\bigcup_{m \in \mathbb{N}, m \neq n} F_{k_m}} = \emptyset$ *for each* $n \in \mathbb{N}$.

This result leads us to some sufficient conditions for the limitedness of sequences in C(K) with pairwise disjoint supports which depend only on topological properties of K (Theorem (3.3.1)). Since they are rather technical, we present a special case:

Suppose that $(f_n : n \in \mathbb{N})$ *is a normed sequence of positive elements in* C(K) *with pairwise disjoint supports, and suppose moreover that it is subsequentially complete, i.e. that each subsequence of* $(f_n : n \in \mathbb{N})$ *contains a subsequence which admits a supremum in* C(K). *Then* $(f_n : n \in \mathbb{N})$ *is limited in* C(K).

It is easy to see that C(K) has the Gelfand-Phillips property if K contains a dense sequentially pre-compact subset D. The question of L. Drewnowski as to whether or not the Gelfand-Phillips property of a C(K) implies, conversely, that K contains such a D will be answered negatively by the example in Theorem (5.3.4). Under the continuum hypothesis, we will even construct a C(K)-space enjoying the Gelfand-Phillips property such that each convergent sequence of K is eventually stationary (Theorem (5.4.7)). These two examples indicate that the relationship between the Gelfand-Phillips property of a C(K)-space and the topological properties of K are not obvious.

Limitedness in combinations of Banach spaces

It is well known that a bounded and pointwise converging net $(T_i : i \in I)$ of operators between two Banach spaces X and Y converges uniformly on compact subsets of X. In Proposition (1.1.2), we note that the same is true for pointwise convergent sequences of operators and X-limited sets. It is also well known that, given a bounded net $(T_i : i \in I) \subset L(X, X)$ which converges pointwise to the identity on X, a set $A \subset X$ is relatively compact if and only if $T_i(A)$ is relatively compact for each $i \in I$ and if, moreover, T_i converges uniformly on A. This argument may not, in general, be transferred to limitedness (Example (4.1.4)). However, we will show that if, moreover, $(T_i : i \in I)$ is sequentially complete, i.e. if for each increasing $(i_n : n \in \mathbb{N}) \subset I$ the sequence (T_{i_n}) converges pointwise, then a set $A \subset X$ is limited in X if and only if T_i converges uniformly on A and if, for each $i \in I$, $T_i(A)$ is limited in $\overline{T_i(X)}$. This observation leads in some cases to satisfying characterizations of limited sets (c.f. section (4.2)). So we can characterize the limited sets of $\mathrm{L}_p(\mu, X)$, $1 \leq p < \infty$, and $\mathrm{M}(\mu, X)$ (see Example (4.2.4)) by the limitedness in X. In spaces admitting a Schauder decomposition, we can characterize the limited sets by limitedness in the components. We also arrive at the corresponding hereditary results for the Gelfand-Phillips property.

We also deal with the problem of characterizing limited sets in tensor products, in particular, in injective tensor products. We recall the known result that a subset A of $X\check{\otimes}Y$ is relatively compact if and only if $A(\mathrm{B}_1(Y')) := \{z(y') \,|\, z \in A, y' \in \mathrm{B}_1(Y')\}$ $(\subset X)$ and $A(\mathrm{B}_1(X'))$ $(\subset Y)$ are relatively compact. We will show that this result about compactness can only be transferred to limitedness in special cases (Proposition (4.4.2) and Examples (4.5.5)) and leads, in general, only to a necessary condition for limitedness in $X\check{\otimes}Y$. Thus, we are interested in additional necessary conditions for limitedness in $X\check{\otimes}Y$. To do this, we need the following two results.

In section (1.1) (Proposition (1.1.10)), we formulate a necessary condition for limitedness in X by boundedness with respect to other norms defined on dense subspaces:

Let V be a dense subspace of X and let $|\!|\!|\cdot|\!|\!|$ be a norm on V which is finer then $\|\cdot\|$. Then each limited set A in X $(= (X, \|\cdot\|))$ is nearly bounded with respect to $|\!|\!|\cdot|\!|\!|$, i.e. for each $\varepsilon > 0$, there is a $|\!|\!|\cdot|\!|\!|$-bounded $A_\varepsilon \subset V$ such that $A \subset A_\varepsilon + \mathrm{B}_\varepsilon(X, \|\cdot\|)$.

Applying this argument to $V := X \otimes Y$ and letting $|\!|\!|\cdot|\!|\!|$ be the projective norm on $X \otimes Y$, we deduce that

a) *limited sets in the injective tensor product $X\check{\otimes}Y$ are nearly bounded in the projective tensor norm.*

Secondly, we demonstrate (Theorem (4.3.2)) the following argument concerning sequences in $\mathrm{L}^c_\infty(\mu, X)$ (the space of the μ-essentially bounded and μ-measurable functions $f : \Omega \to X$ having μ-essentially separable images, where (Ω, Σ, μ) is a positive measure-space):

Let $(f_n : n \in \mathbb{N}) \subset \mathrm{L}^c_\infty(\mu, X)$. Then (at least) one of the following cases *will hold:*

Case 1: There is a subsequence $(f_n : n \in N)$, $N \in \mathcal{P}_\infty(\mathbb{N})$, *such that for any* $\varepsilon > 0$ *there is a countable* Σ*-partition* π *of* Ω *for which the essential oscillation of each* f_n, $n \in N$, *on each* $B \in \pi$ *is not greater than* ε.

Case 2: There is an $\varepsilon > 0$, *a subsequence* $(k_n : n \in \mathbb{N})$ *of* $\mathbb{N}$, *and a tree* $(A(n,i) n \in \mathbb{N}_0, i \in \{1,..,2^n\}) \subset \Sigma$ *of sets with strictly positive measure such that the essential oscillation of* f_{k_n} *on* $A(n,i)$ *is not greater than* $\varepsilon/4$ *but the essential distance between* $A(n,2j)$ *and* $A(n,2j-1)$ *under* f_{k_n} *is at least* ε *(compare Definition (4.3.1)(b)).*

This result is related to Rosenthal's ℓ_1 theorem and its proof: If we assume that $X = \mathbb{R}$ and that $(f_n : n \in \mathbb{N})$ is bounded, the first case implies that $(f_n : n \in \mathbb{N})$ contains a weak Cauchy subsequence, while the second case implies that $(f_n : n \in \mathbb{N})$ contains a subsequence equivalent to the unit-basis of ℓ_1. It leads us to the following necessary condition for limitedness in $X \check{\otimes} Y$:

b) *Let* K_X *and* K_Y *be two compacts such that* X *and* Y *can be embedded in* $C(K_X)$ *and* $C(K_Y)$ *respectively, and consider* $X \check{\otimes} Y$ *as a subspace of* $L_\infty(\Sigma_X \times \Sigma_Y)$, *where* Σ_X *and and* Σ_Y *are the Borel sets of* K_X *and* K_Y *respectively. Then an* $X \check{\otimes} Y$*-limited set* A *has the following property:*
For each $(f_n : n \in \mathbb{N}) \subset A$, *there is a subsequence* $(f_n : n \in N)$, $N \in \mathcal{P}_\infty(\mathbb{N})$, *such that for each* $\varepsilon > 0$ *there exists a countable* Σ_X*-partition* π^X *and a countable* Σ_Y *partition* π^Y *of* K_X *and* K_Y *respectively, such that the oscillation of each* f_n, $n \in N$, *on each rectangle* $B \times \tilde{B} \in \pi^X \times \pi^Y$ *is not greater than* ε.

In section (4.5.), we will show that, in the case where X and Y are $C(K)$-spaces with the Grothendieck property, (a) and (b) are already sufficient for limitedness in $X \check{\otimes} Y$.

Acknowledgements

I wish to thank my advisor, Professor Jürgen Batt, for his guidance and patience. Especially, I wish to thank him for suggesting this subject which I enjoyed very much.

I wish to thank Professor Richard Haydon for our fruitful discussions, and in particular, for drawing my attention to the systems $\mathcal{R} \subset \mathcal{P}_\infty(\mathbb{N})$ satisfying (F) and (FM) (compare (5.1.1)). This was the key to constructing most of the examples in chapter 5.

I am particularly indebted to Dr. Georg Schlüchtermann for his constant willingness to discuss points of detail and content.

Finally, I want to thank Mark Gruenwald for reading and correcting my English text.

0. Preliminaries

0.1 Notations

Let $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. $\mathbb{R}$ represents the real numbers, while $\mathbb{R}^+$ and $\mathbb{R}_0^+$ are the positive and non negative real numbers respectively.

The cardinality of a set Γ is denoted by $|\Gamma|$. $\mathcal{P}_f(\Gamma)$ and $\mathcal{P}_\infty(\Gamma)$ denote the set of all finite and infinite subsets of Γ respectively, whereas $\mathcal{P}(\Gamma)$ denotes the power set of Γ. For $A \subset \Gamma$, we denote the complement of A in Γ by $\Gamma \setminus A$ or, if there are no ambiguities, by A^C . $\chi_A : \Gamma \to \mathbb{R}$ will denote the characteristic function on A.

Ord is the class of all ordinals. We make use of the principle of transfinite induction [55, p.195, Theorem schema 1], the elements of ordinal arithmetic [55, p.205, §7.2], the order topology on ordinals [47, p.53, Beispiel 5.3], and the fact that, by the axiom of choice, every set can be well-ordered [55, p.242, Theorem 6] and that, conversely, every well-ordered set is order isomorphic to a unique ordinal [55, p.234, Theorem 81]. The first infinite ordinal is denoted by ω_0, the first uncountable ordinal by ω_1, and the first ordinal with the cardinality of the continuum by ω_c.

In a topological space $(T, \mathcal{T})$, $\overset{\circ}{A}^{\mathcal{T}}$ is the open kernel and $\overline{A}^{\mathcal{T}}$ is the closed hull of an $A \subset T$; if the context is clear, they are also denoted by $\overset{\circ}{A}$ and $\overline{A}$ respectively. Compact spaces are always assumed to be Hausdorff. A set $A \subset T$ is *relatively compact*, respectively *relatively sequentially compact*, if $\overline{A}$ is compact, respectively sequentially compact; A is called *$\mathcal{T}$-sequentially pre-compact* if every sequence in A has a $\mathcal{T}$-convergent subsequence. Moreover, if $(T, \mathcal{T})$ is a uniform space, A is called *conditionally $\mathcal{T}$-compact* if every sequence in A contains a Cauchy subsequence (in fact, it would be more precise to say "conditionally sequentially $\mathcal{T}$- compact", but we reserve this terminology for the weak topology where "conditionally compact" is more common).

In a normed space $(X, \|\,.\,\|)$, X' denotes the continuous dual of X. The dual norm on X', as well as the operator-norm of linear and bounded operators $T : X \to X$, is denoted by $\|\,.\,\|$. The closed ball in X with radius $r > 0$ and center 0, i.e. the set $\{x \in X \mid \|x\| \leq r\}$, is denoted by $B_r(X, \|\,.\,\|)$ or $\mathrm{B}_r(X)$.

For a subspace V of X', $\sigma(X, V)$ is the coarsest topology on X such that the elements of V are continuous. We also call $\sigma(X, X')$ the weak topology on X and $\sigma(X', X)$ the weak* topology on X' and abbreviate them by w and w*. All topological notations on normed spaces, when no topology is specifically mentioned, refer to the norm topology.

From now on X and Y always denote Banach spaces with the norm $\|.\|$. All Banach spaces are taken to be linear spaces over the real field $\mathbb{R}$. For an $\mathbb{R}$-linear space V and $A \subset V$, co(A) and aco(A) are the convex hull and the absolutely convex hull of A respectively. The linear space generated by A is denoted by span(A). We say that a Banach space X is generated by $A \subset X$ if $\overline{\text{span}(A)} = X$.

0.2 Special spaces

The Banach spaces c_o, $c_o(\Gamma)$, ℓ_p, $\ell_1(\Gamma)$, $L_p(\mu)$ and $C(K)$, where Γ is any set, $1 \leq p \leq \infty$, (Ω, Σ, μ) is a measure space, and K is a compact space, are defined as usual (e.g.[9]).

We denote the usual unconditional unit-basis (compare [54, p.577, Definition 17.4]) of $\ell_1(\Gamma)$ and $c_o(\Gamma)$ by $(e_\gamma^{(1)}:\gamma\in\Gamma)$ and $(e_\gamma^{(0)}:\gamma\in\Gamma)$ respectively. They have the following properties:

0.2.1 Proposition: *Let Γ be a set.*

a) *A bounded family $(x_\gamma:\gamma\in\Gamma) \subset X$ is equivalent to $(e_\gamma^{(1)}:\gamma\in\Gamma)$, i.e. there exists an isomorphic embedding $T : \ell_1(\Gamma) \to X$ which maps $e_\gamma^{(1)}$ to x_γ, iff there exists a $c > 0$, such that for every $F \in \mathcal{P}_f(\Gamma)$ and every family $(a_\gamma : \gamma \in F) \subset \mathbb{R}$:*

$$\|\sum_{\gamma\in F} a_\gamma x_\gamma\| \geq c \sum_{\gamma\in F} |a_\gamma| \quad .$$

b) *A bounded family $(x_\gamma:\gamma\in\Gamma) \subset X$ is equivalent to $(e_\gamma^{(0)}:\gamma\in\Gamma)$ iff there exist $C > c > 0$, such that for every $F \in \mathcal{P}_f(\Gamma)$ and every family $(a_\gamma : \gamma \in F) \subset \mathbb{R}$:*

$$c \max_{\gamma\in F} |a_\gamma| \leq \|\sum_{\gamma\in F} a_\gamma x_\gamma\| \leq C \max_{\gamma\in F} |a_\gamma| \quad .$$

The following result, due to H. P. Rosenthal, is frequently used and is especially important for our purpose.

0.2.2 Theorem: *[9, p.201, Rosenthal's ℓ_1 Theorem]*
If a bounded sequence $(x_n: n \in \mathbb{N}) \subset X$ has no $\sigma(X,X')$-Cauchy-subsequence, then it has a subsequence which is equivalent to $(e_n^{(1)}:n \in \mathbb{N})$.

For a compact space K, $M(K)$ denotes the Banach space of all regular Borel measures with the variation norm. By the representation theorem of Riesz [15, p.265, Theorem 3], the map $T : M(K) \to C(K)'$, defined by $T(\mu)(f) := \int f d\mu$ for $\mu \in M(K)$ and $f \in C(K)$, is an isometric isomorphism; in this way, $M(K)$ may be identified with the dual of $C(K)$. For $\xi \in K$, δ_ξ is the Dirac measure in ξ. Further notations for the space $M(K)$ $(= M(K, \mathbb{R}))$ can be found in (0.3)(c).

0.3 Combinations of Banach spaces

a) $C(K, X)$

For a compact space K, let $C(K, X)$ be the space of all continuous functions $f : K \to X$ with the norm $\| f \| := \sup_{\xi \in K} \| f(\xi) \|$. This space is generated by the set $G := \{gx \mid x \in X, g \in C(K)\}$ and is isometrically isomorphic to the injective tensor product (comp. (0.3)(d)) of $C(K)$ and X [53, p.357, Theorem 20.5.6]. Thus, the notation $g \otimes x$ has meaning for elements of G and it avoids ambiguities if X is also a space of continuous functions. If $X = C(\tilde{K})$, where $\tilde{K}$ is a compact space, we recall [53, p.357, Theorem 20.5.6.] that the spaces $C(K, C(\tilde{K}))$, $C(\tilde{K}, C(K))$ and $C(K \times \tilde{K})$ are isometrically isomorphic.

For $f \in C(K, X)$, the support of f is defined by $\mathrm{supp}(f) := \overline{\{\xi \in K \mid f(\xi) \neq 0\}}$, the norm-function of f by $\| f(.) \| : K \ni \xi \mapsto \| f(\xi) \|$, and for $x' \in X'$ we define $\langle x', f \rangle : K \ni \xi \mapsto \langle x', f(\xi) \rangle$.

b) The spaces $L_p(\mu, X)$ and $L^c_\infty(\mu, X)$

For a positive measure space (Ω, Σ, μ) and $1 \leq p < \infty$, let $L_p(\mu, X)$ be the Banach space of all Bochner integrable functions on (Ω, Σ, μ) with values in X (see [11, p.17 and p.222]) and let $L_\infty(\mu, X)$ be the space of all μ-measurable, essentially bounded functions with values in X (see [11, p.161]).

The subspace of $L_\infty(\mu, X)$ consisting of all members with essentially seperable range, i.e. the Banach space generated by

$$V := \left\{ \sum_{n \in \mathbb{N}} x_n \chi_{B_n} \;\middle|\; \begin{array}{l} (x_n : n \in \mathbb{N}) \subset X \text{ is bounded and} \\ (B_n : n \in \mathbb{N}) \subset \Sigma \text{ pairwise disjoint} \end{array} \right\},$$

will be denoted by $L^c_\infty(\mu, X)$. We remark that $L_\infty(\mu, X) = L^c_\infty(\mu, X)$ whenever μ is σ-finite. If μ is the counting-measure on (Ω, Σ) ($\mu(A) = |A|$ if $|A| < \infty$ and $\mu(A) := \infty$ if not), we put $L^c_\infty(\Sigma, X) := L^c_\infty(\mu, X)$. Finally, we remark that for a compact space K, $C(K, X)$ is a subspace of $L^c_\infty(\Sigma, X)$ provided Σ is the σ- algebra of the Borel sets of K.

c) $M(K, X)$

Vector measures are always assumed to be σ-additive and to be defined on σ-algebras [10, p.2, Definition 1]. If μ is an X-valued measure on a σ-algebra Σ on a set Ω, $|\mu| : \Sigma \to \mathbb{R}$ is the variation of μ [10, p.2, Definition 4]; μ is said to have finite variation if $|\mu|(\Omega) < \infty$. In this case, $|\mu|$ is a finite positive measure [10, p.3, Proposition 9]. For an X-valued measure μ on (Ω, Σ) and an $f \in L_1(|\mu|)$ define

$$f.\mu : \Sigma \to X, \quad A \mapsto \int_A f d\mu.$$

Since $|f.\mu|(\Omega) = \| f \| < \infty$, $f.\mu$ is an X valued vector measure of finite variation.

Now let K be a compact space and let μ be an X-valued measure of bounded variation on the Borel σ-algebra Σ of K. The support of μ is defined by

$$\text{supp}(\mu) := \text{supp}(|\mu|) := \bigcap \{C \mid C \subset K \text{ compact and } |\mu|(C) = |\mu|(K)\}.$$

μ is said to be regular if $|\mu|$ is regular, i.e. if for every $\varepsilon > 0$ and $A \in \Sigma$ there exists a compact $C \subset A$ and an open $O \supset A$ such that $|\mu|(O \setminus C) < \varepsilon$.

We denote the space of all X-valued regular Borel measures on K by $M(K, X)$, a Banach space under the variation norm. If $X = Y'$, then the operator $T : M(K, X) \to C(K, Y)'$, with $T(\mu)(f) := \int f d\mu$ for $\mu \in M(K, X)$ and $f \in C(K, Y)$, defines an isometric isomorphism; thus $C(K, Y)'$ can be identified with $M(K, Y')$ [19, p.735].

d) Tensor products

The algebraic tensor product of two Banach spaces X and Y is denoted by $X \otimes Y$ (see [53, p.344, Definition 20.1.4. and Proposition 20.1.5]). For two bounded and linear operators $T : X_1 \to Y_1$ and $S : X_2 \to Y_2$, between Banach spaces X_1, X_2, Y_1, Y_2, $T \otimes S$ is the linear mapping $T \otimes S : X_1 \otimes X_2 \to Y_1 \otimes Y_2$,with

$$(T \otimes S)(z) := \sum_{i=1}^{n} T(x_i) \otimes S(\tilde{x}_i) \quad \text{whenever} \quad z = \sum_{i=1}^{n} x_i \otimes \tilde{x}_i \in X_1 \otimes X_2$$

(note that, by defintion of the algebraic tensor product, the bilinear map

$$X_1 \times X_2 \ni (x_1, x_2) \mapsto T(x_1) \otimes S(x_2) \in Y_1 \otimes Y_2$$

is uniquely extendable to a linear $T \otimes S : X_1 \otimes X_2 \to Y_1 \otimes Y_2$).

In particular we defined by this $x' \otimes y' : X \otimes Y \to \mathbb{R}$ for $x' \in X'$ and $y' \in Y'$ (note that $\mathbb{R} \otimes \mathbb{R} = \mathbb{R}$).

For a norm α on $X \otimes Y$, the completition of $X \otimes Y$ with respect to α is denoted by $X \overset{\alpha}{\otimes} Y$. Let us consider the following properties of a norm α on $X \otimes Y$:

(T_1) (crossnorm) $\alpha(x \otimes y) = \| x \| \| y \|$ if $x \in X$ and $y \in Y$,

(T_2) (reasonable norm)

$$x' \otimes y' \in (X \overset{\alpha}{\otimes} Y)' \text{ and } \alpha(x' \otimes y') = \| x' \| \| y' \| \quad \text{whenever} \quad x' \in X', y' \in Y',$$

(T_3) for every $S \in \mathrm{L}(X, X)$ and $T \in \mathrm{L}(Y, Y)$, $S \otimes T$ is bounded with respect to α and $\alpha(S \otimes T) \leq \| S \| \| T \|$. In this case, $S \overset{\alpha}{\otimes} T$ denotes the (unique) extension of $S \otimes T$ to an element of $\mathrm{L}(X \overset{\alpha}{\otimes} Y, X \overset{\alpha}{\otimes} Y)$.

A class of norms on tensor products satisfying (T_1), (T_2) and (T_3) are the so called $\otimes$-norms [23, p.8', Definition 2] or tensor norms [30, p.15, Definition 1.4 and p.18, Definition 1.9]. They contain the projective tensor norm which will be denoted by $\| . \|_{\wedge}$, and the injective tensor norm, which will be denoted by $\| . \|_{\vee}$ [23, p.10, Théorème 3].

e) Spaces of Operators

As usual, $\mathrm{L}(X, Y)$ denotes the Banach space of all linear bounded operators on X with values in Y, with the operator-norm. A $T \in \mathrm{L}(X, Y)$ is called *compact*, respectively *weakly compact*, respectively *Rosenthal*, respectively *limited*, if $T(\mathrm{B}_1(X))$ is relatively compact, respectively relatively weakly compact, respectively conditionally weakly compact, respectively Y-limited.

$\mathrm{K}_{w^*}(X', Y)$ denotes the subspace of $\mathrm{L}(X', Y)$ of all compact and $\sigma(X', X)$-$\sigma(Y, Y')$-continuous operators.

The mapping $\tilde{T} : X \otimes Y \to \mathrm{K}_{w^*}(X', Y)$, $\sum x_i \otimes y_i \mapsto (X' \ni x' \mapsto \sum \langle x', x_i \rangle y_i)$ is well defined and is an isometry with respect to the injective tensor norm as one can see by the following equations:

$$\begin{aligned} \| \tilde{T}(\sum_{i=1}^{n} x_i \otimes y_i) \| &= \sup_{x' \in \mathrm{B}_1(X')} \| \sum_{i=1}^{n} \langle x_i, x' \rangle y_i \| \\ &= \sup_{x' \in \mathrm{B}_1(X'), y' \in \mathrm{B}_1(Y')} \| \sum_{i=1}^{n} \langle x_i, x' \rangle \langle y_i, y' \rangle \| \\ &= \| \sum_{i=1}^{n} x_i \otimes y_i \|_{\vee} \quad \text{if} \quad \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y . \end{aligned}$$

Thus, $\tilde{T}$ is extendable to an isometric embedding $T : X \check{\otimes} Y \to \mathrm{K}_{w^*}(X', Y)$. We remark that T is surjective if X or Y has the approximation property, since in this case every element of $\mathrm{K}_{w^*}(X', Y)$ can be approximated by finite rank operators.

0.4 The Grothendieck property and the Dunford-Pettis property

0.4.1 Definition: X is said to have the *Grothendieck property* if every $\sigma(X', X)$-zero sequence of X' converges in $\sigma(X', X'')$ (to zero).

This is equivalent [10, p.179, Theorem] to the property that every linear and bounded operator $T : X \to c_o$ is weakly compact.

0.4.2 Example: The following Banach spaces have the Grothendieck property:

a) reflexive spaces,

b) $C(K)$-spaces, where K is an extremely disconnected compact space ([10, p.154, Definition 7] and [10, p.156, Corollary 12 and p.179 Theorem]). For example, ℓ_∞, $\ell_\infty(\Gamma)$ and $L_\infty(\mu)$, where Γ is a set and (Ω, Σ, μ) is a measure space, are representable in this way.

0.4.3 Definition: X enjoys the *Dunford-Pettis property* if, given weakly null sequences $(x_n : n \in \mathbb{N})$ and $(x'_n : n \in \mathbb{N})$ in X and X', respectively, then $\lim_{n\to\infty}\langle x'_n, x_n\rangle = 0$.

0.4.4 Examples: The following Banach spaces enjoy the Dunford-Pettis property:

a) $c_o(\Gamma)$, for a set Γ [9, p.113, Exercise 1 (ii)],

b) $C(K)$-spaces [9, p. 113, Exercise 1 (ii)],

c) $L_1(\mu)$-spaces [9, p.113, Exercise 1 (iii)].

0.5 Stone compacts

Some examples of Banach spaces which will be constructed are C(K)-spaces, where K is a Stone compact corresponding to an algebra on $\mathbb{N}$. Therefore, we want to recall the necessary definitions and results.

0.5.1 Definition: Let $\mathcal{A}$ be an algebra on a set Γ.

a) $\mathrm{Hom}_b(\mathcal{A},2)$ denotes the set of all Boolean homomorphisms (see [28, p.35, §9]) on $\mathcal{A}$ with values in $\{0,1\}$, i.e. the set of all mappings $h : \mathcal{A} \to \{0,1\}$ with the following properties
   i) $h(\Gamma) = 1$ and $h(\emptyset) = 0$,
   ii) $h(A \cup B) = \max\{h(A), h(B)\}$ and $h(A \cap B) = \min\{h(A), h(B)\}$ if $A, B \in \mathcal{A}$,
   iii) $h(A) = 1 \iff h(A^C) = 0$ if $A \in \mathcal{A}$.

b) We shall call the set $\mathrm{Hom}_b(\mathcal{A},2)$, endowed with the topology generated by the system
$$\overline{\mathcal{A}} := \{\{h \in \mathrm{Hom}_b(\mathcal{A},2) \mid h(A) = 1\} \mid A \in \mathcal{A}\},$$
the *Stone space* corresponding to $\mathcal{A}$, and denote it by $X(\mathcal{A})$.

c) For a compact space K, $\mathcal{A}(K)$ denotes the algebra of all clopen (closed and open) subsets of K. K is called *Boolean* (compare [28, p.72, §17]) or *zero-dimensional* (compare [53, p.138, Definition 8.2.1.]) if $\mathcal{A}(K)$ is a base for the topology on K.

0.5.2 Proposition: *(Representation Theorem of Stone)*
Let $\mathcal{A}$ be an algebra on the set Γ. Then the space $X(\mathcal{A})$ defined in (0.5.1)(b) is Boolean (in particular, compact) and the map
$$i : \mathcal{A} \to \mathcal{A}(X(\mathcal{A})), \quad A \mapsto \{h \in \mathrm{Hom}_b(\mathcal{A},2) \mid h(A) = 1\}$$
is well defined and an isomorphism in the Boolean sense.

Proof of (0.5.2) :
By [28, p.77, §18, Lemma 2], $\mathrm{Hom}_b(\mathcal{A},2)$ is a closed subset of the Cantor space $\{0,1\}^{\mathcal{A}}$ ($\{0,1\}^{\mathcal{A}}$ is furnished with the product of the discrete topology on $\{0,1\}$). By the theorem of Tychanoff, $\{0,1\}^{\mathcal{A}}$ is a compact space. Since its topology is generated by the system $\{\{f \in \{0,1\}^{\mathcal{A}} \mid f(A) = 1\} \mid A \in \mathcal{A}\}$ and since the topology defined in (0.5.1)(b) is just the restriction of this system to $\mathrm{Hom}_b(\mathcal{A},2)$, $X(K)$ must be compact.

For an $A \in \mathcal{A}$ we get

$$(1) \qquad \begin{aligned} i(A^{\mathsf{C}}) &= \{h \in \mathrm{Hom}_b(\mathcal{A},2) \,|\, h(A^{\mathsf{C}}) = 1\} \\ &= \{h \in \mathrm{Hom}_b(\mathcal{A},2) \,|\, h(A) = 0\} \\ &= \{h \in \mathrm{Hom}_b(\mathcal{A},2) \,|\, h(A) = 1\}^{\mathsf{C}} = (i(A))^{\mathsf{C}}\,. \end{aligned}$$

Thus, the set $i(A)$, which is open in $X(\mathcal{A})$ by definition, is also closed. We deduce that $i(A) \in \mathcal{A}(X(\mathcal{A}))$ and thus i is well defined.
Similary one shows that

$$(2) \qquad \begin{aligned} &i(A \cup B) = i(A) \cup i(B), \\ &i(A \cap B) = i(A) \cap i(B), \quad \text{whenever} \quad A, B \in \mathcal{A} \\ \text{and} \quad &i(\emptyset) = \emptyset \quad \text{and} \quad i(\Gamma) = \mathrm{Hom}_b(\mathcal{A},2)\,. \end{aligned}$$

Thus, i is a homorphism on $\mathcal{A}$ to $\mathcal{A}(K(\mathcal{A}))$.

The map i is injective: Let $A \neq B$ be elements of $\mathcal{A}$; we may assume $A \setminus B \neq \emptyset$. By [28, p.77, §18, Lemma 1], there exists an $h \in \mathrm{Hom}_b(\mathcal{A},2)$ such that $h(A) = 1$ and $h(B) = 0$, which implies that $h \in i(A)$ and $h \notin i(B)$, and thus, $i(A) \neq i(B)$. To show the surjectivity of i, we remark that $i(\mathcal{A})$ separates the points of $\mathrm{Hom}_b(\mathcal{A},2)$ (if $h \neq \tilde{h}$ are in $\mathrm{Hom}_b(\mathcal{A},2)$, there exists an $A \in \mathcal{A}$ with $h(A) = 1$ and $\tilde{h}(A) = 0$) and that, by [28, p.74, §17 Lemma1], an algebra of clopen subsets of a compact K which separates points must be the entire collection of all clopen sets of K.

◇

Since $X(\mathcal{A})$ is a compact space by Proposition (0.5.2), we shall call it the *Stone compact corresponding to* $\mathcal{A}$, provided $\mathcal{A}$ is an algebra, and denote it by $K(\mathcal{A})$.

0.5.3 Proposition: *Let $\mathcal{A}$ be an algebra on a set Γ and let $\mathcal{D} \subset \mathcal{A}$ be $\mathcal{A}$-generating, closed under taking finitely many intersections, and containing Γ as element.*
Then $C(K(\mathcal{A}))$ is generated by $\{\chi_{i(D)} \,|\, D \in \mathcal{D}\}$, where $i : \mathcal{A} \to \mathcal{A}(K(\mathcal{A}))$ is defined as in Proposition (0.5.2).

Proof of (0.5.3) :
Since $i(D)$ is clopen in $K := K(\mathcal{A})$, $\chi_{i(D)}$ lies in $C(K)$ for every $D \in \mathcal{D}$.

Since span($\chi_{i(D)}$: $D \in \mathcal{D}$) is closed under taking produkts ($\mathcal{D}$ is stable under taking intersections) and contains $1 = \chi_{i(\Gamma)}$, it is sufficient to show, by the theorem of Stone and Weierstraß that $i(\mathcal{D})$ separates the points of K. To see this, let $\xi_1, \xi_2 \in K$ and suppose $\xi_1 \in i(D) \iff \xi_2 \in i(D)$ for every $D \in \mathcal{D}$; we have to show that $\xi_1 = \xi_2$. The system

$$\tilde{\mathcal{A}} := \{C \subset K \mid C \quad \text{is clopen and} \quad \xi_1 \in C \iff \xi_2 \in C\}$$

is an algebra and contains $i(\mathcal{D})$. By Proposition (0.5.2) and the assumption, $i(\mathcal{D})$ generates $\mathcal{A}(K(\mathcal{A}))$, and thus, $\tilde{\mathcal{A}}$ is all of $\mathcal{A}(K(\mathcal{A}))$. Since the topology of $K(\mathcal{A})$ is generated by $\mathcal{A}(K(\mathcal{A}))$ (Proposition (0.5.2)), it follows that $\xi_1 = \xi_2$, which finishes the proof.

◇

In the following propositions, we consider the case where $\mathcal{A}$ is an algebra on $\mathbb{N}$ which contains $\mathcal{P}_f(\mathbb{N})$.

0.5.4 Proposition: *Let $\mathcal{A}$ be an algebra on $\mathbb{N}$ containing $\mathcal{P}_f(\mathbb{N})$.*

a) *For an $n \in \mathbb{N}$ and an $A \in \mathcal{A}$ let*

$$h_n(A) := \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if not}. \end{cases}$$

Then $h_n \in Hom_b(\mathcal{A}, 2)$ and the set $\{h_n\}$ is open and closed in $K(\mathcal{A})$.

b) *The mapping $j : \mathbb{N} \to K(\mathcal{A})$, $n \mapsto h_n$, is injective and has a dense and open image in $K(\mathcal{A})$.*

In the sequel we identify $j(\mathbb{N})$ with $\mathbb{N}$. Considering this identification we get:

c) $\overline{A}^{K(\mathcal{A})} = \{f \in Hom_b(\mathcal{A}, 2) \mid f(A) = 1\}$, *for $A \in \mathcal{A}$;*
in particular, we deduce from Proposition (0.5.2) that

$$\{C \subset K(\mathcal{A}) \mid C \text{ is clopen }\} = \{\overline{A}^{K(\mathcal{A})} \mid A \in \mathcal{A}\} \quad .$$

d) *The mappings*

$$E_1 : c_\circ \to C(K(\mathcal{A})), \quad x = (x_n) \mapsto \sum_{n \in \mathbb{N}} x_n \chi_{\{n\}}, \quad \text{and}$$

$$E_2 : C(K(\mathcal{A})) \to \ell_\infty, \quad f \mapsto (f(n) : n \in \mathbb{N}),$$

are isometric embeddings and $E_1 \circ E_2$ is the inclusion of $c_\circ$ in ℓ_∞.

Thus, we identify in the sequel the subspace $E_1(c_\circ)$ of $C(K(\mathcal{A}))$ with $c_\circ$.

Proof of (0.5.4) :

Proof of (a): For $n \in \mathbb{N}$, h_n is a Boolean homomorphism on $\mathcal{A}$ and, since $\{n\} \in \mathcal{A}$ and

$$i(\{n\}) = \{h \in \mathrm{Hom}_b(\mathcal{A}, 2) \mid h(\{n\}) = 1\} = \{h_n\},$$

the set $\{h_n\}$ is clopen in $K(\mathcal{A})$.

Proof of (b): Since for $n \neq m$ it follows that $h_n(\{n\}) = 1 \neq 0 = h_m(\{n\})$, j is injective and has, by (a), an open image.

For an arbitrary $h \in \mathrm{Hom}_b(\mathcal{A}, 2)$, $\mathcal{U}_h := \{i(A) \mid A \in \mathcal{A}, h(A) = 1\}$ is a base for the neighborhoods of h (Proposition (0.5.2)). Since for every $A \in \mathcal{A}$ with $h(A) = 1$ (which implies that $A \neq \emptyset$) and for every $n \in A$ it follows that $h_n \in i(A)$, we deduce that $\{h_n \mid n \in \mathbb{N}\}$ is dense in $K(\mathcal{A})$.

Proof of (c): Since $\mathbb{N}$ is dense in $K(\mathcal{A})$ and $i(A)$ is open for $A \in \mathcal{A}$, $\mathbb{N} \cap i(A) = A$ must be dense in $i(A)$. Since $i(A)$ is also closed, the assertion follows.

Proof of (d): obvious.

◇

Finally, we want to mention some remarks on the space $K(\mathcal{A}) \setminus \mathbb{N}$ for an algebra $\mathcal{A}$ on $\mathbb{N}$ containing $\mathcal{P}_f(\mathbb{N})$.

0.5.5 Proposition: *Let $\mathcal{A}$ be an algebra on $\mathbb{N}$ containing $\mathcal{P}_f(\mathbb{N})$. We set $\tilde{K}(\mathcal{A}) := K(\mathcal{A}) \setminus \mathbb{N}$ (note that by Proposition (0.5.4)(b), $\tilde{K}(\mathcal{A})$ is compact in $K(A)$). Then*

a) *For $A, B \in \mathcal{P}(\mathbb{N})$ with $|A \setminus B| < \infty$ ($\Longleftrightarrow : A \overset{a}{\subset} B$), it follows that $\overline{A} \cap \tilde{K}(\mathcal{A}) \subset \overline{B} \cap \tilde{K}(\mathcal{A})$.*

b) *The mapping $T : \mathrm{C}(K(\mathcal{A}))/c_o \to \mathrm{C}(\tilde{K}(\mathcal{A}))$, $f + c_o \mapsto f|_{\tilde{K}(\mathcal{A})}$ is welldefined and an isometric isomorphism.*

Proof of (0.5.5) :

Proof of (a): Let $A, B \in \mathcal{P}(\mathbb{N})$ and $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that $A \setminus B = \{n_1, \ldots, n_k\}$. Then

$$\begin{aligned} \overline{A} \cap \tilde{K}(\mathcal{A}) &= \overline{(A \cap B) \cup (A \setminus B)} \setminus \mathbb{N} \\ &= (\overline{A \cap B} \cup \{n_1, \ldots, n_k\}) \setminus \mathbb{N} \\ &= \overline{A \cap B} \setminus \mathbb{N} \\ &\subset \overline{B} \cap \tilde{K}(\mathcal{A}). \end{aligned}$$

Proof of (b): If $f, g \in \mathrm{C}(K(\mathcal{A}))$ with $f - g \in c_o$ ($\equiv E_1(c_o)$), it follows from the definition of E_1 in Proposition (0.5.4) that $f|_{\tilde{K}(\mathcal{A})} = g|_{\tilde{K}(\mathcal{A})}$. Thus, T is well defined and for an arbitrary $f \in C(\tilde{K}(\mathcal{A}))$ we have

$$\begin{aligned}\|T(f+c_o)\| &= \sup_{\xi\in\tilde{K}(\mathcal{A})} |f(\xi)| \\ &= \inf_{g\in c_o} \sup_{\xi\in\tilde{K}(\mathcal{A})} |(f-g)(\xi)| \\ &[g(\xi)=0 \text{ for } g\in c_o \text{ and } \xi\in\tilde{K}(\mathcal{A})] \\ &= \| f+c_o \| \quad .\end{aligned}$$

Since every $\tilde{f} \in C(\tilde{K}(\mathcal{A}))$ is extendable to an $f \in C(K(\mathcal{A}))$ ($K(\mathcal{A})$ is normal and $\tilde{K}(\mathcal{A}) \subset K(\mathcal{A})$ is closed), the assertion follows.

◇

0.5.6 Examples:

a) Let $\mathcal{A} := \{A \subset \mathbb{N} \,|\, |A| < \infty \text{ or } |\mathbb{N} \setminus A| < \infty\}$. Then $K(\mathcal{A})$ is homeomorphic to the Alexandrof-compactification of $\mathbb{N}$ and $C(K(\mathcal{A}))$ is isometrically isomorphic to the space c of all convergent sequences.

b) If $\mathcal{A} := \mathcal{P}(\mathbb{N})$, then $K(\mathcal{A})$ is homeomorphic to the Stone-Čech compactification and the embedding E_2 of Proposition (0.5.4)(d) is surjective. Thus, we will identify the spaces ℓ_∞ and $C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ denotes the Stone-Čech compactification of $\mathbb{N}$ (where $\mathbb{N}$ is endowed with the discrete topoloy).

c) There exists an $\mathcal{R} \subset \mathcal{P}_\infty(\mathbb{N})$ with the following properties:
   i) $|\mathcal{R}| = |\omega_c|$,
   ii) If $A \neq B$ are in $\mathcal{R}$, then $|A \cap B| < \infty$,
   iii) $\mathcal{R}$ is maximal in the following sense: there is no $A \in \mathcal{P}_\infty(\mathbb{N}) \setminus \mathcal{R}$ for which $\mathcal{R} \cup \{A\}$ satisfies condition (ii).

If $\mathcal{A}$ is the algebra generated by $\mathcal{R}$ and $\mathcal{P}_f(\mathbb{N})$, then $K(\mathcal{A})$ has the following properties:
   iv) $K(\mathcal{A})$ is sequentially compact,
   v) $E_1(c_o)$, with E_1 defined as in Proposition (0.5.4), is not complemented in $C(K(\mathcal{A}))$.

Proof of (0.5.6) :

Proof of (b): For $\mathcal{A} = \mathcal{P}(\mathbb{N})$, the operator E_2 has a dense image, thus E_2 is an isometric isomorphism. By [47, p.142, A 12.9] the Stone-Čech compactification of $\mathbb{N}$ can be represented by the set of all ultrafilters on $\mathbb{N}$ endowed with the topology generated by the system

$$\{\{\mathcal{U} \,|\, \mathcal{U} \subset \mathcal{P}(\mathbb{N}) \text{ is ultra filter with } N \in \mathcal{U}\} \,|\, N \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}\}\,.$$

Since every $h \in \mathrm{Hom}_B(\mathcal{P}_\infty(\mathbb{N}), 2)$ defines the ultrafilter

$$\mathcal{U}_h := \{N \in \mathcal{P}(\mathbb{N}) \,|\, h(N) = 1\},$$

and since, conversely, each ultra filter $\mathcal{U}$ defines an $h \in \mathrm{Hom}_B(\mathcal{P}_\infty(\mathbb{N}), 2)$ by

$$h(N) := \begin{cases} 1 & \text{if } N \in \mathcal{U} \\ 0 & \text{if } N \notin \mathcal{U} \end{cases} \qquad \text{for } N \in \mathcal{P}(\mathbb{N}),$$

the assertion follows.

Proof of (c): The set

$$\mathcal{I} := \{\mathcal{R} \subset \mathcal{P}_\infty(\mathbb{N}) \,|\, \mathcal{R} \text{ satisfies (i) and (ii)}\}$$

is not empty. Indeed, if $(q_n : n \in \mathbb{N})$ are the rational numbers of $[0,1]$ we choose for every $r \in [0,1]$, a sequence $(q_{n(r,k)} : k \in \mathbb{N})$ which converges to r. Then $\mathcal{R}_\circ := \{\{n(k,r) : k \in \mathbb{N}\} \,|\, r \in [0,1]\}$ satisfies (i) and (ii).
Moreover, since every subset $\tilde{\mathcal{I}}$ of $\mathcal{I}$ which is linearly ordered by inclusion has as upper bound $\bigcup \tilde{\mathcal{I}}$, the existence of a maximal $\mathcal{R}$ follows from Zorn's lemma.

To show (iv), we remark that $\mathcal{D} := \mathcal{R} \cup \mathcal{P}_f(\mathbb{N}) \cup \{\mathbb{N}\}$ is closed under taking finite intersections and it generates $\mathcal{A}$. Thus, by Proposition (0.5.3), it is enough to show that for a given sequence $(\xi_n : n \in \mathbb{N}) \subset K(\mathcal{A})$ there exists a subsequence $(\xi_{n(k)} : k \in \mathbb{N})$ such that $(\delta_{\xi_{n(k)}}(\overline{D}) : k \in \mathbb{N})$ converges for every $D \in \mathcal{D}$.
We can assume that the elements of $(\xi_n : n \in \mathbb{N})$ are pairwise distinct and that they are either all in $\mathbb{N}$ or all in $K(\mathcal{A}) \setminus \mathbb{N}$. In the first case, we deduce from the maximality of $\mathcal{R}$ that there exists an $R \in \mathcal{R}$ such that $R \cap \{\xi_n : n \in \mathbb{N}\}$ is infinite and from (ii) we deduce that the subsequence (ξ_{n_k}) consisting of the elements which are in R satisfies the desired property. For the second case, set

$$\tilde{\mathcal{R}} := \{R \in \mathcal{R} \,|\, \text{ there exists an } \ n \in \mathbb{N} \text{ with } \ \xi_n \in \overline{R}\}.$$

From Proposition (0.5.5)(a) and from condition (ii) we conclude that the intersection of two distinct elements of $\mathcal{R}$ with $\tilde{K}(\mathcal{A})$ is disjoint. Thus, $\tilde{\mathcal{R}}$ is countable and we can find a subsequence $(n(k) : k \in \mathbb{N})$ of $\mathbb{N}$ such that $\delta_{\xi_{n(k)}}(\overline{R})$ converges for every $R \in \tilde{\mathcal{R}}$. Since $\lim_{k\to\infty} \delta_{\xi_{n(k)}}(\overline{D}) = 0$, for every $D \in (\mathcal{R} \setminus \tilde{\mathcal{R}}) \cup \mathcal{P}_f(\mathbb{N})$, we have completed the proof of (iv).

To show (v), we suppose that there exists a linear and bounded mapping

$$P : \mathrm{C}(K(\mathcal{A})) \to c_\circ, \quad f \mapsto (\langle \mu_n, f \rangle : n \in \mathbb{N}),$$

(thus $(\mu_n : n \in \mathbb{N})$ is a $\sigma(\mathrm{M}(K(\mathcal{A})), \mathrm{C}(K(\mathcal{A})))$-zero sequence),
such that $P(\chi_{\{n\}}) = (\mu_m(\{n\}) : m \in \mathbb{N}) = e_n^{(0)}$ if $n \in \mathbb{N}$.
Since the set $\{\overline{R} \setminus R \,|\, R \in \mathcal{R}\}$ is uncountable and has, by (0.5.5) and (i), pairwise disjoint elements, there exists an $R \in \mathcal{R}$ such that $|\mu_n|(\overline{R} \setminus R) = 0$ for every $n \in \mathbb{N}$.
It follows that

$$\mu_n(\overline{R}) = \mu_n(R) = \sum_{m \in R} \mu_n(\{m\}) = 1, \quad \text{if } n \in R,$$

which contradicts the w*-zero convergence of (μ_n).

◇

1. Introduction to the notion of limited sets

The aim of this chapter is to formulate some easy arguments about limited sets and the Gelfand-Phillips property.

1.1 Elementary properties of limited sets, first examples

Proposition (1.1.1) gives two equivalent conditions for a set A to be limited in X. The first is a trivial reformulation of the definition, while the second is an easy consequence of the Theorem of Arzelà-Ascoli [15, p.266, Theorem 7].

1.1.1 Proposition: *For $A \subset X$ the following conditions are equivalent:*

a) *A is limited in X.*

b) *For every sequence $(x_n : n \in \mathbb{N}) \subset A$ and for every $\sigma(X', X)$-zero sequence, $\lim_{n\to\infty}\langle x'_n, x_n\rangle = 0$.*

c) *$T(A)$ is relatively compact for every $T \in L(X, c_\circ)$.*

In particular, we conclude that limitedness is countably determined.

Proof of (1.1.1) :

(a) $\iff$ (b): obvious.

(a) $\iff$ (c): Since every $T \in L(X, c_\circ)$ defines in an obvious way a $\sigma(X', X)$-zero sequence and, conversely, every $\sigma(X', X)$-zero sequence $(x'_n : n \in \mathbb{N})$ defines the linear and bounded operator $T : X \to c_\circ, x \mapsto (\langle x'_n, x\rangle : n \in \mathbb{N})$, the assertion follows from the following characterization of relative compactness in $c_\circ$ [15, p. 389, 13.9]: A bounded $A \subset c_\circ$ is relatively compact iff $\sup_{x\in A} |\langle e_n^{(1)}, x\rangle| \underset{n\to\infty}{\longrightarrow} 0$.

◇

The following proposition shows that limitedness could also be defined by the uniform convergence of sequences of pointwise converging operators.

1.1.2 Proposition: *For $A \subset X$, the following conditions are equivalent:*

a) *A is limited in X.*

b) *For any Banach space Y every pointwise convergent sequence $(T_n : n \in \mathbb{N}) \subset L(X, Y)$ (i.e., there exists a $T \in L(X, Y)$ such that $\| T_n(x) - T(x)\| \underset{n\to\infty}{\longrightarrow} 0$ for every $x \in X$) converges uniformly on A.*

Proof of (1.1.2) :

(a) $\Rightarrow$(b): Let $(T_n : n \in \mathbb{N}) \subset L(X, Y)$ be pointwise convergent to $T \in L(X, Y)$ and let $A \subset X$ be limited in X.

Since A is bounded, we can choose for every $n \in \mathbb{N}$ an $x_n \in A$ such that

$$\sup_{x\in A} \| T(x) - T_n(x) \| \leq 2 \| T(x_n) - T_n(x_n) \|$$

and, by the theorem of Hahn-Banach, a $y'_n \in Y'$ of norm 1 with

$$\| T(x_n) - T_n(x_n) \| = \langle T(x_n) - T_n(x_n), y'_n \rangle.$$

For an arbitrary $x \in X$ we have

$$\begin{aligned} \limsup_{n\to\infty} |\langle T'(y'_n) - T'_n(y'_n), x\rangle| &= \limsup_{n\to\infty} |\langle y'_n, T(x) - T_n(x)\rangle| \\ &\leq \limsup_{n\to\infty} \; \|T(x) - T_n(x)\| = 0. \end{aligned}$$

Thus, $(T'(y'_n) - T'_n(y'_n) : n \in \mathbb{N})$ converges in $\sigma(X', X)$ to 0, and we can deduce (b) from the assumption that A is X-limited in the following way:

$$\begin{aligned} \limsup_{n\to\infty} \sup_{x\in A} \|T(x) - T_n(x)\| &\leq 2 \limsup_{n\to\infty} \|T(x_n) - T_n(x_n)\| \\ &= 2 \limsup_{n\to\infty} |\langle y'_n, T(x_n) - T_n(x_n)\rangle| \\ &\leq 2 \limsup_{n\to\infty} \sup_{x\in A} |\langle T'(y'_n) - T'_n(y'_n), x\rangle| = 0. \end{aligned}$$

(b)$\Rightarrow$(a): obvious

◇

1.1.3 Proposition: *Let A and B be subsets of X.*

a) If A and B are limited in X, then the sets $A \cup B$, $A + B$, $\overline{A}$ and aco(A) *have the same property [4, Proposition 1 and 2].*

b) Let $A \subset B$. If B is limited in X, so is A [4, Proposition 3].

c) Let $T \in$ L(X, Y); if A is X-limited, then $T(A)$ is Y-limited.

Proof of (1.1.3) : obvious

A result due to Grothendieck [9, p.227, Lemma 2] states that a $C \subset X$ is relatively weakly compact if for every $\varepsilon > 0$ there exists a weakly compact $C_\varepsilon \subset X$ such that $C \subset C_\varepsilon + \mathrm{B}_\varepsilon(X)$. An analogous statement is true for limited sets:

1.1.4 Proposition: *Let $A \subset X$ and assume that for every $\varepsilon > 0$ there exists an X-limited set A_ε with $A \subset A_\varepsilon + \mathrm{B}_\varepsilon(X)$.*
Then A is limited in X.

Proof of (1.1.4) :

For an arbitrary $\sigma(X', X)$-zero sequence $(x'_n : n \in \mathbb{N}) \subset \mathrm{B}_1(X')$ and an $\varepsilon > 0$ we have

$$\begin{aligned} \limsup_{n\to\infty} \sup_{x\in A} |\langle x'_n, x\rangle| &\leq \limsup_{n\to\infty} \sup_{x\in A_\varepsilon + \mathrm{B}_\varepsilon(X)} |\langle x'_n, x\rangle| \\ &\leq \limsup_{n\to\infty} \sup_{x\in A_\varepsilon} \varepsilon + |\langle x'_n, x\rangle| = \varepsilon, \end{aligned}$$

which implies the assertion.

◇

In the introduction, we remarked that limited sets are always bounded. The following lemma, from [4, Proposition 4] shows that they are even conditionally weakly compact; the proof uses essentially the Rosenthal's ℓ_1-theorem (see Theorem(0.2.2)).

1.1.5 Lemma: *Every in X limited set is conditionally $\sigma(X,X')$-compact.*

Proof of (1.1.5) : (We follow the proof of [12, Proposition 1.6.])
Using Rosenthal's ℓ_1 theorem, it is enough to show that a given sequence $(x_n : n \in \mathbb{N}) \subset X$ which is equivalent to the ℓ_1-basis $(e_n^{(1)} : n \in \mathbb{N})$ is not limited in X. For every $n \in \mathbb{N}$ let $r_n : [0,1] \to \mathbb{R}$, with $r_n(t) := \sin(2\pi n t)$ if $t \in [0,1]$. The operator $\tilde{S} : L_1([0,1]) \to c_o$, $f \mapsto (\int_0^1 r_n f dt \; : n \in \mathbb{N})$, is well defined, bounded, and linear [10, p.60, Example 1']. Thus, the same is true for $S := \tilde{S} \circ I$ where I is the inclusion of $L_\infty([0,1])$ in $L_1([0,1])$. Since $(x_n : n \in \mathbb{N})$ is equivalent to $(e_n^{(1)} : n \in \mathbb{N})$, the operator

$$\tilde{T} : \overline{\operatorname{span}(x_n : n \in \mathbb{N})} \to L_\infty([0,1]),\ \tilde{T}(\sum_{n=1}^{\infty} \xi_n x_n) = \tilde{T}(\sum_{n=1}^{\infty} \xi_n r_n) \text{ if } (\xi_n : n \in \mathbb{N}) \in \ell_1,$$

is linear and bounded. By the injectivity of $L_\infty([0,1])$ [40, p.111, remarks] it is extendable to a linear and bounded $T : X \to L_\infty([0,1])$.
For the image of $(x_n : n \in \mathbb{N})$ under $S \circ T$, we have

$$S \circ T(x_n) = (\int_0^1 r_m(t) r_n(t) dt \; : m \in \mathbb{N}) = \frac{1}{2} e_n^{(0)} .$$

Thus, $S \circ T(\{x_n : n \in \mathbb{N}\})$ is not relatively compact, which implies, by (1.1.1), that $(x_n : n \in \mathbb{N})$ cannot be limited in X.

◇

1.1.6 Corollary: *For $A \subset X$ the following conditions are equivalent:*

a) *A is limited in X.*

b) i) *A is conditionally $\sigma(X,X')$-compact,*

ii) *each $\sigma(X,X')$-zero sequence in $\operatorname{aco}(A)$ is limited in X.*

In particular, we conclude that X is Gelfand-Phillips iff every normed, weakly to zero converging sequence is not limited in X.

Proof of (1.1.6) :

(a) $\Rightarrow$(b): (1.1.3) and (1.1.5)

$\neg$(a)$\Rightarrow\neg$(b): Suppose $A \subset X$ is not limited in X but is conditionally $\sigma(X,X')$-compact. Then there exists a sequence $(x_n : n \in \mathbb{N})$ in A, an $\varepsilon > 0$, and a $\sigma(X',X)$-zero sequence $(x'_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\langle x_n, x'_n \rangle \geq \varepsilon$. W.l.o.g. we may assume that $(x_n : n \in \mathbb{N})$ is weakly Cauchy and that $|\langle x'_n, x_m \rangle| < \varepsilon/2$ whenever $n > m$ (if not, take subsequences). Thus, $((x_{n+1} - x_n)/2 : n \in \mathbb{N})$ is a weak-zero sequence in aco(A) satisfying

$$\langle x'_{n+1}, (x_{n+1} - x_n)/2 \rangle \geq \varepsilon/4 \text{ , for each } n \in \mathbb{N},$$

which implies that $((x_{n+1} - x_n)/2 : n \in \mathbb{N})$ is not limited in X.

◇

Proposition (1.1.7) and Examples (1.1.8) point out that there exist Banach spaces containing limited sets, which are not relatively compact.

1.1.7 Proposition: *If X enjoys the Grothendieck and the Dunford-Pettis properties, then every conditionally $\sigma(X,X')$-compact set is limited. Thus, by (1.1.5), the limited sets are just the conditionally $\sigma(X,X')$-compact sets.*

Proof of (1.1.7) :

By (1.1.6), it is enough to show that every $\sigma(X,X')$-zero sequence in X is limited. But this follows from the assumption that every $\sigma(X',X)$-zero sequence converges in $\sigma(X',X'')$ and that X has the Dunford-Pettis property.

◇

1.1.8 Examples: For every infinite compact space K, C(K) contains conditionally weakly compact sets which are not relatively norm compact. Moreover, it contains weakly conditionally compact subsets which are not relatively weakly compact. Since every C(K)-space enjoys the Dunford-Pettis property, it follows from (1.1.7) that a Grothendieck C(K)-space does not have the Gelfand Phillips property, and that, moreover, it has limited sets which are not relatively weakly compact.

An example for such a space is $\ell_\infty(\Gamma)$, where Γ is an infinite set. In the literature one can find two other examples of infinite dimensional C(K)-spaces wich enjoy the Grothendieck property and which, moreover, do not contain a copy of ℓ_∞:

R. Haydon [33] constructed one without assuming any further set theoretical axiom. By assuming the continuum hypothesis, M. Talagrand [56] found another one which does not even admit a quotient isomorphic to ℓ_∞ (the results cited in chapter 2 show that such an example cannot be constructed without additional set theoretical hypotheses).

From Proposition (1.1.7) and Examples (1.1.8), we deduce the following analog of the Krein-Smulian Theorem.

1.1.9 Corollary: *The absolutely convex hull of a conditionally weakly compact subset of X is conditionally weakly compact also.*

Proof of (1.1.9) :

Let Γ be a set such that there exists an isomorphic embedding E from X into $\ell_\infty(\Gamma)$ (for example $\Gamma := \mathrm{B}_1(X'))$. For $A \subset X$ the following implications hold:

A is conditionally $\sigma(X, X')$-compact

$\iff$ $E(A)$ is conditionally weakly compact in $\ell_\infty(\Gamma)$

$\iff$ $E(A)$ is limited in $\ell_\infty(\Gamma)$ [(1.1.5), (1.1.7)]

$\iff$ $\mathrm{aco}(E(A)) = E(\mathrm{aco}(A))$ is limited in $\ell_\infty(\Gamma)$ [(1.1.3)]

$\iff$ $E(\mathrm{aco}(A))$ is conditionally weakly compact in $\ell_\infty(\Gamma)$ [(1.1.5), (1.1.7)]

$\iff$ $\mathrm{aco}(A)$ is conditionally $\sigma(X, X')$-compact.

◇

Finally we want to present a necessary condition for limitedness in a Banach space X which uses other norms defined on dense subspaces of X. It will be a useful tool to investigate limited sets in tensor products.

1.1.10 Proposition: *Let $V \subset X$ be a dense subspace of X and let $|\!|\!|\cdot|\!|\!|$ be a norm on V which is finer then $\|\cdot\|$. We denote the completition of V corresponding to $|\!|\!|\cdot|\!|\!|$ by $\tilde{X}$.*

Then every in X $(= (X, \|\cdot\|))$ limited set A is "almost bounded corresponding to $|\!|\!|\cdot|\!|\!|$", by this we mean that for each $\varepsilon > 0$ there is a $|\!|\!|\cdot|\!|\!|$-bounded $A^{(\varepsilon)} \subset V$ such that

$$A \subset \bigcap_{\varepsilon>0} A^{(\varepsilon)}.$$

Proof of (1.1.10) :

We have to show the following:

Let $A \subset X$ be $\|\cdot\|$-bounded, $(x_n : n \in \mathbb{N}) \subset A$ and $\varepsilon > 0$, such that

(1) $$r_n := \inf\{|\!|\!|y|\!|\!| \mid y \in V \cap (x_n + \mathrm{B}_\varepsilon(X, \|\cdot\|))\} \underset{n\to\infty}{\longrightarrow} \infty,$$

then A is not limited in $(X, \|\cdot\|)$.

For $n \in \mathbb{N}$ define

$$A_n := \overline{(r_n/2) \cdot \mathrm{B}_1(\tilde{X}, \|\!|\cdot|\!\|)}^{\|\cdot\|} \quad \text{and} \quad B_n := x_n + \mathrm{B}_{\varepsilon/2}(X, \|\cdot\|).$$

We first show that $A_n \cap B_n = \emptyset$ for each $n \in \mathbb{N}$:

Let $y \in A_n$, then there is a $\tilde{y} \in (r_n/2) \cdot \mathrm{B}_1(\tilde{X}, \|\!|\cdot|\!\|)$ with $\| \tilde{y} - y \| < \varepsilon/4$, and a $\hat{y} \in V$ with $\|\!|\hat{y} - \tilde{y}|\!\| < \min(\varepsilon/4, r_n/2)$. Thus, $\|\!|\hat{y}|\!\| \le \|\!|\tilde{y}|\!\| + \|\!|\hat{y} - \tilde{y}|\!\| < r_n$, and we conclude from (1) that $\| \hat{y} - x_n \| > \varepsilon$ and finally

$$\| x_n - y \| \ge \| x_n - \hat{y} \| - \| \hat{y} - \tilde{y} \| - \| \tilde{y} - y \| > \varepsilon - 2\varepsilon/4 = \varepsilon/2,$$

which implies the assertion.

Since A_n and B_n are convex and $\|\cdot\|$-closed, and since, A_n is absolutely convex, we find, by the separation theorem, for each $n \in \mathbb{N}$ an $x'_n \in X'$, with $\| x'_n \| = 1$, and an $a_n \ge 0$ such that

$$\text{(2)} \qquad \langle x'_n, y \rangle \le a_n \le \langle x'_n, x \rangle, \quad \text{whenever } y \in A_n \text{ and } x \in B_n.$$

For $n \in \mathbb{N}$ we choose $y_n \in \mathrm{B}_{\varepsilon/2}(X, \|\cdot\|)$ with $\langle x'_n, y_n \rangle \ge \varepsilon/4$ and we conclude

$$\text{(3)} \qquad \begin{aligned} \langle x'_n, x_n \rangle &= \langle x'_n, x_n - y_n \rangle + \langle x'_n, y_n \rangle \\ &\ge a_n + \varepsilon/4 \ge \varepsilon/4 . \\ &[x_n - y_n \in B_n \text{ and } a_n \ge 0] . \end{aligned}$$

Thus, we are finished, if we have proven that $(x'_n : n \in \mathbb{N})$ is a weak*-zero sequence in $(X', \|\!|\cdot|\!\|)$.

To this end, we first observe that by (2) $\sup_{n\in\mathbb{N}} |a_n| \le \sup_{n\in\mathbb{N}, x\in A} |\langle x'_n, x \rangle| < \infty$, and secondly that for each $v \in V$ it follows from (1) and (2):

$$|\langle x'_n, v \rangle| = \frac{2\|\!|v|\!\|}{r_n} |\langle x'_n, \frac{r_n}{2\|\!|v|\!\|} v \rangle| \le \frac{2\|\!|v|\!\|}{r_n} \cdot a_n \underset{n\to\infty}{\longrightarrow} 0;$$

Since V is dense in X and $(x'_n : n \in \mathbb{N})$ is bounded this implies the assertion.

◇

1.2 Gelfand-Phillips spaces

Proposition (1.2.2) develops some topological conditions on the compact space $(B_1(X'), \sigma(X', X))$ which imply that X has the Gelfand-Phillips property, similar considerations were done by L. Drewnowski in [13].

Then some classes of Banach spaces are investigated which correspond to these conditions, and so we get a first inventory of Gelfand-Phillips spaces. Besides the easy fact that every subspace of a Gelfand-Phillips space enjoys this property also, and that the complemented sum of two Gelfand-Phillips spaces is again Gelfand-Phillips (Proposition (1.2.2) (a)⇒(b) and (d)⇒(a)), we do not consider hereditary properties of the Gelfand-Phillips property (see chapter 4).

1.2.1 Proposition: *The following are equivalent:*

a) X is Gelfand-Phillips.

b) Every subspace of X is Gelfand-Phillips.

c) Every separable subspace Z of X is contained in a subspace $Y \subset X$ which has the Gelfand-Phillips property and is complemented in X.

d) X is the complemented sum of two Gelfand-Phillips spaces.

e) For every Banach space Y, the limited operators from Y to X are compact.

f) Every limited operator from ℓ_1 to X is compact.

Proof of (1.2.1) :

(a)⇒(b): follows from (1.1.3)(c)

(b)⇒(c): obvious

(c)⇒(a): By (1.1.1) it is enough to show that a non compact sequence $(x_n : n \in \mathbb{N})$ in X is not limited, whenever X satisfies (c). But by (c) there exists a complemented subspace Y of X enjoying the Gelfand-Phillips property and containing $(x_n : n \in \mathbb{N})$. Since $(x_n : n \in \mathbb{N})$ is not limited in Y, it follows from (1.1.3)(c) that it cannot be limited in X, since it is the preimage of the projection from X onto Y of a non limited set.

(a)⇒(d): $X = X \oplus \{0\}$

(d)⇒(a): (1.1.3)(c)

(a)⇒(e): obvious

(e)⇒(f): obvious

(f)⇒(a): Let $(x_n : n \in \mathbb{N})$ be limited (in particular bounded) in X and assume that X satisfies (f); we have to show that $(x_n : n \in \mathbb{N})$ is relatively compact. The operator

$$T : \ell_1 \to X\,, \quad y = (y_n) \mapsto \sum_{n \in \mathbb{N}}^{n} y_n x_n$$

is bounded and linear and $T(\mathrm{B}_1(\ell_1)) \subset \overline{\mathrm{aco}(x_n : n \in \mathbb{N})}$ is limited in X. Since every limited operator $T : \ell_1 \to X$ is compact, and since $(x_n : n \in \mathbb{N}) \subset T(\mathrm{B}_1(\ell_1))$, $(x_n : n \in \mathbb{N})$ is relatively compact.

◇

1.2.2 Proposition: *The following implications hold:*

$$(a) \Rightarrow (b_1) \Longleftrightarrow (b_2) \Rightarrow (c_1) \Longleftrightarrow (c_2) \Longleftrightarrow (c_3) \Rightarrow (d)$$

a) $\mathrm{B}_1(X')$ *is* $\sigma(X', X)$*-sequentially compact,*

b₁) $\mathrm{B}_1(X')$ *contains a* $\sigma(X', X)$*-sequentially pre-compact subset* C *which norms* X*, i.e.* $\| x \| = \sup_{x' \in C} |\langle x', x\rangle|$ *for each* $x \in X$.

b₂) There exists a compact K *containing a sequentially pre-compact and dense subset such that* X *can be isometrically embedded in* $\mathrm{C}(K)$.

c₁) There exists an equivalent norm $|\!|\!|.|\!|\!|$ *on* X *such that* $(X, |\!|\!|.|\!|\!|)$ *satisfies* (b_1).

c₂) There exists an equivalent norm $|\!|\!|.|\!|\!|$ *on* X *such that* $(X, |\!|\!|.|\!|\!|)$ *satisfies* (b_2).

c₃) There exists an $r > 0$ *and a sequentially* $\sigma(X', X)$ *pre-compact subset* C *of* $\mathrm{B}_1(X')$*, such that* $\| x \| \leq r \sup_{x' \in C} |\langle x', x\rangle|$.

d) X *has the Gelfand-Phillips property.*

(The implication (c_1) $\Rightarrow$(d) *is also observed in [13, Theorem 2.2].)*

Proof of (1.2.2) :

(a)$\Rightarrow$(b_1): obvious

(b_1)$\Rightarrow$(b_2): Let $C \subset \mathrm{B}_1(X')$ be as in (b_1) and set $K := \overline{C}^{\sigma(X',X)}$. Then K, furnished with $\sigma(X', X) \cap K$, has the desired properties and the operator $E : X \to \mathrm{C}(K)$, $x \mapsto (K \ni x' \mapsto \langle x', x\rangle)$, is an isometric embedding.

(b_2)$\Rightarrow$(b_1): Let K be a compact space which contains a dense sequentially pre-compact subset $\tilde{K}$ and admits an isometric embedding $E : X \to \mathrm{C}(K)$; then the set $C := E'(\{\delta_\xi \mid \xi \in \tilde{K}\})$ satisfies the conditions of (b_1).

(b_1)$\Rightarrow$(c_1): obvious

(c_1) $\Longleftrightarrow$ (c_2): as in ((b_1) $\Longleftrightarrow$ (b_2))

(c_1) $\Longleftrightarrow$ (c_3): obvious

(c_3)$\Rightarrow$(d): By (1.1.6), it is enough to show that a given normed $\sigma(X, X')$-zero sequence $(x_n : n \in \mathbb{N})$ is not limited in X. Suppose that $r > 0$ and $C \subset \mathrm{B}_1(X')$ are as in (c_3). Then there exists for each $n \in \mathbb{N}$ an $x'_n \in C$ with $|\langle x'_n, x_n\rangle| > r/2$. By

(c_3), we may assume that $(x'_n : n \in \mathbb{N})$ converges in $\sigma(X', X)$ to an $x'_0 \in X'$ and, since $(x_n : n \in \mathbb{N})$ converges in $\sigma(X, X')$ to 0, we may assume that $|\langle x'_0, x_n\rangle| < r/4$ for $n \in \mathbb{N}$; thus we have

$$\limsup_{n\to\infty} |\langle x'_n - x'_0, x_n\rangle| \geq r/4,$$

which implies the assertion.

◇

1.2.3 Notation: Condition (a) from Proposition (1.2.2) will be denoted by (w*-sc) ("w*-sequentially compact"), (b_1) by (w*-spcn) ("w*-sequentially pre-compact and norming subset"), and (c_1) by (w*-spcnc) ("w*-sequentially precompact up to a constant norming subset"). These will be considered as properties of the Banach space X.

1.2.4 Examples: The following Banach spaces are (w*-sc):

a) By a result of D. Amir and J. Lindenstrauss all subspaces of weakly generated spaces have the property (w*-sc)[9, p.228, Theorem]. For eaxample:
   - separable Banach spaces,
   - reflexive Banach spaces,
   - $c_o(\Gamma)$, Γ any set, (note that $(e^{(0)}_\gamma : \gamma \in \Gamma)$ is relatively $\sigma(c_o(\Gamma), \ell_1(\Gamma))$-compact),
   - $C(K)$, if K is an Eberlein compact [1, p.37, Theorem 2],
   - $L_1(\mu)$-spaces, for σ-finite measures μ.

b) If X is a Banach space whose dual X' does not contain ℓ_1, we deduce from Rosenthal's ℓ_1 theorem that $B_1(X')$ must be conditionally weakly compact and thus, by the theorem of Alaoglu-Bourbaki, sequentially weak*-compact. Examples for this situation (not satisfying (a)) are the non separable versions of the James and James-tree spaces as introduced in [16] and [5] respectively.

1.2.5 Examples: The following spaces are (w*-spcn)but in general not (w*-sc):

a) $C(K)$-spaces, where K contains a dense sequential pre-compact subset as for example $C(\{0,1\}^\Gamma)$ (the set $\{(\theta_\gamma : \gamma \in \Gamma) \subset \{0,1\} \mid |\{\gamma \in \Gamma \mid \theta_\gamma = 1\}| < \infty\}$ is a sequentially pre-compact dense subset of $\{0,1\}^\Gamma$).

b) Spaces X which can be isometrically embedded in dual spaces Y' whose predual does not contain ℓ_1.

c) Banach lattices not containing c_o, for example AL-spaces (by [51, p.114, Theorem 8.5], AL-spaces are representable as $L_1(\mu)$-spacess, where μ is a positive measure).

Proof of (1.2.5) :

Proof of (a): Proposition (1.1.2)((b_1) $\Longleftrightarrow$ (b_2))

Proof of (b): Let $E : X \to Y'$ be an isometric embedding and $\ell_1 \not\subset Y$. By Rosenthal's ℓ_1 theorem we deduce that $B_1(Y)$ is a $\sigma(Y'', Y')$-sequentially pre-compact subset of $B_1(Y'')$ and by Goldstine's theorem it is w* dense in $B_1(Y'')$. Thus, $C := E'(B_1(Y))$ satisfies the conditions of (b_1).

Proof of (c): For a Banach lattice $(X, \leq)$, we use the notations of chapter 1 in [41]. The proof will use several basic results about Banach lattices and a theorem of [42].

If X has no copy of c_o, it follows from [41, p.6, Theorem 1.a.5.] that X is σ-complete (every increasing and bounded sequence in X has a supremum in X). Thus, the operator P_x given by

$$P_x : X \to X, y \mapsto \bigvee_{n=1}^{\infty}(nx \wedge y^+) - \bigvee_{n=1}^{\infty}(nx \wedge y^-)$$

is well defined for every $x \in X$, $x \geq 0$, and is by the remarks in [41, p.12f], a continuous projection of norm 1 onto the closed subspace generated by the lattice-interval $[0, x]$.

From [42, Theorem 2 ($\neg$(d)$\Rightarrow$ $\neg$(e))] it follows that for each $x \geq 0$ of X, the set $[0, x]$ is conditionally weakly compact. Since c_o is not in X, X must be sequentially complete; thus, by the theorem of Eberlein-Smulian, $[0, x]$ must be weakly compact for every non negative $x \in X$. Thus, the image of every P_x is weakly compactly generated and is (w*-sc)by (1.2.4)(a).

To finish the proof, we show that $C := \bigcup_{x \geq 0} P'_x(B_1(X'))$ is $\sigma(X', X)$-sequentially pre-compact and X-norming.

Since $x = P_{|x|}(x)$ for each $x \in X$, we can deduce for an x-norming $x' \in B_1(X')$ that $\langle x, P'_{|x|}(x')\rangle = \langle P_{|x|}(x), x'\rangle = \| x \|$; by this we have shown that C is X-norming.

Let $(x'_n : n \in \mathbb{N}) \subset C$ be arbitrary. By definition of C, there exists for each $n \in \mathbb{N}$ a non negative $x_n \in X$ such that $P'_{x_n}(x'_n) = x'_n$ and we set

$$x := \sum_{n \in \mathbb{N}} \frac{x_n}{2^n \, \| x_n \|}.$$

For any non negative $z \in X$ and any $n \in \mathbb{N}$ we have:

$$\begin{aligned} P_{x_n} \circ P_x(z) &= \bigvee_{m\in\mathbb{N}} (mx_n \wedge \bigvee_{\ell\in\mathbb{N}} (\ell x \wedge z)) \\ &= \bigvee_{m\in\mathbb{N},\ell\in\mathbb{N}} (mx_n \wedge \ell x \wedge z) \\ &= \bigvee_{m\in\mathbb{N}} (mx_n \wedge z) = P_{x_n}(z) \end{aligned}$$

[note that $(mx_n) \wedge (2^n m \,\|x_n\|\, x) = mx_n$].

It follows that for any $z \in X$

$$\langle x'_n - P'_x(x'_n), z\rangle = \langle P'_{x_n}(x'_n) - P'_{x_n} \circ P'_x(x'_n), z\rangle = \langle x'_n, P_{x_n}(z) - P_{x_n}P_x(z)\rangle = 0\,,$$

which means that $(x'_n : n \in \mathbb{N})$ lies in $P'_x(\mathrm{B}_1(X'))$ and has a $\sigma(P'_x(X), P_x(X))$ converging subsequence. Since

$$\langle x'_n, z\rangle = \langle P'_x(x'_n), z\rangle = \langle x'_n, P_x(z)\rangle \quad \text{for } n \in \mathbb{N} \text{ and } z \in X\,,$$

this subsequence converges also in $\sigma(X, X')$, which finishes the proof.

◇

1.2.6 Example: The Schur spaces are of course also Gelfand-Phillips spaces (there, relatively compactness and conditionally weakly compactness are the same). We do not know if in general they enjoy one of the stronger properties introduced in (1.2.2).

1.2.7 Remark: The examples in (1.2.5)(a) show that the implication "(a)⇒(b)" in (1.2.2) is not reversible. Examples that "(b)⇒(c)" and "(c)⇒(d)" are strict, will be given in section (5.3) (Theorem (5.3.3)).

We showed in (1.2.2) that C(K)-spaces, where K is a compact space containing a sequentially pre-compact subset, enjoy the Gelfand-Phillips property. In (5.3) (Theorem (5.3.4)) we will show that the converse is not true. Under the continuum hypothesis, we can even construct an infinite compact space K such that C(K) enjoys the Gelfand-Phillips property and such that every convergent sequence of K is eventually stationary (Theorem (5.4.7) in (5.4)).

1.3 Characterization of non limited sets by biorthogonal sequences, complemented c_0-subspaces

Let $A \subset X$ be bounded but not X-limited. By definition, there exists a sequence $(x_n : n \in \mathbb{N}) \subset A$ and a $\sigma(X', X)$-zero sequence such that $\langle x'_n, x_n \rangle = 1$ for all $n \in \mathbb{N}$. Lemma (1.3.1) shows that $(x_n : n \in \mathbb{N})$ and $(x'_n : n \in \mathbb{N})$ can be chosen to be biorthogonal. Using this fact, one can deduce some results concerning complemented copies of c_0.

1.3.1 Lemma: *Let $A \subset X$ be bounded but not limited.*

Then there exists a sequence $(x_n : n \in \mathbb{N})$ in A and a $\sigma(X', X)$-zero sequence, such that:

$$\langle x'_n, x_m \rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if not} \end{cases} \quad \text{for all } m, n \in \mathbb{N}.$$

Proof of (1.3.1) :

Since A is not limited in X, there exists a sequence $(x_n : n \in \mathbb{N})$ in A and a $\sigma(X', X)$-zero sequence with

$$(1) \qquad \langle z'_n, x_n \rangle = 1 \quad \text{for } n \in \mathbb{N}.$$

Since A is bounded, we can assume that $a_n := \lim_{m \to \infty} \langle x_n, z'_n \rangle$ exists for each $n \in \mathbb{N}$ (otherwise we pass to a subsequence). By taking another subsequence we can assume that one of the following three cases is satisfied:

case 1: $a_n = 0$ for $n \in \mathbb{N}$;

case 2: $a_n \neq 0$ for $n \in \mathbb{N}$ and $\lim_{n \to \infty} a_n = 0$;

case 3: There exists an $\varepsilon > 0$ such that $|a_n| \geq \varepsilon$ for $n \in \mathbb{N}$.

In the first case we take $y'_n = z'_n$ for $n \in \mathbb{N}$.

In the second case we may assume that

$$r_n := \langle z'_n - (a_n/a_{n-1}) z'_{n-1}, x_n \rangle \geq 1/2.$$

If we now set

$$y'_n := (1/r_n)(z'_n - (a_n/a_{n-1}) z'_{n-1}), \text{ for } n \in \mathbb{N},$$

the sequence $(y'_n : n \in \mathbb{N})$ is a $\sigma(X', X)$-zero sequence also. We still have for every $n \in \mathbb{N}$, $\langle z'_n, x_n \rangle = r_n / r_n = 1$. Moreover, we deduce that, for each $n \in \mathbb{N}$,

$$\lim_{m \to \infty} \langle y'_n, x_m \rangle = \lim_{m \to \infty} \langle (1/r_n)(z'_n - (a_n/a_{n-1}) z'_{n-1}), x_m \rangle = 0.$$

In the third case, $(1/a_n: n \in \mathbb{N})$ is bounded and, since $(z'_n: n \in \mathbb{N})$ is a weak*-zero sequence, we can assume that for each $n \in \mathbb{N}$

$$r_n := \langle z'_n - (a_n/a_{n+1})z'_{n+1}, x_n \rangle \geq 1/2.$$

If we choose

$$y'_n := (1/r_n)(z'_n - (a_n/a_{n+1})z'_{n+1}),$$

the sequence $(y'_n: n \in \mathbb{N})$ is a $\sigma(X', X)$-zero sequence also. For each $n \in \mathbb{N}$ we get, as in the second case, $\langle z'_n, x_n \rangle = r_n/r_n = 1$ and

$$\lim_{m\to\infty} \langle y'_n, x_m \rangle = \lim_{m\to\infty} \langle (1/r_n)(z'_n - (a_n/a_{n+1})z'_{n+1}), x_m \rangle = 0.$$

So in all three cases we have, for each $n \in \mathbb{N}$,

$$\lim_{m\to\infty} \langle y'_m, x_n \rangle = \lim_{m\to\infty} \langle y'_n, x_m \rangle = 0\,.$$

Thus, by taking subsequence, we can assume that

$$\sum_{m \in \mathbb{N}\setminus\{n\}} |\langle y'_n, x_m \rangle| \leq \frac{1}{2}$$

holds for each $n \in \mathbb{N}$. This implies that the image of $(x_n: n \in \mathbb{N})$ under $T : X \to c_\circ$, $x \mapsto (\langle x, y'_n \rangle : n \in \mathbb{N})$, is equivalent to $(e_n^{(0)}: n \in \mathbb{N})$, as can be seen by the following equations:

$$\|\sum_{i\in\mathbb{N}} a_i T(x_i)\| = \sup_{j\in\mathbb{N}} |\langle y'_j, \sum_{i\in\mathbb{N}} a_i T(x_i)\rangle|$$

$$\begin{cases} \leq \sup_{j\in\mathbb{N}} \left(|a_j| + \sum_{i\neq j} |\langle y'_j, x_i \rangle a_i|\right) \\ \geq \sup_{j\in\mathbb{N}} \left(|a_j| - \sum_{i\neq j} |\langle y'_j, x_i \rangle a_i|\right) \end{cases}$$

$$\begin{cases} \leq (1 + \frac{1}{2}) \sup_{j\in\mathbb{N}} |a_j| \\ \geq (1 - \frac{1}{2}) \sup_{j\in\mathbb{N}} |a_j| \end{cases}$$

for every sequence $(a_n: n \in \mathbb{N}) \subset \mathbb{R}$ such that $|\{n \in \mathbb{N} \mid a_n \neq 0\}| < \infty$. By the separable injectivity of $c_\circ$ [9, p.71, Theorem 4], we can extend the isomorphism $\tilde{S} : \overline{\text{span}(T(x_n) : n \in \mathbb{N})} \to c_\circ$, which assigns $T(x_n)$ the value $e_n^{(0)}$ if $n \in \mathbb{N}$, to a linear bounded operator $S : c_\circ \to c_\circ$. We deduce that $S \circ T(x_n) = e_n^{(0)}$ for $n \in \mathbb{N}$, which means that the components $(x'_n: n \in \mathbb{N})$ of $S \circ T$ have the desired properties.

◇

With Lemma (1.3.1) we can characterize the property that a given sequence $(x_n: n \in \mathbb{N}) \subset X$, which is equivalent to the $c_\circ$-basis, is limited in X.

1.3.2 Theorem: *Let $(x_n{:}n \in \mathbb{N}) \subset X$ be equivalent to $(e_n^{(0)}{:}n \in \mathbb{N})$. Then the following are equivalent:*

a) $\{x_n \,|\, n \in \mathbb{N}\}$ is limited in X.

b) For every $N \in \mathcal{P}_\infty(\mathbb{N})$, $\overline{\mathrm{span}(\{x_n \,|\, n \in N\})}$ is not complemented in X.

Proof of (1.3.2) :

(a) $\Rightarrow$(b): If $(x_n{:}\, n \in \mathbb{N})$ is limited in X, then every bounded linear operator T: $X \to c_0$ maps $\{x_n \,|\, n \in \mathbb{N}\}$ to a relatively compact set (Proposition (1.1.1)(a) $\Longleftrightarrow$ (c)). Thus, for no $N \in \mathcal{P}_\infty(\mathbb{N})$ there is a projection of X onto $\overline{\mathrm{span}(\{x_n \,|\, n \in N\})}$.
$\neg$(a) $\Rightarrow$ $\neg$(b): By taking a subsequence of $(x_n{:}\, n \in \mathbb{N})$ and by using (1.3.1), we can assume that there exists a $\sigma(X',X)$-zero sequence $(x'_n : n \in \mathbb{N})$ such that $\langle x'_n, x_m\rangle = \delta_{(n,m)}$ for $n, m \in \mathbb{N}$. This means that the operator $T : X \to c_0$, $x \mapsto (\langle x'_n, x\rangle : n \in \mathbb{N})$, maps each x_n to $e_n^{(0)}$. Since $(x_n{:}\, n \in \mathbb{N})$ is equivalent to $(e_n^{(0)}{:}n \in \mathbb{N})$ and thus, T restricted to $\overline{\mathrm{span}(\{x_n \,|\, n \in N\})}$ is an isomorphism, T is a projection, which finishes the proof.

◇

With Theorem (1.3.2) we get the following variants of the separable injectivity of c_0 (compare [9, p.71, Theorem 4]):

1.3.3 Corollary:

a) If X is a Gelfand-Phillips space, then every sequence $(x_n{:}n \in \mathbb{N}) \subset X$ which is equivalent to $(x_n{:}n \in \mathbb{N})$ contains an infinite subsequence $(x_n{:}n{\in}N)$ such that $\overline{\mathrm{span}(x_n : n \in \mathbb{N})}$ is complemented in X.

b) If every limited set of X is relatively weakly compact, then every copy of c_0 in X contains a subspace still isomorphic to c_0 which is complemented in X. (Sufficient conditions that every limited set in X is relatively compact will be formulated in chapter 2.)

Proof of (1.3.3) :

Proof of (a): Theorem (1.3.2).

Proof of (b): Let $(x_n{:}n \in \mathbb{N}) \subset X$ be equivalent to the c_0-basis and set, for $n \in \mathbb{N}$, $y_n := \sum_{i=1}^n x_i$. Then $(y_n{:}n \in \mathbb{N})$ is bounded but not relatively weakly compact; thus, by the assumption it is not limited in X. Using Lemma (1.3.1) we find a $\sigma(X',X)$-zero sequence $(y'_n{:}n \in \mathbb{N})$ and an increasing sequence $(k_n{:}n \in \mathbb{N})$ in $\mathbb{N}$ such that

$$\langle y'_n, y_{k_m}\rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if not} \end{cases} \qquad \text{for all } m, n \in \mathbb{N}.$$

It follows that for each $n \in \mathbb{N}$ (setting: $k_0 := 0$ and $y_0 := 0$):

$$1 = \langle y'_n, y_{k_n} - y_{k_{n-1}} \rangle = \langle y'_n, \sum_{i=k_{n-1}+1}^{k_n} x_i \rangle .$$

Thus, the sequence $(\sum_{i=k_{n-1}+1}^{k_n} x_i : n \in \mathbb{N})$ is not limited and is equivalent to $(e_n^{(0)}: n \in \mathbb{N})$ and we deduce from (1.3.2) the assertion.

◇

1.3.4 Remark: Let us consider the following properties of a Banach space X:

i) Every copy of c_o is complemented in X (by [10, p.71, Theorem 4], this is true if X is separable).

ii) Every sequence $(x_n: n \in \mathbb{N})$ which is equivalent to $(e_n^{(0)}: n \in \mathbb{N})$ contains a subsequence such that the space generated by this subsequence is complemented in X.

iii) Every copy of c_o contains a subspace isomorphic to c_o which is complemented in X.

Of course (i) ⇒(ii) and (ii) ⇒(iii) are true.

a) In (0.5.6)(c), we constructed a compact and sequentially compact space K such that $C(K)$ contained a copy of c_o which was not complemented in $C(K)$. By (1.2.2), $C(K)$ is a Gelfand-Phillips space and thus enjoys property (ii). Hence the implication (i) ⇒(ii) is not reversible.

In section (5.1) we shall construct a $C(K)$-space which contains a limited sequence which is equivalent to $(e_n^{(0)}: n \in \mathbb{N})$ and which is conditionally weakly compact generated. This implies, as will be shown in (2.3.3), that all limited sets in $C(K)$ are relatively weakly compact; we deduce, using (1.3.3)(b), that (iii) ⇒(ii) does not hold.

b) Corollary (1.3.3) leads one to ask whether the Gelfand-Phillips property is characterized by property (ii). The answer would be yes if one could show that Banach spaces not enjoying the Gelfand-Phillips property contain a limited sequence which is equivalent to $(e_n^{(0)}: n \in \mathbb{N})$. In chapter 3 it will be shown that this is true for $C(K)$-spaces. But, in general, it seems to be unknown if Banach spaces without the Gelfand-Phillips property contain any copy of c_o, a question which is solved for lattices in (1.2.5)(c).

At the end of this section we want to cite two known results which can be proven with Theorem (1.3.2) . Corollary (1.3.5) describes the Grothendieck property by limitedness ((a) ⇒(c)) and leads to the characterization of Grothendieck spaces, as given in [48, p.18., Satz 3.2]. Corollary (1.3.6) was first proven in [6] for the space $C(K,X)(= C(K)\check{\otimes}X)$ and has been generalized for any injective tensor products by E. Saab [50].

1.3.5 Corollary: *The following properties (a), (b), and (c) are equivalent:*

a) X is a Grothendieck space.

b) i) Every $T : X \to c_o$ which is not weakly compact fixes a copy of c_o, i.e. there exists a copy of c_o on which T acts as an isomorphism.

ii) X does not contain a complemented copy of c_o.

c) i) As in (b).

ii) Every sequence in X which is equivalent to $(e_n^{(0)} : n \in \mathbb{N})$ is limited in X.

Proof of (1.3.5) :

(a) ⇒(b): A Grothendieck space X does not admit any operator $T : X \to c_o$ which is not weakly compact.

(a)(ii) ⇒(b)(ii) Theorem (1.3.2)

(c) ⇒(a) Let $T : X \to c_o$ be linear and bounded. By (c)(ii), T cannot fix any copy of c_o; thus, by (c)(i), it must be weakly compact.

◇

1.3.6 Corollary: *Suppose that X and Y are of infinite dimension and that X contains a copy of c_o. Then $X\check{\otimes}Y$ contains an isomorphic copy of c_o which is complemented in $X\check{\otimes}Y$.*

Proof of (1.3.6) : As in [6], we use the theorem of Josefson and Nissenzweig (compare Corollary (2.4.6)).

Let $(x_n : n \in \mathbb{N}) \subset X$ be equivalent to $(e_n^{(0)} : n \in \mathbb{N})$. By the Theorem of Josefson and Nissenzweig, there exists a normed $\sigma(Y',Y)$-zero sequence $(y'_n : n \in \mathbb{N})$ in Y. For each $n \in \mathbb{N}$, choose $y_n \in B_2(Y)$ such that $\langle y'_n, y_n\rangle = 1$.

By (1.3.2), it is enough to show that $(x_n \otimes y_n : N \in \mathbb{N})$ is equivalent to $(e_n^{(0)} : n \in \mathbb{N})$ and is not limited in $X\check{\otimes}Y$.

If we choose $C > c > 0$ such that

$$c \max_{n\leq k} |a_n| \leq \| \sum_{n\leq k} a_n x_n \| \leq C \max_{n\leq k} |a_n|$$

if $a_1, a_2, \ldots, a_k \in \mathbb{R}$ and $k \in \mathbb{N}$, we get

$$\begin{aligned}
c\max_{i\le k}|a_i| &\le c \sup_{y'\in B_1(Y')}\max_{i\le k}|a_i\langle y',y_i\rangle| \\
&[\|y_i\|\ge 1] \\
&\le \sup_{y'\in B_1(Y')}\|\sum_{i=1}^{n} a_i\langle y',y_i\rangle x_i\| = \|\sum_{i=1}^{n} a_i y_i\otimes x_i\| \\
&\le C \sup_{y'\in B_1(Y')}\max_{i\le k}|a_i\langle y',y_i\rangle| \\
&\le 2C\max_{i\le k}|a_i| \\
&[\|y_i\|\le 2]
\end{aligned}$$

if $a_1, a_2, \ldots, a_k \in \mathbb{R}$ and $k \in \mathbb{N}$,

which implies the first assertion.
If we choose for each $n \in \mathbb{N}$ an $x'_n \in B_1(X')$ which norms x_n then the sequence $x'_n \otimes y'_n$ is weak*-convergent to 0 in $(X\check{\otimes}Y)'$ (note that $(x'_n \otimes y'_n : n \in \mathbb{N})$ is bounded and that $\langle x'_n \otimes y'_n, x\otimes y\rangle$ converges to zero for each $x \in X$ and $y \in Y$). Moreover, we have for $n \in \mathbb{N}$

$$\langle x'_n \otimes y'_n, x_n \otimes y_n\rangle = \langle x'_n, x_n\rangle\langle y'_n, y_n\rangle = 1,$$

which verifies the second assertion and finishes the proof.

◇

2 A result about dual spaces which contain bounded sequences without any weak*-convergent convex blocks

In this chapter we want to prove the following result and use it to answer a question posed by R. Haydon in [34, p.11, Remarks]:

If the dual of a Banach space X contains a bounded sequence $(x'_n : n \in \mathbb{N})$ for which no convex block converges in $\sigma(X', X)$, then X' contains an isometric copy of $L_1(\{0,1\}^{\omega_1})$.

In section (2.1) we will formulate the theorem and the necessary definitions and then cite some results related to this topic. The proof will be given in (2.2). In the last part of this chapter we will show, following a proof of J. Bourgain and J. Diestel [4], that for spaces containing limited sets which are not relatively weakly compact the assumption of the main theorem holds. We will also deduce a generalization of a result in [4] (cf. corollary (2.3.3)) which says that, in Banach spaces not containing a copy of ℓ_1, all limited sets are relatively weakly compact.

2.1 Formulation of the main theorem and review of related results

For a set Γ, let μ_Γ be the product measure $\otimes_{\gamma\in\Gamma}\frac{1}{2}(\delta_0 + \delta_1)$ on the set $\{0,1\}^\Gamma$ furnished with the product σ-algebra $\otimes_{\gamma\in\Gamma}\mathcal{P}(\{0,1\})$. As usual, we denote the spaces $L_1(\mu_\Gamma)$ by $L_1(\{0,1\}^\Gamma)$ and $L_\infty(\mu_\Gamma)$ by $L_\infty(\{0,1\}^\Gamma)$. Since μ is finite, $L_\infty(\{0,1\}^\Gamma)$ can be viewed as a subspace of $L_1(\{0,1\}^\Gamma)$. To avoid ambiguities, we denote the usual norm on $L_1(\{0,1\}^\Gamma)$ by $\|.\|_1$ and the norm on $L_\infty(\{0,1\}^\Gamma)$ by $\|\cdot\|_\infty$.

2.1.1 Definition:

a) Let $(x_n : n \in \mathbb{N}) \subset X$ be bounded. A sequence $(\sum_{i=k_n}^{k_{n+1}-1} a_i x_i : n \in \mathbb{N})$ is called a convex block (respectively an absolutely convex block basis) of $(x_n : n \in \mathbb{N})$ if $(k_n : n \in \mathbb{N})$ is increasing in $\mathbb{N}$, $(a_n : n \in \mathbb{N}) \subset \mathbb{R}_0^+$ (respectively $(a_n : n \in \mathbb{N}) \subset \mathbb{R}$), and $\sum_{i=k_n}^{k_{n+1}-1} a_i = 1$ (respectively $\sum_{i=k_n}^{k_{n+1}-1} |a_i| = 1$) for each $n \in \mathbb{N}$.

b) We say that the Banach space X satisfies

(CBH) (convex block hypothesis) if X' contains a bounded sequence $(x'_n : n \in \mathbb{N})$ which has no $\sigma(X', X)$-convergent convex block;

(ACBH) (absolutely convex block hypothesis) if X' contains a bounded sequence $(x'_n : n \in \mathbb{N})$ which has no $\sigma(X', X)$-convergent absolutely convex block basis.

We remark that the definition of (ACBH) in (2.1.1) is equivalent to the condition, considered by J. Hagler and W.B. Johnson [26] and by R. Haydon [34], that X' contains an infinite dimensional subspace Y on which $\sigma(X',X)$-convergence of sequences implies norm convergence. This equivalence will be shown in the following proposition.

2.1.2 Proposition: *The following are equivalent:*

a) *X has property (ACBH).*

b) *X' contains an infinite dimensional subspace Y in which $\sigma(X',X)$-convergence of sequences implies norm-convergence.*

Proof of (2.1.2) :

(a) $\Rightarrow$(b): Let $(x'_n: n \in \mathbb{N})$ be a sequence in X' without a w*-convergent absolutely convex block basis. In particular, $(x'_n: n \in \mathbb{N})$ has no $\sigma(X',X'')$-Cauchy subsequence and thus we can assume, by Rosenthal's ℓ_1 theorem, that $(x'_n: n \in \mathbb{N})$ is equivalent to $(e_n^{(1)}: n \in \mathbb{N})$.

We are finished if we can show that a given sequence

$$(y'_n: n \in \mathbb{N}) = (\sum_{i=1}^{\infty} a_i^{(n)} x'_i : n \in \mathbb{N}),$$

with $\sum_{i=1}^{\infty} |a_i^{(n)}| \leq 1$ if $n \in \mathbb{N}$, is not $\sigma(X',X)$-convergent if there exists an $\varepsilon > 0$ such that $\| y'_n - y'_m \| \geq \varepsilon$ for all $n, m \in \mathbb{N}$ with $n \neq m$.

Suppose that such a sequence is w*-convergent. By taking a subsequence, we may assume that $a_i := \lim_{n\to\infty} a_i^{(n)}$ exists for each $i \in \mathbb{N}$. Setting $z'_n := y'_n - y'_{n+1}$, we find an increasing sequence $(m_n: n \in \mathbb{N})$ and a sequence $(\tilde{z}'_n: n \in \mathbb{N}) \subset X'$, with $\tilde{z}'_n = \sum_{i=k_n}^{k_{n+1}-1} b_i x'_i$ for $n \in \mathbb{N}$, (k_n) increasing in $\mathbb{N}$, and $(b_i) \subset \mathbb{R}$, such that $\lim_{n\to\infty} \| \tilde{z}'_n - z'_{m_n} \| = 0$.

Since $\| y'_n - y'_{n+1} \| \geq \varepsilon$ for $n \in \mathbb{N}$, we deduce that $\liminf_{n\to\infty} \| \tilde{z}'_n \| \geq \varepsilon$ and by this

$$\liminf_{n\to\infty} \sum_{i=k_n}^{k_{n+1}-1} |b_i| > 0 .$$

As a consequence, there exists an $n_0 \in \mathbb{N}$ such that

$$((\sum_{i=k_n}^{k_{n+1}-1} b_i x'_i)/ \sum_{i=k_n}^{k_{n+1}-1} |b_i|) : n_0 \leq n \in \mathbb{N})$$

is an absolutely convex block basis of $(x'_n: n \in \mathbb{N})$ which converges in $\sigma(X',X)$ to zero. This contradicts the assumption.

(b) ⇒(a): Let $Y \subset X'$ be as in (b). Since Y is of infinite dimension, we find a bounded sequence in Y without any norm-convergent, and thus, without any $\sigma(X', X)$-convergent subsequence. By Rosenthal's ℓ_1 theorem, it must contain a subsequence $(x'_n : n \in \mathbb{N})$ which is equivalent to $(e_n^{(1)} : n \in \mathbb{N})$. Since no absolutely convex block basis of $(e_n^{(1)} : n \in \mathbb{N})$ is norm-convergent, we deduce (a).

◇

Now we can formulate the main theorem of this chapter:

2.1.3 Theorem: *If X has property (CBH), then X' contains an isometric copy of* $L_1(\{0,1\}^{\omega_1})$.

In the sequel we need the following ordinal:

2.1.4 Definition: (compare [34, p.2] and [35, p.3])

The ordinal ω_p is the smallest of all $\alpha \in$ Ord such that:

There exists a family $(M_\beta : \beta < \alpha) \subset \mathcal{P}_\infty(\mathbb{N})$ with

a) $|\bigcap_{\beta \in F} M_\beta| = \infty$ for each $F \in \mathcal{P}_f(\alpha)$, and

b) There is no $M \in \mathcal{P}_\infty(\mathbb{N})$ with $M \overset{a}{\subset} M_\beta$ for every $\beta < \alpha$.

2.1.5 Remark:

a) ω_p is an initial ordinal, i.e. $\omega_p = \min\{\alpha \in \text{Ord} \mid |\alpha| = |\omega_p|\}$; and thus, it could be considered as a cardinal.

b) For every countable ordinal α there is no family $(M_\beta : \beta < \alpha)$ satisfying (a) and (b) of (2.1.4). Indeed, assuming that $(M_\beta : \beta < \alpha) \subset \mathcal{P}_\infty(\mathbb{N})$ satisfies (a), we can choose an increasing sequence $(F_n : n \in \mathbb{N}) \subset \mathcal{P}_f(\alpha)$ with $\bigcup_{n \in \mathbb{N}} F_n = \alpha$, and we can choose for each $n \in \mathbb{N}$ $m_n \in \bigcap_{\beta \in F_n} M_\beta$ such that $(m_n : n \in \mathbb{N})$ increases. As $M := \{m_n \mid n \in \mathbb{N}\}$ is almost contained in $\bigcap_{\beta \in F} M_\beta$, provided that $F \in \mathcal{P}_f(\alpha)$, we conclude that $(M_\beta : \beta < \alpha)$ does not satisfy (b).
On the other hand, if $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$ (i.e. $\mathcal{U} \subset \mathcal{P}_\infty(\mathbb{N})$ is a maximal filter) and $(M_\alpha : \alpha < \omega_c)$ a well-ordering of $\mathcal{U}$ (every ultrafilter on $\mathbb{N}$ has the cardinality of the continuum), then $(M_\alpha : \alpha < \omega_c)$ satisfies (a) and (b) of Definition (2.1.4).
We conclude that $\omega_1 \leq \omega_p \leq \omega_c$.

c) In [35, p.3] it is remarked that under Martin's axiom we have $\omega_p = \omega_c$. Thus, under the assumption that Martin's axiom holds but the continuum hypothesis does not, $\omega_1 < \omega_p = \omega_c$.

The following proposition collects some known results which are related to the topic of this chapter.

2.1.6 Proposition: *For a Banach space X we consider the following properties:*

(E_1) *X has a quotient isomorphic to ℓ_∞.*

(E_2) *X has a non-reflexive Grothendieck space as a quotient.*

(E_3) *X has the property (ACBH).*

(E_4) *X has the property (CBH).*

And for $\omega \in \mathrm{Ord}$ we consider:

(E_5)(ω) *X contains an isomorphic copy of $\ell_1(\omega)$.*

(E_6)(ω) *X' contains an isomorphic copy of $L_1(\{0,1\}^\omega)$.*

(E_7)(ω) *There exists a bounded and linear $S : X \to L_\infty(\{0,1\}^\omega)$, a $\Delta > 0$, and a bounded family $(x_\alpha : \alpha < \omega)$, such that*

$$\| S(x_\alpha) - S(x_\beta) \|_1 \geq \Delta \text{ whenever } \alpha, \beta \in [0, \omega[\text{ with } \alpha \neq \beta$$

Then the following relationships between these properties hold:

a) *[34, p.2, Theorem 1]: (E_3) implies (E_6)(ω_p).*

b) i) *[35, p.6, Corollary 3 C]: If X is a $C(K)$-space then (E_4) implies (E_6)(ω_p) (in [35] it has been shown that if (E_4) is satisfied for a $C(K)$-space, then there exists a positive measure $\mu \in M(K)$ such that $L_1(\mu)$ is isometrically isomorph to $L_1(\{0,1\}^{\omega_p})$).*

ii) *[35, p.6, Theorem 3 D]: Under the (set theoretical) assumption that $\omega_p > \omega_1$, the property (E_4) implies (E_5)(ω_p).*

c) *[43, p.1083, Theorem 4.7]: For any $\omega \in \mathrm{Ord}$ (E_5)(ω) implies (E_6)(ω).*

d) *From [3, p.80, Theorem 1.2] we easily deduce (see the proof below) that if $\omega > \omega_1$, then (E_7)(ω) implies (E_5)(ω).*

From (a)-(d) it is easy to deduce (e),(f) and (g).

e) *Without any further set theoretical assumption,*

$$(E_1) \iff (E_5)(\omega_c) \Rightarrow (E_2) \Rightarrow (E_3) \Rightarrow (E_6)(\omega_p),$$

$$(E_3) \Rightarrow (E_4), \text{ and } (E_6)(\omega) \Rightarrow (E_7)(\omega) \text{ for any } \omega \in \mathrm{Ord}.$$

f) *Under the assumption that $\omega_1 < \omega_p$, we deduce moreover*

$$(E_4) \Rightarrow (E_5)(\omega_p) \iff (E_6)(\omega_p) \iff (E_7)(\omega_p).$$

g) *If we assume $\omega_1 < \omega_p = \omega_c$, then*

$$(E_1) \iff (E_2) \iff (E_3) \iff (E_4) \iff (E_5)(\omega_c) \iff (E_6)(\omega_c) \iff (E_7)(\omega_c).$$

h) *The Grothendieck $C(K)$, constructed by Talagrand under the continuum hypothesis [56, p.189, Théorème 4], does not satisfy (E_1) but it does satisfy*

(E_2) and thus (E_3),(E_4),$(E_6)(\omega_c)$ and $(E_7)(\omega_c)$.

Proof of (2.1.6)(d), (e), (f) and (g) :

Proof of (d): [3, p.80, Theorem 1.2] states the following:

(1) Let $\omega > \omega_1$ and suppose $(f_\alpha : \alpha < \omega)$ is a family in $L_\infty(\{0,1\}^\omega)$ such that there exists a $\Delta > 0$ with:

$$\| f_\alpha - f_\beta \|_1 \geq \Delta \quad \text{whenever } \alpha, \beta \in [0, \omega, [\text{ with } \alpha \neq \beta.$$

Then there exists an $I \subset \omega$ with $|I| = |\omega|$ such that $(f_\alpha : \alpha \in I)$ is equivalent to $(e_\alpha^{(1)} : \alpha \in I)$.

Now let S, $(x_\alpha : \alpha < \omega)$, and Δ be as prescribed in $(E_7)(\omega)$. Then the family $(S(x_\alpha) : \alpha < \omega)$ satisfies the assumption of (1) and we find $I \subset \omega$ with $|I| = |\omega|$ such that $(S(x_\alpha) : \alpha \in I)$ is equivalent to the corresponding ℓ_1-basis. We deduce from the lifting-property of $\ell_1(I)$ [40, p.107, Proposition 2.f.7. and following remarks], that $(x_\alpha : \alpha \in I)$ is also equivalent to $(e_\alpha^{(1)} : \alpha \in I)$.

Proof of (e): It remains to prove $(E_1) \iff (E_5)(\omega_c)$, $(E_2) \Rightarrow (E_3)$, and $(E_6)(\omega) \Rightarrow (E_7)(\omega)$ for any $\omega \in$ Ord ($(E_1) \Rightarrow (E_2)$ and $(E_3) \Rightarrow (E_4)$ are obvious).

$(E_1) \Rightarrow (E_5)(\omega_c)$: Let $Q : X \to \ell_\infty$ be a quotient mapping. Since ℓ_∞ contains a copy of $\ell_1(\omega_c)$ [9, p.211, Exercise (1) (i)] and since $\ell_1(\omega_c)$ is projective [40, p.107, Proposition 2.f.7. and remarks] X contains a copy of $\ell_1(\omega_c)$ as well.

$(E_5)(\omega_c) \Rightarrow (E_1)$: Let $(x_\alpha : \alpha < \omega_c) \subset X$ be equivalent to $(e_\alpha^{(1)} : \alpha < \omega_c)$ and choose an algebraic basis B of ℓ_∞ in $B_1(\ell_\infty)$ ($\text{span}(B) = \ell_\infty$). Since $|B| = |\ell_\infty| = |\omega_c|$, we can well-order B by $(b_\alpha : \alpha < \omega_c)$. From the property of an ℓ_1-basis, we deduce that the mapping

$$\tilde{Q} : \text{span}(x_\alpha : \alpha < \omega_c) \to \ell_\infty, \quad \sum_{\alpha \in F} r_\alpha x_\alpha \mapsto \sum_{\alpha \in F} r_\alpha b_\alpha \quad \text{whenever } F \in \mathcal{P}_f(\omega_c),$$

is linear, bounded; since $\text{span}(B) = \ell_\infty$, it is also surjective and can be extended to a linear and bounded, and still surjective $Q : X \to \ell_\infty$ by the injectivity of ℓ_∞. By the open mapping theorem, the spaces $X/\text{Ker}(Q)$ and ℓ_∞ are isomorphic.

$(E_2) \Rightarrow (E_3)$: Let Z be a quotient of X which is Grothendieck and not reflexive. $B_1(Z')$ cannot be weakly compact and according to the theorem of Eberlein and Smulian, $B_1(Z')$ contains a sequence $(z'_n : n \in \mathbb{N})$ without any $\sigma(Z', Z'')$-convergent subsequence. Thus, by the Grothendieck property, it has no $\sigma(Z', Z)$-convergent subsequences and, by Rosenthal's ℓ_1 theorem, we can assume that it is equivalent to $(e_n^{(1)}: n \in \mathbb{N})$. Thus, no absolutely convex block basis of $(z'_n: n \in \mathbb{N})$ is norm-convergent, hence, by the theorem of Schur, also not $\sigma(Z', Z'')$-convergent, and finally, by the Grothendieck property of Z, not even $\sigma(Z', Z)$-convergent. Since Z' can be $\sigma(Z', Z)$-$\sigma(X', X)$-embedded in X', we deduce (E_3).

$(E_6)(\omega) \Rightarrow (E_7)(\omega)$ for $\omega \in \mathrm{Ord}$:

Let $E : L_1(\{0,1\}^\omega) \to X'$ be an isomorphic embedding. For $\alpha < \omega$ we define $y_\alpha := \chi_{\{p_\alpha = 1\}} - \chi_{\{p_\alpha = 0\}}$, where $p_\alpha : \{0,1\}^\omega \to \{0,1\}$ is the α-coordinate projection for $\alpha < \omega$, and consider y_α as an element of $L_1(\{0,1\}^\omega)$ as well as an element of $L_\infty(\{0,1\}^\omega)$ (in both spaces it is of norm 1). For $I \subset \omega$ we denote the σ-algebra on $\{0,1\}^\omega$, generated by $(p_\alpha : \alpha \in I)$, by Σ_I and note that Σ_I and $\Sigma_{\tilde{I}}$ are independent if $I, \tilde{I} \subset \omega$ are disjoint.

If the cardinality of ω is not equal to ω_1 we deduce from [43, p.1084, Theorem 4.9] that there exists an isomorphic embedding $T : \ell_1(\omega) \to X$. Thus, the operator

$$\tilde{S} : \ell_1(\omega) \to L_\infty(\{0,1\}^\omega), \quad (\xi_\alpha : \alpha < \omega) \mapsto \sum_{\alpha < \omega} \xi_\alpha y_\alpha$$

is extendable to a linear and bounded operator $S : X \to L_\infty(\{0,1\}^\omega)$, by the injectivity of $L_\infty(\{0,1\}^\omega)$. We observe that

$$\| S \circ T(e_\alpha^{(1)}) - S \circ T(e_\beta^{(1)}) \|_1 = \| y_\alpha - y_\beta \|_1 = 1 \qquad \text{if } 0 \le \beta < \alpha$$

and deduce the assertion.

If $\omega = \omega_1$, we set $S := E'|_X$ (note that $E' : X'' \to L_\infty(\{0,1\}^\omega)$) and, in order to show the assertion, we choose by transfinite induction $x_\alpha \in X$, for each $\alpha < \omega$, such that

$$\| S(x_\alpha) - S(x_\beta) \|_1 \ge \Delta := \frac{1}{2} \inf_{\lambda < \omega} \| E(y_\lambda) \| \quad \text{for } \beta < \alpha .$$

Assuming that $(x_\beta : \beta < \alpha)$ has been chosen for $\alpha < \omega$ we note that there is a $\gamma < \omega$ such that for each $\beta < \alpha$, $S(x_\beta)$ is measurable with respect to Σ_γ (note that β is countable and that each $f \in L_\infty(\{0,1\}^\omega)$ is measurable with respect to Σ_λ where $\lambda < \omega$ is sufficiently large). Then we choose $x_\alpha \in B_1(X)$ satisfying $\langle E(y_\gamma), x_\alpha \rangle \ge \Delta$ (which is possible by the definition of Δ). Since Σ_γ and $\Sigma_{\{\gamma\}}$ are independent we deduce for each $\beta < \alpha$ that

$$\| S(x_\alpha) - S(x_\beta) \|_1 \ge \langle S(x_\alpha), y_\gamma \rangle - \langle S(x_\beta), y_\gamma \rangle = \langle S(x_\alpha), y_\gamma \rangle = \langle x_\alpha, E(y_\gamma) \rangle \ge \Delta .$$

This implies the assertion.

Proof of (f): (b)(ii), (c), (e), and (d).

Proof of (g): (e) and (f), for $\omega = \omega_c$.

◇

If we combine the result cited in Proposition (2.1.6)(b)(ii) with that of Theorem (2.1.3), we get the following generalization of (2.1.6)(a).

2.1.7 Corollary: *If X satisfies property (CBH), then X' contains an isometric copy of* $L_1(\{0,1\}^{\omega_p})$. *(Under the assumption "*$\omega_p = \omega_1$*" take (2.1.3),* and under *the assumption "*$\omega_p > \omega_1$*", use (2.1.6)(e)(ii)* and *(2.1.6)(e))*

2.1.8 Remark: The proof of (2.1.6)(b)(ii) depends in an essential way on the equivalence "$(E_7)(\omega_p) \Longleftrightarrow (E_5)(\omega_p)$", which is not true without any additional set-theoretical assumption. The proof of (2.1.3) uses the countability of all $\alpha < \omega_1$. Thus, neither proof can be dispensed with.

2.2 Proof of Theorem (2.1.3)

The following lemma is due to H. P. Rosenthal [49], who used it to show that if X satisfies (CBH), then X contains ℓ_1:

2.2.1 Lemma: *(cited from [35, p.4, Lemma 3A])*

Let X satisfy (CBH). *Then there exists a bounded sequence $(x'_n: n \in \mathbb{N})$ in X' and $c \in \mathbb{R}$ such that*

(a)
$$\sup_{x \in B_1(X)} \operatorname{osc}(\langle x'_n, x\rangle) \quad = \quad 1\,,$$

where $\operatorname{osc}(r_n) := \limsup_{n\to\infty} r_n - \liminf_{n\to\infty} r_n$ *for a bounded* $(r_n: n \in \mathbb{N}) \subset \mathbb{R}$,

(b) for every convex block $(y'_n: n \in \mathbb{N})$ of $(x'_n: n \in \mathbb{N})$ and every $\eta < \frac{1}{2}$ there exists an $x \in B_1(X)$ such that

$$\limsup_{n\to\infty}\langle y'_n, x\rangle > c+\eta \qquad \text{and} \qquad \liminf_{n\to\infty}\langle y'_n, x\rangle < c-\eta\,.$$

Proof of (2.2.1) :(Since the original paper of H. P. Rosenthal was not available, we include a proof.)

For a bounded sequence $(y'_n: n \in \mathbb{N}) \subset X'$, let $CB(y'_n)$ be the set of all convex blocks of $(y'_n: n \in \mathbb{N})$ and set

$$\delta(y'_n) := \sup_{x\in B_1(X)} \operatorname{osc}\langle y'_n, x\rangle \quad \text{and} \quad \varepsilon(y'_n) := \inf_{(z'_n)\in CB(y'_n)} \delta(z'_n).$$

We remark that $\delta(y'_n) = 0$ if and only if $(y'_n: n \in \mathbb{N})$ is w*-Cauchy and thus convergent. Since for each $(z'_n: n \in \mathbb{N}) \in CB(y'_n)$ we have

$$\limsup_{n\to\infty}\langle y'_n, x\rangle \ge \limsup_{n\to\infty}\langle z'_n, x\rangle \quad \text{and} \quad \liminf_{n\to\infty}\langle y'_n, x\rangle \le \liminf_{n\to\infty}\langle z'_n, x\rangle \quad \text{for } x \in X\,,$$

and since $CB(z'_n) \subset CB(y'_n)$, whenever $(z'_n: n \in \mathbb{N})$ is a convex block of $(y'_n: n \in \mathbb{N})$ we conclude that for every bounded $(y'_n: n \in \mathbb{N}) \subset X'$,

(1) $$\delta(y'_n) \ge \delta(z'_n) \ge \varepsilon(z'_n) \ge \varepsilon(y'_n) \quad \text{whenever } (z'_n: n \in \mathbb{N}) \in CB(y'_n).$$

In the first step we want to show the existence of a bounded sequence $(\tilde{x}'_n: n \in \mathbb{N})$ in X' which satisfies

(2) $$1 = \delta(\tilde{x}'_n) = \varepsilon(\tilde{x}'_n)\quad .$$

For this, let $(y'_n : n \in \mathbb{N}) \subset X'$ be bounded with no $\sigma(X', X)$-convergent convex block. We set $(z'^{(0)}_n : n \in \mathbb{N}) := (y'_n : n \in \mathbb{N})$ and inductively, assuming that for a $k \in \mathbb{N}$, $(z'^{(k-1)}_n : n \in \mathbb{N}) \in \mathrm{CB}(y'_n)$ has already been chosen, we choose $z'^{(k)} \in \mathrm{CB}(z'^{(k-1)}_n)$ such that

$$(3) \qquad \delta(z'^{(k)}_n) \le \varepsilon(z'^{(k-1)}_n) + \frac{1}{k}.$$

Finally, we define $\tilde{y}'_n := z'^{(n)}_n$ for each $n \in \mathbb{N}$. Thus, $(\tilde{y}'_n : n \ge k)$ is a convex block of $(z'^{(k)}_n : n \in \mathbb{N})$ for any $k \in \mathbb{N}$. From the property of $(y'_n : n \in \mathbb{N})$ we deduce that $\delta := \delta(\tilde{y}'_n) > 0$. Since $\delta(.)$ and $\varepsilon(.)$ do not change if we pass to cofinite subsequences, we deduce from (1) and (3) for every $k \in \mathbb{N}$ that

$$\delta = \delta(\tilde{y}'_n) \le \delta(z'^{(k)}_n) \le \varepsilon(z'^{(k-1)}_n) + \frac{1}{k} \le \varepsilon(\tilde{y}'_n) + \frac{1}{k} \le \delta(\tilde{y}'_n) + \frac{1}{k},$$

and thus, $\delta = \delta(\tilde{y}'_n) = \varepsilon(\tilde{y}'_n)$.
If we define $\tilde{x}'_n := \tilde{y}'_n/\delta$, (2) follows.

Now we define for $(y'_n : n \in \mathbb{N}) \in \mathrm{CB}(\tilde{x}'_n)$ and $\varepsilon > 0$

$$\mathrm{A}(\varepsilon, y'_n) := \{x \in \mathrm{B}_1(X) \,|\, \mathrm{osc}(\langle y'_n, x\rangle) > 1 - \varepsilon\}$$

and

$$\mathrm{R}(y'_n) := \inf_{\varepsilon > 0} \sup_{x \in \mathrm{A}(\varepsilon, y'_n)} \limsup_{n \to \infty} \langle y'_n, x\rangle \qquad \text{and} \qquad \mathrm{r}(y'_n) := \inf_{(\tilde{y}'_n) \in \mathrm{CB}(y'_n)} \mathrm{R}(\tilde{y}'_n),$$

By (2), $\mathrm{A}(\varepsilon, y'_n)$ is not empty and one has $\mathrm{A}(\tilde{\varepsilon}, y'_n) \subset \mathrm{A}(\varepsilon, y'_n)$ for $0 < \tilde{\varepsilon} < \varepsilon$. This implies that

$$(4) \qquad \mathrm{R}(y'_n) = \liminf_{\varepsilon \to 0} \sup_{x \in \mathrm{A}(\varepsilon, y'_n)} \limsup_{n \to \infty} \langle y'_n, x\rangle \quad \text{for each } (y'_n : n \in \mathbb{N}) \in \mathrm{CB}(\tilde{x}'_n).$$

Secondly we note that $\mathrm{A}(\varepsilon, \tilde{y}'_n) \subset \mathrm{A}(\varepsilon, y'_n)$, whenever $(\tilde{y}'_n : n \in \mathbb{N}) \in \mathrm{CB}(y'_n)$ and we conclude

$$(5) \qquad \mathrm{R}(y'_n) \ge \mathrm{R}(\tilde{y}'_n) \ge \mathrm{r}(\tilde{y}'_n) \ge \mathrm{r}(y'_n), \text{whenever } (\tilde{y}'_n : n \in \mathbb{N}) \in \mathrm{CB}(y'_n)$$

in the same way as was shown (1). Now we can proceed in a similar way as in step 1 to show that there exists a sequence $(x'_n : n \in \mathbb{N}) \in \mathrm{CB}(\tilde{x}'_n)$ such that

$$(6) \qquad r := \mathrm{r}(x'_n) = \mathrm{R}(x'_n).$$

Defining $c := r - \frac{1}{2}$, we deduce (a) of the assertion from (1) and (2), while we get (b) as follows:

Let $(y'_n : n \in \mathbb{N}) \in \mathrm{CB}(x'_n)$ and $\eta < \frac{1}{2}$ be arbitrary. From (5) and (6) we deduce that $r = \mathrm{R}(x'_n) = \mathrm{R}(y'_n)$ and by (4) we find an $0 < \tilde{\varepsilon} \in]0, \varepsilon/2]$, where $\varepsilon := \frac{1}{2} - \eta$, such that

$$(7) \qquad \sup_{x \in \mathrm{A}(\tilde{\varepsilon}, y'_n)} \limsup_{n \to \infty} \langle y'_n, x \rangle - \varepsilon/2 \le r \le \sup_{x \in \mathrm{A}(\tilde{\varepsilon}, y'_n)} \limsup_{n \to \infty} \langle y'_n, x \rangle.$$

Thus, we find an $x \in \mathrm{A}(\tilde{\varepsilon}, y'_n)$ with

$$\limsup_{n \to \infty} \langle y'_n, x \rangle > r - (\frac{1}{2} - \eta) = c + \eta.$$

On the other hand, we deduce from the definition of $\mathrm{A}(\tilde{\varepsilon}, y'_n)$ and (7) that

$$\begin{aligned} \liminf_{n \to \infty} \langle y'_n, x \rangle &= \limsup_{n \to \infty} \langle y'_n, x \rangle - \mathrm{osc}(\langle y'_n, x \rangle) \\ &< r + \frac{\varepsilon}{2} - (1 - \tilde{\varepsilon}) \le r - \frac{1}{2} - \frac{1}{2} + \varepsilon = c - \eta \end{aligned}$$

which completes the proof.

◇

For the sequel, we assume that X has property (CBH) and that we have chosen $(x'_n : n \in \mathbb{N}) \subset X'$ and $c \in \mathbb{R}$ as in Lemma (2.2.1). To handle the space $\mathrm{L}_1(\{0,1\}^\Gamma)$, we need the following notations: For a finite set A, the set of all mappings $\varphi : A \to \{0,1\}$ will be denoted by 2^A; for $A' \subset A$ and $\varphi' \in 2^{A'}$, the set of all extensions of φ' onto the whole of A will be denoted by $2^{\varphi', A}$. For any set Γ, the union $\bigcup \{2^A | A \in \mathcal{P}_f(\Gamma)\}$ is denoted by S_Γ and for the domain of $\varphi \in \mathrm{S}_\Gamma$ we write $\mathrm{D}(\varphi)$.

R. Haydon [34, p.6, Lemma 3] provided the following characterization for a Banach space Y to contain an isometric copy of $\mathrm{L}_1(\{0,1\}^\Gamma)$.

2.2.2 Lemma: *Let Y be a Banach space and Γ a set. Then Y contains an isometric copy of $\mathrm{L}_1(\{0,1\}^\Gamma)$ if and only if there exists a family $(y_\varphi : \varphi \in \mathrm{S}_\Gamma)$ in Y satisfying (a) and (b) as given below:*

(a) $\quad y_{\varphi'} = 2^{|A'| - |A|} \sum_{\varphi \in 2^{\varphi', A}} y_\varphi$ *for any* $A \in \mathcal{P}_f(\Gamma), A' \subset A$ *and* $\varphi' \in 2^{A'}$

(note, that $|2^{\varphi',A}| = 2^{|A|-|A'|}$, and hence, that $y_{\varphi'}$ is the arithmetic mean of $(y_\varphi : \varphi \in 2^{\varphi',A})$).

(b) $$\| \sum_{\varphi \in 2^A} a_\varphi y_\varphi \| = \sum_{\varphi \in 2^A} |a_\varphi| \text{ for any } A \in \mathcal{P}_f(\Gamma) \text{ and } (a_\varphi : \varphi \in 2^A) \subset \mathbb{R}.$$

In this case, there exists an isometry $T : \mathrm{L}_1(\{0,1\}^\Gamma) \to Y$ such that $T(e_\varphi) = y_\varphi$ for $\varphi \in S_\Gamma$, where $e_\varphi \in \mathrm{L}_1(\{0,1\}^\Gamma)$ is defined by

$$e_\varphi := 2^{|\mathrm{D}(\varphi)|} \chi_{\{(\theta_\gamma : \gamma \in \Gamma) \in \{0,1\}^\Gamma \,|\, \theta_\gamma = \varphi(\gamma) \text{ if } \gamma \in \mathrm{D}(\varphi)\}} \ .$$

Another sufficient condition, for X' to contain $\mathrm{L}_1(\{0,1\}^\Gamma)$ can be formulated using the following definition.

2.2.3 Definition: Let Γ be a set. A family $F = (x(A,B) : A \in \mathcal{P}_f(\Gamma),\ B \subset 2^A)$ in $\mathrm{B}_1(X)$ is said to satisfy $(\mathcal{F}_\Gamma)$ if the following condition hold:

$(\mathcal{F}_\Gamma)$ For every $A \in \mathcal{P}_f(\Gamma)$ and $n \in \mathbb{N}$ there exists a family $(x'(\varphi,n) : \varphi \in 2^A) \subset X'$ such that

(a) $$x'(\varphi,n) \in \mathrm{co}(\{x'_m | m \geq n\}), \text{ if } \varphi \in 2^A, \quad \text{and}$$

(b) $$(2^{|A'|-|A|} \sum_{\varphi \in 2^{\varphi',A}} x'(\varphi,n), x(A',B')) - c \begin{cases} \geq \ \frac{1}{2}(1 - \frac{1}{|A'|+1} - \frac{1}{n}) & \text{if } \varphi' \in B' \\ \leq -\frac{1}{2}(1 - \frac{1}{|A'|+1} - \frac{1}{n}) & \text{if } \varphi' \notin B' \end{cases}$$

whenever $A' \subset A$, $\varphi' \in 2^{A'}$ and $B' \subset 2^{A'}$.

For the sake of brevity, we will denote the set $\{(A,B) | A \in \mathcal{P}_f(\Gamma), B \subset 2^A\}$ by I_Γ, the set of all families $F = (x(A,B) : (A,B) \in \mathrm{I}_\Gamma)$ which satisfy $(\mathcal{F}_\Gamma)$ by $\mathcal{F}_\Gamma$; and the values $\frac{1}{2}(1 - \frac{1}{|A|+1} - \frac{1}{n})$ and $\frac{1}{2}(1 - \frac{1}{|A|+1})$ by $\Delta(A,n)$ and $\Delta(A)$ respectively for $A \in \mathcal{P}_f(\Gamma)$ and $n \in \mathbb{N}$.

With this definition we are in a position to state the following result.

2.2.4 Lemma: *Let Γ be an infinite set. If $\mathcal{F}_\Gamma \neq \emptyset$, then there exists an isometric copy of $\mathrm{L}_1(\{0,1\}^\Gamma)$ in X'.*

Proof of (2.2.4) :

Let $F = (x(A,B) : (A,B) \in \mathrm{I}_\Gamma) \subset \mathrm{B}_1(X')$ satisfy property $(\mathcal{F}_\Gamma)$. For each $\varphi \in \mathrm{S}_\Gamma$ and each $n \in \mathbb{N}$ choose $x'(\varphi, n) \in \mathrm{B}_1(X')$ as prescribed in $(\mathcal{F}_\Gamma)$ and define for each $\psi \in \mathrm{S}_\Gamma$ and each $A \in \mathcal{P}_f(\Gamma)$

$$(1) \qquad y'(\psi, A) := 2^{|\mathrm{D}(\psi)\cap A| - |A|} \sum_{\varphi \in 2^{(\psi|\mathrm{D}(\psi)\cap A), A}} x'(\varphi, |A|+1) \quad ,$$

The net $(y'(\psi, A) : \psi \in \mathrm{S}_\Gamma)_{A \in \mathcal{P}_f(\Gamma)}$ has an accumulation point $(y'(\psi) : \psi \in \mathrm{S}_\Gamma)$ in the product $K := \prod_{\varphi \in \mathrm{S}_\Gamma} \overline{\mathrm{co}(\{x'_n : n \in \mathbb{N}\})}^{w^*}$, endowed with the product of the weak*-topology on $\overline{\mathrm{co}(\{x'_n : n \in \mathbb{N}\})}^{w^*}$, (the elements of $\mathcal{P}_f(\Gamma)$ are ordered by inclusion).

From (1) and $(\mathcal{F}_\Gamma)$(a) we have

$$(2) \qquad y'(\psi) \in \bigcap_{n \in \mathbb{N}} \overline{\mathrm{co}(\{x'_m : m \geq n\})}^{w^*} \quad \text{for each } \psi \in \mathrm{S}_\Gamma,$$

Since for $A, A', \tilde{A} \in \mathcal{P}_f(\Gamma)$, with $A' \subset A \subset \tilde{A}$, and $\psi' \in 2^{A'}$ we get from (1)

$$\begin{aligned} 2^{|A'|-|A|} \sum_{\psi \in 2^{\psi', A}} y'(\psi, \tilde{A}) &= 2^{|A'|-|A|} \sum_{\psi \in 2^{\psi', A}} 2^{|A|-|\tilde{A}|} \sum_{\varphi \in 2^{\varphi, \tilde{A}}} x'(\psi, |\tilde{A}|+1) \\ &= 2^{|A'|-|\tilde{A}|} \sum_{\varphi \in 2^{\psi', \tilde{A}}} x'(\varphi, |\tilde{A}|+1) \\ &= y'(\psi', \tilde{A}), \end{aligned}$$

we deduce that

$$(3) \qquad y'(\psi') = 2^{|A'|-|A|} \sum_{\psi \in 2^{\psi', A}} y'(\psi)$$

whenever $A' \subset A \in \mathcal{P}_f(\Gamma)$ and $\psi' \in 2^{A'}$.

From $(\mathcal{F}_\Gamma)$(b) we conclude for $A, \tilde{A} \in \mathcal{P}_\infty(\Gamma)$, with $A \subset \tilde{A}$, for $\psi \in 2^A$ and $B \subset 2^A$

$$\begin{aligned} \langle y'(\psi, \tilde{A}), x(A,B) \rangle - c &= 2^{|A|-|\tilde{A}|} \langle \sum_{\varphi \in 2^{\psi, \tilde{A}}} x'(\varphi, |\tilde{A}|+1), x(A,B) \rangle - c \\ &\begin{cases} \geq \ \frac{1}{2}(1 - \frac{1}{|A|+1} - \frac{1}{|\tilde{A}|+1}) & \text{if } \psi \in B \\ \leq -\frac{1}{2}(1 - \frac{1}{|A|+1} - \frac{1}{|\tilde{A}|+1}) & \text{if } \psi \notin B. \end{cases} \end{aligned}$$

Since $y'(\psi)$ is a w*-accumulation-point of the net

$$(y'(\psi,\tilde{A}) : \tilde{A} \in \mathcal{P}_f(\Gamma), \text{ with } \mathrm{D}(\psi) \subset \tilde{A})$$

for every $\psi \in \mathrm{S}_\Gamma$, it follows that

$$(4) \qquad \langle y'(\psi), x(A,B)\rangle - c \begin{cases} \geq \ \ \Delta(A) & \text{if } \psi \in B \\ \leq -\Delta(A) & \text{if } \psi \notin B \end{cases}$$

whenever $A \in \mathcal{P}_f(\Gamma)$, $\psi \in 2^A$ and $B \subset 2^A$.

We now choose a fixed $\gamma \in \Gamma$. Since Γ is infinite, it suffices to show that the family $((y'(\psi^1) - y'(\psi^0)) : \psi \in \mathrm{S}_{\Gamma\setminus\{\gamma\}})$, satsifies (a) and (b) of Lemma (2.2.2), where $\psi^\theta : \mathrm{D}(\psi) \cup \{\gamma\} \to \{0,1\}$, is given by $\psi^\theta|_{\mathrm{D}(\psi)} = \psi$ and $\psi^\theta(\gamma) = \theta$ if $\theta \in \{0,1\}$. and $\psi \in \mathrm{S}_{\Gamma\setminus\{\gamma\}}$.
Condition (a) follows from (3). To show (b), let $A \in \mathcal{P}_f(\Gamma \setminus \{\gamma\})$ and $(a_\varphi : \varphi \in 2^A) \subset \mathbb{R}$. From (2) and the choice of $(x'_n : n \in \mathbb{N})$, it follows (compare (2.2.1)) that for any $x \in \mathrm{B}_1(X)$ and $\varphi \in 2^A$

$$\langle x, y'(\varphi^1) - y'(\varphi^0)\rangle \leq \mathrm{osc}\langle x, x'_n\rangle \leq 1 \quad ,$$

which implies that

$$\Big\| \sum_{\varphi \in 2^A} a_\varphi(y'(\varphi^1) - y'(\varphi^0))\Big\| \ \leq \sum_{\varphi \in 2^A} |a_\varphi|.$$

To show "$\geq$" let $\varepsilon > 0$. Without loss of generality, assume $2\Delta(A) \geq 1 - \varepsilon$. Otherwise replace A by an $\tilde{A} \in \mathcal{P}_f(\Gamma \setminus \{\gamma\})$ with $A \subset \tilde{A}$ and $2\Delta(\tilde{A}) \geq 1 - \varepsilon$ and note that by (3) we have

$$\sum_{\tilde{\varphi} \in 2^{\tilde{A}}} 2^{|A|-|\tilde{A}|} a_{(\tilde{\varphi}|_A)}(y'(\tilde{\varphi}^1) - y'(\tilde{\varphi}^0)) \ = \ \sum_{\varphi \in 2^A} a_\varphi(y'(\varphi^1) - y'(\varphi^0))$$

and

$$\sum_{\tilde{\varphi} \in 2^{\tilde{A}}} 2^{|A|-|\tilde{A}|}|a_{(\tilde{\varphi}|_A)}| \ = \ \sum_{\varphi \in 2^A} |a_\varphi|.$$

Now take $x := x(A \cup \{\gamma\}, B)$, where

$$B := \{\varphi^1 | \varphi \in 2^A \text{and } a_\varphi \geq 0\} \cup \{\varphi^0 | \varphi \in 2^A \text{and } a_\varphi < 0\}.$$

By (4) we have

$$\Big\| \sum_{\varphi \in 2^A} a_\varphi (y'(\varphi^1) - y'(\varphi^0)) \Big\| \geq \sum_{\varphi \in 2^A} a_\varphi \langle x, y'(\varphi^1) - y(\varphi^0) \rangle$$
$$\geq \sum_{\varphi \in 2^A} a_\varphi \mathrm{sign}(a_\varphi) 2\Delta(A)$$
$$\geq (1-\varepsilon) \sum_{\varphi \in 2^A} |a_\varphi| .$$

The assertion follows since $\varepsilon \geq 0$ was arbitrary.

◇

By (2.2.4), it is enough to show that $\mathcal{F}_{\omega_1} \neq \emptyset$. As we will see from Lemma (2.2.5), it is sufficient to show that for every $\alpha \in [1, \omega_0]$ each $F \in \mathcal{F}_{[1,\alpha[}$ can be extended to an $F_0 \in \mathcal{F}_{[0,\alpha[}$.

2.2.5 Lemma: *Suppose that for every $\alpha \in [1, \omega_0]$, each family $F = (x(A,B) : (A,B) \in I_{[1,\alpha[}) \subset \mathrm{B}_1(X)$ satisfying $(\mathcal{F}_{[1,\alpha[})$ can be extended to an $F_0 = (x(A,B) : (A,B) \in I_{[0,\alpha[})$ which satsifies $(\mathcal{F}_{[0,\alpha[})$.*
Then $\mathcal{F}_{\omega_1}$ is not empty; in particular, $\mathrm{L}_1(\{0,1\}^{\omega_1})$ can be embedded in X'.

Proof of (2.2.5) :
In order to show that there exists an $F \in \mathcal{F}_{\omega_1}$, we define an $F_\beta \in \mathcal{F}_\beta$ by transfinite induction for every $\beta \in [0, \omega_1]$ such that $F_\beta|_{I_{\tilde\beta}} = F_{\tilde\beta}$ whenever $\tilde\beta < \beta$.
For $\beta = 0$, we remark that $\mathrm{I}_0 = \{(\emptyset, \{\emptyset\}), (\emptyset, \emptyset)\}$ and choose $x = x(\emptyset, \{\emptyset\}) = x(\emptyset, \emptyset)$ in $\mathrm{B}_1(X)$ with $\limsup_{n\to\infty} \langle x, x'_n \rangle > c$ and $\liminf_{n\to\infty} \langle x, x'_n \rangle < -c$ (this is possible by Lemma (2.2.1)). Since $\Delta(\emptyset, n) = -\frac{1}{n}$ for $n \in \mathbb{N}$, $(\mathcal{F}_\emptyset)$ follows trivially.
If $\beta = \tilde\beta + 1$, with $\tilde\beta < \omega_1$ and with $F_{\tilde\beta} \in \mathcal{F}_{\tilde\beta}$ having been chosen, one can use the assumption to get an extension F_β of $F_{\tilde\beta}$ in $\mathcal{F}_\beta$ by reordering β into $(\gamma_n : n < \alpha)$ for an $\alpha \leq \omega_o$ and where $\gamma_0 = \tilde\beta$.
If β is a limit ordinal and if we assume that $(F_{\tilde\beta} : \tilde\beta < \beta)$ has already been chosen, we first observe that $\mathrm{I}_\beta = \bigcup_{\tilde\beta < \beta} \mathrm{I}_{\tilde\beta}$.
Since F_{β_2} is an extension of F_{β_1} for $0 < \beta_1 < \beta_2 < \beta$, we get a family

$$F_\beta = (x(A,B) : (A,B) \in \mathrm{I}_\beta)$$

such that $F_\beta|_{\mathrm{I}_{\tilde\beta}} = F_{\tilde\beta}$ whenever $0 < \tilde\beta < \beta$. Since every $A \in \mathcal{P}_f(\beta)$ is already element of $\mathcal{P}_f(\tilde\beta)$, where $\tilde\beta < \beta$ is sufficiently large, F_β satisfies $(\mathcal{F}_\beta)$.

◇

In order to show the assumption of Lemma (2.2.5), one needs the following Lemmas (2.2.6) and (2.2.7). Lemma (2.2.6) can be shown in a similar way as [34, p.3, Lemma 2], where (ACBH) is assumed, while Lemma (2.2.7) involves the classical Ramsey theorem as presented in [44, Theorem 1.1].

2.2.6 Lemma: *Let $A_1, A_2, \dots A_k \subset X'$ be sets cotaining convex blocks of $(x'_n : n \in \mathbb{N})$ and let $\delta > 0$.*
Then there exists $\tilde{A}_1 \subset A_1, \tilde{A}_2 \subset A_2, \dots, \tilde{A}_k \subset A_k$, still containing convex blocks of $(x'_n : n \in \mathbb{N})$, and for every $B \subset \{1,\dots,k\}$ there exists $x(B) \in B_1(X)$ with

$$\langle x', x(B)\rangle - c \begin{cases} \geq \ (\frac{1}{2} - \delta) & \text{if } i \in B \\ \leq -(\frac{1}{2} - \delta) & \text{if } i \notin B \end{cases}$$

whenever $i \in \{1,\dots,k\}$, $x' \in \tilde{A}_i$ and $B \subset \{1,\dots,k\}$.

Proof of (2.2.6) :
By assumption we can choose for every $i \leq k$ a convex block $(y_n'^{(i)})$ of $(x'_n : n \in \mathbb{N})$ in A_i. By passing to subsequences if necessary, we can assume that $(y'_n : n \in \mathbb{N})$, where $y'_n := \frac{1}{k}\sum_{i=1}^k y_n'^{(i)}$ for $n \in \mathbb{N}$, is a convex block of $(y'_n : n \in \mathbb{N})$ also.
Using Lemma (2.2.1), we find $x \in B_1(X)$ and infinite, disjoint $N_1, N_2 \in \mathbb{N}$ with

$$(1) \qquad \langle y'_n, x\rangle \geq c + \frac{1}{2} - \frac{\delta}{4k} \text{ if } n \in N_1 l \text{ and } \langle y'_n, x\rangle \leq c - \frac{1}{2} + \frac{\delta}{4k} \quad \text{if } n \in N_2.$$

From the properties of $(x'_n : n \in \mathbb{N})$ (compare Lemma (2.2.1)), we deduce for each $i \leq k$ that

$$\begin{aligned} &\limsup_{n\to\infty, n\in N_1} \langle y_n'^{(i)}, x\rangle \\ &\qquad = (\limsup_{n\to\infty, n\in N_1} \langle y_n'^{(i)}, x\rangle - \liminf_{n\to\infty, n\in N_2} \langle y'_n, x\rangle) + \liminf_{n\to\infty, n\in N_2} \langle y'_n, x\rangle \\ &\qquad \leq 1 + c - \frac{1}{2} + \frac{\delta}{4k} \\ &\qquad = c + \frac{1}{2} + \frac{\delta}{4k}. \end{aligned}$$

By passing to a cofinite subset of N_1, we may assume that

$$(2) \qquad \langle y_n'^{(i)}, x\rangle \leq c + \frac{1}{2} + \frac{\delta}{2k} \qquad \text{if} \quad n \in N_1.$$

Similary, we may assume that

$$(3) \qquad \langle y_n'^{(i)}, x\rangle \geq c - \frac{1}{2} - \frac{\delta}{2k} \qquad \text{if } n \in N_2.$$

We deduce from (1) and (2) that for each $i \leq k$ and $n \in N_1$

$$\begin{aligned}\langle y_n'^{(i)}, x\rangle &= k\langle y_n', x\rangle - \sum_{j\leq k,\, j\neq i} \langle y_n'^{(j)}, x\rangle \\ &\geq k(c+\frac{1}{2}-\frac{\delta}{4k}) - (k-1)(c+\frac{1}{2}+\frac{\delta}{2k}) \\ &> c+\frac{1}{2}-\delta .\end{aligned}$$

Similary, we deduce from (1) and (3):

$$\langle y_n'^{(i)}, x\rangle < c-\frac{1}{2}+\delta \qquad \text{for } i\leq k \text{ and } n\in N_2 .$$

Now let $B \subset \{1,\ldots,k\}$. If we define for each $i \in \{1,\ldots,k\}$

$$A_i' := \begin{cases} \{y_n'^{(i)} | n \in N_1\} & \text{if } i \in B \\ \{y_n'^{(i)} | n \in N_2\} & \text{if } i \notin B \end{cases}$$

and $x(B) := x$, then $A_1',\ldots,A_k'$ still contain convex blocks of $(x_n' : n \in \mathbb{N})$. Moreover, for $i \leq k$ and $y' \in A_i'$,

$$(4) \qquad \langle y', x(B)\rangle - c \begin{cases} \geq \ (\frac{1}{2}-\delta) & \text{if } i\in B \\ \leq -(\frac{1}{2}-\delta) & \text{if } i \notin B \end{cases}$$

Repeating this process for every $B \in \{B_1,\ldots,B_{2^k}\} = \mathcal{P}(\{1,\ldots,k\})$ we get sets $A_i \supset A_i^{(1)} \supset \ldots \supset A_i^{(2^k)}$ for every $i \leq k$ and elements $x(B_1), x(B_2),\ldots,x(B_{2^k}) \in B_1(X)$ such that for every $\ell \in \{1,\ldots,2^k\}$, $i \leq k$, and $y' \in A_i^{(\ell)}$, (4) holds for $B := B_\ell$. Taking $\tilde{A}_i := A_i^{(2^k)} = \bigcap_{\ell\leq 2^k} A_i^{(\ell)}$ for $i \in \{1,\ldots,k\}$, we note that the assertion holds for the chosen $x(B_1),\ldots,x(B_{2^k})$.

◇

2.2.7 Lemma: *Let $(J_m : m \in \mathbb{N})$ be a sequence of finite sets; for every $m \in \mathbb{N}$ and $j \in J_m$ let $L_{(m,j)}$ again be a finite set. For every $m \in \mathbb{N}$, $j \in J_m$, and $\ell \in L_{(m,j)}$, let $f_{(m,j)}^{(\ell)} : \mathbb{N}_0 + m \to \mathbb{R}$.*
Also, assume that

$$\sum_{\ell\in L_{(m,j)}} f_{(m,j)}^{(\ell)}(k) \geq 0 \quad \text{for } m \in \mathbb{N}_0, j \in J_m\,, \text{ and } k \in \mathbb{N}_0 + m.$$

Then there exists a subsequence (k_m) of $\mathbb{N}$, and for each $m \in \mathbb{N}$ and $j \in J_m$, a bijection

$$b(m,j) : \{1,2,\ldots,|L_{(m,j)}|\} \to L_{(m,j)}\,,$$

such that

$$\sum_{\ell=1}^{|L_{(m,j)}|} f_{(m,j)}^{b(m,j)(\ell)}(k_{m_\ell}) \geq 0\,, \text{ whenever } \ m \leq m_1 < m_2 < \ldots < m_{|L_{(m,j)}|}.$$

Proof of (2.2.7) :
First let be $f^{(\ell)} : \mathbb{N} \to \mathbb{R}$, for $\ell \in \{1, \ldots, k\}$, $k \in \mathbb{N}$, be such that $\sum_{\ell=1}^{k} f^{(\ell)}(n) \geq 0$ if $n \in \mathbb{N}$.
We show that, for given infinite set $N \subset \mathbb{N}$, there exists an infinite $M \subset N$ and a bijection $b : \{1, \ldots, k\} \to \{1, \ldots, k\}$ such that

$$(1) \qquad \sum_{\ell=1}^{k} f^{b(\ell)}(m_\ell) \geq 0 \quad \text{whenever } m_1 < \ldots < m_k \text{ lie in } M.$$

The classical Ramsey theorem (compare [44, Theorem 1.1 and following remarks]) states that for any infinite $\tilde{N} \subset \mathbb{N}$ and any

$$\mathcal{A} \subset [\tilde{N}]_k := \{(n_1, \ldots, n_k) \in \tilde{N}^k \,|\, n_1 < \ldots < n_k\}$$

there exists an infinite $\tilde{M} \subset \tilde{N}$ such that

$$\text{either } [\tilde{M}]_k \subset \mathcal{A} \quad \text{or } \mathcal{A} \subset [\tilde{N}]_k \setminus [\tilde{M}]_k.$$

Let $\Pi = \{\pi_1, \ldots, \pi_{k!}\}$ be the set of all permutations on $\{1, 2, \ldots, k\}$. Setting $M^{(0)} := N$ and using Ramsey's theorem, we can choose successively for each $i \in \{1, \ldots, k!\}$ an infinite $M^{(i)} \subset N$ with $M^{(i)} \subset M^{(i-1)}$ such that the set

$$\mathcal{A}^{\pi_i} := \{(n_1, \ldots, n_k) \in [M^{(i-1)}]_k \mid \sum_{\ell=1}^{k} f^{\pi_i(\ell)}(n_\ell) \geq 0\}$$

either contains $[M^{(i)}]_k$ or does not meet it. Now we have to show that there exists at least one $i \leq k!$ with $[M^{(i)}]_k \subset \mathcal{A}_{\pi_i}$. This can be seen as follows:
Assuming that no $\mathcal{A}^{\pi_i}$ contains $[M^{(i)}]_k$, we conclude that $\mathcal{A}^{\pi} \cap [M^{(k!)}]_k = \emptyset$ for every $\pi \in \Pi$. This means that for any $m_1 < m_2 < \ldots < m_k$ in $M^{(k!)}$ and any permutation $\pi \in \Pi$, $\sum_{\ell=1}^{k} f^{\pi(\ell)}(m_\ell) < 0$. But this would imply, that for any $m_1 < m_2 \ldots < m_k$ of $M^{(k!)}$,

$$\begin{aligned}
0 &> \sum_{\pi \in \Pi} \sum_{\ell=1}^{k} f^{\pi(\ell)}(m_\ell) \\
&= \sum_{\ell=1}^{k} \sum_{\pi \in \Pi} f^{\pi(\ell)}(m_\ell) \\
&= \sum_{\ell=1}^{k} \sum_{j=1}^{k} |\{\pi \in \Pi | \pi(\ell) = j\}| \cdot f^{(j)}(m_\ell) \\
&= (k-1)! \sum_{\ell=1}^{k} \sum_{j=1}^{k} f^{(j)}(m_\ell),
\end{aligned}$$

which contradicts the assumption. Thus, we have verified the assertion stated at the beginning of the proof.

Applying the same reasoning, for a fixed $m \in \mathbb{N}$ and for an infinite $N \subset \mathbb{N}_0 + m$, $|J_m|$ times, we get an infinite $M_m \subset N$ and, for every $j \in J_m$, a bijection $b(m,j)$: $\{1,\dots,|L_{(m,j)}|\} \to L_{(m,j)}$, such that

$$\sum_{\ell=1}^{L_{(m,j)}} f_{(m,j)}^{b(m,j)(\ell)}(n_\ell) \geq 0 \tag{2}$$

whenever $j \in J_m$ and $n_1 < \ldots < n_{|L_{(m,j)}|}$ are in M_m.

It can be assumed that $(M_m : m \in \mathbb{N})$ decreases. For an increasing sequence $(k_m : m \in \mathbb{N})$, with $k_m \in M_m$ if $m \in \mathbb{N}$, the assertion is then satisfied.

◇

Now we can state and show the last step of the proof of Theorem (2.1.3.).

2.2.8 Lemma: *Suppose $\alpha \in [1,\omega_0]$ and that $F = (x(A,B) : (A,B) \in I_{[1,\alpha[})$ satisfies condition $(\mathcal{F}_{[1,\alpha[})$.*

Then there exists an extension $F_0 = (x(A,B) : (A,B) \in I_{[0,\alpha[})$ of F, which satisfies $(\mathcal{F}_{[0,\alpha[})$.

Proof of (2.2.8) :

By induction, we will choose for every $\beta \in [0,\alpha] \cap \omega_0$ a family

$$(x(A,B) : A \subset \beta, \text{with } 0 \in A \text{ and, if } \beta > 0,\ \beta - 1 \in A;\ B \subset 2^A)$$

such that the following condition $(1)(\beta)$ is satisfied:

$(1)(\beta)$ For each $\gamma \in [\beta,\alpha] \cap \omega_0$ and $n \in \mathbb{N}$ there exists a family $(z'(\varphi,n) : \varphi \in 2^\gamma)$ in X' such that

a) $z'(\varphi,n) \in \text{co}(\{x'_m : m \geq n\})$ if $\varphi \in 2^\gamma$, and

b) $$\langle 2^{|A|-|\gamma|} \sum_{\varphi \in 2^{\psi,\gamma}} z'(\varphi,n), x(A,B)\rangle - c \begin{cases} \geq \ \ \Delta(A,n) & \text{if } \psi \in B \\ \leq -\Delta(A,n) & \text{if } \psi \notin B \end{cases}$$

whenever $A \in \mathcal{P}(\beta) \cup \mathcal{P}([1,\gamma[)$, $\psi \in 2^A$ and $B \subset 2^A$.

(Since for every $\beta \in [0,\alpha] \cap \omega_0$:

$$\bigcup_{0 \leq \beta' \leq \beta} \{A \subset \beta' \,|\, 0 \in A \text{ and,if } 0 < \beta',\ \beta' - 1 \in A\} = \{A \subset \beta \,|\, 0 \in A\},$$

the value $x(A,B)$ is defined for each $A \in \mathcal{P}_f([1,\alpha[) \cup \mathcal{P}(\beta)$ and each $B \subset 2^A$ in the induction step β.)

Having done this, we get an extension $(x(A,B) : A \in \mathcal{P}_f(\alpha), B \subset 2^A)$ of F satisfying $(\mathcal{F}_\alpha)$, which can be seen as follows: For an arbitrary $A \in \mathcal{P}_f(\alpha)$ and an $n \in \mathbb{N}$, one chooses $\beta \in [0,\alpha] \cap \omega_0$ with $A \subset \beta$ and a family $(z'(\varphi,n) : \varphi \in 2^\beta)$ as in (1)(β). Then one observes that $(x'(\varphi,n) : \varphi \in 2^A)$, can be defined by

$$x'(\varphi,n) := 2^{|A|-|\beta|} \sum_{\tilde\varphi \in 2^{\varphi,\beta}} z'(\tilde\varphi,n) \quad \text{for } \varphi \in 2^A ;$$

this family satisfies (a) of $(\mathcal{F}_\alpha)$ because of (1)(β)(a) and from (1)(β)(b) we deduce $(\mathcal{F}_\alpha)$(b) by the following equations:

$$\begin{aligned}
&\langle 2^{|A'|-|A|} \sum_{\varphi \in 2^{\varphi',A}} x'(\varphi,n), x(A',B')\rangle - c \\
&= \langle 2^{|A'|-|A|} \sum_{\varphi \in 2^{\varphi',A}} 2^{|A|-|\beta|} \sum_{\tilde\varphi \in 2^{\varphi,\beta}} z'(\tilde\varphi,n), x(A',B')\rangle - c \\
&= \langle 2^{|A'|-|\beta|} \sum_{\tilde\varphi \in 2^{\varphi',\beta}} z'(\tilde\varphi,n), x(A',B')\rangle - c \\
&\begin{cases} \geq \ \ \Delta(A',n) & \text{if } \varphi' \in B' \\ \leq -\Delta(A',n) & \text{if } \varphi' \notin B' \end{cases}
\end{aligned}$$

whenever $A' \subset A$, $\varphi' \in 2^{A'}$ and $B' \subset 2^{A'}$.

If $\beta = 0$, no $x(A,B)$ has to be defined. To verify (1)(0), we chose for $\gamma \in [0,\alpha] \cap \omega_0$ and $n \in \mathbb{N}$ a family $(x'(\varphi,n) : \varphi \in 2^{[1,\gamma[}) \subset X'$ as in $\mathcal{F}_{[1,\alpha[}$ (taking $A := [1,\gamma[$) and set, for each $\varphi \in 2^\gamma$, $z'(\varphi,n) := x'(\varphi|_{[1,\gamma[},n)$. It follows, that $(z'(\varphi,n) : \varphi \in 2^\gamma)$ satisfies (a) and (b) of (1)(0). Indeed, (1)(0)(a) follows from $(\mathcal{F}_{[1,\alpha[})$(a), and (1)(0)(b) follows from $(\mathcal{F}_{[1,\alpha[})$(b) which can be shown in the following way:

$$\begin{aligned}
&\langle 2^{|A|-|\gamma|} \sum_{\varphi \in 2^{\psi,\gamma}} z'(\varphi,n), x(A,B)\rangle - c \\
&= \langle 2^{|A|-|[1,\gamma[|} \sum_{\varphi \in 2^{\psi,[1,\gamma[}} x'(\varphi,n), x(A,B)\rangle - c \\
&\begin{cases} \geq \ \ \Delta(A,n) & \text{if } \psi \in B \\ \leq -\Delta(A,n) & \text{if } \psi \notin B \end{cases}
\end{aligned}$$

whenever $A \in \mathcal{P}([1,\gamma[)$, $\psi \in 2^A$ and $B \subset 2^A$.

Suppose now that for $\beta > 0$, $x(A,B)$ has been chosen for each $A \subset \beta - 1$ with $0 \in A$ and each $B \subset 2^A$.

For $n \in \mathbb{N}$ we set $\gamma(n) := \max\{\gamma \leq \alpha : |\gamma| \leq n\}$ (thereby concluding that $\gamma(1) = 1, \gamma(2) = 2, \ldots$ and if $\alpha < \omega_0$, then $\alpha = \gamma(|\alpha|) = \gamma(|\alpha + 1|)\ldots$), for $A \in \mathcal{P}_f(\alpha)$ we set $\ell(A) := \max(A)+1$ (so we have $A \subset \ell(A) \subset \alpha$ for an $A \in \mathcal{P}_f(\alpha)$) and, for $\psi \in \mathrm{S}_\alpha$, $\ell(\psi) := \ell(\mathrm{D}(\psi))$.

Choosing for every $n \in \mathbb{N}$, $(\tilde{z}'(\varphi, n) : \varphi \in 2^{\gamma(n)\cup\beta})$ as in (1)($\beta - 1$), and setting for each $\psi \in \mathrm{S}_\alpha$ and $n \in \mathbb{N}$ with $\gamma(n) \geq \ell(\psi)$,

$$\tilde{y}'(\psi, n) := 2^{|\mathrm{D}(\psi)|-|\gamma(n)\cup\beta|} \sum_{\varphi \in 2^{\psi,\gamma(n)\cup\beta}} \tilde{z}'(\varphi, n)$$

we get a family $(\tilde{y}'(\psi, n) : \psi \in \mathrm{S}_\alpha,\ \gamma(n) \geq \ell(\psi))$ with properties (2),(3) and (4) as stated and verified below.

By (1)($\beta - 1$)(a)),

$$\text{(2)} \qquad \tilde{y}'(\varphi, n) \in \mathrm{co}(\{x'_m : m \geq n\}) \quad \text{if } \varphi \in \mathrm{S}_\alpha \text{ and } \gamma(n) \geq \ell(\psi).$$

From the definition of $\tilde{y}'(\psi, n)$ we have

$$\begin{aligned}
\text{(3)} \qquad 2^{|A'|-|A|} & \sum_{\psi \in 2^{\psi',A}} \tilde{y}'(\psi, n) \\
&= 2^{|A'|-|A|} \sum_{\psi \in 2^{\psi',A}} 2^{|\mathrm{D}(\psi)|-|\gamma(n)\cup\beta|} \sum_{\varphi \in 2^{\psi,\gamma(n)\cup\beta}} \tilde{z}'(\varphi, n) \\
&= 2^{|A'|-|\gamma(n)\cup\beta|} \sum_{\varphi' \in 2^{\psi',\gamma(n)\cup\beta}} \tilde{z}'(\varphi', n) \\
&= \tilde{y}'(\psi', n),
\end{aligned}$$

whenever $A \in \mathcal{P}_f(\alpha), A' \subset A, \psi' \in 2^{A'}$ and $\gamma(n) \geq \ell(A)$.

Finally (1)($\beta - 1$)(b) implies

$$\begin{aligned}
\text{(4)}\ & \langle \tilde{y}'(\psi, n), x(A,B)\rangle - c \\
&= \langle 2^{|A|-|\gamma(n)\cup\beta|} \sum_{\varphi \in 2^{\psi,\gamma(n)\cup\beta}} \tilde{z}'(\varphi, n), x(A,B)\rangle - c \begin{cases} \geq \ \ \Delta(A,n) & \text{if } \psi \in B \\ \leq -\Delta(A,n) & \text{if } \psi \notin B \end{cases}
\end{aligned}$$

whenever $A \in \mathcal{P}_f([1,\alpha]) \cup \mathcal{P}(\beta - 1)$, $\psi \in 2^A$ and $\gamma(n) \geq \ell(A)$.

Define now for $m \in \mathbb{N}$

$$J_m := \{(A,B,\psi) | A \in \mathcal{P}_f([1,\gamma(m)[) \cup \mathcal{P}((\beta-1)\cap\gamma(m)), B \subset 2^A \text{ and } \psi \in 2^A\},$$

for $(A,B,\psi) \in J_m$

$$L_{(m,A,B,\psi)} := 2^{\psi, A\cup\beta},$$

and for $\varphi \in L_{(m,A,B,\psi)}$ and $k \geq m$:

$$f^{\varphi}_{(m,A,B,\psi)}(k) := \begin{cases} 2^{|A|-|A\cup\beta|}(\ \langle \tilde{y}'(\varphi,k), x(A,B)\rangle - c - \Delta(A,k)) & \text{if } \psi \in B \\ 2^{|A|-|A\cup\beta|}(-\langle \tilde{y}'(\varphi,k), x(A,B)\rangle + c - \Delta(A,k)) & \text{if } \psi \notin B. \end{cases}$$

We conclude from (3) and (4), that the assumption of Lemma (2.2.7) is satisfied. Indeed, we have

$$\begin{aligned} &\sum_{\varphi \in L_{(m,A,B,\psi)}} f^{\varphi}_{(m,A,B,\psi)}(k) \\ &\qquad = (2^{|A|-|A\cup\beta|} \sum_{\varphi \in 2^{\psi,A\cup\beta}} \pm\langle \tilde{y}'(\varphi,k), x(A,B)\rangle) \mp c - \Delta(A,k) \\ &\qquad = \pm\langle \tilde{y}'(\psi,k), x(A,B)\rangle \mp c - \Delta(A,k) \quad \geq \quad 0 \end{aligned}$$

whenever $m \in \mathbb{N}$, $k \geq m$ and $(A,B,\psi) \in J_m$.

So we can find a subsequence $(k_n : n \in \mathbb{N})$ of $\mathbb{N}$ such that the family $(y'(\varphi,n) : \varphi \in S_\alpha,\ \gamma(n) \geq \ell(\varphi))$, where $y'(\varphi,n) := \tilde{y}'(\varphi,k_n)$ if $\varphi \in S_\alpha$ and $\gamma(n) \geq \ell(\varphi)$, still satisfies (2),(3) and (4), and such that, moreover, the following property holds:

(5) For every $n \in \mathbb{N}$, $A \in \mathcal{P}([1,\gamma(n)[) \cup \mathcal{P}((\beta-1)\cap\gamma(n))$, $B \subset 2^A$, and $\psi \in 2^A$, there exists a bijection $b(A,B,\psi,n) : \{1,\dots 2^{|A\cup\beta|-|A|}\} \to 2^{\psi,A\cup\beta}$ such that

$$\begin{aligned} &\langle 2^{|A|-|A\cup\beta|} \sum_{i=1}^{2^{|A\cup\beta|-|A|}} y'(b(A,B,\psi,n)(i), n_i), x(A,B)\rangle - c \\ &\qquad = \begin{cases} \sum_{i=1}^{2^{|A\cup\beta|-|A|}} f^{b(A,B,\psi,n)(i)}(k_{n_i}) + \Delta(A,k_n) & \text{if } \psi \in B \\ -\sum_{i=1}^{2^{|A\cup\beta|-|A|}} f^{b(A,B,\psi,n)(i)}(k_{n_i}) - \Delta(A,k_n) & \text{if } \psi \notin B \end{cases} \\ &\qquad \begin{cases} \geq \ \ \Delta(A,n) & \text{if } \psi \in B \\ \leq -\Delta(A,n) & \text{if } \psi \notin B \end{cases} \end{aligned}$$

whenever $n \leq n_1 < \ldots < n_{2^{|A\cup\beta|-|A|}}$.

Now we apply (2.2.6) to $(A_\varphi : \varphi \in 2^\beta)$, where $A_\varphi := \{y'(\varphi,n) | \ \gamma(n) \geq \beta\}$ for $\varphi \in 2^\beta$, and to $\delta = \frac{1}{2(|\beta|+1)}$ to find for every $\varphi \in 2^\beta$ an $N(\varphi) \in \mathcal{P}_\infty(\mathbb{N}_0 + |\beta|)$ and for every $B \subset 2^\beta$ an $x(\beta, B) \in \mathrm{B}_1(X)$ such that

$$(6) \qquad \langle y'(\varphi,n), x(\beta,B)\rangle - c \begin{cases} \geq \ \ \Delta(\beta) & \text{if } \varphi \in B \\ \leq -\Delta(\beta) & \text{if } \varphi \notin B \end{cases}$$

whenever $B \subset 2^\beta$, $\varphi \in 2^\beta$ and $n \in N(\varphi)$.

For an arbitrary $A \subset \beta$ with 0, $(\beta - 1) \in A$ and $B \subset 2^A$ we set:

$$(7) \qquad x(A,B) := x(\beta, \bigcup_{\psi \in B} 2^{\psi,\beta}).$$

Now we have to verify (1)(β). Toward this end let $n \in \mathbb{N}$ and $\gamma \in [\beta, \alpha] \cap \omega_0$ be arbitrary.

We may assume that $\gamma(n) \geq \gamma$, otherwise we replace n by a sufficiently large $\tilde{n} \in \mathbb{N}$. We choose $\ell \in \mathbb{N}$ such that

$$(8) \qquad \ell \geq 12 \cdot n \cdot 2^{2|\beta|} \cdot \sup_{j \in \mathbb{N}}(\| x'_j \| + 1).$$

Next we choose for each $i \in \{1, \ldots \ell\}$ and $\varphi \in 2^\beta$ an $n(\varphi, i)$ with

(9)(a) $n(\varphi,i) \geq 2n$ and $n(\varphi,i) \in N(\varphi)$,

(b) $\max(\{n(\varphi, i-1) \,|\, \varphi \in 2^\beta\}) < \min(\{n(\varphi,i) \,|\, \varphi \in 2^\beta\})$, if $1 < i \leq \ell$.

Finally, we define for each $\varphi \in 2^\gamma$:

$$(10) \qquad z'(\varphi, n) := \frac{1}{\ell} \sum_{i=1}^{\ell} y'(\varphi, n(\varphi|_\beta, i)).$$

By (9)(a) and (2), the family $(z'(\varphi,n) : \varphi \in 2^\gamma)$ satisfies (1)(β)(a). To show (1)(β)(b), let $A \in \mathcal{P}([1,\gamma[) \cup \mathcal{P}(\beta)$, $B \subset 2^A$, and $\psi \in 2^A$, it remains to show

$$(11) \qquad \langle 2^{|A|-|\gamma|} \sum_{\varphi \in 2^{\psi,\gamma}} z'(\varphi,n), x(A,B)\rangle - c \begin{cases} \geq \ \ \Delta(A,n) & \text{if } \psi \in B \\ \leq -\Delta(A,n) & \text{if } \psi \notin B \end{cases}.$$

To do this, we consider two cases:

Case 1 : $0 \in A$ and $(\beta - 1) \in A$ (thus $A \subset \beta$ and $x(A,B)$ was defined in the present induction step).

For this case we remark first that, by (10) and (3),

$$
\begin{aligned}
2^{|A|-|\gamma|} & \sum_{\varphi\,\in\,2^{\psi,\gamma}} z'(\varphi,n) \\
&= \frac{1}{\ell}\sum_{i=1}^{\ell} 2^{|A|-|\gamma|} \sum_{\varphi\,\in\,2^{\psi,\gamma}} y'(\varphi, n(\varphi|_\beta, i)) \\
&= \frac{1}{\ell}\sum_{i=1}^{\ell} 2^{|A|-|\beta|} \sum_{\varphi'\,\in\,2^{\psi,\beta}} 2^{|\beta|-|\gamma|} \sum_{\varphi''\,\in\,2^{\varphi',\gamma}} y'(\varphi'', n(\varphi', i)) \\
&= \frac{1}{\ell}\sum_{i=1}^{\ell} 2^{|A|-|\beta|} \sum_{\varphi'\,\in\,2^{\psi,\beta}} y'(\varphi', n(\varphi', i)).
\end{aligned}
$$

Moreover,

$$\psi\in B \iff 2^{\psi,\beta}\subset \bigcup_{\tilde\psi\in B} 2^{\tilde\psi,\beta} \quad\text{and}\quad \psi\notin B \iff 2^{\psi,\beta}\cap \bigcup_{\tilde\psi\in B} 2^{\tilde\psi,\beta}=\emptyset,$$

thus,

$$
\begin{aligned}
&\langle 2^{|A|-|\beta|} \sum_{\varphi'\,\in\,2^{\psi,\beta}} y'(\varphi', n(\varphi', i)), x(A,B)\rangle - c \\
&= 2^{|A|-|\beta|} \sum_{\varphi'\,\in\,2^{\psi,\beta}} \langle y'(\varphi', n(\varphi', i)), x(A, \bigcup_{\tilde\psi\in B} 2^{\tilde\psi})\rangle - c \\
&[\text{by } (7)] \\
&\begin{cases} \geq \ \frac{1}{2}(1-\frac{1}{|\beta|+1}) & \text{if } \psi\in B \\ \leq -\frac{1}{2}(1-\frac{1}{|\beta|+1}) & \text{if } \psi\notin B \end{cases} \\
&\begin{cases} \geq \ \Delta(A,n) & \text{if } \psi\in B \\ \leq -\Delta(A,n) & \text{if } \psi\notin B, \end{cases}
\end{aligned}
$$

which implies (11).

Case 2: $A\in\mathcal{P}(\beta-1)\cup\mathcal{P}([1,\gamma[)$ (thus, $x(A,B)$ was chosen in a previous induction step or was given by the assumption).

First we want to introduce the following notations:
For $\varphi\in S_\alpha$ and $A'\in\mathcal{P}_f(\alpha)$ we set: $\hat\varphi := \varphi|_{D(\varphi)\cap\beta}$, $\hat A' := A'\cap\beta$,
$b := b(A,B,\psi,2n)$ (compare (5) and remark that $A\in\mathcal{P}_f([1,\gamma[)\cup\mathcal{P}_f(\beta-1)\subset\mathcal{P}_f([1,\gamma(2n)[)\cup\mathcal{P}_f(\beta-1))$, and

$$y' := \frac{1}{\ell+1-2^{\beta-|\hat A|}} \sum_{i=1}^{\ell+1-2^{|\beta|-|\hat A|}} 2^{|\hat A|-|\beta|} \sum_{j=1}^{2^{|\beta|-|\hat A|}} y'(b(j), n(\hat b(j), i-1+j)).$$

(Note, that $b(j) \in 2^{\beta \cup A}$ if $j \le 2^{|\beta|-|\hat{A}|} = 2^{|A\cup\beta|-|A|}$ and therefore $\hat{b}(j) \in 2^{\beta}$.) To finish the proof we will show first that

$$\| 2^{|A|-|\gamma|} \sum_{\varphi \in 2^{\varphi,\gamma}} z'(\varphi, n) - y' \|$$

is sufficiently small and secondly that y' satisfies the statement in (11) for $2n$ instead of n. We first do the following calculation:

$$2^{|A|-|\gamma|} \sum_{\varphi \in 2^{\psi,\gamma}} z'(\varphi, n)$$

$$= \frac{1}{\ell} \sum_{i=1}^{\ell} 2^{|A|-|\gamma|} \sum_{\varphi \in 2^{\psi,\gamma}} y'(\varphi, n(\varphi|_{\beta}, i))$$

[by(10)]

$$= \frac{1}{\ell} \sum_{i=1}^{\ell} 2^{|A|-|A\cup\beta|} \sum_{\varphi' \in 2^{\psi,A\cup\beta}} 2^{|A\cup\beta|-|\gamma|} \sum_{\varphi'' \in 2^{\varphi',\gamma}} y'(\varphi'', n(\hat{\varphi}', i))$$

$$= \frac{1}{\ell} \sum_{i=1}^{\ell} 2^{|A|-|A\cup\beta|} \sum_{\varphi' \in 2^{\psi,A\cup\beta}} y'(\varphi', n(\hat{\varphi}', i))$$

[by (3)]

$$= \frac{1}{\ell} \sum_{i=1}^{\ell} 2^{|\hat{A}|-|\beta|} \sum_{j=1}^{2^{|\beta|-|\hat{A}|}} y'(b(j), n(\hat{b}(j), i)$$

[the image of b is $2^{\psi,A\cup\beta}$]

$$= \frac{1}{\ell} 2^{|\hat{A}|-|\beta|} \Bigg[\sum_{i=1}^{\ell+1-2^{|\beta|-|\hat{A}|}} \sum_{j=1}^{2^{|\beta|-|\hat{A}|}} y'(b(j), n(\hat{b}(j), i-1+j))$$

$$+ \sum_{i=1}^{2^{|\beta|-|\hat{A}|}} \sum_{j=i+1}^{2^{|\beta|-|\hat{A}|}} y'(b(j), n(\hat{b}(j), i))$$

$$+ \sum_{i=\ell+2-2^{|\beta|-|\hat{A}|}}^{\ell} \sum_{j=1}^{i-(\ell+1-2^{|\beta|-|\hat{A}|})} y'(b(j), n(\hat{b}(j), i)) \Bigg]$$

[for a proof see below]

$$= \frac{\ell+1-2^{|\beta|-|\hat{A}|}}{\ell} y' + \frac{\Sigma_2 + \Sigma_3}{\ell}$$

[note the definition of y'; Σ_2 and Σ_3 are defined below] [the second to the last equation can be seen in the following way: Setting $a(i,j) = y'(b(j), n(\hat{b}(j), i)$ for $i \leq \ell$ and $j \leq 2^{|\beta||\hat{A}|}$, we have

$$\begin{aligned}
\sum_{i=1}^{\ell} \sum_{j=1}^{2^{|\beta|-|\hat{A}|}} a(i,j)
&= a(1,1) + a(1,2) + \ldots + a(1, 2^{|\beta|-|\hat{A}|}) \\
&\quad + a(2,1) + a(2,2) + \ldots + a(2, 2^{|\beta|-|\hat{A}|}) \\
&\quad \vdots \qquad\qquad\qquad \vdots \\
&\quad + a(\ell,1) + a(\ell,2) + \ldots + a(\ell, 2^{|\beta|-|\hat{A}|}) \\
&= a(1,1) + a(2,2) + \ldots + a(2^{|\beta|-|\hat{A}|}, 2^{|\beta|-|\hat{A}|}) \\
&\quad + a(2,1) + a(3,2) + \ldots + a(1 + 2^{|\beta|-|\hat{A}|}, 2^{|\beta|-|\hat{A}|}) \\
&\quad \vdots \qquad\qquad\qquad \vdots \\
&\quad + a(\ell + 1 - 2^{|\beta|-\hat{A}}, 1) + \ldots + a(\ell, 2^{|\beta|-|\hat{A}|}) \\
&\quad \left.\begin{aligned}
&+ a(1,2) + a(1,3) + \ldots + a(1, 2^{|\beta|-|\hat{A}|}) \\
&\qquad + a(2,3) + \ldots + a(2, 2^{|\beta|-|\hat{A}|}) \\
&\qquad\qquad \ddots \quad \vdots \\
&\qquad\qquad + a(2^{|\beta|-|\hat{A}|} - 1, 2^{|\beta|-|\hat{A}|})
\end{aligned}\right\} =: \Sigma_2 \\
&\quad \left.\begin{aligned}
&+ a(\ell + 2 - 2^{|\beta|-\hat{A}}, 1) \\
&+ a(\ell + 3 - 2^{|\beta|-\hat{A}}, 1) + a(\ell + 3 - 2^{|\beta|-\hat{A}}, 2) \\
&\quad \vdots \qquad\qquad\qquad \ddots \\
&+ a(\ell,1) + a(\ell,2) + \quad \ldots \quad + a(\ell, 2^{|\beta|-\hat{A}} - 1)
\end{aligned}\right\} =: \Sigma_3
\end{aligned}$$

from which we conclude the assertion.]
It follows that

$$(12) \quad \| y' - 2^{|A|-|\gamma|} \sum_{\varphi \in 2^{\psi,\gamma}} z'(\varphi, n) \|$$
$$\leq \frac{2^{|\hat{A}|-|\beta|}}{\ell} \| \Sigma_2 + \Sigma_3 \| + (1 - \frac{\ell + 1 - 2^{|\beta|-|\hat{A}|}}{\ell}) \| y' \|$$

$$\leq \frac{1}{\ell} 2 \cdot 2^{2|\beta|} \sup_{j \in \mathbb{N}} \| x'_j \| + \frac{2^{|\beta|}}{\ell} \sup_{j \in \mathbb{N}} \| x'_j \|$$

[note that $y'(\varphi, n), y' \in \mathrm{co}(\{x'_j \mid j \in \mathbb{N}\})$ and that Σ_2 and Σ_3 have less than $2^{|\beta|}2^{|\beta|}$ summands]

$$\leq 3 \cdot 2^{2|\beta|} \sup_{j \in \mathbb{N}} \| x'_j \| / \ell$$

$$\leq 1/(4n).$$

[(8)]

From (9)(b) we deduce that

$$2n \leq n(b(1), i-1+1) < n(b(2), i-1+2) \ldots < n(b(2^{|A \cup \beta| - |A|}), i-1+2^{|A \cup \beta| - |A|}),$$

whenever $i \in \{1, \ldots, \ell + 1 - 2^{|A \cup \beta| - |A|}\}$

and it follows from (5) for each $i \in \{1, \ldots, \ell + 1 - 2^{|A \cup \beta| - |A|}\}$ that

$$\langle 2^{|\hat{A}| - |\beta|} \sum_{j=1}^{2^{|\beta| - |\hat{A}|}} y'(b(j), n(\hat{b}(j), i-1+j)), x(A, B) \rangle - c \begin{cases} \leq \ \ \Delta(A, 2n) & \text{if } \psi \in B \\ \geq -\Delta(A, 2n) & \text{if } \psi \notin B. \end{cases}$$

Since y' is a convex combination of

$$\left(2^{|\hat{A}| - |\beta|} \sum_{j=1}^{2^{|\beta| - |\hat{A}|}} y'(b(j), n(\hat{b}(j), i-1+j)) \ : \ i \leq \ell + 1 - 2^{|\beta| - |\hat{A}|} \right),$$

we have

$$\langle y', x(A, B) \rangle - c \begin{cases} \geq \ \ \Delta(A, 2n) & \text{if } \psi \in B \\ \leq -\Delta(A, 2n) & \text{if } \psi \notin B \end{cases}.$$

So we deduce (11) from (12) and the fact that $\| x(A, B) \| \leq 1$, which finishes the proof.

◇

2.3 Corollaries

First we want to deduce two properties of a Banach space which satisfies (CBH). Both are derived from the fact that a Banach space having (CBH) admits a linear and bounded operator $S : X \to L_\infty(\{0,1\}^{\omega_p})$, a family $(x_\alpha : \alpha < \omega_p)$, and a $\Delta > 0$ such that

$$\| S(x_\alpha) - S(x_\beta)\|_1 \geq \Delta \text{ for } \alpha, \beta \in [0, \omega_p[, \text{ with } \alpha \neq \beta$$

(Proposition (2.1.5) $(E_6)(\omega_p) \Rightarrow (E_7)(\omega_p)$). The first property shows that such a Banach space "nearly contains " $\ell_1(\omega_p)$ (but only nearly, as shown by the example cited in (2.1.5)(i)) and the second says that such a space cannot be generated by a conditionally weakly compact set.

Then we show, following a proof of J. Bourgain and J. Diestel [4, p.55, Proposition 7], that Banach spaces admitting limited sets which are not relatively weakly compact enjoy property (CBH).

2.3.1 Corollary: *Let X satisfy (CBH). Then:*

a) *There exists a family $(x_\alpha : \alpha < \omega_p) \subset X$ and a $\delta > 0$ with the following properties:*
   For every infinite $I \subset \omega_p$ and every familiy $(y_\alpha : \alpha \in I) \subset X$ for which $\| x_\alpha - y_\alpha \| < \delta$ for $\alpha \in I$, there exists an infinte $\tilde{I} \subset I$ such that $(y_\alpha : \alpha \in \tilde{I})$ is equivalent to $(e_i^{(1)} : i \in \tilde{I})$.

b) *X is not generated by a weakly conditionally compact set.*

Proof of (2.3.1) :

Proof of (a):

By Proposition (2.1.6)(b)(ii) and (g) (in the case $\omega_p > \omega_1$) and by Theorem (2.1.3) and (2.1.6)(f)($(E_6)(\omega_p) \Rightarrow (E_7)(\omega_p)$) (in the case $\omega_p = \omega_1$), we get a bounded operator $S : X \to L_\infty(\{0,1\}^{\omega_p})$, a bounded family $(x_\alpha : \alpha < \omega_p) \subset X$ and a $\Delta > 0$ as in $(E_7)(\omega_p)$. We want to verify the assertion for $\delta := \Delta/(3(\| S \| + 1))$. To this end, let $I \in \mathcal{P}_\infty(\omega_p)$ and $(y_\alpha : \alpha \in I) \subset X$, such that $\| y_\alpha - x_\alpha \| < \delta$, whenever $\alpha \in I$; and denote the inclusion from $L_\infty(\{0,1\}^{\omega_p})$ into $L_1(\{0,1\}^{\omega_p})$ by T. From $(E_7)(\omega_p)$ and the fact that $\| T \| = 1$, we have

$$\| T \circ S(y_\alpha) - T \circ S(y_\beta)\|_1 \geq \| S(x_\alpha) - S(x_\beta)\|_1 - 2\delta \| S \| \| T \| \geq \frac{1}{3}\Delta \geq \delta.$$

Consequently, the set $T \circ S(\{x_\alpha \mid \alpha \in I\})$ is not relatively compact in $L_1(\{0,1\}^{\omega_p})$, thus not limited in $L_1(\{0,1\}^{\omega_p})$ (Examples (1.2.4)(a)). We deduce that the preimage of this set corresponding to T cannot be limited in $L_\infty(\{0,1\}^{\omega_p})$. Since

$L_\infty(\{0,1\}^{\omega_p})$ is a Grothendieck space and enjoys the Dunford-Pettis property we deduce from (1.1.7) and the Rosenthal's ℓ_1 theorem that $\{S(x_\alpha) \mid \alpha \in I\}$ contains a sequence $(S(x_{\alpha_n}) : n \in \mathbb{N})$, which is equivalent to $(e_n^{(1)} : n \in \mathbb{N})$. Since ℓ_1 is projective, (x_{α_n}) is equivalent to $(e_n^{(1)} : n \in \mathbb{N})$ as well.

Proof of (b): Suppose that X is generated by a conditionally $\sigma(X,X')$-compact set C. For every α we find $k_\alpha \in \mathbb{N}$, $r(\alpha,1),\dots,r(\alpha,k_\alpha) \in \mathbb{R}$ and $y(\alpha,1),\dots,y(\alpha,k_\alpha) \in C$ such that for every $\alpha < \beta$,

$$\| x_\alpha - \sum_{i=1}^{k_\alpha} r(\alpha,i) y(\alpha,i) \| < \delta,$$

where (x_α) and δ are as in (a).

Since ω_p is uncountable, we find an infinite subset I of ω_p such that:

$$k := \sup\{k_\alpha \mid \alpha \in I\} < \infty \text{ and } r := \sup_{\alpha \in I} \max\{|r(\alpha,i)| \mid i \le k_\alpha\} < \infty.$$

But this implies that the family $(\sum_{i=1}^{k_\alpha} r(\alpha,i) y(\alpha,i) : \alpha \in I)$ is conditionally weakly compact, which cannot be true by (a).

◇

2.3.2 Proposition: *If X contains limited sets which are not relatively weakly compact, then X enjoys the property (CBH).*

Proof of (2.3.2) :(We follow the first part of the proof in [4, p.55, Proposition 7])

Let $A \subset X$ be limited but not relatively $\sigma(X,X')$-compact. By the theorem of Eberlein,Smulian, and James [36, p.103, Theorem 1], we find a bounded sequence $(x_n : n \in \mathbb{N}) \subset A$ and $(x'_n : n \in \mathbb{N}) \subset X'$ such that

$$(1) \qquad \langle x'_n, x_m \rangle = \begin{cases} 1 & \text{if } n \le m \\ 0 & \text{if } n > m. \end{cases}$$

We want to show that $(x'_n : n \in \mathbb{N})$ has no w*-convergent convex block.

Let $(y'_n : n \in \mathbb{N})$ be a convex block of $(x'_n : n \in \mathbb{N})$. Thus, there exists an increasing sequence $(k_n : n \in \mathbb{N}) \subset \mathbb{N}$ and a sequence of non-negative numbers $(a_i : i \in \mathbb{N})$ such that:

$$(2) \qquad y'_n = \sum_{i=k_n}^{k_{n+1}-1} a_i x'_i \quad \text{and} \quad \sum_{i=k_n}^{k_{n+1}-1} a_i = 1 \text{ if } n \in \mathbb{N}.$$

From (1) and (2) we have for each $n \in \mathbb{N}$

$$\begin{aligned}\langle x_{k_{n+1}-1}, y'_n - y'_{n+1}\rangle &= \sum_{i=k_n}^{k_{n+1}-1} a_i\langle x_{k_{n+1}-1}, x'_i\rangle - \sum_{i=k_{n+1}}^{k_{n+2}-1} a_i\langle x_{k_{n+1}-1}, x'_i\rangle \\ &= \sum_{i=k_n}^{k_{n+1}-1} a_i = 1.\end{aligned}$$

Since $\{x_{k_{n+1}-1} \mid n \in \mathbb{N}\}$ is limited in X, the sequence $(y'_n - y'_{n+1} : n \in \mathbb{N})$ does not converge in $\sigma(X', X)$ to 0, therefore $(y'_n : n \in \mathbb{N})$ is not $\sigma(X', X)$-convergent.

◇

The following Corollary collects the conditions which imply that in a given Banach space all limited sets are relatively weakly compact. They are all weaker then the property that X contains a copy of ℓ_1, thus we have generalized the result of J. Bourgain and J. Diestel.

2.3.3 Corollary: *The following conditions imply that the limited sets of X are relatively weakly compact:*

a) *X' does not contain an copy of $L_1(\{0,1\}^{\omega_p})$.*
b) *No family $(x_\alpha : \alpha < \omega_p) \subset X$ exists such that every infinite subfamily contains a ℓ_1-basis.*
c) *X is conditionally weakly compactly generated.*
d) *Under the set theoretical assumption that $\omega_p > \omega_1$, X does not contain a copy of $\ell_1(\omega_p)$.*

2.3.4 Corollary:

a) *A non-separable Banach space X cannot be generated by an X-limited set.*
b) *(Theorem of Josefson and Nissenzweig) In the dual of an infinite dimensional Banach space X there exist normed sequences which converges in $\sigma(X', X)$ to zero.*

Proof of (2.3.4) :

Proof of (a): Suppose $X = \text{span}(A)$ where A is limited in X. Since A must be conditionally weakly compact (1.1.5), all limited sets of X are relatively weakly compact by (2.3.3)(c), which implies that X is weakly compactly generated. Thus, X is a Gelfand-Phillips space and must be generated by the relatively compact set A, which means that X is separable.

Proof of (b): If there does not exist a normed $\sigma(X', X)$-zero sequence in X', $B_1(X)$ is limited in X. Following the arguments in (a), $B_1(X)$ must be compact, which implies that X is finite dimensional. ◇

2.3.5 Remark:

a) The Grothendieck C(K)- space constructed by M. Talagrand under the continuum hypothesis (compare (2.1.5)(h)) is a space not containing $\ell_1(\omega_1)$ (Proposition (2.1.5)((E_1) $\Longleftrightarrow$ (E_5)(ω_c))). By (1.1.7) and (1.1.8) it contains limited sets which are not relatively weakly compact. Thus, without any further set axioms, we cannot deduce from the fact that X has limited subsets which are not relatively weakly compact, that X contains a copy of $\ell_1(\omega_1)$.

b) In (5.2) we shall construct a Banach space not containing ℓ_1 and not satisfying the Gelfand-Phillips property. Thus, the result of J. Bourgain and J. Diestel (that $\ell_1 \not\subset X \Rightarrow$ all limited subsets of X are relatively weakly compact) cannot be sharpened in the following sense: from $\ell_1 \not\subset X$ we cannot deduce that all limited sets are already relatively (norm-) compact.

3 Limited sets in C(K)-spaces

An easy argument shows that in order to characterize limited sets in any Banach space X, it would be sufficient to investigate limited sets in C(K)-spaces; namely, we have for an $A \subset X$,

$$A \text{ is } X\text{-limited} \iff A \text{ is limited in } \mathrm{C}(\mathrm{B}_1(X')),$$

where $\mathrm{B}_1(X')$ is endowed with $\sigma(X',X) \cap \mathrm{B}_1(X')$ and X is embedded in $\mathrm{C}(\mathrm{B}_1(X'))$ in the canonical way.

(" $\Rightarrow$ " : obvious

"$\neg \Rightarrow \neg$" : If $(x'_n : n \in \mathbb{N})$ is a $\sigma(X',X)$-zero sequence, then $(\delta_{x'_n} - \delta_0 : n \in \mathbb{N})$ is w*-zero in $\mathrm{M}(\mathrm{B}_1(X'))$.)

Thus, the investigation of limited sets is of special interest.

In section (3.1), we show how to construct, from a limited but not relatively compact subset in a C(K)-space, a normed and limited sequence $(f_n : n \in \mathbb{N}) \subset \mathrm{C}(K)$ of functions with pairwise disjoint supports. Since, in particular, such a sequence is equivalent to $(e_n^{(0)} : n \in \mathbb{N})$, we deduce that (1.3.3)(a) is reversible for C(K)-spaces and so we can characterize the Gelfand-Phillips property of a C(K)-space by the property that every sequence $(x_n : n \in \mathbb{N})$ which is equivalent to $(e_n^{(0)} : n \in \mathbb{N})$ contains a subsequence $(x_n : n \in N)$ for which $\overline{\mathrm{span}(x_n : n \in \mathbb{N})}$ is complemented in C(K).

In the other sections, we formulate sufficient conditions for limitedness, which will be applied in chapter 5 to construct spaces without the Gelfand-Phillips property.

In the sequel, K is always a compact space.

3.1 The existence of limited and normed sequences in C(K) with pairwise disjoint supports if C(K) is not Gelfand-Phillips

Theorem (3.1.3) shows how to construct, from a limited and not relatively compact set $A \subset C(K)$, a limited normed sequence $(f_n : n \in \mathbb{N})$ whose elements have pairwise disjoint supports. We begin with two lemmas which contain the substance of the proof of (3.1.3).

3.1.1 Lemma: *Let A be limited in $C(K)$ and let $k \in \mathbb{N}$. Then for every $g \in C([-c,c]^k)$, where $c := \sup_{f\in A} \|f\|$, the set*

$$A_g := \{g(f_1(\cdot), f_2(\cdot), \ldots, f_k(\cdot)) \mid f_1, f_2, \ldots, f_k \in A\}$$

is also limited in $C(K)$.

Proof of (3.1.1) :

By induction, we first show that for each $\ell \in \mathbb{N}_0$ and each $\overline{n} := (n_1, \ldots, n_k) \in \mathbb{N}_0^k$ with $\sum_{i=1}^k n_i = \ell$, the set

$$A_{\overline{n}} := \{f_1^{n_1} \cdot f_2^{n_2} \cdots f_k^{n_k} \mid f_1, f_2, \ldots, f_k \in A\}$$

is limited in $C(K)$.

For $\ell = 0$ the assertion is trivial ($A_{(0,\ldots,0)} = \{1\}$).

Supposing that the assertion has been proven for $\ell - 1$, $\ell \geq 1$, we have to show that for $\overline{n} = (n_1, n_2, \ldots, n_k) \in \mathbb{N}_0^k$ with $\sum_{i=1}^k n_i = \ell$, a sequence $(f_{(1,m)}^{n_1} \cdot f_{(2,m)}^{n_2} \cdots f_{(k,m)}^{n_k} : m \in \mathbb{N}) \subset A_{\overline{n}}$ (i.e. $f_{(j,m)} \in A$ for $j \leq k$ and $m \in \mathbb{N}$), and a w*-zero sequence $(\mu_n : n \in \mathbb{N})$ in $M(K)$ it follows that

$$(1) \qquad \lim_{m\to\infty} \langle \mu_m, f_{(1,m)}^{n_1} \cdot f_{(2,m)}^{n_2} \cdots f_{(k,m)}^{n_k} \rangle = 0 \quad .$$

Since $\ell \geq 1$, we can assume w.l.o.g. that $n_1 \geq 1$. Since $(h.\mu_n : n \in \mathbb{N})$ is a w*-zero sequence for each $h \in C(K)$, we deduce from the assumption that the assertion has been proven for $\ell - 1$ that

$$\langle h, (f_{(1,m)}^{n_1-1} \cdot f_{(2,m)}^{n_2} \cdots f_{(k,m)}^{n_k}).\mu_m \rangle = \langle f_{(1,m)}^{n_1-1} \cdot f_{(2,m)}^{n_2} \cdots f_{(k,m)}^{n_k}, h.\mu_m \rangle \underset{m\to\infty}{\longrightarrow} 0$$

for each $h \in C(K)$.

Thus, $((f_{(1,m)}^{n_1-1} \cdot f_{(2,m)}^{n_2} \cdots f_{(k,m)}^{n_k}).\mu_m : m \in \mathbb{N})$ is a $\sigma(C(K), M(K))$-zero sequence also and we deduce (1) from the $C(K)$-limitedness of $(f_{(1,m)} : m \in \mathbb{N})$ since

$$\langle f_{(1,m)}^{n_1} \cdot f_{(2,m)}^{n_2} \cdots f_{(k,m)}^{n_k}, \mu_n \rangle = \langle f_{(1,m)}, (f_{(1,m)}^{n_1-1} \cdot f_{(2,m)}^{n_2} \cdots f_{(k,m)}^{n_k}).\mu_m \rangle \underset{m\to\infty}{\longrightarrow} 0 .$$

This finishes the proof of the induction step.

We conclude that the set $G := \{g \in C([-c,c]^k) \mid A_g$ is $C(K)$ – limited$\}$ contains all polynomials. In particular it follows from the theorem of Stone and Weierstraß that it is dense in $C([-c,c]^k)$, so we can deduce the assertion from Proposition (1.1.4).

◇

3.1.2 Lemma: *Let $(f_n : n \in \mathbb{N})$ be a normed, non negative, weakly to zero converging sequence in $C(K)$.*

Then there exist $k \in \mathbb{N}_0$, $g \in C([0,1]^{k+1})$, with $0 \le g \le 1$, $N \in \mathcal{P}_\infty(\mathbb{N})$, and $m_1, m_2, \ldots, m_k \in \mathbb{N}$ such that the sequence

$$(g_n : n \in N), \text{ where } g_n := g(f_n(\cdot), f_{m_k}(\cdot), f_{m_{k-1}}(\cdot), \cdots f_{m_1}(\cdot)) \text{ for } n \in N,$$

is normed and its elements have pairwise disjoint supports.

Proof of (3.1.2) :

For the sequel, we choose a fixed $h \in C([0,1])$ with

$$(1) \qquad 0 \le h \le 1, \quad h|_{[0,\frac{1}{4}]} = 0, \quad \text{and} \quad h|_{[\frac{1}{2},1]} = 1.$$

For $f \in C(K)$ and $\varepsilon > 0$ we set

$$A^{(\varepsilon)}(f) := \{\xi \in K \mid f(\xi) \ge 1 - \varepsilon\}$$

We will say that a sequence $(\tilde{f}_n : n \in \mathbb{N}) \subset C(K)$ satisfies (2) if

(2) there exists an $N \in \mathcal{P}_\infty(\mathbb{N})$ such that $(A^{(1/4)}(\tilde{f}_n) : n \in N)$ is pairwise disjoint.

If a sequence $(\tilde{f}_n : n \in \mathbb{N}) \subset C(K)$ does not satisfy (2), it follows that

(3) there exists an $n \in \mathbb{N}$ for which the set

$$\{m \in \mathbb{N}, m \ge n \mid A^{(1/4)}(\tilde{f}_n) \cap A^{(1/4)}(\tilde{f}_m) \ne \emptyset\}$$

is of infinite cardinality.

This can be seen in the following way:

Assume that for each $n \in \mathbb{N}$ the set $\{m \in \mathbb{N}, m \ge n \mid A^{(1/4)}(\tilde{f}_n) \cap A^{(1/4)}\tilde{f}_m \ne \emptyset\}$ is finite. Then we can choose for each $n \in \mathbb{N}$ an $m(n) \in \mathbb{N}$ such that

$$A^{(1/4)}(\tilde{f}_n) \cap A^{(1/4)}(\tilde{f}_m) = \emptyset \qquad \text{whenever } m \ge m(n).$$

Setting $n_1 := 1$ and, inductively, $n_{k+1} := m(n_k)$, the sets $A^{(1/4)}(\tilde{f}_{n_k})$, $k \in \mathbb{N}$, are pairwise disjoint, and thus, (3) is fulfilled.

By induction, we choose now for every $k \in \mathbb{N}_0$ an $N_k \in \mathcal{P}_\infty(\mathbb{N})$, a $g^{(k)}$ in $C([0,1]^{k+1})$ with $\| g^{(k)} \| = 1$ and $0 \leq g^{(k)} \leq 1$, and an $m_k \in \mathbb{N}_0$ such that one of the following two cases occur:
Either $k \geq 1$ and the sequence $(g^{(k-1)}(f_n(\cdot), f_{m_{k-1}}(\cdot), \cdots f_{m_1}(\cdot)) : n \in N_{k-1})$ has property (2), in which case
(4)(k) $N_k := N_{k-1}$, $m_k := m_{k-1}$ and
$$g^{(k)}(\xi_1, \xi_2, \ldots, \xi_{k+1}) := g^{(k-1)}(\xi_1, \xi_2, \ldots, \xi_k) \text{ if } \xi_1, \xi_2, \ldots, \xi_{k+1} \in [0,1],$$
or, this is not true and we have
(5)(k) $m_k = \min(N_k) > m_{k-1}$ and $N_k \subset N_{k-1}$ if $k \geq 1$, and
(6)(k) the sequence $(g_n^{(k)} : n \in N_k)$, with
$$g_n^{(k)} := g^{(k)}(f_n(\cdot), f_{m_k}(\cdot), f_{m_{k-1}}(\cdot), \cdots f_{m_1}(\cdot)) \quad \text{for } n \in \mathbb{N},$$
is a normed weak-zero sequence and for each $n \in N_k$ we have
$$\{g_n^{(k)} \geq \frac{1}{4}\} \subset \bigcap_{i=1}^{k} \{f_{m_i} \geq \frac{1}{4}\} \cap \{f_n \geq \frac{1}{4}\} \quad .$$

If $k = 0$, we set $N_0 := \mathbb{N}$, $m_0 := 0$ and take for $g^{(0)}$ the identity on $[0,1]$. Then (5)(0) is an empty condition while (6)(0) follows from the assumption on $(f_n : n \in \mathbb{N})$.

We suppose now that for $k \in \mathbb{N}$, $(N_\ell : \ell < k)$, $(m_\ell : \ell < k)$, and $(g_\ell : \ell < k)$ have been chosen.
In the case that the sequence $(g_n^{(k-1)} : n \in N_{k-1})$ (where $g_n^{(k-1)}$ is defined in (6)($k-1$)) has property (2), we choose N_k, g_k, and m_k as prescribed in (4)(k).
If it does not satisfy (2), the sequence $(g_n^{(k-1)} : n \in N_{k-1}, n > m_{k-1})$ does not satisfy it either and we conclude from the observation at the beginning of the proof that it satisfies (3). Thus, there exists $m_k > m_{k-1}$ such that the set
$$\{n \in N_{k-1}, n \geq m_k \mid A^{(1/4)}(g_{m_k}^{k-1}) \cap A^{(1/4)}(g_n^{(k-1)}) \neq \emptyset\}$$
is infinite. Hence, if we take
$$N_k := \{n \in N_{k-1}, n \geq m_k \mid A^{(1/4)}(g_{m_k}^{k-1}) \cap A^{(1/4)}(g_n^{(k-1)}) \neq \emptyset\},$$

(5)(k) is satisfied.
For $\xi_1, \xi_2 \ldots \xi_{k+1} \in [0,1]$, we set

$$g^{(k)}(\xi_1, \xi_2 \ldots \xi_{k+1}) := h(g^{(k-1)}(\xi_1, \xi_3 \ldots \xi_{k+1}) \cdot g^{(k-1)}(\xi_2, \xi_3 \ldots \xi_{k+1})).$$

Then $g^{(k)} \in C([0,1]^{k+1})$ and, by (1) and the fact that $g^{(k-1)}$ was assumed to be of norm 1, we deduce that $g^{(k)}$ takes its values in $[0,1]$ and is of norm 1.
By the choice of N_k, there exists for each $n \in N_k$ an $\omega \in K$ with $g_{m_k}^{(k-1)}(\omega) \geq 3/4$ and $g_n^{(k-1)}(\omega) \geq 3/4$. From (1) we deduce that

$$\begin{aligned} g_n^{(k)}(\omega) &= h(g^{(k-1)}(f_n, f_{m_{k-1}}, \ldots, f_{m_1}) \cdot g^{(k-1)}(f_{m_k}, f_{m_{k-1}}, \ldots, f_{m_1}))(\omega) \\ &= h(g_n^{(k-1)}(\omega) \cdot g_{m_k}^{(k-1)}(\omega)) = 1, \end{aligned}$$

which implies that $\| g_n^{(k)} \| = 1$ for each $n \in N_k$. Furthermore, $(g_n^{(k)} : n \in N_k)$ is weak-zero convergent since $(g_n^{(k-1)} : n \in N_k)$ has this property and h is continuous and vanishes in 0.
It remains to show the inclusion in (6)(k).
First we remark that, for an $\omega \in K$ and an $n \in N_k$ with $g_n^{(k)}(\omega) \geq 1/4$, it follows from the fact that $g_n^{(k)}(\omega) = h(g_n^{(k-1)}(\omega) g_{m_k}^{(k-1)}(\omega))$ and from (1) that $g_n^{(k-1)}(\omega) \geq 1/4$ and $g_{m_k}^{(k-1)}(\omega)) \geq 1/4$. Thus we deduce from (6)($k-1$) that

$$\begin{aligned} \{g_n^{(k)} \geq \tfrac{1}{4}\} &\subset \{g_{m_k}^{(k-1))} \geq \tfrac{1}{4}\} \cap \{g_n^{(k-1))} \geq \tfrac{1}{4}\} \\ &\subset \bigcap_{i=1}^{k-1} \{f_{m_i} \geq \tfrac{1}{4}\} \cap \{f_{m_k} \geq \tfrac{1}{4}\} \cap \bigcap_{i=1}^{k-1} \{f_{m_i} \geq \tfrac{1}{4}\} \cap \{f_n \geq \tfrac{1}{4}\} \\ &= \bigcap_{i=1}^{k} \{f_{m_i} \geq \tfrac{1}{4}\} \cap \{f_n \geq \tfrac{1}{4}\}, \end{aligned}$$

which verifies the last assertion and finishes the induction step.
We now want to show that there is a $k \in \mathbb{N}$ for which $(g_n^{(k)} : n \in \mathbb{N})$ satisfies condition (2).
Assuming that this is not true, we deduce that for all $k \in \mathbb{N}$, (5)(k) and (6)(k) are satisfied; in particular, the set $\bigcap_{i=1}^{k} \{f_{m_i} \geq \frac{1}{4}\}$ is non empty for every $k \in \mathbb{N}$, and thus, by compactness of K,

$$\bigcap_{i=1}^{\infty} \{f_{m_i} \geq \frac{1}{4}\} \neq \emptyset.$$

But this is a contradiction to the assumption that $(f_n : n \in \mathbb{N})$ is weakly zero convergent.

Thus, we can choose a $k \in \mathbb{N}$ such that $(g_n^{(k)} : n \in N_k)$ satisfies (2) and we find $N \in \mathcal{P}_\infty(N_k)$ for which the elements of $(\mathrm{A}^{(1/4)}(g_n^{(k)}) : n \in N)$ are pairwise disjoint. Choosing now $\tilde{h} \in \mathrm{C}([0,1])$ with

$$0 \le \tilde{h} \le 1, \quad \tilde{h}|_{[0,7/8]} \equiv 0, \quad \text{and} \quad \tilde{h}|_{[8/9,1]} \equiv 1,$$

we deduce that $g := \tilde{h} \circ g^{(k)}$, $m_1, m_2, \dots, m_k$ and N satisfy the desired conditions (note that

$$\begin{aligned}\overline{\{g_n > 0\}}^K &= \overline{\{\tilde{h} \circ g^{(k)}(f_n, f_{m_k}, \dots, f_{m_1}) > 0\}}^K \\ &\subset \{g^{(k)}(f_n, f_{m_k}, \dots, f_{m_1}) \ge \frac{3}{4}\} = \mathrm{A}^{(1/4)}(g_n^{(k)})\end{aligned}$$

whenever $n \in N$).

◇

3.1.3 Theorem: *Let $A \subset \mathrm{C}(K)$ be limited but not relatively compact.*

Then there exists a sequence $(f_n : n \in \mathbb{N}) \subset \mathrm{aco}(A)$, finitely many $h_1, h_2, \dots, h_k$ in $\mathrm{aco}(A)$, and a $g \in \mathrm{C}([-c,c]^{k+1})$, where $c := \sup_{f \in A} \|f\|$, such that the sequence $(g_n : n \in \mathbb{N})$, with

$$g_n := g(f_n, h_k, h_{k-1}, \dots, h_1) \quad \text{for } n \in \mathbb{N},$$

is normed, non negative, is still limited in $\mathrm{C}(K)$, and has pairwise disjoint supports.

Proof of (3.1.3) :

Since A is limited in $\mathrm{C}(K)$, thus weakly conditionally compact by (1.1.5), but not relatively compact, we find a $\sigma(\mathrm{C}(K), \mathrm{M}(K))$-zero sequence $(\tilde{f}_n : n \in \mathbb{N}) \subset \mathrm{aco}(A)$ with

$$r := \inf_{n \in \mathbb{N}} \|\tilde{f}_n\| > 0.$$

Defining for each $n \in \mathbb{N}$, $\hat{f}_n := \frac{1}{r}\min(|\tilde{f}_n(\cdot)|, r)$, the sequence $(\hat{f}_n : n \in \mathbb{N})$ satisfies the assumptions of (3.1.2) and we deduce the existence of $k \in \mathbb{N}_0$, $\hat{g} \in \mathrm{C}([0,1]^{k+1})$, $m_1, m_2, \dots, m_k \in \mathbb{N}$, and $N \in \mathcal{P}_\infty(\mathbb{N})$ such that the sequence $(\hat{g}(\hat{f}_n, \hat{f}_{m_k}, \dots, \hat{f}_1) : n \in N)$ is normed and such that its element have pairwise disjoint supports.

Taking now

$$g(\xi_1,\xi_2,\ldots,\xi_{k+1}) := \hat{g}\Big(\frac{1}{r}(\min(|\xi_1|,r),\min(|\xi_2|,r),\ldots,\min(|\xi_k|,r))\Big)$$
$$\text{whenever } \xi_1,\ldots,\xi_{k+1} \in [\sup_{f\in A}\|f\|, -\sup_{f\in A}\|f\|],$$

$h_j := \tilde{f}_{m_j}$ for $1 \leq j \leq k$, and

$f_j := \tilde{f}_{n_j}$ for $j \in \mathbb{N}$,

where $(n_j : j \in \mathbb{N})$ is strictly increasing and contains just the elements of $N \setminus \{m_1,\ldots,m_k\}$, we deduce the assertion from (3.1.1).

◇

3.1.4 Corollary: *The following are equivalent:*

a) $C(K)$ *enjoys the Gelfand-Phillips property.*

b) *Each* $(f_n : n \in \mathbb{N}) \subset C(K)$, *which is equivalent to* $(e_n^{(0)} : n \in \mathbb{N})$, *contains a subsequence whose closed span is complemented in* $C(K)$.

c) *Each* $(f_n : n \in \mathbb{N}) \subset C(K)$, *consisting of non negative elements of norm 1 with pairwise disjoint supports, contains a subsequence whose closed span is complemented in* $C(K)$.

Proof of (3.1.4) :

(a) ⇒(b): (1.3.3) (a)

(b) ⇒(c): obvious

¬(a) ⇒ ¬(c): (3.1.3) and (1.3.2)

◇

3.2 Decompositions of sequences of measures. Auxiliary results to investigate limited sets in C(K).

Starting from the fact that a $\sigma(\mathrm{C}(K), \mathrm{M}(K))$-zero sequence $(f_n: n \in \mathbb{N})$ is not limited if and only if there exists a $\sigma(\mathrm{C}(K), \mathrm{M}(K))$-converging (not necessarily to zero) sequence $(\mu_n: n \in \mathbb{N})$ such that

$$(+) \qquad \limsup_{n \to \infty} \langle f_n, \mu_n \rangle > 0,$$

we will investigate, which additional properties can be required from a weak*-zero sequence $(\mu_n: n \in \mathbb{N}) \subset \mathrm{M}(K)$ which satisfies (+) for a given $\sigma(\mathrm{C}(K), \mathrm{M}(K))$-zero sequence $(f_n: n \in \mathbb{N})$ which is not limited in C(K).

Closely following a part of a proof of [9, p.94, Theorem], we will first show that $(\mu_n: n \in \mathbb{N})$ can be chosen to have pairwise disjoint supports (Lemma (3.2.1) and Corollary (3.2.2)).

Secondly, we show ((3.2.3) and (3.2.4)) that for a given sequence $(F_n: n \in \mathbb{N})$ of closed subsets of K which is "σ-disjoint" (compare condition (3.2.3.1) in (3.2.3)), we find $N \in \mathcal{P}_\infty(\mathbb{N})$ and a $\sigma(\mathrm{C}(K), \mathrm{M}(K))$-converging sequence $(\mu_n : n \in N)$ satisfying (+) and with supports having pairwise disjoint neighborhoods O_n for which $O_n \cap \overline{\bigcup_{n' \in N, n' \neq n} F_n} = \emptyset$ $(n \in N)$.

The necessary Lemma (3.2.3) will be formulated in the vector-valued setting (by considering $\mathrm{M}(K, X)$ instead of $\mathrm{M}(K)$), because we will need it in this form in chapter 4.

3.2.1 Lemma: *Let (Ω, Σ, μ) be a finite measure space and let $(f_n: n \in \mathbb{N})$ be bounded in $\mathrm{L}_1(\mu)$.*

Then there is a subsequence $(n_k: k \in \mathbb{N})$ of $\mathbb{N}$ and for each $k \in \mathbb{N}$, g_k and h_k in $\mathrm{L}_1(\mu)$ such that

a) $(h_k: k \in \mathbb{N})$ *converges weakly,*

b) $\{g_k \neq 0\} \cap \{g_{k'} \neq 0\} = \emptyset$ μ–almost everywhere for each $k, k' \in \mathbb{N}$, with $k \neq k'$, and

c) $f_{n_k} = g_k + h_k$ μ–almost everywhere for each $k \in \mathbb{N}$.

Proof of (3.2.1) :

W.l.o.g. we can assume that $(f_n: n \in \mathbb{N})$ is not relatively weakly compact; otherwise we find, according to the theorem of Eberlein and Smulian, a subsequence $(f_{n_k}: k \in \mathbb{N})$ which converges weakly and we can take $h_k := f_{n_k}$ and $g_k := 0$ for each $k \in \mathbb{N}$.

For $r > 0$ and $f \in L_1(\mu)$ we define

$$\eta(r,f) := \sup_{E\in\Sigma,\mu(E)\le r} \int_E |f| d\mu ,$$

and for a bounded $A \subset L_1(\mu)$

$$\eta(r,A) := \sup_{f\in A} \eta(r,f) = \sup_{E\in\Sigma,\mu(E)\le r, f\in A} \int_E |f| d\mu .$$

By the theorem of Dunford and Pettis (compare [9, p.93, Theorem]), a bounded $A \subset L_1(\mu)$ is relatively weakly compact if and only if

$$\lim_{r\searrow 0} \eta(r,A) = 0 .$$

Thus, we have

(1)
$$\eta^* := \lim_{r\searrow 0} \eta(r, \{f_n \mid n \in \mathbb{N}\}) > 0$$

(note that the limit exist, since $\eta(r,A)$ is decreasing for decreasing r).
We find therefore a strictly increasing $(m_k : k \in \mathbb{N}) \subset \mathbb{N}$, a decreasing sequence $(r_k : k \in \mathbb{N}) \subset \mathbb{R}^+$, and $(E_k : k \in \mathbb{N}) \subset \Sigma$ such that

(2)
$$\mu(E_k) = r_k \xrightarrow[k\to\infty]{} 0$$

(3)
$$\left|\eta^* - \int_{E_k} |f_{m(k)}| d\mu\right| \le 2^{-k}\eta^* .$$

We want to show that the sequence $(\chi_{\Omega\setminus E_k} f_{m(k)} : k \in \mathbb{N})$ is relatively weakly compact.
Assuming that this is not the case, we find again an increasing $(k_\ell : \ell \in \mathbb{N}) \subset \mathbb{N}$, a decreasing sequence $(\tilde{r}_\ell : \ell \in \mathbb{N}) \subset \mathbb{R}$, and $(\tilde{E}_\ell : \ell \in \mathbb{N}) \subset \Sigma$ such that

(4)
$$\mu(\tilde{E}_\ell) = \tilde{r}_k \xrightarrow[k\to\infty]{} 0 \quad \text{and}$$

(5)
$$\left|\tilde{\eta}^* - \int_{\tilde{E}_\ell} |\chi_{\Omega\setminus E_{k(\ell)}} f_{m(k(\ell))}| d\mu\right| \le 2^{-\ell}\tilde{\eta}^* ,$$

where $\tilde{\eta}^* := \lim_{r\searrow 0} \eta(r, \{\chi_{\Omega\setminus E_k} \mid k \in \mathbb{N}\}) > 0$.

From the definition of $\eta(.,.)$ and from (2), (3), (4), and (5), we deduce for each $\ell \in \mathbb{N}$ that

$$\begin{aligned}\eta(\tilde{r}_\ell + r_{k(\ell)}, \{f_n \,|\, n \in \mathbb{N}\}) &\geq \int_{\tilde{E}_\ell \cup E_{k(\ell)}} |f_{m(k(\ell))}| d\mu \\ &= \int_{E_{k(\ell)}} |f_{m(k(\ell))}| d\mu + \int_{\tilde{E}_\ell} |\chi_{\Omega \setminus E_{k(\ell)}} f_{m(k(\ell))}| d\mu \\ &\geq \eta^* - 2^{-k(\ell)}\eta^* + \tilde{\eta}^* - 2^{-\ell}\tilde{\eta}^* \quad .\end{aligned}$$

Since $\eta^* = \lim_{\ell\to\infty} \eta(r_{k(\ell)} + r_\ell, \{f_n \,|\, n \in \mathbb{N}\})$ and since $\tilde{\eta}^*$ was assumed to be strictly positive, we have a contradiction.

By the theorem of Eberlein and Smulian, we find a subsequence $(k_\ell: \ell \in \mathbb{N})$ of $\mathbb{N}$ such that

(6) $$\tilde{h}_\ell := \chi_{\Omega \setminus E_{k(\ell)}} f_{m(k(\ell))} \quad \text{is weakly convergent.}$$

For $\ell \in \mathbb{N}$ we set

(7) $$\tilde{g}_\ell := f_{m(k(\ell))} - \tilde{h}_\ell = \chi_{E_{k(\ell)}} f_{m(k(\ell))} \, .$$

Since (2) implies that $\mu(\{|\tilde{g}_\ell| > 0\}) \leq \mu(E_{k(\ell)}) = r_{k(\ell)} \underset{\ell\to\infty}{\longrightarrow} 0$, there is a subsequence $(\ell(j) : j \in \mathbb{N})$ of $\mathbb{N}$ with

(8) $$\int_{\bigcup_{j'>j+1} \{|\tilde{g}_{\ell(j')}|>0\}} |\tilde{g}_{\ell(j)}| d\mu \leq 2^{-j} \quad \text{for each } j \in \mathbb{N}\,.$$

If we define

$$g_j := \chi_{A_j} \tilde{g}_{\ell_j} \text{ , where } A_j := \{|\tilde{g}_{\ell(j)}| > 0\} \setminus \bigcup_{j'>j} \{|\tilde{g}_{\ell(j')}| > 0\} \text{ for } j \in \mathbb{N},$$

then the g_j's have pairwise disjoint supports and the sequence $(h_j: j \in \mathbb{N})$, where

(9) $$h_j := \tilde{h}_{\ell(j)} + \chi_{\Omega \setminus A_j} \tilde{g}_{\ell(j)} \quad \text{for } j \in \mathbb{N}\,,$$

converges weakly according to (6) and (8). Thus (a) and (b) of the assertion are satisfied; (c) follows for $n_j := m(k(\ell(j)))$, $j \in \mathbb{N}$, from (7) and (9).

◇

3.2.2 Corollary:

a) *Let* $(\nu_n : n \in \mathbb{N}) \subset \mathrm{M}(K)$ *be bounded. Then there is a subsequence* $(n_k : k \in \mathbb{N})$ *of* $\mathbb{N}$ *and, for each* $k \in \mathbb{N}$, $\nu_k^{(1)}$ *and* $\nu_k^{(2)}$ *in* $\mathrm{M}(K)$ *such that*
   i) $(\nu_k^{(1)} : k \in \mathbb{N})$ *converges weakly in* $\mathrm{M}(K)$,
   ii) $\mathrm{supp}(\nu_k^{(2)}) \cap \mathrm{supp}(\nu_{k'}^{(2)}) = \emptyset$ *for* $k, k' \in \mathbb{N}$ *with* $k \neq k'$, *and*
   iii) $\nu_{n_k} = \nu_k^{(1)} + \nu_k^{(2)}$ *for each* $k \in \mathbb{N}$.

b) *Let* $A \subset \mathrm{C}(K)$ *be conditionally weakly compact but not limited. Then there exists a sequence* $(f_n : n \in \mathbb{N}) \subset A$ *and a normed* $\sigma(\mathrm{M}(K), \mathrm{C}(K))$*-zero sequence* $(\mu_n : n \in \mathbb{N})$ *whose elements have pairwise disjoint support such that*

$$\inf_{n \in \mathbb{N}} \langle \mu_n, f_n \rangle > 0 .$$

Proof of (3.2.2) :

Proof of (a): Let $\mu := \sum_{n \in \mathbb{N}} 2^{-n} |\nu_n|$. Then for every $n \in \mathbb{N}$, ν_n is μ-continuous and has a density $f_n \in \mathrm{L}_1(\mu)$. Since $(f_n : n \in \mathbb{N})$ is bounded, we find by (3.2.1) a subsequence $(n_k : k \in \mathbb{N})$ of $\mathbb{N}$ and, for each $k \in \mathbb{N}$, $g_k, h_k \in \mathrm{L}_1(\mu)$ satisfying (a),(b), and (c) of (3.2.1). Since μ is regular, there are compact and pairwise disjoint $C_k \subset \{|g_k| > 0\}$, for $k \in \mathbb{N}$, with

$$\int_{K \setminus C_k} |g_k| \mathrm{d}\mu < 2^{-k} .$$

Taking for each $k \in \mathbb{N}$

$$\nu_k^{(1)} := h_k . \mu + (\chi_{K \setminus C_k} \cdot g_k) . \mu \qquad \text{and} \qquad \nu_k^{(2)} := (\chi_{C_k} \cdot g_k) . \mu ,$$

we deduce (i), (ii), and (iii).

Proof of (b): If A is conditionally $\sigma(\mathrm{C}(K), \mathrm{M}(K))$-compact but not limited, we find by Lemma (1.3.1) a $\sigma(\mathrm{M}(K), \mathrm{C}(K))$-zero sequence $(\tilde{\mu}_n : n \in \mathbb{N})$ and a weak Cauchy sequence $(\tilde{f}_n : n \in \mathbb{N}) \subset A$ such that $\langle \tilde{\mu}_n, \tilde{f}_m \rangle = \delta_{(n,m)}$ for $n, m \in \mathbb{N}$. By (a), there is a subsequence $(n_k : k \in \mathbb{N})$ of $\mathbb{N}$ and, for each $k \in \mathbb{N}$, $\nu_k^{(1)}$ and $\nu_k^{(2)}$ in $\mathrm{M}(K)$ satisfying (i), (ii), and (iii) of (a). From the Dunford-Pettis property of $\mathrm{C}(K)$ we have

$$\begin{aligned} &\liminf_{n \to \infty} \langle \nu_{2k}^{(2)} - \nu_{2k-1}^{(2)}, \tilde{f}_{n_{2k}} \rangle \\ &\qquad = \liminf_{n \to \infty} [\langle \tilde{\mu}_{n_{2k}} - \tilde{\mu}_{n_{2k-1}}, \tilde{f}_{n_{2k}} \rangle + \langle \nu_{2k-1}^{(1)} - \nu_{2k}^{(1)}, \tilde{f}_{n_{2k}} \rangle] = 1 . \end{aligned}$$

Since $(\nu_{2k}^{(2)} - \nu_{2k-1}^{(2)} : k \in \mathbb{N})$ is a weak*-zero sequence also, there is an $n_0 \in \mathbb{N}$ such that the sequences $(\mu_j : j \in \mathbb{N})$ and $(f_j : j \in \mathbb{N})$ with,

$$\mu_j := \frac{(\nu_{2(j+n_0)}^{(2)} - \nu_{2(j+n_0)-1}^{(2)})}{\| \nu_{2(j+n_0)}^{(2)} - \nu_{2(j+n_0)-1}^{(2)} \|} \quad \text{and} \quad f_j := \tilde{f}_{n_{2(j+n_0)}}, \quad \text{for } j \in \mathbb{N},$$

satisfies the desired conditions.

◇

3.2.3 Lemma: *Let $(\mu_n : n \in \mathbb{N})$ be a bounded sequence in $M(K,X)$ whose elements have pairwise disjoint supports and let $(F_n : n \in \mathbb{N})$ be a sequence of closed and pairwise disjoint subsets of K with the following property:*

(3.2.3.1) For any two disjoint $N_1, N_2 \in \mathcal{P}_\infty(\mathbb{N})$, there are $\tilde{N}_1 \in \mathcal{P}_\infty(N_1)$ and $\tilde{N}_2 \in \mathcal{P}_\infty(N_2)$ such that

$$\overline{\bigcup_{n \in \tilde{N}_1} F_n} \cap \overline{\bigcup_{n \in \tilde{N}_2} F_n} = \emptyset .$$

Let $\varepsilon > 0$.

Then there exists a subsequence $(n_k : k \in \mathbb{N})$ of $\mathbb{N}$, two sequences $(f_k : k \in \mathbb{N})$ and $(h_k : k \in \mathbb{N})$ in $C(K)$, both non negative and normed, such that the following properties hold:

(3.2.3.2) For each $k \in \mathbb{N}$ there is a neighborhood O_k of $\operatorname{supp}((g_k \cdot h_1 \cdot \ldots \cdot h_{k-1}).\mu_{n_k})$ with

i) $O_k \cap O_{k'} = \emptyset$ for $k, k' \in \mathbb{N}$ with $k \neq k'$, and

ii) $O_k \cap \overline{\bigcup_{k' \in \mathbb{N} \setminus \{k\}} F_{n_{k'}}} = \emptyset$ for each $k \in \mathbb{N}$.

(3.2.3.3) $\| \mu_{n_k} - (g_k \cdot h_1 \cdot \ldots \cdot h_{k-1}).\mu_{n_k} \| \leq \varepsilon$.

(3.2.3.4) $\lim_{\ell \to \infty} \sup_{k \geq \ell+1} \| (h_1 \cdot \ldots \cdot h_\ell) \cdot (1 - g_k \cdot h_{\ell+1} \cdot \ldots \cdot h_{k-1}).\mu_{n_k} \| = 0$.

Proof of (3.2.3) :(The proof uses ideas from the proof of the Lemma of Rosenthal [9, p.82, Rosenthal's Lemma] and the proof of a result of Pełczyński [45, p.643, Lemma 1])

W.l.o.g we may assume that $\| \mu_n \| \leq 1$ for $n \in \mathbb{N}$. First we choose a sequence $(m_k : k \in \mathbb{N}) \subset \mathbb{N}$ with

(1) $$\sum_{k=1}^{\infty} \frac{1}{m_k} \leq \varepsilon .$$

Then we choose inductively, for each $k \in \mathbb{N}$, $g_k, h_k \in C(K)$ with $\| g_k \| = \| h_k \| = 1$, $0 \leq g_k \leq 1$ and $0 \leq h_k \leq 1$, an open $U_k \subset K$, a strictly increasing $(n_j^{(k)} : j \in \mathbb{N})$ in $\mathbb{N}$, and an $n_k \in \mathbb{N}_0$ such that the following properties are satisfied:

(2)(k) $(n_j^{(k)} : j \in \mathbb{N}) \subset (n_j^{(k-1)} : j > 2m_k)$ and $n_k \in (n_j^{(k-1)} : 1 \le j \le 2m_k)$ if $k > 1$,

(3)(k) $\overline{U_k} \cap \overline{U_{k'}} = \emptyset$ whenever $1 \le k' < k$ and $\overline{U_k} \cap \bigcup_{k'<k} F_{n_{k'}} = \emptyset$,

(4)(k) $\overline{\{h_k > 0\}} \cap (\overline{U_k} \cup F_{n_k}) = \emptyset$,

(5)(k) $\overline{\{g_k > 0\}} \cap \overline{\bigcup_{j\in\mathbb{N}} F_{n_j^{(k)}}} = \emptyset$,

(6)(k) $\text{supp}((h_1 \cdot h_2 \ldots h_{k-1}).\mu_{n_k}) \subset U_k$,

(7)(k) $\| h_1 \cdot h_2 \ldots h_{k-1}(1 - h_k).\mu_{n_j^{(k)}} \| \le 1/m_k$, and

(8)(k) $\| h_1 \cdot h_2 \ldots h_{k-1}(1 - g_k).\mu_{n_k} \| \le 1/m_k$.

For $k = 1$ we set $n_j^{(1)} := j$ if $j \in \mathbb{N}$, $h_1 := 1$, $g_1 := 0$ $U_1 := \emptyset$, and $n_1 := 0$, where $\mu_0 := 0$ and $F_0 := \emptyset$, and we deduce easily that the desired conditions are satisfied.

We assume now that for $k > 1$ and for all $r \in \{1, \ldots, k-1\}$, $(n_j^{(r)} : j \in \mathbb{N})$, h_r, g_r, and n_r have been chosen.

We define

(9)
$$A_0 := \bigcup_{r<k} \overline{U_r} \cup F_{n_r}$$

and for each $i \in \{1, 2, \ldots 2m_k\}$

(10)
$$A_i := \text{supp}((h_1 \cdot h_2 \ldots h_{k-1}).\mu_{n_i^{(k)}}).$$

From (4)(r), $r < k$, we deduce for each $i \in \{1, 2, \ldots 2m_k\}$ that

$$A_i \subset \bigcap_{r<k} \overline{\{h_r > 0\}} \subset \bigcap_{r<k} (\overline{U_r} \cup F_{n_r})^{\mathsf{C}} = (\bigcup_{r<k} \overline{U_r} \cup F_{n_r})^{\mathsf{C}} = A_{i_0}^{\mathsf{C}}$$

Since, by assumption, the supports of the elements of $\{\mu_{n_i^{(k-1)}} \mid 1 \le 2m_k\}$ are pairwise disjoint, we may apply the normality of K to find, for each $0 \le i \le 2m_k$, an open $G_i \subset K$ such that

(11)
$$A_i \subset G_i \quad \text{and} \quad \overline{G_i} \cap \overline{G_j} = \emptyset \quad \text{for } i, j \in \{0, 1, \ldots, 2m_k\} \text{ with } i \ne j.$$

Since the elements of $\{F_{n_i^{(k-1)}} \mid 1 \le i \le 2m_k\}$ are closed and pairwise disjoint also, we find open $V_i \subset K$, $1 \le i \le 2m_k$, such that

(12)
$$F_{n_i^{(k-1)}} \subset V_i \quad \text{and} \quad \overline{V_i} \cap \overline{V_j} = \emptyset \text{ for } i, j \in \{1, 2, \ldots, 2m_k\} \text{ with } i \ne j.$$

For $i \in \{1, 2, \ldots, 2m_k\}$ we set

(13) $$M_i := \{j > 2m_k \mid |\mu_{n_j^{(k-1)}}|(\overline{V_i} \cup \overline{G_i}) \leq 1/m_k\}.$$

Since $\mu_{n_j^{(k-1)}}$ is of norm not greater than one, we deduce from (11) and (12) that

$$\begin{aligned} 2 &\geq |\mu_{n_j^{(k-1)}}|(\bigcup_{i=1}^{2m_k} \overline{V_i}) + |\mu_{n_j^{(k-1)}}|(\bigcup_{i=1}^{2m_k} \overline{G_i}) \\ &\geq \sum_{i=1}^{2m_k} |\mu_{n_j^{(k-1)}}|(\overline{V_i} \cup \overline{G_i}) \end{aligned}$$

for each $j \in \mathbb{N}$. Thus, for each $j \in \mathbb{N}$, at least one of the above summands must not be greater than $1/m_k$, and it follows that there exists an $i_0 \in \{1, 2, \ldots, 2m_k\}$ for which M_{i_0} is infinite. Now we decompose M_{i_0} into m_k pairwise disjoint and infinite sets, denoted by $M_{i_0}^{(1)}, \ldots M_{i_0}^{(m_k)}$, and we deduce from (3.2.3.1) and the normality of K that there exist $\tilde{M}_{i_0}^{(j)} \in \mathcal{P}_\infty(M_{i_0}^{(j)})$ and open $W_j \subset K$, for $1 \leq j \leq m_k$, such that

(14) $$\overline{\bigcup_{\ell \in \tilde{M}_{i_0}^{(j)}} F_{n_\ell^{(k-1)}}} \subset W_j \quad \text{and} \quad \overline{W_i} \cap \overline{W_j} = \emptyset$$

for $i, j \in \{1, 2, \ldots, m_k\}$ with $i \neq j$.

Since $\mu_{n_{i_0}^{(k-1)}}$ is of norm not greater than one, there is a $j_0 \in \{1, 2, \ldots, m_k\}$ such that

(15) $$|\mu_{n_{i_0}^{(k-1)}}|(W_{j_0}) \leq 1/m_k\,.$$

Now we take $n_k := n_{i_0}^{(k-1)}$ and $n_j^{(k)} := n_{\ell(j)}^{(k)}$, for $j \in \mathbb{N}$, where $(\ell_j : j \in \mathbb{N}) \subset \mathbb{N}$ is increasing and consists of the elements of $\tilde{M}_{i_0}^{(j_0)}$. With this choice, (2)(k) follows. Since A_{i_0} is closed and since G_{i_0} is open and contains A_{i_0}, there is an open $U_k \subset K$ with

(16) $$A_{i_0} \subset U_k \subset \overline{U_k} \subset G_{i_0}\,.$$

Using (10) and the definition of n_k, we deduce (6)(k), while (11), (9), and (16) imply condition (3)(k).

Now we choose an h_k and a g_k, both in $C(K)$ and satisfying (17) and (18) as listed below:

(17) $0 \le h_k \le 1$, $h_k|_{G_{i_0}^C \cap V_{i_0}^C} = 1$, and h_k vanishes on a neighborhood of $\overline{U_k} \cup F_{n_k}$
(note that $F_{n_k} = F_{n_{i_0}^{(k-1)}} \subset V_{i_0}$ by (12), and that $\overline{U_k} \subset G_{i_0}$ by (16))

(18) $0 \le g_k \le 1$, $g_k|_{W_{j_0}^C} = 1$, and g_k vanishes on a neighborhood of
$\tilde{F} := \overline{\bigcup_{\ell \in \tilde{M}_{i_0}^{j_0}} F_{n_\ell^{(k-1)}}}$
(note that $W_{j_0}^C$ and $\tilde{F}$ are disjoint by (14)).

With this choice, we deduce (4)(k) from (17) and (5)(k) from (18).

Moreover, we deduce for each $j \in \mathbb{N}$ that

$$\begin{aligned}\| h_1 \cdot h_2 \ldots h_{k-1}(1-h_k).\mu_{n_j^{(k)}} \| &\le |h_1 \cdot h_2 \ldots h_{k-1}.\mu_{n_j^{(k)}}|(\{h_k \neq 1\}) \\ &\le |\mu_{n_j^{(k)}}|(G_{i_0} \cup V_{i_0}) \\ &[\text{ by (17)}] \\ &\le 1/m_k \\ &[n_j^{(k)} \in \{n_\ell^{(k-1)} \,|\, \ell \in M_{i_0}\} \text{ and (13)}],\end{aligned}$$

which verifies (7)(k). Also,

$$\begin{aligned}\| h_1 \cdot h_2 \ldots h_{k-1}(1-g_k).\mu_{n_k} \| &\le |h_1 \cdot h_2 \ldots h_{k-1}.\mu_{n_k}|(\{g_k \neq 1\}) \\ &\le |\mu_{n_k}|(W_{j_0}) \\ &[\text{ by (18)}] \\ &\le 1/m_k \\ &[n_k = n_{i_0}^{(k-1)} \text{ and (15)}],\end{aligned}$$

which verifies (8)(k) and finishes the induction step.

To prove the assertion, we choose $O_k := U_k \setminus \overline{\bigcup_{j \in \mathbb{N}} F_{n_j^{(k)}}}$ and n_k, h_k, and g_k as in the induction.

From (6)(k) and (5)(k), we deduce that O_k is a neighborhood of the support of $(g_k h_1 h_2 \ldots h_{k-1}).\mu_{n_k}$. (3.2.3.2)(i) follows from (3)(k) and, since from (2)(k) it follows that $(n_{k'} : k' > k) \subset (n_j^{(k)} : j \in \mathbb{N})$, we deduce (3.2.3.2)(ii) from (3)(k) and the above definition of O_k as follows:

$$O_k \cap \overline{\bigcup_{k' \in \mathbb{N} \setminus \{k\}} F_{n_k}} = (U_k \setminus \overline{\bigcup_{j \in \mathbb{N}} F_{n_j^{(k)}}}) \cap (\overline{\bigcup_{k' > k} F_{n_k}} \cup \overline{\bigcup_{k' < k} F_{n_k}}) = \emptyset,$$

for each $k \in \mathbb{N}$.

Finally, we deduce (3.2.3.3) and (3.2.3.4) respectively from the following inequalities:

$$\begin{aligned}
&\| \mu_{n_k} - (g_k h_1 h_2 \ldots h_{k-1}).\mu_{n_k} \| \\
&\quad \le \| (1-g_k) h_1 h_2 \ldots h_{k-1}.\mu_{n_k} \| + \| \mu_{n_k} - (h_1 h_2 \ldots h_{k-1}).\mu_{n_k} \| \\
&\quad \le \frac{1}{m_k} + \sum_{j=1}^{k-1} \| (1-h_j) h_1 h_2 \ldots h_{j-1}.\mu_{n_k} \| \\
&\quad [\text{by } (8)(k)] \\
&\quad \le \frac{1}{m_k} + \sum_{j=1}^{k-1} \frac{1}{m_j} < \varepsilon \\
&\quad [\text{by } (7)(j) \text{ and } (2)(j), \text{ and } (1)]
\end{aligned}$$

and

$$\begin{aligned}
&\| (h_1 \cdot \ldots \cdot h_\ell) \cdot (1 - g_k \cdot h_{\ell+1} \cdot \ldots \cdot h_{k-1}).\mu_{n_k} \| \\
&\qquad \le \| (h_1 \cdot \ldots \cdot h_{k-1}) \cdot (1-g_k).\mu_{n_k} \| \\
&\qquad\quad + \| (h_1 \cdot \ldots \cdot h_\ell) \cdot (1 - h_{\ell+1} \cdot \ldots \cdot h_{k-1}).\mu_{n_k} \| \\
&\qquad \le \frac{1}{m_k} + \sum_{j=\ell+1}^{k-1} \| (1-h_j) h_1 h_2 \ldots h_{j-1}).\mu_{n_k} \| \\
&\qquad [\text{by } (8)(k)] \\
&\qquad \le \frac{1}{m_k} + \sum_{j=\ell-1}^{k-1} \frac{1}{m_j} \le \sum_{j=\ell-1}^{\infty} \frac{1}{m_j} \xrightarrow[\ell \to \infty]{} 0 \\
&\qquad [\text{by } (7)(j) \text{ and } (2)(j), \text{ and } (1)].
\end{aligned}$$

This completes the proof.

◇

3.2.4 Proposition: *Let $(\mu_k : k \in \mathbb{N}) \subset \mathrm{M}(K, X')$ be normed, $\varepsilon > 0$, and let $(h_k : k \in \mathbb{N})$, $(g_k : k \in \mathbb{N}) \subset \mathrm{C}(K)$, satisfy (3.2.3.3) and (3.2.3.4) of Lemma (3.2.3) (with $n_k := k$ and X' instead of X).*
We define $\tilde{\mu}_k := (h_1 h_2 \ldots g_k).\mu_k$ for $k \in \mathbb{N}$.

Then

a) *if $(\mu_k : k \in \mathbb{N})$ is a $\sigma(\mathrm{M}(K, X'), \mathrm{C}(K, X))$-zero sequence, then $(\tilde{\mu}_k : k \in \mathbb{N})$ has the same property, and*

b) *for a sequence* $(f_n : n \in \mathbb{N}) \subset B_1(C(K,X))$ *it follows that*

$$\inf_{k\in\mathbb{N}} |\langle\tilde{\mu}_k, f_k\rangle| \geq \inf_{k\in\mathbb{N}} |\langle\mu_k, f_k\rangle| - \varepsilon .$$

Proof of (3.2.4) :
In order to prove (a), let $f \in B_1(C(K))$, $x \in B_1(X)$, and $\delta > 0$ be arbitrary. By (3.2.3.4), we choose an $\ell \in \mathbb{N}$, with

$$\sup_{k\geq\ell} \|(h_1 \cdot \ldots \cdot h_\ell) \cdot (1 - g_k \cdot h_{\ell+1} \cdot \ldots \cdot h_{k-1}).\mu_k \| \leq \frac{\delta}{2}$$

and $k_0 \geq \ell + 1$ with

$$|\langle xh_1h_2 \ldots h_\ell f, \mu_k\rangle| \leq \frac{\delta}{2} \quad \text{whenever } k \geq k_0 .$$

Consequently,

$$|\langle xf, \mu_k\rangle| \leq |\langle xfh_1..h_\ell, \mu_k\rangle| + \|(h_1 \cdot .. \cdot h_\ell) \cdot (1 - g_k \cdot h_{\ell+1} \cdot \ldots \cdot h_{k-1}).\mu_k \| \leq \delta,$$

which implies the assertion since $C(K,X)$ is generated by $\{xf \mid x \in X, f \in C(K)\}$. The assertion (b) follows directly from (3.2.3.3).

◇

Combining Corollary (3.2.2)(b), Lemma (3.2.3) and Proposition (3.2.4), we have
3.2.5 Corollary: *Let* $(f_n : n \in \mathbb{N}) \subset C(K)$ *be conditionally weakly compact but not limited and let* $(F_n : n \in \mathbb{N})$ *be a sequence of pairwise disjoint closed subsets of* K *which satisfies (3.2.3.1) of Lemma (3.2.3).*

Then there is a subsequence $(n_k : k \in \mathbb{N})$ *of* $\mathbb{N}$ *and a* $\sigma(C(K), M(K))$*-zero sequence* $(\mu_k : k \in \mathbb{N})$ *such that*

a) $\inf_{k\in\mathbb{N}} \langle f_{n_k}, \mu_k\rangle > 0$, *and*
b) *for each* $k \in \mathbb{N}$, *the support of* μ_k *has a neighborhood* O_k *with*
   i) $O_k \cap O_{k'} = \emptyset$ *for* $k, k' \in \mathbb{N}$ *with* $k \neq k'$, *and*
   ii) $O_k \cap \overline{\bigcup_{k'\in\mathbb{N}\setminus\{k\}} F_{n_{k'}}} = \emptyset$ *for each* $k \in \mathbb{N}$.

3.3 Sufficient conditions for limitedness in $C(K)$

In section (3.1) it was shown, that a limited and normed sequence of non negative functions of $C(K)$ with pairwise disjoint supports can be constructed from a given limited but not relatively compact subset of $C(K)$. By this, we reduced the limitedness of subsets in $C(K)$ to the limitedness of normed sequences of non negative elements of $C(K)$ with pairwise disjoint support. For such sequences, we now want to find sufficient conditions for $C(K)$-limitedness using only topological properties of K. This will be done in Theorem (3.3.1), which uses essentially the results in section (3.2). Proposition (3.3.2) formulates an easy special case of the rather technical conditions in (3.3.1).

3.3.1 Theorem: *Let $(f_n: n \in \mathbb{N}) \subset C(K)$ be normed and consisting of non negative elements with pairwise disjoint supports.*

This sequence is limited in $C(K)$ if the following is true:

For any $\delta > 0$, there exists an $\ell = \ell(\delta) \in \mathbb{N}$ and a sequence $(F_n^{(\delta)}: n \in \mathbb{N})$ of pairwise disjoint closed subsets of K such that the following conditions (3.3.1.1) and (3.3.1.2) are satisfied:

(3.3.1.1) *i)* $\{f_n \geq \delta\} \subset F_n^{(\delta)}$ *for* $n \in \mathbb{N}$,

ii) For any two disjoint $N_1, N_2 \in \mathcal{P}_\infty(\mathbb{N})$, there are $\tilde{N}_1 \in \mathcal{P}_\infty(N_1)$ and $\tilde{N}_2 \in \mathcal{P}_\infty(N_2)$ such that

$$\overline{\bigcup_{n\in\tilde{N}_1} F_n^{(\delta)}} \cap \overline{\bigcup_{n\in\tilde{N}_2} F_n^{(\delta)}} = \emptyset .$$

(3.3.1.2) Let $N \in \mathcal{P}_\infty(\mathbb{N})$, $\rho_1, \rho_2 \in [\delta, 1]$ with $\rho_1 < \rho_2$, and let $(A_n: n\in N)$ be a sequence of pairwise disjoint closed subsets of K satisfying the following property (i):

i) For each $n \in N$, there is a neighborhood O_n of A_n, with $O_n \cap O_{n'} = \emptyset$ for $n, n' \in N$ with $n \neq n'$, and with $O_n \cap \bigcup_{n'\in N\setminus\{n\}} F_{n'}^{(\delta)} = \emptyset$ for each $n \in N$.

Then there exists, for each $n \in N$, an open $O_n^{(i)} \subset K$, $i \in \{1, \dots, \ell\}$, with

ii) $A_n \cap \{f_n \leq \rho_1\} \subset \bigcup_{i=1}^{\ell} O_n^{(i)}$, for $n \in N$, and

iii) for each sequence $(\theta_n: n \in N) \subset \{1, 2, \dots, \ell\}$, there exists an $M \in \mathcal{P}_\infty(N)$ such that

$$\overline{\bigcup_{n\in M} \{f_n \geq \rho_2\} \cap A_n} \cap \overline{\bigcup_{n\in M} \{f_n \leq \rho_1\} \cap A_n \cap O_n^{\theta_n}} = \emptyset .$$

Proof of (3.3.1) :
Let $(f_n : n \in \mathbb{N}) \subset C(K)$ satisfy the assumption. To show that $(f_n : n \in \mathbb{N})$ is limited in $C(K)$, we consider an $N \in \mathcal{P}_\infty(\mathbb{N})$ and a normed sequence $(\mu_n : n \in N) \subset M(K)$ with pairwise disjoint supports satisfying

$$(1) \qquad r := \inf_{n\in N} \langle \mu_n, f_n \rangle > 0 .$$

We have to show that $(\mu_n : n \in N)$ does not converge w* to zero.
(By assumption, $(f_n : n \in \mathbb{N})$ is weakly zero convergent and, from (3.2.2), we deduce that for any non limited but weakly conditionally compact sequence $(\tilde{f}_n : n \in \mathbb{N})$ there exists an $N \in \mathcal{P}_\infty(\mathbb{N})$ and a weak*-zero sequence $(\mu_n : n \in N) \subset M(K)$ having pairwise disjoint supports and satisfying (1).)

We set $\varepsilon := r/2$ and, for each $n \in N$, $F_n := F_n^{(1/m)}$, where $m \in \mathbb{N}$ is chosen to be greater than $24/r$.

Now the assumptions of Lemma (3.2.3) are satisfied and we find a subsequence $N_1 = (n_k : k \in \mathbb{N})$ of N and, for each $k \in \mathbb{N}$, g_k and h_k in $C(K)$ for which (3.2.3.2),(3.2.3.3), and (3.2.3.4) of Lemma (3.2.3) are satisfied.
By Proposition (3.2.4)(a), it is enough to show that $(\nu_n : n \in N_1)$, with

$$\nu_n := (g_k \cdot h_1 \dots h_{k-1}).\mu_{n_k} \quad \text{for } n = n_k \in N_1,$$

is not weak*-zero convergent.

From (3.2.3.2) of (3.2.3) it follows that, for $A_n := \operatorname{supp}(\nu_n)$, $n \in N_1$, open neighborhoods O_n can be chosen such that (i) of (3.3.1.2) is satisfied (with $\delta = 1/m$).
Moreover, from Proposition (3.2.4)(b) we have

$$(2) \qquad \langle \nu_n, f_n \rangle \geq r/2 \quad \text{for each } n \in N_1 .$$

We now want to show the following, which is central for the rest of the proof:
(3) Let $\rho_2 > \rho_1 \geq 1/m$, $\tilde{\delta} > 0$, and $M \in \mathcal{P}_\infty(N_1)$. Then there is a $g \in C(K)$ and an $\tilde{M} \in \mathcal{P}_\infty(M)$ with

$$0 \leq g \leq 1, \quad g|_{A_n \cap \{f_n \geq \rho_2\}} = 1 \quad \text{and} \quad \| g.\nu_n|_{\{f_n \leq \rho_1\}} \| \leq \tilde{\delta} \quad \text{for } n \in \tilde{M}.$$

In order to show (3) for given $1/m \leq \rho_1 < \rho_2$ and $M \in \mathcal{P}_\infty(N_1)$, we choose recursively, for each $k \in \mathbb{N}_0$, an $M_k \in \mathcal{P}_\infty(M)$ and a $g^{(k)} \in C(K)$ with

(4)(k) $M_k \subset M_{k-1}$ if $k > 0$, and

(5)(k) $0 \leq g^{(k)} \leq 1$, $g^{(k)}|_{A_n \cap \{f_n \geq \rho_2\}} = 1$, and $\| g^{(k)}.\nu_n|_{\{f_n \leq \rho_1\}} \| \leq (\frac{\ell-1}{\ell})^k$ for each $n \in \tilde{M}_k$, where $\ell := \ell(1/m)$ is as in (3.3.11).

(note that $(\frac{\ell-1}{\ell})^k \underset{k\to\infty}{\longrightarrow} 0$ and set $0^0 := 1$).

For $k = 0$ we define $g^{(0)} := 1$ and $M^{(0)} := M$ and deduce (5)(0), since $\| \nu_n \| \leq \| \mu_n \|$ for $n \in M$.

Assuming that $g^{(k-1)} \in C(K)$ and $M_{(k-1)} \in \mathcal{P}_\infty(M)$ have been chosen for $k > 0$, let $\tilde{N} := M_{(k-1)}$ and, for $n \in \tilde{N}$, $\tilde{A}_n := \mathrm{supp}(g^{(k-1)}.\nu_n)$. Then $(\tilde{A}_n : n \in \tilde{N})$ satisfies (i) of (3.3.1.2) (note that $\tilde{A}_n \subset A_n$) and we deduce from (3.3.1.2) the existence of open $O_n^{(i)}$, for $i \leq \ell$, $n \in \tilde{N}$, satisfying (ii) and (iii) of (3.3.1.2).

For $n \in \tilde{N}$ we choose $\theta_n \in \{1, \dots \ell\}$ such that

$$|g^{(k-1)}.\nu_n|(\{f_n \leq \rho_1\} \cap O_n^{\theta_n}) = \max_{\theta \in \{1,\dots,\ell\}} |g^{(k-1)}.\nu_n|(\{f_n \leq \rho_1\} \cap O_n^{\theta}).$$

Thus by (3.3.1.2)(ii),

$$\begin{aligned}
(6)\quad & |g^{(k-1)}.\nu_n|(\{f_n \leq \rho_1\} \setminus O_n^{(\theta_n)}) \\
&= |g^{(k-1)}.\nu_n|(\{f_n \leq \rho_1\}) - |g^{(k-1)}.\nu_n|(\{f_n \leq \rho_1\} \cap O_n^{(\theta_n)}) \\
&\leq |g^{(k-1)}.\nu_n|(\{f_n \leq \rho_1\}) - \frac{1}{\ell}\sum_{\theta=1}^{\ell} |g^{(k-1)}.\nu_n|(\{f_n \leq \rho_1\} \cap O_n^{(\theta)}) \\
&\leq |g^{(k-1)}.\nu_n|(\{f_n \leq \rho_1\}) - \frac{1}{\ell}|g^{(k-1)}.\nu_n|(\{f_n \leq \rho_1\}) \\
&= \frac{\ell-1}{\ell}|g^{(k-1)}.\nu_n|(\{f_n \leq \rho_1\}).
\end{aligned}$$

Using (3.3.1.2), we find an $M_k \in \mathcal{P}_\infty(\tilde{N})$ such that

$$\overline{\bigcup_{n\in M_k} \{f_n \geq \rho_2\} \cap \tilde{A}_n} \cap \overline{\bigcup_{n\in M_k} \{f_n \leq \rho_1\} \cap \tilde{A}_n \cap O_n^{\theta_n}} = \emptyset.$$

Therefore, from the normality of K, we deduce the existence of $\tilde{g}^{(k)} \in C(K)$ with

$$0 \leq \tilde{g}^{(k)} \leq 1, \quad \tilde{g}^{(k)}|_{\tilde{A}_n \cap \{f_n \geq \rho_2\}} = 1, \quad \text{and} \quad \tilde{g}^{(k)}|_{\tilde{A}_n \cap \{f_n \leq \rho_1\} \cap O_n^{(\theta_n)}} = 0$$

for each $n \in M_k$.

Now taking $g^{(k)} := g^{(k-1)}\tilde{g}^{(k)}$ and observing that by (5)($k-1$) we have

$$\tilde{A}_n = \text{supp}(g^{(k-1)}.\nu_n) \supset A_n \cap \{f_n \geq \rho_2\} \quad \text{for } n \in M_k,$$

we deduce that

$$0 \leq g^{(k)} \leq 1 \quad \text{and} \quad g^{(k)}|_{A_n \cap \{f_n \geq \rho_2\}} = 1 \text{ for each } n \in \tilde{M}_k.$$

The last condition of (5)(k) follows from the following inequalities

$$\begin{aligned}
\| g^{(k)}.\nu_n|_{\{f_n \leq \rho_1\}} \| &= \| \tilde{g}^{(k)} g^{(k-1)}.\nu_n|_{\{f_n \leq \rho_1\}} \| \\
&\leq |g^{(k-1)}.\nu_n|(\{f_n \leq \rho_1\} \cap \{\tilde{g}^{(k)} \neq 0\} \cap \tilde{A}_n) \\
&\leq |g^{(k-1)}.\nu_n|(\{f_n \leq \rho_1\} \setminus O_n^{(\theta_n)}) \\
&\leq \frac{\ell-1}{\ell} \| g^{(k-1)}.\nu_n|_{\{f_n \leq \rho_1\}} \| \\
&[\text{by}(6)] \\
&\leq (\frac{\ell-1}{\ell})^k, \quad \text{for } n \in N_k \\
&[\text{by}(5)(k-1)],
\end{aligned}$$

which finishes the induction step and the proof of (3).

Applying (3) successively for each $j \in \{1, 2, \dots, m-1\}$ to $\rho_1^{(j)} := j/m$, $\rho_2^{(j)} := (j+1)/m$, and $\tilde{\delta} := 1/m$, we find $N_1 \supset M^{(1)} \supset M^{(2)} \dots M^{(m-1)} =: N_2$ and $g_j \in C(K)$ as in (3). Therefore we have

$$(7) \qquad 0 \leq g_j \leq 1, \; g_j|_{A_n \cap \{f_n \geq (j+1)/m\}} = 1, \text{ and } \| g_j.\nu_n|_{\{f_n \leq (j/m)\}} \| \leq \frac{1}{m}$$

for each $n \in N_2$.

Defining $g := \sum_{j=1}^{m-1}(1/m)g_j$, we deduce for each $n \in N_2$ that

$$
\begin{aligned}
&|\langle \nu_n, g - f_n\rangle| \\
&= \Big|\Big(\sum_{j=1}^{m-1} \frac{1}{m}\langle \nu_n, g_j\rangle\Big) - \langle \nu_n, f_n\rangle\Big| \\
&\le \Big|\Big(\sum_{j=1}^{m-1} \frac{1}{m}\langle \nu_n, g_j \chi_{\{f_n \ge (j+1)/m\}}\rangle\Big) - \langle \nu_n, f_n\rangle\Big| \\
&\quad + \Big|\sum_{j=1}^{m-1} \frac{1}{m}\langle \nu_n, g_j \chi_{\{j/m < f_n < (j+1)/m\}}\rangle\Big| \\
&\quad + \Big|\sum_{j=1}^{m-1} \frac{1}{m}\langle \nu_n, g_j \chi_{\{f_n \le j/m\}}\rangle\Big| \\
&\le \Big|\Big(\sum_{j=1}^{m-1} \frac{1}{m}\langle \nu_n, \chi_{\{f_n \ge (j+1)/m\}}\rangle\Big) - \langle \nu_n, f_n\rangle\Big| + \frac{1}{m}\,\|\nu_n\| + \frac{m-1}{m^2}
\end{aligned}
$$

[by (7) and since $A_n = \operatorname{supp}(\nu_n)$]

$$
\begin{aligned}
&= \Big|\Big(\sum_{j=1}^{m-1} \frac{j}{m}\langle \nu_n, \chi_{\{(j+2)/m > f_n \ge (j+1)/m\}}\rangle\Big) - \langle \nu_n, f_n\rangle\Big| \\
&\quad + \frac{1}{m}\,\|\nu_n\| + \frac{m-1}{m^2} \\
&= \Big|\Big(\sum_{j=1}^{m-1} \int_{\{(j+2)/m > f_n \ge (j+1)/m\}} \frac{j}{m}\,\mathrm{d}\nu_n\Big) \\
&\quad - \Big(\sum_{j=1}^{m-1} \int_{\{(j+2)/m > f_n \ge (j+1)/m\}} f_n\,\mathrm{d}\nu_n\Big) - \int_{\{f_n < 2/m\}} f_n\,\mathrm{d}\nu_n\Big| \\
&\quad + \frac{1}{m}\,\|\nu_n\| + \frac{m-1}{m^2} \\
&\le \sum_{j=1}^{m-1} \int_{\{(j+2)/m > f_n \ge (j+1)/m\}} \Big|\frac{j}{m} - f_n\Big|\,\mathrm{d}\nu_n \\
&\quad + \frac{2}{m}\,\|\nu_n\| + \frac{1}{m}\,\|\nu_n\| + \frac{m-1}{m^2} \\
&\le \sum_{j=1}^{m-1} \frac{2}{m}|\nu|(\{(j+2)/m > f_n \ge (j+1)/m\}) + (3/m)\,\|\nu_n\| + \frac{m-1}{m^2} \\
&\le \frac{6}{m} \quad \le \frac{r}{4}
\end{aligned}
$$

[$\|\nu_n\| \le 1$ and $1/m \le r/24$].

This implies, together with (2), that

$$\langle \nu_n, g\rangle \geq \langle \nu_n, f_n\rangle - r/4 \geq r/4\,.$$

Thus, $(\nu_n : n \in N)$ does not converge to 0 in $\sigma(\mathrm{M}(K), \mathrm{C}(K))$, which finishes the proof.

◇

3.3.2 Proposition: *Let $(f_n : n \in \mathbb{N}) \subset \mathrm{C}(K)$ be a normed sequence of non negative functions with pairwise disjoint supports.*

Suppose that that $(f_n : n \in \mathbb{N})$ is "subsequentially complete" (c.f.[33]), i.e.

(3.3.2.1) for all $N \in \mathcal{P}_\infty(\mathbb{N})$ there exists an $M \in \mathcal{P}_\infty(N)$ such that $(f_m : m \in M)$ has a supremum f_M in $\mathrm{C}(K)$.
($f_m \leq f_M$ for all $m \in M$ and for each $g \in \mathrm{C}(K)$ with $f_m \leq g$ for every $m \in M$ it follows that $f_M \leq g$.)

Then conditions (3.3.1.2) and (3.3.1.1), with $\ell := 1$ and $F_n^{(\delta)} := \{f_n \geq \delta\}$ for $\delta > 0$ and $n \in \mathbb{N}$, are satisfied; in particular, it follows that $(f_n : n \in \mathbb{N})$ is limited in $\mathrm{C}(K)$.

Proof of (3.3.2) :

For an $M \in \mathcal{P}_\infty(\mathbb{N})$ for which the supremum f_M of $(f_m : m \in M)$ exists in $\mathrm{C}(K)$, it follows, for each $\rho > 0$, that $\bigcup_{n \in M}\{f_n \geq \rho\} \subset \{f_M \geq \rho\}$. Thus,

$$(1) \qquad \overline{\bigcup_{n \in M}\{f_n \geq \rho\}} \subset \{f_M \geq \rho\} \subset \{f_M > \rho - \delta\} \text{ for all } \delta > 0\,.$$

Verification of (3.3.1.1):

For two disjoint N_1, $N_2 \in \mathcal{P}_\infty(\mathbb{N})$, we choose $\tilde{N}_1 \in \mathcal{P}_\infty(N_1)$ and $\tilde{N}_2 \in \mathcal{P}_\infty(N_2)$ such that the suprema $f_{\tilde{N}_1}$ and $f_{\tilde{N}_2}$ exist in $\mathrm{C}(K)$.

Let $\delta > 0$ be arbitrary.

For any $n \in \tilde{N}_2$ and $\xi \in \{f_n \geq \delta\}$, we choose a $g \in \mathrm{C}(K)$, with $0 \leq g \leq 1$, $g(\xi) = 0$, and $g|_{\{f_n \leq \delta/2\}} = 1$. Since the supports of f_n, $n \in \mathbb{N}$, are pairwise disjoint, it follows that $g \geq f_n$ for each $n \in \tilde{N}_1$. Thus, $g \geq f_{\tilde{N}_1}$, so that, in particular, $f_{\tilde{N}_1}(\xi) = 0$.

Since $n \in \tilde{N}_2$ and $\xi \in \{f_n \geq \delta\}$ were assumed to be arbitrary, we deduce that

$$\{f_{\tilde{N}_1} > \delta/2\} \cap \bigcup_{n \in \tilde{N}_2}\{f_n \geq \delta\} = \emptyset$$

and, since $\{f_{\tilde N_1} > \delta/2\}$ is open, that

$$\{f_{\tilde N_1} > \delta/2\} \cap \overline{\bigcup_{n\in\tilde N_2} \{f_n \geq \delta\}} = \emptyset .$$

Then we deduce (3.3.1.1) from (1).

Verification of (3.3.1.2) with $\ell = 1$:

Let ρ_1, ρ_2 be in $[\delta, 1]$ with $\rho_1 < \rho_2$, let $N \in \mathcal{P}_\infty(\mathbb{N})$, and let $(A_n : n\in N)$ be a sequence of closed and pairwise disjoint subsets of K which satisfies (3.3.1.2)(i).

To verify (3.3.1.2)(ii) and (iii), let $O_n^{(1)} := O_n$ for $n \in N$, where O_n is as in (3.3.1.2)(i).

Choosing $M \in \mathcal{P}_\infty(N)$ such that f_M exists in $C(K)$, it follows from (3.3.1.2)(i) that for each $n \in M$

$$O_n \subset \left(\overline{\bigcup_{n'\in M\setminus\{n\}} \{f_{n'} \geq \rho_1\}}\right)^c \subset \left(\bigcup_{n'\in M\setminus\{n\}} \{f_{n'} \geq \rho_1\}\right)^c \subset \bigcap_{n'\in M\setminus\{n\}} \{f_{n'} < \rho_1\} .$$

Consequently,

(2) $$O_n \cap \{f_n < \frac{\rho_1+\rho_2}{2}\} \subset \bigcap_{n'\in M} \{f_{n'} < \frac{\rho_1+\rho_2}{2}\} \quad \text{for } n \in M.$$

For $n \in M$ and $\xi \in O_n \cap \{f_n < (\rho_1+\rho_2)/2\}$, we can choose $g \in C(K)$ with $(\rho_1+\rho_2)/2 \leq g \leq 1$, $g(\xi) = (\rho_1+\rho_2)/2$, and $g|_{O_n^c \cup \{f_n < (\rho_1+\rho_2)/2\}^c} = 1$. By (2) it follows that $f_m \leq g$ for any $m \in M$. Thus, $f_M \leq g$, so that, in particular, $f_M(\xi) \leq (\rho_1+\rho_2)/2$. Since $n \in M$ and an $\xi \in O_n \cap \{f_n < (\rho_1+\rho_2)/2\}$ were assumed to be arbitrary, we have shown that

$$\bigcup_{n\in M} O_n \cap \{f_n < \frac{\rho_1+\rho_2}{2}\} \subset \{f_M \leq \frac{\rho_1+\rho_2}{2}\} .$$

This implies that

$$\begin{aligned}
&\overline{\bigcup_{n\in M} O_n \cap \{f_n \leq \rho_1\}} \cap \overline{\bigcup_{n\in M} O_n \cap \{f_n \geq \rho_2\}} \\
&\qquad \subset \overline{\bigcup_{n\in M} O_n \cap \{f_n < \frac{\rho_1+\rho_2}{2}\}} \cap \{f_M > \frac{\rho_1+\rho_2}{2}\} \\
&\qquad \text{[by (1)]} \\
&\qquad \subset \{f_M \leq \frac{\rho_1+\rho_2}{2}\} \cap \{f_M > \frac{\rho_1+\rho_2}{2}\} = \emptyset \\
&\qquad \text{[by (3)]},
\end{aligned}$$

which implies (3.3.1.2)(iii); (3.3.1.2)(ii) follows from the choice of $O_n^{(1)}$.

◇

4 Lifting results

In this chapter, we want to discuss problems of the following type: Supposing that we know the limited sets of certain subspaces of X, we want to characterize the limited sets of the whole space.

In the first section we utilize for this a net $(T_i : i \in I) \subset L(X, X)$ which approximates the identity on X and deduce an analogue of the well known result that a set $C \subset X$ is relatively compact if and only if $(T_i : i \in I)$ converges uniformly on C and $T_i(C)$ is relatively compact for each $i \in I$. Situations in which this leads to a sactisfactory characterization of limitedness are presented in section (4.2).

In the last three sections we will discuss limited sets in tensor products and, in particular, in injective tensor products. We will see that the known equivalence

$$A \subset X \check{\otimes} Y \text{ is rel. compact} \iff A(B_1(X')) \text{ and } A(B_1(Y')) \text{ are rel. compact}$$

cannot in general be transferred to limitedness and leads only to a necessary condition for limitedness in $X \check{\otimes} Y$. Thus, we will formulate other necessary conditions for a set $A \subset X \check{\otimes} Y$ to be limited for which we show that they are also sufficient if X and Y are Grothendieck $C(K)$-spaces.

4.1 Characterization of limitedness by nets of operators which approximate the identity

Proposition (4.1.3) recalls a well known result [15, p.259, Lemma 4] which characterizes relatively compact sets in Banach spaces by a directed family of operators $(T_i : i \in I) \subset L(X, X)$ which approximates the identity on X. Example (4.1.3) shows that this characterization cannot be transferred to limited sets. However, Theorem (4.1.6) formulates additional conditions on $(T_i : i \in I)$ which make an analogous characterization of limitedness possible.

4.1.1 Definition: Let I be a set with a directed order denoted by "$\leq$"; i.e. "$\leq$" is a partial order on I under which each finite $F \subset I$ has an upper bound. We will call a bounded family $(T_i : i \in I) \subset L(X, X)$

a) an *approximation of the identity on* X if, for each $x \in X$, the net $(T(x) : i \in I)$ converges to x, i.e. if

$$\forall x \in X, \varepsilon > 0 \exists i = i(x, \varepsilon) \in I : \| T_j(x) - x \| \leq \varepsilon \text{ for any } j \in I \text{ with } j \geq i.$$

b) a *sequentially complete approximation of the identity on* X, if (a) is satisfied and if, moreover, for every increasing sequence $(i_n : n \in \mathbb{N})$ in I and every $x \in X$, $(T_{i_n}(x) : n \in \mathbb{N})$ converges.

4.1.2 Proposition: *Let $(T_i : i \in I) \subset L(X, X)$ be bounded, $(I, \leq)$ directed, and $D \subset X$ with $\overline{\text{span}(D)} = X$.*

a) *$(T_i : i \in I)$ satisfies condition (a) of Definition (4.1.1) if the convergence of $(T_i(x) : i \in I)$ to x holds for each $x \in D$, and it satisfies (b) if, moreover, $T_{i_n}(x)$ converges for each $x \in D$ and each increasing $(i_n : n \in \mathbb{N}) \subset I$.*

b) *If $(T_i : i \in I)$ satisfies (b) of (4.1.1), then for each increasing $\vec{i} = (i_n : n \in \mathbb{N})$ in I, the map $T_{\vec{i}} : X \ni x \mapsto \lim_{n\to\infty} T_{i_n}(x)$ is linear and bounded.*

Proof of (4.1.2) : obvious.

◇

4.1.3 Proposition: *(c.f. [15, p.259, Lemma 4])*
Let $(T_i : i \in I) \subset L(X, X)$ be an approximation of the identity on X.
Then for a bounded $K \subset X$, the following conditions (a) and (b) are equivalent:

a) *K is relatively compact.*

b) i) *$T_i(K)$ is relatively compact for each $i \in I$, and*

ii) *$T_i \xrightarrow[i \in I]{} \text{Id}_X$ uniformly on K, where Id_X denotes the identity on X.*

It is easy to see that the implication (b) $\Rightarrow$(a) is still true if we replace in (a) and in (b)(i) "relatively compact" with "limited in X". In fact, if for $A \subset X$ $T_i(A)$ is X-limited for each $i \in I$ and if (b)(ii) holds, then it follows that for each $\varepsilon > 0$ there is an $i(\varepsilon) \in I$ with $A \subset \mathrm{B}_\varepsilon(X) + T_{i(\varepsilon)}(A)$; this implies by (1.1.4) that A is limited in X. The following example shows that (b) $\Rightarrow$(a) is not true for limitedness.

4.1.4 Example: Let $\xi \in \beta\mathbb{N} \setminus \mathbb{N}$ and $X := \{f \in \mathrm{C}(\beta\mathbb{N}) \,|\, f(\xi) = 0\}$. The neighborhood basis $\mathcal{U}$ of ξ, consisting of all clopen subsets of $\beta\mathbb{N}$ containing ξ, (i.e. the system $\mathcal{U} = \{\overline{N}^{\beta\mathbb{N}} \,|\, N \in \mathcal{P}_\infty(\mathbb{N}),\, \xi \in \overline{N}^{\beta\mathbb{N}}\}$), will be ordered by: $U \geq V :\Longleftrightarrow U \subset V$, $U, V \in \mathcal{U}$.
The family $(T_V : V \in \mathcal{U})$ defined by

$$T:\ X \to X, \quad f \mapsto \chi_{\beta\mathbb{N}\setminus V} f\,, \quad \text{for } V \in \mathcal{U}\,,$$

is an approximation of the identity on X.
Secondly, we remark that the sequence $(\chi_{\{n\}} : n \in \mathbb{N})$ is limited in X, because $(\chi_{\{n\}} : n \in \mathbb{N})$ is limited in $\mathrm{C}(\beta\mathbb{N})$ and $E(\chi_{\{n\}}) = \chi_{\{n\}}$ for $n \in \mathbb{N}$, where the operator $E : \mathrm{C}(\beta\mathbb{N}) \to X$ is defined by $f \mapsto f - f(\xi) \cdot \chi_{\beta\mathbb{N}}$.
But (b)(ii) is not satisfied since

$$\sup_{n\in\mathbb{N}} \| T_V(\chi_{\{n\}}) - \chi_{\{n\}} \| = \sup_{n\in\mathbb{N}} \| \chi_{\{n\}}\chi_V \| = 1 \ \text{ whenever } V \in \mathcal{U}\,.$$

◇

The following Lemma represents the essential part of the proof of Theorem (4.1.6), which characterizes limitedness by sequentially complete approximations of the identity. Since we will need its result in section (4.2) also, we formulate it independently.

4.1.5 Lemma: *Let $A \subset X$ be bounded and $(T_i : i \in I)$ a sequentially complete approximation of the identity. For an increasing $\vec{\imath} = (i_n : n \in \mathbb{N}) \subset I$, let $T_{\vec{\imath}} \in \mathrm{L}(X, X)$ be as in (4.1.2)(b).*

If for each increasing $\vec{\imath} = (i_n : n \in \mathbb{N}) \subset I$, T_{i_n} converges uniformly on A, then $(T_i : i \in I)$ converges uniformly on A to the identity.

Proof of (4.1.5) :
Without loss of generality we assume $A \neq \emptyset$.
By induction we choose, for each $k \in \mathbb{N}$, an $x_k \in A$ and $i_k, j_k \in I$ such that the following conditions are satisfied:

(1)(k) $j_k \le i_k$,
(2)(k) $i_{k-1} \le j_k$ if $k > 1$,
(3)(k) $\| T_j(x_m) - x_m \| \le \frac{1}{k}$ whenever $j \ge j_k$ and $m < k$, and
(4)(k) $\| T_{i_k}(x_k) - x_k \| \ge \frac{1}{2} \sup_{i \ge j_k, x \in A} \| T_i(x) - x \|$.

For $k = 1$ we choose $x_1 \in A$ and $i_1 \in I$ with

$$\| T_{i_1}(x_1) - x_1 \| \ge \frac{1}{2} \sup_{i \in I, x \in A} \| T_i(x) - x \|$$

and set $j_1 := i_1$; we deduce (1)(1) and (4)(1) while (2)(1) and (3)(1) are empty.

If i_m, j_m, and x_m are choosen for all $m < k$, where $k > 1$, we find by Definition (4.1.1)(a) $j_k \ge i_{k-1}$ in I which satisfies (3)(k). Then we choose $i_k \ge j_k$ in I and $x_k \in A$ satisfying (4)(k), which finishes the induction step.

For $\bar{i} := (i_k : k \in \mathbb{N})$ we deduce from (3)(k) and (1)(k) that

$$T_{\bar{i}}(x_n) = \lim_{k \to \infty} T_{i_k}(x_n) = x_n \quad \text{for each } n \in \mathbb{N}\,.$$

Since, by assumption, $(T_{i_k} : k \in \mathbb{N})$ converges uniformly on A to $T_{\bar{i}}$, we find for an arbitrary $\varepsilon > 0$ an $n = n(\varepsilon) \in \mathbb{N}$ with

$$\| T_{i_m}(x_k) - x_k \| \le \varepsilon/2\,, \text{ whenever } m \ge n \text{ and } k \in \mathbb{N}\,.$$

Thus, by (4)(k),

$$\sup_{i \ge j_n, x \in A} \| T_i(x) - x \| \le 2 \| T_{i_n}(x_n) - x_n \| \le \varepsilon,$$

which implies the assertion.

◇

4.1.6 Theorem: *Let $(T_i : i \in I)$ be a sequentially complete approximation of the identity on X.*

Then the following are equivalent for a bounded $A \subset X$:

a) *A is limited, respectively relatively compact, in X.*

b) i) *$T_i(A)$ is limited, respectively relatively compact, in $\overline{T_i(X)}$, for each $i \in I$, and*

ii) *$T_{i_n} \xrightarrow[n \to \infty]{} T_{\bar{i}}$ uniformly on A for each increasing $\bar{i} = (i_n : n \in \mathbb{N}) \in I$.*

c) i) *as in (b), and*

ii) *$T_i \xrightarrow[i \in I]{} \mathrm{Id}_X$ uniformly on A.*

Proof of (4.1.6) :

(a) ⇒(b)(i) : (1.1.3)(c), respectively (4.1.3),

(a) ⇒(b)(ii): (1.1.2)

(b)(ii)⇒(c)(ii): (4.1.5)

(c) ⇒(a) : (1.1.4), respectively (4.1.3)

◇

From (4.1.6) we deduce

4.1.7 Corollary: *Let $(T_i; i \in I)$ be a sequentially complete approximation of the identity on X.*

Then X is a Gelfand-Phillips space if and only if $\overline{T_i(X)}$ is a Gelfand-Phillips space, for each $i \in I$.

4.2 Applications of Theorem (4.1.6) to several situations

Using Theorem (4.1.6), we can characterize the limited sets of several Banach spaces by the limited sets of certain subspaces.

Thus we can characterize the limited sets of

- $L_p(\mu, X)$, *where* $1 \leq p < \infty$ *and* (Ω, Σ, μ) *is a measure space, by the limited sets of* X *(Corollary (4.2.2)),*
- $Y \overset{\alpha}{\otimes} X$ *by the limited sets of* X, *where* α *is a tensor norm and* Y *is a Banach space which admits a sequentially complete approximation of the identity* $(T_i : i \in I)$ *with* $\dim(T_i X) < \infty$ *for each* $i \in I$ *(Proposition (4.2.3)),*
- $C(\Pi_{j \in J} K_j)$, *where* K_j *is compact for each* j *in a set* J, *by the limited sets of all* $C(\Pi_{j \in \tilde{J}} K_j)$, *with* $\tilde{J} \in \mathcal{P}_f(J)$ *(Proposition (4.2.5)),*
- *spaces having a transfinite Schauder decomposition* $(X_\alpha : \alpha < \eta)$, *where* $\eta \in$ Ord, *by the limited sets of all* X_α *(Proposition (4.2.6)).*

Moreover, we deduce the corresponding hereditary results of the Gelfand-Phillips property.

4.2.1 Proposition: *Let* (Ω, Σ, μ) *be a positive measure space,* $1 \leq p < \infty$ *and* Π *be the set of all finite* Σ*-partitions of* Ω, *ordered by fineness.*

For $\pi \in \Pi$ *we define:*

$$\tilde{E}_\pi : L_p(\mu, X) \to L_p(\mu, X), \quad f \mapsto \sum_{B \in \pi, \mu(B) < \infty} \chi_B \int_B f \, d\mu / \mu(B) \quad (\text{where } \frac{0}{0} := 0)$$

(note that in the case $\mu(\Omega) < \infty$, $\tilde{E}_\pi$ *is just the conditional expectation corresponding to* π*).*

Then $(\tilde{E}_\pi : \pi \in \Pi)$ *is a sequentially complete approximation of the identity on* $L_p(\mu, X)$.

Proof of (4.2.1) :

For $A \in \Sigma$, let $T_A : L_p(\mu, X) \to L_p(\mu|_A, X)$ and $S_A : L_p(\mu|_A, X) \to L_p(\mu, X)$ be the restriction and the embedding respectively and observe that the norm of both operators is not greater than 1.

In the case $\mu(\Omega) < \infty$, we deduce that $(\tilde{E}_\pi : \pi \in \Pi)$ is bounded from [10, p.122, Lemma 3]. In the general case, we remark that

$$\tilde{E}_\pi = S_{A(\pi)} \circ \tilde{E}_{\pi \cap A(\pi)} \circ T_{A(\pi)}, \text{ where } A(\pi) := \bigcup \{A \mid A \in \pi, \mu(A) < \infty\}, \text{for } \pi \in \Pi$$

and deduce that $(\tilde{E}_\pi : \pi \in \Pi)$ is bounded.

Since $\{x\chi_B \mid B \in \Sigma,\ \mu(\Sigma) < \infty\}$ generates $L_p(\mu, X)$, we deduce condition (a) of (4.1.1) from (4.1.2).
Condition (b) follows from the martingale convergence theorem [10, p.125, Theorem 1] in the finite and, thus, in the σ-finite case. In general, we observe that for an increasing $(\pi_n : n \in \mathbb{N}) \subset \Pi$ the set

$$A := \bigcup\{A \mid \exists n \in \mathbb{N} : A \in \pi_n \text{ and } \mu(A) < \infty\}$$

is σ-finite. Thus, condition (b) follows also in the general case, noting that

$$\tilde{E}_{\pi_n} = S_A \circ \tilde{E}_{\pi_n \cap A} \circ T_A .$$

◇

4.2.2 Corollary: *(Limited sets in $L_p(\mu, X)$)*
Let (Ω, Σ, μ) be a positive measure space and $1 \leq p < \infty$. Then the following conditions are equivalent for a bounded $A \subset L_p(\mu, X)$:
- *a) A is limited, respectively relatively compact, in $L_p(\mu, X)$.*
- *b) i) For each $B \in \Sigma$, with $\mu(B) < \infty$, the set $\{\int_B f d\mu \mid f \in A\}$ is limited, respectively relatively compact, in X, and*
  - *ii) for each increasing $(\pi_n : n \in \mathbb{N}) \subset \Pi$, $\tilde{E}_{\pi_n}$ converges uniformly on A.*
- *c) i) as in (b), and*
  - *ii) $(E_\pi : \pi \in \Pi)$ converges uniformly on A to the identity on $L_p(\mu, X)$.*

In particular, it follows that the elements of limited sets of $L_p(\mu, X)$ are measurable with respect to the same countable generated σ-algebra and have a common σ-finite support. Moreover, $L_p(\mu, X)$ is a Gelfand-Phillips space if X is a Gelfand-Phillips space.

Proof of (4.2.2) : Theorem (4.1.6) and Proposition (4.2.1)
(note that the image of $\tilde{E}_\pi$ is a finite complemented sum of copies of X, and thus, in this case, (b)(i) of (4.2.2) is equivalent to (b)(i) of (4.1.6)).

◇

In the case $p = 1$, $L_p(\mu, X)$ can be represented as the projective tensor product of $L_1(\mu)$ and X [10, p.228, Example 10] and $L_1(\mu)$ admits a sequentially complete approximation of the identity whose elements have finite dimensional range (this follows from (4.2.1) taking $X := \mathbb{R}$). This situation can be generalized in the following way:

4.2.3 Proposition: *(Limited sets in some tensor products)*

Let $\|.\|$ be a norm on $Y \underset{\alpha}{\otimes} X$ enjoying the properties (T_1), (T_2), and (T_3) (compare (0.3)(d)) and suppose that Y admits a sequentially complete approximation of the identity on Y, $(P_i : i \in I)$ with each P_i having finite-dimensional range.

Then the family $(P_i \overset{\alpha}{\otimes} \mathrm{Id}_X : i \in I)$ is a sequentially complete approximation of the identity on $Y \overset{\alpha}{\otimes} X$. Thus, Theorem (4.1.6) is applicable to $T_i := P_i \overset{\alpha}{\otimes} \mathrm{Id}_X$, $i \in I$, and $Y \overset{\alpha}{\otimes} X$ is a Gelfand-Phillips space if Y is a Gelfand-Phillips space.

Proof of (4.2.3) :

From (T_3) we deduce that $(P_i \overset{\alpha}{\otimes} \mathrm{Id}_X : i \in I)$ is bounded, while (4.1.1)(a) and (b) follow from Proposition (4.1.2) and the fact that $D := \{y \otimes x \,|\, y \in Y, x \in X\}$ generates $Y \overset{\alpha}{\otimes} X$.

$\diamond$

4.2.4 Example: If (Ω, Σ, μ) is a measure space, an example which satisfies the conditions of (4.2.3) is the space $K(\mu, X)$ of all μ-continuous X-valued measures on Σ with relatively compact range, endowed with the semi variation. This space is isometrically isomorphic to $L_1(\mu)\check{\otimes}X$ [10, p.223, Example 4]).

4.2.5 Proposition: *(Limited sets in $C(\Pi_{j \in J} K_j)$)*

Let $(K_j : j \in J)$ be a family of non-empty compact spaces and let the product $\Pi_{j \in J} K_j$ be endowed with the product topology.

We define $I := \mathcal{P}_f(J)$ and order it by inclusion.

For each $j \in J$, we choose a fixed $s_j \in K_j$ and define for each $E \in I$

$$T_E : C(\Pi_{j \in J} K_j) \to C(\Pi_{j \in J} K_j) \text{ by } T(f)(t_j) := f(t_j^{(E)}) \text{ for } (t_j) \in \Pi_{j \in J} K_j,$$

$$\text{where } t_j^{(E)} := \begin{cases} tj & \text{if } j \in E \\ s_j & \text{if not} \end{cases} .$$

Then $T_E(C(\Pi_{j \in J} K_j)) = C(\Pi_{j \in E} K_j)$ for each $E \in I$ and the family $(T_E : E \in I)$ is a sequentially complete approximation of the identity.

Thus, Theorem (4.1.6) is applicable and $C(\Pi_{j \in J} K_j)$ is a Gelfand-Phillips space if, for each $E \in \mathcal{P}_f(J)$, $C(\Pi_{j \in E} K_j)$ is a Gelfand-Phillips space. This is equivalent to the the condition that $C(K_j)$ is a Gelfand-Phillips space for each $j \in J$, which will be shown in Proposition (4.4.3).

Proof of (4.2.5) :
The familiy $(T_E : E \in I)$ is bounded and $C(\Pi_{j\in J}K_j)$ is generated by $D := \bigcup_{E\in I} C(\Pi_{j\in E}K_j)$.

◇

Finally, we want to apply Theorem (4.1.6) to transfinite Schauder decompositions.

4.2.6 Proposition: *Let $(X_\lambda : 1 \le \lambda < \vartheta)$ be a transfinite Schauder decomposition of X (compare [54, p.622, Definition 19.2 and p.623, Definition 19.3]; in particular, X_λ is a closed subspace of X [54, p.624, Theorem 19.1]). Let $(v_\lambda : 1 \le \lambda < \vartheta)$ be the family of the coordinate projections, i.e. the projections onto X_λ which assign to every $x \in X$ the unique $v_\lambda(x)$ such that $x = \sum_{1\le\lambda<\vartheta} v_\lambda(x)$ (for the definition of $\sum_{1\le\lambda<\vartheta}$, compare [54, p.580]).*

Then the following conditions (a),(b), and (c) are equivalent for a bounded $A \subset X$:

a) *A is limited, respectively relatively compact, in X.*

b) i) *$v_\lambda(A)$ is limited, repectively relatively compact, in X_λ for each $1 \le \lambda < \vartheta$, and*

ii) *for increasing $(\lambda_n : n \in \mathbb{N}) \subset [1, \vartheta[$, $\sum_{\alpha<\lambda_n} v_\alpha$ converges uniformly on A.*

c) i) *as in (b), and*

ii) *for each $\lambda \in [1, \vartheta[$, $\sum_{\alpha<\beta} v_\alpha \xrightarrow[\beta\nearrow\lambda]{} \sum_{\alpha<\lambda} v_\alpha$ uniformly on A.*

In particular, X is a Gelfand-Phillips space if, for each $1 \le \lambda < \vartheta$, X_λ is a Gelfand-Phillips space (for example, if X has a transfinite Schauder basis, i.e. if $\dim(X_\lambda) = 1$, $\lambda < \vartheta$).

Proof of (4.2.6) :
For $1 \le \lambda < \vartheta$ we set

$$u_\lambda : X \to X\,, \quad x \mapsto \sum_{\alpha<\lambda} v_\alpha .$$

By [54, p.625, Theorem 19.2], u_λ is a continuous projection onto the space $X^{(\lambda)} := \overline{\operatorname{span}(\bigcup_{\alpha<\lambda} X_\alpha)}$ and the function $S_x : [0, \vartheta] \ni \lambda \mapsto u_\lambda(x) \in X$ is continuous for each $x \in X$. This implies that, for each $1 \le \lambda < \vartheta$, $(u_\alpha|_{X^{(\lambda)}} : \alpha < \lambda)$ is a sequentially complete approximation of the identity on $X^{(\lambda)}$.
We now show the desired implications:
(a) ⇒(b): Theorem (4.1.6)
(b)(ii) ⇒(c)(ii): If λ is a successor, the assertion is trivial; if not, we deduce it

from Lemma (4.1.5).
(c) ⇒(a): By transfinite induction we show that for each $\lambda \leq \vartheta$, $u_\lambda(A)$ is limited in $X^{(\lambda)}$.
Assuming that for a given λ $u_\alpha(A)$ is limited in $X^{(\alpha)}$ whenever $\alpha < \lambda$, we deduce from (c)(i), in the case that λ is a successor, that $u_\lambda(A)$ is limited in $X^{(\lambda)}$. In the case that λ is a limit ordinal, we deduce it from Theorem (4.1.6) (c) ⇒(a) applied to $X := X^{(\lambda)}$ and $(u_\alpha : \alpha < \lambda)$.

◇

4.2.9 Remark:

a) Independently it was proven in [14, p.4, Theorem 3.1] that $\mathrm{L}_p(\mu, X)$, where $1 \leq p < \infty$, inherits the Gelfand-Phillips property from X.
b) The result that $\mathrm{K}(\mu, X)$ has the Gelfand-Phillips property if $\mathrm{B}_1(X')$ is weak* sequentially compact was shown in [17, Theorem 1]; the generalization for any Gelfand-Phillips space X follows from [13, p.407, Theorem 3.1.].
c) The statement that $\mathrm{C}(\Pi_{j\in J} K_j)$ is a Gelfand-Phillips space if $\mathrm{C}(K_j)$ is a Gelfand-Phillips space for each $j \in J$ was first shown in [14, p.8, Theorem 4.2].

4.3 A combinatorial result for $L^c_\infty(\mu, X)$

This section serves to prepare the next one, where we want to investigate limited sets in injective tensor products, in particular in $C(K, X)$. We will show the following result:

Let $(f_n : n \in \mathbb{N})$ be a sequence in $L^c_\infty(\mu, X)$, where (Ω, Σ, μ) is a positive measure space. Then (at least) one of the following two cases happens:
Case 1:
There exists a subsequence $(f_{n_k} : k \in \mathbb{N})$ admitting, for every $\varepsilon > 0$, a countable Σ-partition π^ε of Ω such that the essential oscillation of every f_{n_k} on every $B \in \pi^\varepsilon$ is not greater than ε.
Case 2:
There exists a subsequence $(f_{n_k} : k \in \mathbb{N})$, an $\varepsilon > 0$, and a tree of sets $(A(k,j) : k \in \mathbb{N}_0, j \in \{1, \ldots, 2^k\}) \subset \Sigma$ whose elements have strictly positive measure, such that the essential oscillation of f_{n_k} on $A(k,j)$ is not greater than $\varepsilon/4$ and the essential distance between $A(k, 2i-1)$ and $A(k, 2i)$ under f_{n_k} is at least ε whenever $k, i, j \in \mathbb{N}$ and $1 \le i \le 2^{k-1}$ and $1 \le j \le 2^k$.

Since this result leads in the scalar case to Rosenthal's ℓ_1 theorem we could call it a vector-valued Rosenthal result.

In the sequel, (Ω, Σ, μ) always denotes a measure space.

4.3.1 Definition:

a) We denote by Π the set of all countable Σ-partitions of Ω, where a countable $\pi \subset \Sigma$ is called a *countable Σ-partition of* Ω if the elements of π are almost (always corresponding to μ) pairwise disjoint and if their union is almost Ω. For $A \in \Sigma$ we set $\pi|_A := \{B \cap A \mid B \in \pi\}$ if $\pi \in \Pi$ and $\Pi(A) := \{\pi|_A \mid \pi \in \Pi\}$.

b) For two bounded subsets $A, B \subset X$, let $D(A)$ be the diameter of A, i.e.

$$D(A) := \sup_{x,y \in A} \|x - y\|, \quad \text{with } D(\emptyset) := 0,$$

and $d(A, B)$ the distance between A and B, i.e.

$$d(A, B) := \inf_{x \in A, y \in B} \|x - y\| \quad \text{with } d(\emptyset, \cdot) = d(\cdot, \emptyset) := \infty.$$

If $f : \Omega \to X$ is bounded, measurable, and has separable image, and if $A, B \in \Sigma$, we define the *essential oscillation of f on A* by

$$\operatorname{ess\,osc}(f, A) := \inf_{\tilde{A} \equiv A} D(f(\tilde{A}))$$

and the *essential distance between A and B under f* by:

$$\operatorname{ess\,dist}(f,A,B) := \sup_{\tilde{A}\equiv A,\tilde{B}\equiv B} d(f(\tilde{A})f(\tilde{B})) \quad ,$$

where "$\inf_{\tilde{A}\equiv A}$" and "$\sup_{\tilde{A}\equiv A,\tilde{B}\equiv B}$" mean, that the infimum and the supremum are taken over all $\tilde{A}$ and $\tilde{B} \in \Sigma$ with

$$\mu((A\cup\tilde{A})\setminus(A\cap\tilde{A})) = \mu((B\cup\tilde{B})\setminus(B\cap\tilde{B})) = 0 .$$

For two almost equal, measurable, and bounded $f,\tilde{f}:\Omega\to X$ with separable image, there is an $\tilde{\Omega}\in\Sigma$, $\tilde{\Omega}\equiv\Omega$, for which $f(\omega)=\tilde{f}(\omega)$ whenever $\omega\in\tilde{\Omega}$. Thus,

$$\begin{aligned}\operatorname{ess\,osc}(f,A) &= \inf_{\tilde{A}\equiv A} D(f(A)) = \inf_{\tilde{A}\equiv A,\tilde{A}\subset\tilde{\Omega}\cap A} D(f(A)) \\ &= \inf_{\tilde{A}\equiv A,\tilde{A}\subset\tilde{\Omega}\cap A} D(\tilde{f}(A)) = \operatorname{ess\,osc}(\tilde{f},A)\end{aligned}$$

for each $A\in\Sigma$. Similary

$$\operatorname{ess\,dist}(f,A,B) = \operatorname{ess\,dist}(\tilde{f},A,B)\,,\ \text{whenever } A,B\in\Sigma\,.$$

Thus, $\operatorname{ess\,dist}(\cdot,A)$ and $\operatorname{ess\,osc}(\cdot,A,B)$ are well defined on $L^c_\infty(\mu,X)$.
In the case that we consider $L^c_\infty(\Sigma_K,X)$ (which is by the observation in (0.3)(b) representable as an $L^c_\infty(\mu,X)$-space) we write osc instead of ess osc and dist instead of ess dist.

c) Let $A\in\Sigma$, $\pi\in\Pi(A)$, and $\varepsilon>0$. An $f\in L^c_\infty(\mu,X)$ is said to be *(π,ε)-compatible on A* if, for each $B\in\pi$,

$$\operatorname{ess\,osc}(f,B)\le\varepsilon \quad .$$

$F\subset L^c_\infty(\mu,X)$ is called *(π,ε)-compatible on A* if every $f\in F$ has this property. A sequence $(f_n\colon n\in\mathbb{N})\subset L^c_\infty(\mu,X)$ is called *totally ε-incompatible on A* if, for every $\pi\in\Pi(A)$, no subsequence is (ε,π)-compatible, i.e. if

$$\forall\pi\in\Pi(A),N\in\mathcal{P}_\infty(\mathbb{N}),\exists B\in\pi,n\in N \text{ such that } \operatorname{ess\,osc}(f_n,B)>\varepsilon.$$

Now we are in the position to state the main result of this section.

4.3.2 Theorem: *Let $(f_n: n \in \mathbb{N}) \subset L^c_\infty(\mu, X)$.*
Then at least one of the following two cases happens:

Case 1: *There exists an increasing $(n_k: k \in \mathbb{N}) \subset \mathbb{N}$ such that $(f_{n_k}: k \in \mathbb{N})$ has the following property:*
For all $\varepsilon > 0$ there is a $\pi^\varepsilon \in \Pi$ for which (f_{n_k}) is ~~$(\pi^\varepsilon, \varepsilon)$-compat~~ible.

Case 2: *There exist an $\varepsilon > 0$, an increasing $(n_k: k \in \mathbb{N}) \subset \mathbb{N}$, and a family $(A(k,j): k \in \mathbb{N}_0, j \in \{1, \ldots, 2^k\}) \subset \Sigma$, such that:*

a) For each $k \in \mathbb{N}_0$ and $i \in \{1, \ldots, 2^k\}$, $\mu(A(k,i)) > 0$,
$A(k+1, 2i-1) \cup A(k+1, 2i) \subset A(k,i)$ and
$A(k+1, 2i-1) \cap A(k+1, 2i) = \emptyset$ almost everywhere

b) $\operatorname{ess\,osc}(f_{n_k}, A(k,i)) \le \varepsilon/4$ and $\operatorname{ess\,dist}(f_{n_k}, A(k, 2j-1), A(k, 2j)) \ge \varepsilon$
whenever $k \in \mathbb{N}, i \in \{1, \ldots, 2^k\}$ and $j \in \{1, \ldots, 2^{k-1}\}$.

The following Lemma (4.3.3) collects some frequently used results, while Lemma (4.3.4) represents the central step of the proof of Theorem (4.3.2).

4.3.3 Lemma:

a) Let $f \in L^c_\infty(\mu, X)$.

i) If $A_1, A_2, B_1, B_2 \in \Sigma$, $A_1 \subset A_2$, and $B_1 \subset B_2$, then

$$\operatorname{ess\,osc}(f, A_1) \le \operatorname{ess\,osc}(f, A_2) \text{ and } \operatorname{ess\,dist}(f, A_1, B_1) \ge \operatorname{ess\,dist}(f, A_2, B_2).$$

ii) If $A_n, B_n \in \Sigma$ for $n \in \mathbb{N}$ then

$$\operatorname{ess\,dist}\Big(f, \bigcup_{n\in\mathbb{N}} A_n, \bigcup_{n\in\mathbb{N}} B_n\Big) = \inf_{n\in\mathbb{N}, m\in\mathbb{N}} \operatorname{ess\,dist}(f, A_n, B_m).$$

iii) If $A, B, C \in \Sigma$ and $\mu(C) > 0$, then

$$\operatorname{ess\,dist}(f, A, B) \le \operatorname{ess\,dist}(f, A, C) + \operatorname{ess\,dist}(f, B, C) + \operatorname{ess\,osc}(f, A) + \operatorname{ess\,osc}(f, B) + \operatorname{ess\,osc}(f, C).$$

iv) If $A_n \in \Sigma$ for $n \in \mathbb{N}$ then

$$\operatorname{ess\,osc}\Big(f, \bigcup_{n\in\mathbb{N}} A_n\Big) \le 2 \sup_{n\in\mathbb{N}} \operatorname{ess\,osc}(f, A_n) + \sup_{n\in\mathbb{N}, m\in\mathbb{N}} \operatorname{ess\,dist}(f, A_n, A_m).$$

b) Let A, $\tilde{A} \in \Sigma$, F, $\tilde{F} \subset L^c_\infty(\mu, X)$, and $\varepsilon > 0$.

i) If $|F| < \infty$, then there exists a $\pi \in \Pi(A)$ such that F is (π, ε)-compatible on A.

ii) If, for $\pi \in \Pi(A)$, F is (π, ε)-compatible on A and if $\tilde{A} \subset A$, then F is also $(\pi|_{\tilde{A}}, \varepsilon)$-compatible on $\tilde{A}$.

iii) If F and $\tilde{F}$ are (π, ε)-, respectively $(\tilde{\pi}, \varepsilon)$-compatible on A, then $F \cup \tilde{F}$ is $(\pi \vee \tilde{\pi}, \varepsilon)$-compatible on A, where $\pi \vee \tilde{\pi} := \{A \cap \tilde{A} \,|\, A \in \pi \text{ and } \tilde{A} \in \tilde{\pi}\}$.

iv) If $\pi \in \Pi(A)$ and $\pi^B \in \Pi(B)$ for each $B \in \Pi$ and if F is (π^B, ε)-compatible on B for each $B \in \pi$, then F is $(\bigcup_{B \in \pi} \pi^B, \varepsilon)$-compatible on A.

c) Let $A_n \in \Sigma$ for $n \in \mathbb{N}$, and let $\varepsilon > 0$. If $(f_n : n \in \mathbb{N}) \subset L^c_\infty(\mu, X)$ is totally ε-incompatible on $A := \bigcup_{n \in \mathbb{N}} A_n$, then there exists an $n \in \mathbb{N}$ and a subsequence of $(f_n : n \in \mathbb{N})$ which is totally ε-incompatible on A_n.

Proof of (4.3.3) :

Proof of (a): If we replace f by a fixed representative and $\operatorname{ess\,osc}(f, ., .)$ and $\operatorname{ess\,dist}(f, .)$ by $D(f(.))$ and $d(f(.), f(.))$ the inequalities to be shown are obvious. Next, we remark that for $f = \sum_{i \in \mathbb{N}} \chi_{B_i} x_i$, where $B_i \in \Sigma$ with $\mu(B_i) > 0$ for $i \in \mathbb{N}$ and $\bigcup_{i \in \mathbb{N}} B_i = \Omega$, and where $(x_i) \subset X$ is bounded, we have

$$\operatorname{ess\,osc}(f, A) = D(f(\tilde{A})) \quad \text{and} \quad \operatorname{ess\,dist}(f, A, B) = d(f(\tilde{A}), f(\tilde{B})),$$

where $\tilde{A} := \bigcup\{B_i \,|\, \mu(A \cap B_i) > 0\}$ and $\tilde{B} := \bigcup\{B_i \,|\, \mu(B \cap B_i) > 0\}$.
Finally, we observe that $\operatorname{ess\,osc}(\cdot, A)$ and $\operatorname{ess\,dist}(\cdot, A, B)$ are continuous on $L^c_\infty(\mu, X)$. In fact, if $\varepsilon > 0$ and $f, g \in L^c_\infty(\mu, X)$ with $\| f - g \| \le \varepsilon$, then we find an $\tilde{\Omega} \in \Sigma$ with $\mu(\Omega \setminus \tilde{\Omega}) = 0$ and $\| f(\omega) - g(\omega) \| \le \varepsilon$, for $\omega \in \tilde{\Omega}$. Thus,

$$\operatorname{ess\,osc}(f, A) = \inf_{\tilde{A} \equiv A \cap \tilde{\Omega}} d(f(\tilde{A})) \le \varepsilon + \inf_{\tilde{A} \equiv A \cap \tilde{\Omega}} d(g(\tilde{A})) = \varepsilon + \operatorname{ess\,osc}(g, A),$$

and by symmetry, $\operatorname{ess\,osc}(g, A) \le \varepsilon + \operatorname{ess\,osc}(f, A)$.
For ess dist, we show the continuity in a similar way.
From these three observations we deduce (a).

Proof of (b) : obvious

Proof of (c) : Let $(f_n : n \in \mathbb{N}) \subset L^c_\infty(\mu, X)$ and $(A_n : n \in \mathbb{N}) \in \Sigma$.
Suppose that the conclusion is false, i.e. that

$$(1) \qquad \forall k \in \mathbb{N}, N \in \mathcal{P}_\infty(\mathbb{N}) \, \exists M \in \mathcal{P}_\infty(N), \pi \in \Pi(A_k) \text{ such that } (f_m : m \in M) \text{ is } (\pi, \varepsilon)\text{-compatible on } A_k,$$

we show that the assumption of (c) is not true.

For this we may assume that the A_n's are pairwise disjoint almost everywhere; otherwise we pass to $\tilde{A}_n := A_n \setminus \bigcup_{i<n} A_i$. Using (1), we can define by induction for each $k \in \mathbb{N}$, an infinite $M_k \subset \mathbb{N}$ and $\tilde{\pi}_k \in \Pi(A_k)$ such that $M_k \subset M_{k-1}$ (with $M_0 := \mathbb{N}$) and such that $(f_m : m \in M_k)$ is $(\tilde{\pi}_k, \varepsilon)$-compatible on A_k (we apply in the k-th inductionstep (1) to $N := M_{k-1}$ and k).
Then we choose an increasing sequence $(m_k) \subset \mathbb{N}$ such that $m_k \in M_k$ for $k \in \mathbb{N}$ and we choose, for each $k \in \mathbb{N}$, a refinement $\pi_k \in \Pi(A_k)$ of $\tilde{\pi}_k$ such that $\{f_{m_1}, \dots f_{m_k}\}$ is (ε, π_k)-compatible on A_k. This implies that $(f_{m_k} : k \in \mathbb{N})$ is (π_k, ε)-compatible for each $k \in \mathbb{N}$ and, by (b)(iv), $(f_{m_k} : k \in \mathbb{N})$ is $(\bigcup_{k\in\mathbb{N}} \pi_k, \varepsilon)$-compatible on $A = \bigcup_{k\in\mathbb{N}} A_k$, which finishes the proof.

◇

4.3.4 Lemma: *Let $A \in \Sigma$ and $\varepsilon > 0$, and let $(f_n : n \in \mathbb{N}) \subset L^c_\infty(\mu, X)$ be totally ε-incompatible on A.*

Then there exist $n \in \mathbb{N}$, $M \in \mathcal{P}_\infty(\mathbb{N})$, and $A_1, A_2 \in \Sigma \cap A$ such that

a) *$\operatorname{ess\,dist}(f_n, A_1, A_2) > \varepsilon/4$ (in particular $A_1 \cap A_2 = \emptyset$ almost everywhere),*
b) *$(f_m : m \in M)$ is totally ε-incompatible on A_1 as well as on A_2.*

Proof of (4.3.4) :
Let $(f_n : n \in \mathbb{N}) \subset L^c_\infty(\mu, X)$ be totally ε-incompatible on A.
We suppose that the assertion of the lemma is not true, i.e. that

(1) $\forall n \in \mathbb{N}, M \in \mathcal{P}_\infty(\mathbb{N})$ and $A_1, A_2 \in \Sigma \cap A$ with $\operatorname{ess\,dist}(f_n, A_1, A_2) \geq \varepsilon/4$:

$$[(f_m : m \in M) \text{ is totally } \varepsilon\text{-incompatible on } A_1 \quad \Rightarrow$$
$$\exists \tilde{M} \in \mathcal{P}_\infty(M), \pi \in \Pi(A_2) : \quad (f_m : m \in \tilde{M}) \text{ is } (\pi, \varepsilon)\text{-compatible on } A_2]$$

Under this assumption, we first choose inductively for each $k \in \mathbb{N}_0$ an $m_k \in \mathbb{N}_0$, an $M_k \in \mathcal{P}_\infty(\mathbb{N})$ and $B_k \in \Sigma$ such that

(2)(k) $m_k \in M_{k-1}$, $m_k > m_{k-1}$, and $M_k \subset M_{k-1}$, if $k > 0$,
(3)(k) $B_k \subset B_{k-1}$ if $k > 0$,
(4)(k) $\operatorname{ess\,osc}(f_{m_k}, B_k) \leq \varepsilon$ if $k > 0$,
(5)(k) $(f_m : m \in M_k)$ is totally ε-incompatible on B_k, and
(6)(k) there exists a $\pi \in \Pi(A \setminus B_k)$, such that $(f_m : m \in M_k)$ is (π, ε)-compatible on $A \setminus B_k$.

For $k = 0$, let $m_0 := 0$, $M_0 := \mathbb{N}$, and $B_0 := A$. Then (2)(0),(3)(0), and (4)(0) are empty conditions, while (5)(0) is just the assumption and (6)(0) is trivial.

If we assume that for a $k > 0$, $(m_\ell : 0 \le \ell < k)$, $(M_\ell : 0 \le \ell < k)$, and $(B_\ell : 0 \le \ell < k)$ have already been chosen, we set

$$\text{(7)} \qquad m_k := \min\{m \in M_{k-1} \,|\, m > m_{k-1} \text{ and } \operatorname{ess\,osc}(f_m, B_{m_{k-1}}) > \varepsilon\},$$

remarking that such an m_k exists because of (5)($k-1$).

Since $f_{m_k} \in L^c_\infty(\mu, X)$, we find a $\tilde\pi \in \Pi(B_{k-1})$ for which f_{m_k} is $(\tilde\pi, \varepsilon/16)$-compatible on B_{k-1}. From (4.3.3)(c) and (5)($k-1$) we deduce that there exists a $\tilde B_k \in \tilde\pi$ and an $\tilde M_k \in \mathcal{P}_\infty(M_{k-1})$, such that $(f_m : m \in \tilde M_k)$ is totally ε-incompatible on $\tilde B_k$.

Setting

$$B_k := \bigcup\{B \,|\, B \in \tilde\pi'\}, \text{ where } \tilde\pi' := \{\tilde B \in \tilde\pi \,|\, \operatorname{ess\,dist}(f, \tilde B_k, \tilde B) \le \varepsilon/4\},$$

we deduce (3)(k) and the fact that $(f_m : m \in \tilde M_k)$ is ε-incompatible on B_k (since $\tilde B_k \subset B_k$).

From (4.3.3.)(a)(iii) and (iv) we have

$$\begin{aligned}\text{(8)} \qquad \operatorname{ess\,osc}(f_{m_k}, B_k) &\le \sup_{B, \tilde B \in \tilde\pi'} \operatorname{ess\,dist}(f_{m_k}, B, \tilde B) + 2 \sup_{B \in \tilde\pi'} \operatorname{ess\,osc}(f_{m_k}, B) \\ &\le 2 \sup_{B \in \tilde\pi'} \operatorname{ess\,dist}(f_{m_k}, B, \tilde B_k) + 5 \sup_{B \in \tilde\pi'} \operatorname{ess\,osc}(f_{m_k}, B) \\ &\le \varepsilon/2 + 5\varepsilon/16 < \varepsilon \quad ,\end{aligned}$$

which implies (4)(k).

Defining $A_2 := \bigcup_{B \in \tilde\pi \setminus \tilde\pi'} B = B_{k-1} \setminus B_k$ and applying (4.3.3)(a)(ii), we have

$$\operatorname{ess\,dist}(f_{m_k}, A_2, \tilde B_k) = \inf_{B \in \tilde\pi \setminus \tilde\pi'} \operatorname{ess\,dist}(f_{m_k}, B, \tilde B_k) \ge \varepsilon/4 .$$

Now we are in a position to apply (1) to $M := \tilde M_k$, $n := m_k$, $A_1 := \tilde B_k$, and A_2 as defined above; we deduce that we can choose $M_k \in \mathcal{P}_\infty(\tilde M_k)$ and $\pi \in \Pi(A_2)$ such that $(f_m : m \in M_k)$ is (π, ε)-compatible on A_2.

Since $A \setminus B_k = (A \setminus B_{k-1}) \cup (B_{k-1} \setminus B_k) = (A \setminus B_{k-1}) \cup A_2$, we deduce (6)($k$) from (6)($k-1$) and the choice of M_k. Property (2)(k) follows from (7) and the fact that M_k was chosen to be a subset of M_{k-1}, while (5)(k) follows from the fact that $\tilde B_k \subset B_k$ and that subsequences of totally incompatible sequences inherit this property. This finishes the induction step.

Now we note that, by (3)(k) and the choice of B_0,

$$A = B_0 = \bigcup_{k\in\mathbb{N}} (B_{k-1} \setminus B_k) \cup \bigcap_{k\in\mathbb{N}} B_k .$$

From (6)(k) and (2)(k) we deduce that, for each $k \in \mathbb{N}$, there exists a $\pi_k \in \Pi(B_{k-1} \setminus B_k)$ such that $(f_{m_j} : j \in \mathbb{N})$ is (π_k, ε)-compatible on $B_{k-1} \setminus B_k$. (3) (k) and (4) (k) imply that $(f_{m_j} : j \in \mathbb{N})$ is $(\{\bigcap_{j\in\mathbb{N}} B_j\}, \varepsilon)$-compatible on $B_\infty := \bigcap_{j\in\mathbb{N}} B_j$. From (4.3.3) (b) (iv) we deduce that $(f_{m_j} : j \in \mathbb{N})$ is $(\bigcup_{k\in\mathbb{N}} \pi_k \cup \{B_\infty\}, \varepsilon)$-compatible on A, which contradicts the assumption of this lemma.

◇

Proof of Theorem (4.3.2)

We start the proof by showing that, for a sequence $(f_n : n \in \mathbb{N}) \subset L^c_\infty(\mu, X)$ which does not satisfy case 1, there exists an $\tilde{\varepsilon} > 0$ and a subsequence which is totally $\tilde{\varepsilon}$- incompatible.

We prove this by showing that if $(f_n : n \in \mathbb{N})$ satisfies the property

(1) $\forall N \in \mathcal{P}_\infty(\mathbb{N}), \varepsilon > 0\, \exists M \in \mathcal{P}_\infty(N), \pi \in \Pi : (f_m : m \in M)$ is (π, ε) – compatible,

then case 1 is satisfied.

For this, we choose inductively for each $k \in \mathbb{N}$ an $N_k \in \mathcal{P}_\infty(\mathbb{N})$ and a $\pi_k \in \Pi$ such that $N_k \subset N_{k-1}$ (where $N_0 := \mathbb{N}$) and such that for each $B \in \pi_k$ and each $m \in N_k$, $\operatorname{ess\,osc}(f_m, B) \le \frac{1}{k}$ (this can be done by applying, in every induction step, (1) to $N := N_{k-1}$ and $\varepsilon := \frac{1}{k}$).

Then we choose an increasing sequence $(n_k) \subset \mathbb{N}$ with $n_k \in N_k$ for $k \in \mathbb{N}$. To show that $(f_{n_k} : k \in \mathbb{N})$ satisfies the assertion of case 1, let $\varepsilon > 0$ be arbitrary and choose $k_0 \in \mathbb{N}$ with $\varepsilon > \frac{1}{k_0}$. Since $(f_{n_\ell} : \ell \ge k_0)$ is (π_{k_0}, ε)-compatible, we find a $\pi \in \Pi$ which is finer than π_{k_0} and for which $(f_{n_k} : k \in \mathbb{N})$ is (π, ε)-compatible.

To prove (4.3.2), we assume that $(f_n : n \in \mathbb{N})$ does not satisfy case 1 and we show that it satisfies case 2.

By the above remark, we can assume that there exists an $\tilde{\varepsilon} > 0$ such that $(f_n : n \in \mathbb{N})$ is totally $\tilde{\varepsilon}$-incompatible.

By induction, we choose for each $k \in \mathbb{N}_0$ an $n_k \in \mathbb{N}_0$, an $N_k \in \mathcal{P}_\infty(\mathbb{N})$, and a family $(A(n,j) : 1 \le j \le 2^n) \subset \Sigma$ such that

(1)(k) $n_k \in N_{k-1}$, $n_k > n_{k-1}$, and $N_k \subset N_{k-1}$ if $k > 0$,

(2)(k) $A(k, 2i-1) \cup A(k, 2i) \subset A(k, i)$ and

$A(k,2i-1) \cap A(k,2i) = \emptyset$ almost everywhere if $k > 0$ and $i \in \{1,\dots,2^{k-1}\}$,

(3)(k) $\operatorname{ess\,osc}(f_{n_k}, A(k,i)) \le \tilde{\varepsilon}/16$ and $\operatorname{ess\,dist}(f_{n_k}, A(k,2j-1), A(k,2j)) \ge \tilde{\varepsilon}/4$ if $k > 0$, $i \in \{1,\dots,2^k\}$ and $j \in \{1,\dots,2^{n-1}\}$, and

(4)(k) for each $j \in \{1,\dots,2^k\}$, $(f_m : m \in N_k)$ is totally $\tilde{\varepsilon}$-incompatible on $A(k,j)$.

If $k = 0$, we put $A(0,1) := \Omega$, $n_0 := 0$ and $N_0 := \mathbb{N}$. Then (1)(0),(2)(0), and (3)(0) are empty, while (4)(0) is just the assumption.

We suppose that for a $k > 0$, n_{k-1}, N_{k-1} and $(A(k-1,j) : j \le 2^{k-1})$ has been chosen.

First we want to verify the following:

(6) Let $M \in \mathcal{P}_\infty(N_{k-1})$ and $j \in \{1,\dots,2^{k-1}\}$ be arbitrary.
Then there exists an increasing sequence $(\tilde{n}_\ell : \ell \in \mathbb{N}) \subset \mathbb{N}$ and for each $\ell \in \mathbb{N}$ an $\tilde{M}_\ell \in \mathcal{P}_\infty(M)$ and $\tilde{A}(1,\ell), \tilde{A}(2,\ell) \in \Sigma \cap A(k-1,j)$ such that for each $\ell \in \mathbb{N}$:
a) $\tilde{n}_\ell \in \tilde{M}_{\ell-1}$ $(\tilde{M}_0 := M)$, $\tilde{M}(\ell) \subset \tilde{M}(\ell-1)$,
b) the conditions (a) and (b) of Lemma (4.3.4) hold for

$$n = \tilde{n}_\ell, M := \tilde{M}_\ell \text{ and } A_1 := \tilde{A}(1,\ell), A_2 := \tilde{A}(2,\ell).$$

We prove this by applying (4.3.4) successively for each $\ell \in \mathbb{N}$ to the sequence $(f_m : m \in \tilde{M}_{\ell-1}, m > \tilde{n}_{\ell-1})$ and $A := A(k-1,j)$.

Applying (6) successively to each $j \le 2^{k-1}$, we find sequences $(\tilde{n}(j,\ell) : \ell \in \mathbb{N}) \subset \mathbb{N}$, $(\tilde{A}(j,1,\ell) : \ell \in \mathbb{N})$, $(\tilde{A}(j,2,\ell) : \ell \in \mathbb{N}) \subset \Sigma \cap A(k-1,j)$, and decreasing $(\tilde{M}_\ell^{(j)} : \ell \in \mathbb{N}_0) \in \mathbb{N}$, such that

(7) the condition (6) holds for each $j \in \{1,\dots,2^{k-1}\}$
and, moreover,

$$\begin{aligned} N^{k-1} &= M_0^{(1)} \supset (\tilde{n}(1,\ell) : \ell \in \mathbb{N}) = M_0^{(2)} \supset (\tilde{n}(2,\ell) : \ell \in \mathbb{N}) \\ &= M_0^{(3)} \dots \supset (\tilde{n}(2^{k-1}-1,\ell) : \ell \in \mathbb{N}) = M_0^{(2^{k-1})} \supset (\tilde{n}(2^{k-1},\ell) : \ell \in \mathbb{N}). \end{aligned}$$

Now we are in a position to choose $n_k \in \{\tilde{n}(2^{k-1},\ell) \mid \ell \in \mathbb{N}\}$ with $n_k > n_{k-1}$ and we set $\tilde{N}_k := \{\tilde{n}(2^{k-1},\ell) \mid \ell \in \mathbb{N}\}$. By (7), we find for each $j \in \{1,\dots,2^{k-1}\}$ an $\ell_j \in \mathbb{N}$ such that $n_k = \tilde{n}(j,\ell_j)$.

Then we choose, for each $j \in \{1,\dots,2^{k-1}\}$, a $\pi_j^1 \in \Pi(\tilde{A}(j,1,\ell_j)$ and a $\pi_j^2 \in \Pi(\tilde{A}(j,2,\ell_j)$ such that f_{n_k} is $(\pi_j^i, \tilde{\varepsilon})$-compatible on $\tilde{A}(j,i,\ell_j)$ for $i = 1,2$. Applying (4.3.2)(c) 2^k times, we deduce from (7) that we can choose $A(k,2j-1) \in \pi_j^1$ and $A(k,2j) \in \pi_j^2$ respectively and $N_k \in \mathcal{P}_\infty(\tilde{N}_k)$, such that $(f_m : m \in N_k)$ is totally $\tilde{\varepsilon}$-incompatible on each $A(k,j)$ with $j \in \{1,\dots,2^k\}$. Thus, we deduce (4)(k) and the first part of (3)(k). (1)(k) and (2)(k) follows from this choice also, while the second part of (3)(k) follows from (7). Thus, we completed the proof of the induction step. The assertion of the theorem follows from (1)(k),(2)(k), (3)(k) and (4)(k) if we take $\varepsilon := \tilde{\varepsilon}/4$ and remark that from (4)(k) it follows that $\mu(A(k,i)) > 0$ for $k \in \mathbb{N}$ and $i \in \{1,\ldots,2^k\}$.

◇

4.3.5 Proposition: *Let $A \subset L^c_\infty(\mu, X)$. Then the following are equivalent:*

a) *Every sequence in A fulfills the conditions of case 1 in Theorem (4.3.2).*

b) *Every sequence in A contains a subsequence $(f_n : n \in \mathbb{N})$ such that for all $\varepsilon > 0$ there exists $\pi^\varepsilon \in \Pi$ and $(x(B,n) : B \in \pi^\varepsilon, n \in \mathbb{N}) \subset X$ such that*

$$\| f_n - \sum_{B\in\pi^\varepsilon} \chi_B x(B,n) \| \leq \varepsilon \quad \text{for all } n \in \mathbb{N}\,.$$

Proof of (4.3.5) : : obvious

◇

4.3.6 Proposition: *Let $A \subset L^c_\infty(\mu, X)$ be bounded and such that every sequence $(f_n : n \in \mathbb{N}) \subset A$ satisfies case 1 of (4.3.2).*

Then for each $x' \in X'$, each sequence in $A(x') := \{\langle f(\cdot), x'\rangle \,|\, f \in A\}$ has a subsequence that converges μ-almost everywhere.

Proof of (4.3.6) :

Let $(f_n : n \in \mathbb{N}) \subset A$ and $x' \in X'$. By assumption and by (4.3.5), we can assume that there exists an increasing sequence $(\pi_k : k \in \mathbb{N}) \subset \Pi$, $\pi^k = (B(k,m) : m \in \mathbb{N})$, and a bounded family $(x(k,m,n) : n,m,k \in \mathbb{N}) \subset X$ such that

$$\text{(1)} \qquad \| f_n - \sum_{m\in\mathbb{N}} \chi_{B(k,m)} x(k,m,n) \| \leq \frac{1}{k} \quad \text{for each } k,n \in \mathbb{N}$$

Since A is bounded, we find by the diagonal method an $N \in \mathcal{P}_\infty(\mathbb{N})$, such that

$$\text{(2)} \qquad r(m,k) := \lim_{n\in N, n\to\infty} \langle x', x(k,m,n)\rangle$$

exists for each $m,k \in \mathbb{N}$.

By the definition of countable Σ-partitions and by (1), we can assume that for each $\omega \in \Omega$ and each $k \in \mathbb{N}$ there exists a unique $m(\omega,k) \in \mathbb{N}$ with $\omega \in B(k, m(\omega,k))$ and that $\| f_n(\omega) - x(k, m(\omega,k), n) \| \leq \frac{1}{k}$ for each $n \in N$; otherwise we pass to a suitable $\tilde{\Omega} \in \Sigma$ with $\mu(\Omega \setminus \tilde{\Omega}) = 0$.

Therefore, we deduce for each $\omega \in \Omega$, $n, n' \in N$, and $k \in \mathbb{N}$ that

$$\begin{aligned}
&|\langle x', f_n(\omega) - f_{n'}(\omega)\rangle| \\
&\quad \leq \| x' \| \| f_n(\omega) - x(k, m(\omega, k), n) \| + \| x' \| \| f_{n'}(\omega) - x(k, m(\omega, k), n') \| \\
&\qquad + |\langle x', x(k, m(\omega, k), n) - x(k, m(\omega, k), n')\rangle| \\
&\quad \leq \frac{2}{k} + |\langle x', x(k, m(\omega, k), n) - x(k, m(\omega, k), n')\rangle|,
\end{aligned}$$

which implies the assertion together with (2).

◇

Finally we want to collect the results as we will need them in the next section, and formulate them for the space $L^c_\infty(\Sigma, X)$.

4.3.7 Corollary: *Let (Ω, Σ) be a measurable space and $A \subset L^c_\infty(\Sigma, X)$.*

a) At least one of the following cases happens:

case 1: For each $(f_n : n \in \mathbb{N}) \subset A$, there is a subsequence $(\tilde{f}_n : n \in \mathbb{N})$ such that: For each $\varepsilon > 0$ there is a countable Σ-partition $\pi^{(\varepsilon)}$ of Ω with

$$\operatorname{osc}(\tilde{f}_n, B) \leq \varepsilon \quad \text{for } n \in \mathbb{N} \text{ and } B \in \pi^{(\varepsilon)} .$$

case 2: There is a sequence $(f_n : n \in \mathbb{N}) \subset A$, an $\varepsilon > 0$, and a tree of non-empty sets $(A(n,j) : n \in \mathbb{N}_0, j \in \{1,..2^n\}) \subset \Sigma$, such that

$$\operatorname{osc}(f_n, A(n,j)) \leq \varepsilon/4 \text{ and } \operatorname{dis}(f_n, A(n, 2i-1), A(n, 2i)) \geq \varepsilon$$

whenever $n \in \mathbb{N}$, $j \in \{1,..2^n\}$ and $i \in \{1,..2^{n-1}\}$.

b) If A is bounded and satisfies the above case 1, then for each $(f_n : n \in \mathbb{N}) \subset A$ and each $x' \in X'$, $(\langle x', f_n\rangle : n \in \mathbb{N})$ has a pointwise converging subsequence.

4.4 Limited sets in $X\check{\otimes}Y$ and $K_{w^*}(X',Y)$

In Proposition (4.4.1) we recall the known result [52, p.22, Propositon 2], which characterizes compactness in $X\check{\otimes}Y$ and $K_{w^*}(X',Y)$ by compactness in X and Y. We will observe that this result may be carried over to limitedness only in special cases (Proposition (4.4.2) and Examples (4.5.5) respectively). In general, it only leads to a necessary condition for a set to be limited in $X\check{\otimes}Y$ and $K_{w^*}(X',Y)$ respectively. We deduce from this observation that $X\check{\otimes}Y$ and $K_{w^*}(X',Y)$ inherit the Gelfand-Phillips property of X and Y (Corollary (4.4.3)).

In addition, we will formulate two other necessary conditions for a set to be limited in $X\check{\otimes}Y$ and $K_{w^*}(X',Y)$. The first one follows from the observations in section (4.3) and states that the limited sequences in $X\check{\otimes}Y$ and $K_{w^*}(X',Y)$, viewed in $L^c_\infty(\Sigma_X,Y)$ and $L^c_\infty(\Sigma_Y,X)$ respectively (where Σ_X and Σ_Y are the the σ-algebra of the weak* Borel sets of $B_1(X')$ and $B_1(Y')$ respectively), must satisfy case 1 of Theorem (4.3.2). Secondly, we will observe that limited sets of $X\check{\otimes}Y$ must be "almost" bounded in the projective tensor norm (compare Proposition (1.1.10)).

In the next section we will show that these two conditions are also sufficient for limitedness if we suppose that X and Y are Grothendieck $C(K)$-spaces.

4.4.1 Proposition: *For $A \subset K_{w^*}(X',Y)$, the following properties (a) and (b) are equivalent.*

a) *A is relatively compact in $K_{w^*}(X',Y)$.*
b) *$A(B_1(X')) := \{T(x') \mid T \in A, x' \in B_1(X')\}$ and $A(B_1(Y')) := \{T'(y') \mid T \in A, y' \in B_1(Y')\}$ are relatively compact subsets of Y and X respectively.*

Since $X\check{\otimes}Y$ is a closed subspace of $K_{w^*}(X',Y)$ (compare (0.3)(e)), we deduce that (a) and (b) are also equivalent for $A \subset X\check{\otimes}Y$.

Proof of (4.4.1) :(We could prove (4.4.1), by showing that we are in a special situation of [52, p.22, Proposition 2], where the locally convex case is described; but it is just as fast to prove it in a direct way.)
(a) $\Rightarrow$(b): Let $(T_n(x'_n) : n \in \mathbb{N})$ be an arbitrary sequence in $A(B_1(X'))$ (i.e. $T_n \in A$ and $x'_n \in B_1(X')$). Since A is assumed to be relatively compact, there is an $N \in \mathcal{P}_\infty(\mathbb{N})$ for which $(T_n : n \in N)$ converges to a $T_0 \in K_{w^*}(X',Y)$. Since T_0 is a compact operator, there is an $M \in \mathcal{P}_\infty(N)$ for which $(T_0(x'_n) : n \in M)$ converges

to a $y_0 \in Y$. This implies that, for $n \in M$,

$$\begin{aligned} \| T_n(x'_n) - y_0 \| &\leq \| T_n(x'_n) - T_0(x'_n) \| + \| T_0(x'_n) - y_0 \| \\ &\leq \| T_n - T_0 \| + \| T_0(x'_n) - y_0 \| \underset{n \in M, n \to \infty}{\longrightarrow} 0. \end{aligned}$$

We deduce that $A(\mathrm{B}_1(X'))$ is relatively compact and it follows that the same is true for $A(\mathrm{B}_1(Y'))$ noting that

$$I : \mathrm{K}_{w^*}(X', Y) \to \mathrm{K}_{w^*}(Y', X), \quad T \mapsto T',$$

is well-defined and an isometric isomorphism, and that $(I(A))(\mathrm{B}_1(Y')) = A(\mathrm{B}_1(Y'))$.

(b) $\Rightarrow$(a): We view for this X, Y, and $\mathrm{K}_{w^*}(X', Y)$ as subspaces of $\mathrm{C}(\mathrm{B}_1(X'))$, $\mathrm{C}(\mathrm{B}_1(Y'))$,and $\mathrm{C}(\mathrm{B}_1(X') \times \mathrm{B}_1(Y'))$ respectively, where $\mathrm{B}_1(X')$, $\mathrm{B}_1(Y')$ and $\mathrm{B}_1(X') \times \mathrm{B}_1(Y')$ are endowed with $\sigma(X', X)$, $\sigma(Y', Y)$, and $\sigma(X', X) \times \sigma(Y', Y)$ respectively.
From

$$\sup_{T \in A} \| T \| = \sup_{x' \in \mathrm{B}_1(X'), T \in A} \| T(x') \| = \sup_{y \in A(\mathrm{B}_1(X'))} \| y \|,$$

we deduce that A is bounded and so, by the theorem of Arzelà-Ascoli, it is sufficient, to prove the equi-continuity of A in $\mathrm{C}(\mathrm{B}_1(X') \times \mathrm{B}_1(Y'))$. For this let $(x', y') \in \mathrm{B}_1(X') \times \mathrm{B}_1(Y')$ and $\varepsilon > 0$.
By (b) we find open sets $U \subset \mathrm{B}_1(X')$ and $V \subset \mathrm{B}_1(Y')$, with $x' \in U$, $y' \in V$ and

$$|f(\hat{x}') - f(\tilde{x}')| \leq \varepsilon/2 \quad \text{and} \quad |g(\hat{y}') - g(\tilde{y}')| \leq \varepsilon/2$$

whenever $\hat{x}', \tilde{x}' \in U$, $\hat{y}', \tilde{y}' \in V$, $f \in A(\mathrm{B}_1(Y'))$, and $g \in A(\mathrm{B}_1(X'))$.
This implies that, for each $h \in A$ and $(\hat{x}', \hat{y}'), (\tilde{x}', \tilde{y}') \in U \times V$,

$$|h(\hat{x}', \hat{y}') - h(\tilde{x}', \tilde{y}')| \leq |h(\hat{x}', \hat{y}') - h(\hat{x}', \tilde{y}')| + |h(\hat{x}', \tilde{y}') - h(\tilde{x}', \tilde{y}')| \leq \varepsilon,$$

which implies the equi-continuity of A and finishes the proof.

◇

4.4.2 Proposition: *For $A \subset K_{w^*}(X', Y)$ or $A \subset X\check{\otimes}Y$ it follows that*

$$(a) \Rightarrow (b) \Rightarrow (c),$$

where

a) *A is limited in $K_{w^*}(X', Y)$, respectively in $X\check{\otimes}Y$,*
b) *$A(B_1(X'))$ and $A(B_1(Y'))$ are limited in Y and X respectively, and*
c) *$A(B_1(X'))$ is limited in Y and $A(\{y'\}) = \{T'(y') | T \in A\}$ is limited in X for each $y' \in Y$.*

If Y is a Gelfand-Phillips space and has the approximation property, then "(c) $\Rightarrow$(a)" is also true.

Proof of (4.4.2) :

Proof of (a) $\Rightarrow$(b): Since $X\check{\otimes}Y$ is a closed subspace of $K_{w^*}(X', Y)$, it is enough to show the assertion for $K_{w^*}(X', Y)$.

For an arbitrary sequence $(T_n(x'_n) : n \in \mathbb{N})$ in $A(B_1(X'))$ (i.e. $T_n \in A$ and $x'_n \in B_1(X')$ for $n \in \mathbb{N}$) and an arbitrary $\sigma(Y', Y)$-zero sequence $(y'_n : n \in \mathbb{N})$, we have to show that $\lim_{n\to\infty}\langle y'_n, T_n(x_n)\rangle = 0$. Since for every $T \in K_{w^*}(X', Y)$ the adjoint T' maps $(y'_n : n \in \mathbb{N})$ to a norm-zero sequence, we deduce that

$$\langle T'(y'_n), x'_n\rangle = \langle T(x'_n), y'_n\rangle \underset{n\to\infty}{\longrightarrow} 0 \text{ for } T \in K_{w^*}(X', Y).$$

Thus, $(x'_n \otimes y'_n : n \in \mathbb{N})$ converges in $\sigma(K_{w^*}(X', Y)', K_{w^*}(X', Y))$ to zero (where $x'_n \otimes y'_n(T) := \langle T(x'_n), y'_n\rangle$ for each $n \in \mathbb{N}$) and we deduce from the assumption that

$$\langle T_n(x'_n), y'_n\rangle = \langle T_n, x'_n \otimes y'_n\rangle \underset{n\to\infty}{\longrightarrow} 0,$$

which proves the assertion.

In the same way we can prove that $A(B_1(Y'))$ is limited in X.

(b) $\Rightarrow$(c): obvious.

(c) $\Rightarrow$(a):(Under the additional assumption that Y is a Gelfand-Phillips space enjoying the approximation property, and thus, by (0.3)(e), $X\check{\otimes}Y = K_{w^*}(X', Y)$.)

Let $A \subset K_{w^*}(X', Y)$ satisfy (c). From the assumptions on Y we deduce that, for each $\varepsilon > 0$, there is a finite-dimensional projection

$$P^{(\varepsilon)} : \quad Y \to Y, \qquad y \mapsto \sum_{i=1}^{n(\varepsilon)} y(\varepsilon, i)\langle y'(\varepsilon, i), y\rangle,$$

where $y(\varepsilon, 1), \ldots, y(\varepsilon, n(\varepsilon)) \in Y$, $y'(\varepsilon, 1), \ldots, y'(\varepsilon, n(\varepsilon)) \in Y'$, and $n(\varepsilon) \in \mathbb{N}$, such that $\| P^{(\varepsilon)}(y) - y \| \leq \varepsilon$ for each $y \in A(B_1(X'))$. Thus,

(1) $$\| P^{(\varepsilon)} \circ T - T \| \leq \varepsilon \quad \text{whenever } T \in A.$$

For each $\varepsilon > 0$, we now want to show that $A(\varepsilon) := \{P^{(\varepsilon)} \circ T \,|\, T \in A\}$ is limited in $\mathrm{K}_{w^*}(X', Y)$. To see this, we remark that, by assumption (c), the set

$$\tilde{A}(\varepsilon, i) := \{T'(y'(\varepsilon, i)) \,|\, T \in A\}$$

is limited in X, for each $i \leq n(\varepsilon)$. Thus, the set

$$A(\varepsilon, i) := \{y(\varepsilon, i) \otimes T'(y'(\varepsilon, i)) \,|\, T \in A\}$$

is limited in $\mathrm{K}_{w^*}(X', Y)$ (it is the image of $\tilde{A}(\varepsilon, i)$ under the operator $X \ni x \mapsto y(\varepsilon, i) \otimes x \in \mathrm{K}_{w^*}(X', Y)$). Since for each $x' \in X'$ and $T \in A$ we have

$$\begin{aligned} P^{(\varepsilon)} \circ T(x') &= \sum_{i=1}^{n(\varepsilon)} y(\varepsilon, i) \langle T(x'), y'(\varepsilon, i) \rangle \\ &= \sum_{i=1}^{n(\varepsilon)} y(\varepsilon, i) \langle x', T'(y'(\varepsilon, i)) \rangle = \Big(\sum_{i=1}^{n(\varepsilon)} y(\varepsilon, i) \otimes T'(y'(\varepsilon, i)) \Big)(x'), \end{aligned}$$

we deduce that $A(\varepsilon) \subset \sum_{i=1}^{n(\varepsilon)} A(\varepsilon, i)$, and thus, that $A(\varepsilon)$ is limited in $\mathrm{K}_{w^*}(X', Y)$. From (2) we deduce finally that $A \subset \bigcap_{\varepsilon > 0} \mathrm{B}_\varepsilon(\mathrm{K}_{w^*}(X', Y)) + A(\varepsilon)$. The assertion follows from (1.1.4).

◇

From Proposition (4.4.1) and the implication "(a) ⇒(b)" of (4.4.2), we deduce that $X \check{\otimes} Y$ and $\mathrm{K}_{w^*}(X', Y)$ have the Gelfand-Phillips property if X and Y enjoy this property. This is a result which is already proven in [18, p.486, Theorem 2.1.] for $X \check{\otimes} Y$ under additional assumptions on X and in [13, p.407, Theorem 3.1] and [14, p.2, Theorem 2.1.] for general $X \check{\otimes} Y$ and $\mathrm{K}_{w^*}(X', Y)$ respectively:

4.4.3 Corollary: *If X and Y are Gelfand-Phillips spaces, then $\mathrm{K}_{w^*}(X', Y)$ and $X \check{\otimes} Y$ have the Gelfand-Phillips property also.*

Proof of (4.4.3) : By (4.4.1) and (4.4.2)((a) ⇒(b)).

◇

To formulate other necessary conditions for limitedness in $\mathrm{K}_{w^*}(X', Y)$ and $X \check{\otimes} Y$, we need the following Lemma which uses essentially the results of section (4.3).

4.4.4 Lemma: *Let K be compact and let Σ_K be the σ-algebra of the Borel sets of K.*
Let $A \subset C(K,X)$ have a common X-limited range $A(K) := \{f(\xi) \mid \xi \in K\}$.
Then the following are equivalent:

a) *Every sequence $(f_n : n \in \mathbb{N}) \subset A$ enjoys, as a sequence in $L^c_\infty(\Sigma_K, X)$, the condition of case 1 in Corollary (4.3.7)(a).*
b) *For each positive and finite Borel measure on K, $T_\mu(A)$ is limited in $L_1(\mu, X)$, where $T_\mu : L^c_\infty(\Sigma_K, X) \to L_1(\mu, X)$, $f \mapsto f$.*

In particular, we deduce that a set $A \subset C(K,X)$ which is limited in $C(K,X)$ satisfies condition (a).

Proof of (4.4.4) :
Proof of (a) $\Rightarrow$(b):
Let μ be a finite measure on Σ_K. It is enough to show that an arbitrary sequence $(f_n : n \in \mathbb{N}) \subset A$ contains a subsequence whose image under T_μ is limited in $L_1(\mu, X)$.
Thus, let $(f_n : n \in \mathbb{N}) \subset A$. By taking a subsequence, we may assume by (a) that there is an increasing sequence of countable Σ_K-partitions $(\pi_k : k \in \mathbb{N})$, $\pi_k := (B(k,m) : m \in \mathbb{N})$, and a family $(x(n,k,m) : n,k,m \in \mathbb{N}) \subset A(K))$ such that

(1) $$\| f_n - \sum_{m \in \mathbb{N}} \chi_{B(k,m)} x(n,k,m) \|_\infty \le \frac{1}{k}, \text{ for each } n,k \in \mathbb{N}.$$

Thus (note that $\| T_\mu \| = |\mu|(K)$),

$$(T_\mu(f_n) : n \in \mathbb{N}) \subset \bigcup_{k \in \mathbb{N}} \{f_n^{(k)} \mid n \in \mathbb{N}\} + B_{\mu(K)/k}(L_1(\mu, X)),$$

where

$$f_n^{(k)} := \sum_{m \in \mathbb{N}} \chi_{B(k,m)} x(n,k,m) \text{ for each } n,k \in \mathbb{N}.$$

By (1.1.4) it is sufficient to show that, for given $k \in \mathbb{N}$, $(f_n^{(k)} : n \in \mathbb{N})$ is limited in $L_1(\mu, X)$. In order to show this, we first remark that for any $B \in \Sigma_K$ and $n \in \mathbb{N}$,

$$\int_B f_n^{(k)} d\mu = \sum_{m \in \mathbb{N}} \mu(B \cap B(k,m)) x(n,k,m) \in \mu(K) \cdot \overline{\mathrm{aco}(A(K))},$$

which implies, together with the assumption, that (b)(i) of Corollary (4.2.2) is satisfied.

Secondly, we observe that for any $\varepsilon > 0$ and for an $m_0 \in \mathbb{N}$ for which

$$\sum_{m>m_0} \mu(B(m,k)) < \frac{\varepsilon}{2}(1 + \sup_{m\in\mathbb{N}} \|x(m,k)\|),$$

it follows that

$$\| f_n^{(k)} - \tilde{E}_\pi(f_n^{(k)})\|_1 \leq 2 \, \| f_n^{(k)}|_{\bigcup_{m>m_0} B(m,k)}\|_1 < \varepsilon,$$

for each $n \in \mathbb{N}$ and each finite Σ_K-partition π which is finer than

$$\pi^\varepsilon := (B(1,k), B(2,k), \ldots, B(m_0,k), \bigcup_{m>m_0} B(m,k)),$$

(where $\tilde{E}_\pi$ is defined as in (4.2.2)).
This implies that $(f_n : n \in \mathbb{N})$ satisfies (c)(ii) of (4.2.2), from which we deduce the assertion.

$\neg$(a) $\Rightarrow$ $\neg$(b):
Using Corollary (4.3.7)(a), it is enough to show that, for a sequence $(f_n : n \in \mathbb{N}) \subset A$ admitting an $\varepsilon > 0$ and a tree of sets $(A(n,j) : n \in \mathbb{N}_0, 1 \leq j \leq 2^n)$ such that the conditions of case 2 are satisfied, there is a finite Borel measure on K for which $(T_\mu(f_n) : n \in \mathbb{N})$ is not limited in $L_1(\mu, X)$.
For this we set

$$\tilde{K} := \bigcap_{n\in\mathbb{N}} \bigcup_{j=1}^{2^n} \overline{A(n,j)}^K \text{ and } C(n,j) := \tilde{K} \cap \overline{A(n,j)}^K \text{ for } n \in \mathbb{N}_0, j \leq 2^n.$$

From the property of $(A(n,j) : n \in \mathbb{N}_0, 1 \leq j \leq 2^n)$ and the compactness of K we deduce that no $C(n,j)$ is empty.
Since f_n is continuous for each $n \in \mathbb{N}$, we deduce from the assumptions that

$$(2) \quad \sup_{\xi,\tilde{\xi}\in C(n,j)} \| f_n(\xi) - f_n(\tilde{\xi}) \| \leq \frac{\varepsilon}{4} \text{ and } \inf_{\xi\in C(n,2i),\tilde{\xi}\in C(n,2i-1)} \| f_n(\xi) - f_n(\tilde{\xi}) \| \geq \varepsilon$$

whenever $n \in \mathbb{N}$, $j \leq 2^n$, and $i \leq 2^{n-1}$.

Moreover, for $m \in \mathbb{N}$ and $i \leq 2^{n-1}$ we have

$$\begin{aligned} C(m-1,i) &= \tilde{K} \cap \overline{A(m-1,i)} \\ &= \Big(\bigcap_{n\in\mathbb{N}} \bigcup_{j=1}^{2^n} \overline{A(n,j)} \Big) \cap \big(\overline{A(m,2i)} \cup \overline{A(m,2i-1)} \big) \\ &= C(m,2i) \cup C(m,2i-1), \end{aligned}$$

and thus (together with the second part of (2)),

(4) $$C(m,2i)\cup C(m,2i-1)=C(m,2i) \text{ and } C(m,2i)\cap C(m,2i-1)=\emptyset.$$

For each $n\in\mathbb{N}$ and $j\le 2^n$ we choose an $\xi(n,j)\in C(n,j)$ and an accumulation point $\mu\in M(\tilde K)$ of the sequence $(\mu_n : n\in\mathbb{N}) := (2^{-n}\sum_{i=1}^{2^n}\delta_{\xi(n,j)} : n\in\mathbb{N})$ in $\sigma(M(\tilde K),C(\tilde K))$. We observe that for $n\in\mathbb{N}$, $m\ge n$, and $j\in\{1,2..2^n\}$,

$$\mu_m(C(n,j))=2^{-m}|\{i\le 2^m\,|\,C(m,i)\subset C(n,j)\}|=2^{-n}.$$

Since $C(n,j)$ is clopen in $\tilde K$ (note that $\tilde K\setminus C(n,j)=\bigcup_{i\in\{1,\dots 2^n\}\setminus\{j\}}C(n,i))$ for each $n\in\mathbb{N}$ and $j\le 2^n$), we deduce that

(5) $$\mu(C(n,j))=2^{-n} \text{ for each } n\in\mathbb{N} \text{ and } j\le 2^n.$$

Taking $\pi_n := \{C(n,j)\,|\,j\le 2^n\}\cup\{K\setminus\tilde K\}$, we have

$$\begin{aligned}
&\|\tilde E_{\pi_n}\circ T_\mu(f_n)-\tilde E_{\pi_{n-1}}\circ T_\mu(f_n)\|_1\\
&=\Big\|\sum_{i=1}^{2^{n-1}}2^{n-1}\Big[2\chi_{C(n,2i)}\int_{C(n,2i)}f_n\mathrm{d}\mu+2\chi_{C(n,2i-1)}\int_{C(n,2i-1)}f_n\mathrm{d}\mu\\
&\qquad-\chi_{C(n-1,i)}\int_{C(n-1,i)}f_n\mathrm{d}\mu\Big]\Big\|_1\\
&=2^{n-1}\Big\|\sum_{i=1}^{2^{n-1}}\Big[\chi_{C(n,2i)}\Big(\int_{C(n,2i)}f_n\mathrm{d}\mu-\int_{C(n,2i-1)}f_n\mathrm{d}\mu\Big)\\
&\qquad+\chi_{C(n,2i-1)}\Big(\int_{C(n,2i-1)}f_n\mathrm{d}\mu-\int_{C(n,2i)}f_n\mathrm{d}\mu\Big)\Big]\Big\|_1\\
&=\sum_{i=1}^{2^{n-1}}\Big\|\int_{C(n,2i)}f_n\mathrm{d}\mu-\int_{C(n,2i-1)}f_n\mathrm{d}\mu\Big\|\\
&\ge\sum_{i=1}^{2^{n-1}}2^{-n}\Big[\|f_n(\xi(n,2i))-f_n(\xi(n,2i-1))\|\\
&\qquad-\Big(\sup_{\xi,\tilde\xi\in C(n,2i)}\|f_n(\xi)-f_n(\tilde\xi)\|+\sup_{\xi,\tilde\xi\in C(n,2i-1)}\|f_n(\xi)-f_n(\tilde\xi)\|\Big)\Big]\\
&\ge\sum_{i=1}^{2^{n-1}}(\varepsilon-2\varepsilon/4)2^{-n}=\varepsilon/4,
\end{aligned}$$

where $\tilde{E}_\pi$ is defined as in (4.2.1). Thus, we have shown that $(\tilde{E}_{\pi_n} : n \in \mathbb{N})$ does not converge uniformly on $\{T_\mu(f_n) \mid n \in \mathbb{N}\}$ and from (4.2.2) we deduce that $T_\mu(A)$ is not limited in $L_1(\mu, X)$.

◇

Using Proposition (1.1.10), Proposition (4.4.2), and Lemma (4.4.4), we are now in the position to give the following necessary conditions for limitedness in $K_{w^*}(X', Y)$ and in $X\check{\otimes}Y$:

4.4.5 Theorem: *Let Z be the space $X\check{\otimes}Y$ or $K_{w^*}(X', Y)$.*

Let K_X and K_Y be two compacta for which X and Y can be isometrically embedded in $C(K_X)$ and $C(K_Y)$ respectively, and thus, Z can be isometrically embedded in $C(K_X \times K_Y)$. Let Σ_X and Σ_Y be the corresponding Borel sets. (For example $K_X := (B_1(X'), \sigma(X', X))$; if $X = C(K)$, then $K_X := K$ is possible also.)

Then a limited set A in Z satisfies the following conditions (a), (b), and (c):

a) *$A(B_1(X'))$ is Y-limited and $A(B_1(Y'))$ is X-limited.*

b) *Every sequence $(z_n : n \in \mathbb{N})$ contains a subsequence $(z_n : n \in N)$, $N \in \mathcal{P}_\infty(\mathbb{N})$, such that*

for any $\varepsilon > 0$ there are countable Σ_X- and Σ_Y-partitions π_ε^X and π_ε^Y of K_X and K_Y respectively such that

$$\operatorname{osc}(z_n, B \times \tilde{B}) \leq \varepsilon \text{ whenever } B \in \pi_\varepsilon^X, \ \tilde{B} \in \pi_\varepsilon^Y, \text{ and } n \in \mathbb{N}$$

(we view Z as a subspace of $L_\infty(\Sigma_X \otimes \Sigma_Y)$).

c) *If $Z = X\check{\otimes}Y$, then A is almost bounded in the projective tensor norm, i.e. for any $\varepsilon > 0$ there is a $\|\cdot\|_\wedge$-bounded $A^\varepsilon \subset X\check{\otimes}Y$ such that*

$$A \subset \bigcap_{\varepsilon > 0} A^\varepsilon + B_\varepsilon(X\check{\otimes}Y).$$

Proof of (4.4.5) :

(a) : Proposition (4.4.2)(a) $\Rightarrow$(b)

(b) : We note that Z can be embedded in $C(K_X, Y)$ as well as in $C(K_Y, X)$ in a canonical way. Applying Lemma (4.4.4) (b) $\Rightarrow$(a) twice to a sequence $(f_n : n \in \mathbb{N}) \subset A$, we get an $N \in \mathcal{P}_\infty(\mathbb{N})$ and, for each $\varepsilon > 0$, countable Σ_X- and Σ_Y- partitions $\pi^{(1)}$ and $\pi^{(2)}$ of K_X and K_Y respectively, such that the sequence $(f_n : n \in N)$ is $(\pi^{(1)}, \varepsilon/2)$ compatible on K_X (viewed in $L_\infty^c(\Sigma_X, Y)$) and

$(\pi^{(2)}, \varepsilon/2)$ compatible on K_Y (viewed in $L^c_\infty(\Sigma_Y, X)$). Thus, for each $n \in N$, for each $B^{(1)} \in \pi^{(1)}$ and $B^{(2)} \in \pi^{(2)}$, and any $\xi(1), \tilde{\xi}(1) \in B^{(1)}$ and $\xi(2), \tilde{\xi}(2) \in B^{(2)}$ we deduce that

$$\begin{aligned} &|z_n(\xi^{(1)},\xi^{(2)}) - z_n(\tilde{\xi}^{(1)},\tilde{\xi}^{(2)})| \\ &\quad \leq |z_n(\xi^{(1)},\xi^{(2)}) - z_n(\xi^{(1)},\tilde{\xi}^{(2)})| + |z_n(\xi^{(1)},\tilde{\xi}^{(2)}) - z_n(\tilde{\xi}^{(1)},\tilde{\xi}^{(2)})| \\ &\quad \leq \| z_n(\cdot,\xi^{(2)}) - z_n(\cdot,\tilde{\xi}^{(2)}) \| + \| z_n(\xi^{(1)},\cdot) - z_n(\tilde{\xi}^{(1)},\cdot) \| \leq \varepsilon, \end{aligned}$$

which implies (b).

(c) : Proposition (1.1.10), applied to $V := X \otimes Y$ and $|\!|\!| \cdot |\!|\!| := \| \cdot \|_\wedge$.

◇

4.4.6 Remark:

a) Let $\| \cdot \|_\alpha$ be a reasonable norm on $X \otimes Y$. By [10, p.223, Proposition 3] it follows that $\| \cdot \|_\alpha \geq \| \cdot \|_\vee$; thus, the identity I on $X \otimes Y$ can be extended to a continuous linear $I_\alpha : X \overset{\alpha}{\tilde{\otimes}} Y \to X \check{\otimes} Y$.
So it follows from ~~(4.4.5)~~ that for any limited set in $X \overset{\alpha}{\tilde{\otimes}} Y$ the image under I_α enjoys the properties (a),(b) and (c) of (4.4.5).

b) Let Z be an arbitrary Banach space, K_Z a compact space for which Z can be isometrically embedded in $C(K_Z)$, and Σ_Z the Borel sets of K_Z. In ~~(1.2.5)~~ we showed that a Z-limited set $A \subset Z$ must be weakly conditionally compact; but this is equivalent to the property that each sequence $(z_n : n \in \mathbb{N}) \subset A$ satisfies in $L_\infty(\Sigma_Z) = L^c_\infty(\Sigma, \mathbb{R})$ the property of case 1 in ~~Corollary (4.3.7)(a)~~, i.e. there is a subsequence $(\tilde{z}_n : n \in \mathbb{N})$ of $(z_n : n \in \mathbb{N})$ and, for each $\varepsilon > 0$, a countable Σ_Z-partition π of K_Z such that for each $n \in \mathbb{N}$ and each $B \in \pi$ the oscillation of $\tilde{z}_n$ on B is not greater than ε.
Now Theorem ~~(4.4.5)~~ sharpens this result in the following sense:
If Z is the injective tensor product of X and Y and if we take $K_Z := K_X \times K_Y$ (K_X, K_Y, Σ_X and Σ_Y as in (4.4.5)), then for each ε, the above partition π can be taken in $\Sigma_X \otimes \Sigma_Y$ having only rectangular sets, i.e. π can be chosen of the form

$$\pi = \pi^{(1)} \times \pi^{(2)} = \{B^{(1)} \times B^{(2)} \mid B^{(1)} \in \pi^{(1)}, \quad B^{(2)} \in \pi^{(2)}\},$$

where $\pi(1)$ and $\pi^{(2)}$ are countable Σ_X- and Σ_Y-partitions of K_X and K_Y respectively.

Moreover, we have shown that in Grothendieck C(K)-spaces the class of conditionally weakly compact sets and the class of limited sets coincide. In the next section, we will prove that in injective tensor products of Grothendieck C(K)-spaces the conditions (b) and (c) of Theorem (4.4.5) characterize the limited sets.

4.5 Characterization of limited sets in $C(K_1 \times K_2)$ if $C(K_1)$ and $C(K_2)$ are Grothendieck spaces

We now want to show the converse of Theorem (4.4.5) in the special case that X and Y are $C(K)$-spaces enjoying the Grothendieck property. In this case, condition (a) of (4.4.5) is superfluous.

4.5.1 Theorem: *Let K_1 and K_2 be two compacta such that $C(K_1)$ and $C(K_2)$ have the Grothendieck property. We denote the Borel sets of K_1 and K_2 by Σ_1 and Σ_2 respectively. For $A \subset C(K_1 \times K_2)$ the following conditions (a) and (b) are equivalent.*

a) *A is limited in $C(K_1 \times K_2)$.*

b) i) *Each sequence $(f_n : n \in \mathbb{N}) \subset A$ contains a subsequence $(f_n : n \in N)$, $N \in \mathcal{P}_\infty(\mathbb{N})$, with the following property:*
For each $\varepsilon > 0$, there are countable Σ_1- and Σ_2- partions π_1 and π_2 of K_1 and K_2 respectively such that

$$\operatorname{osc}(f_n, A \times B) \leq \varepsilon \quad \text{for all } A \in \pi_1, B \in \pi_2 \text{ and } n \in N.$$

ii) *A is almost bounded in $C(K_1)\hat{\otimes}C(K_2)$ (in the sense of Proposition (1.1.10)).*

Before we can show Theorem (4.5.1), we need the following Proposition and Lemmas.

4.5.2 Proposition: *Let α be a cross-norm on $X \otimes Y$ and let $A_1 \subset X$ and $A_2 \subset Y$ be X- and Y-limited respectively.*

a) *For any sequence $(x_n : n \in \mathbb{N}) \subset A_1$ and any $\sigma(X \overset{\alpha}{\otimes} Y, (X \overset{\alpha}{\otimes} Y)')$-zero sequence $(z'_n : n \in \mathbb{N})$, the sequence $(z'_n(x_n) : n \in \mathbb{N})$ is $\sigma(Y', Y)$-zero convergent, where we define for $z' \in (X \overset{\alpha}{\otimes} Y)'$, $x \in X$, and $y \in Y$, $z'(x)(y) = z'(y)(x) := \langle z', x \otimes y \rangle$.*

b) *$A_1 \otimes A_2 = \{x \otimes y \mid x \in A_1, y \in A_2\}$ is limited in $X \overset{\alpha}{\otimes} Y$.*

Proof of (4.5.2) a:

Let $(z'_n : n \in \mathbb{N}) \subset (X \overset{\alpha}{\otimes} Y)'$ be a w*-zero sequence. Then for each $y \in Y$ the sequence $(z'_n(y) : n \in \mathbb{N})$ is $\sigma(X', X)$-zero convergent and, since A_1 is limited in X, we deduce the assertion from

$$\lim_{n \to \infty} z'_n(x_n)(y) = \lim_{n \to \infty} z'_n(y)(x_n) = 0.$$

Part b) follows from (a).

◇

4.5.3 Lemma: *Let K be a compact space such that $C(K)$ is a Grothendieck space. We consider a bounded sequence $(f_n: n \in \mathbb{N}) \subset C(K, X)$ with the following properties (a), (b), and (c):*

a) *The sequence $(f_n(\xi) : n \in \mathbb{N})$ is limited in X for each $\xi \in K$.*
b) *Every subsequence of $(f_n: n \in \mathbb{N})$ satisfies the property of case 1 in Corollary (4.3.7) (we view $C(K, X)$ as a subspace of $L^c_\infty(\Sigma_K, X)$), where Σ_K are the Borel sets of Σ_K).*
c) *$(f_n: n \in \mathbb{N})$ is not limited in $C(K, X)$.*

Then it follows that there is a subsequence $(\tilde{f}_n: n \in \mathbb{N})$ of $(f_n: n \in \mathbb{N})$, a sequence $(C_n: n \in \mathbb{N})$ of closed subsets of K, and a sequence $(O_n: n \in \mathbb{N})$ of open subsets of K such that.

d) *$\emptyset \neq C_n \subset O_n$ and $O_n \cap O_{n'} = \emptyset$ if $n, n' \in \mathbb{N}$, with $n \neq n'$, and*
e) *for each sequence $(h_n: n \in \mathbb{N}) \subset C(K)$ with $0 \leq h_n \leq 1$, $h_n|_{C_n} = 1$ and $h_n|_{O_n^c} = 0$, for $n \in \mathbb{N}$, the sequence $(h_n \tilde{f}_n : n \in \mathbb{N})$ is not limited in $C(K, X)$.*

Proof of (4.5.3) :

By the assumptions in (b) and (c), we can assume that there is an $\varepsilon > 0$ and a normed w*-zero sequence $(\mu_n: n \in \mathbb{N})$ in $C(K, X)' = M(K, X')$ with

$$(1) \qquad \varepsilon \leq \langle \mu_n, f_n \rangle \qquad \text{for all } n \in \mathbb{N}$$

and that, moreover, there is a countable Σ_K-partition $\pi = (B_m: m \in \mathbb{N})$ such that $(f_n: n \in \mathbb{N})$ is $(\varepsilon/6, \pi)$-compatible on K. Choosing for each $m \in \mathbb{N}$ a $\xi_m \in B_m$ and setting $x(n, m) := f_n(\xi_m)$, we deduce that

$$(2) \qquad \Big\| f_n - \sum_{m \in \mathbb{N}} \chi_{B_m} x(n, m) \Big\|_\infty \leq \varepsilon/6 \qquad \text{for each } n \in \mathbb{N}.$$

Since for arbitrary $x \in X$ and $B \in \Sigma_K$ we deduce from (4.5.2) and from the Grothendieck property of $C(K)$ that

$$\langle \mu_n(B), x \rangle = \langle \mu_n, x \rangle(\chi_B) \underset{n \to \infty}{\longrightarrow} 0,$$

it follows that

$$(3) \qquad \mu_n(B) \underset{n \to \infty}{\longrightarrow} 0 \qquad \text{in } \sigma(X', X) \text{ for any } B \in \Sigma_K.$$

Now choose inductively, for each $k \in \mathbb{N}$, n_k and m_k both in $\mathbb{N}$ such that

(4)(k) $n_k < m_k$ and if $k > 1$, then $m_{k-1} < n_k$,

(5)(k) $|\mu_{n_k}|(\bigcup_{m \geq m_k} B_m) < \frac{\varepsilon}{6c}$,
where $c := 1 + \sup\{\| x(n,m) \| \; | n,m \in \mathbb{N}\}$, and

(6)(k) $|\langle \mu_{n_k}, \sum_{m \leq m_{k-1}} \chi_{B_m} x(n_k, m) \rangle| < \frac{\varepsilon}{6}$, if $k > 1$.

If $k = 1$ we choose $n_1 = 1$ and an $m_1 > n_1$ large enough so that (5)(1) holds (note that $\| \mu_j \| = 1$ and that the sets B_j's are pairwise disjoint).

If m_{k-1} and n_{k-1} are already chosen for $k > 1$, we deduce from (3) and the limitedness of $(x(n,m) : n \in \mathbb{N}, m \leq m_{k-1})$ in X (assumption (a)) that there is an $n_k > m_{k-1}$ such that

(7) $$|\langle \mu_{n_k}, x(n_k, m)\chi_{B_m} \rangle| < \frac{\varepsilon}{m_{k-1} 6c} \text{ for each } m \leq m_{k-1},$$

which implies (6)(k). As in the first induction step, we can choose an $m_k > n_k$ satisfying (5)(k).

Setting $D_k := \bigcup_{j=m_{k-1}+1}^{m_k} B_j$, for each $k \in \mathbb{N}$ (with $m_0 := 0$), we deduce that

(8) $$\begin{aligned} \langle \mu_{n_k}, \chi_{D_k} f_{n_k} \rangle &\geq \langle \mu_{n_k}, f_{n_k} \rangle - |\langle \mu_{n_k}, \chi_{D_k^c} f_{n_k} \rangle| \\ &\geq \varepsilon - \varepsilon/6 - |\langle \mu_{n_k}, \chi_{D_k^c} \sum_{m \in \mathbb{N}} \chi_{B_m} x(n_k, m) \rangle| \\ &\quad [(1),\ (2) \text{ and } \| \mu_{n_k} \| = 1] \\ &\geq 5\varepsilon/6 - |\langle \mu_{n_k}, \sum_{m \leq m_{k-1}} \chi_{B_m} x(n_k, m) \rangle| \\ &\qquad - |\langle \mu_{n_k}, \sum_{m > m_k} \chi_{B_m} x(n_k, m) \rangle| \\ &\geq 5\varepsilon/6 - 2\varepsilon/6 = \varepsilon/2 \\ &\quad [\text{by } (5)(k) \text{ and } (6)(k)]. \end{aligned}$$

Since for any $f \in C(K)$ and $x \in X$ the sequence $(\langle f.\mu_{n_k}, x \rangle : k \in \mathbb{N})$ is weak*-zero convergent in $M(K)$, and thus weakly convergent, since by (1.1.7) $(\chi_{D_k} : k \in \mathbb{N})$ is limited in $L_\infty(\Sigma_K)$ (we recall that $L_\infty(\Sigma_K)$ is a Grothendieck space with the Dunford-Pettis property and that $(\chi_{D_k} : k \in \mathbb{N})$ is weakly zero convergent in $L_\infty(\Sigma_K)$), and, finally, since $L_\infty(\Sigma_K)$ is a subspace of $C(K)''$, we deduce:

$$\langle \mu_{n_k}|_{D_k}, f \cdot x \rangle = \langle f.\mu_{n_k}, x \rangle (\chi_{D_k}) \underset{k \to \infty}{\longrightarrow} 0 \text{ whenever } f \in C(K),\ x \in X,$$

which implies that $(\mu_{n_k}|_{D_k} : k \in \mathbb{N})$ converges in $\sigma(M(K, X'), C(K, X))$ to zero. Since μ_{n_k}, is regular for $k \in \mathbb{N}$ there is for each $k \in \mathbb{N}$ a compact $\tilde{D}_k \subset D_k$ with $|\mu_{n_k}|(D_k \setminus \tilde{D}_k) \leq \frac{\varepsilon}{6k\tilde{c}}$, where $\tilde{c} := \sup_{n \in \mathbb{N}}(1 + \| f_n \|)$.

Thus, the sequence $(\nu_k : k \in \mathbb{N})$, with $\nu_k := \mu_{n_k}|_{\tilde{D}_k}$ $(k \in \mathbb{N})$, has pairwise disjoint supports, converges w* to zero, and satisfies, by (8),

$$(9) \qquad \langle \nu_k, f_{n_k} \rangle \geq \langle \mu_{n_k}, \chi_{D_k} f_{n_k} \rangle - |\langle \mu_{n_k}, \chi_{D_k \setminus \tilde{D}_k} f_{n_k} \rangle| \geq \frac{\varepsilon}{2} - \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.$$

This implies that the assumptions of Lemma (3.2.3) are satisfied (taking $F_j := \emptyset$ for $j \in \mathbb{N}$) and, together with (3.2.4), we can find a subsequence $(k_\ell : \ell \in \mathbb{N})$ of $\mathbb{N}$ and a w*-zero sequence $(\tilde{\nu}_\ell : \ell \in \mathbb{N})$ in $\mathrm{M}(K, X')$ such that the supports C_ℓ of $\tilde{\nu}_\ell$ have pairwise disjoint open neighborhoods O_ℓ and such that

$$\| \nu_\ell - \tilde{\nu}_\ell \| \leq \varepsilon / (1 + 6 \sup_{n \in \mathbb{N}} \| f_n \|) \text{ for each } \ell \in \mathbb{N}.$$

Taking $\tilde{f}_\ell := f_{n_{k_\ell}}$ for $\ell \in \mathbb{N}$, we deduce for an arbitrary sequence $(h_\ell : \ell \in \mathbb{N}) \subset \mathrm{C}(K)$ with $0 \leq h_\ell \leq 1$, $h_\ell|_{C_\ell} = 1$, and $h_\ell|_{O_\ell^c} = 0$ that

$$\begin{aligned} \langle h_\ell \tilde{f}_\ell, \tilde{\nu}_\ell \rangle &= \langle \tilde{f}_\ell, \tilde{\nu} \ell \rangle \\ &\geq \langle f_{n_{k_\ell}}, \nu_{k_\ell} \rangle - \| f_{n_{k_\ell}} \| \cdot \| \nu_{k_\ell} - \tilde{\nu}_{k_\ell} \| \\ &\geq \varepsilon / 6, \end{aligned}$$

which implies that $(h_\ell \tilde{f}_\ell : \ell \in \mathbb{N})$ is not limited in $\mathrm{C}(K, X)$ and so the assertion follows.

◇

4.5.4 Lemma: *Let K_1 and K_2 be compact spaces with $\mathrm{C}(K_1)$ and $\mathrm{C}(K_2)$ being Grothendieck spaces.*

We consider a sequence $(f_n : n \in \mathbb{N}) \subset \mathrm{C}(K_1 \times K_2)$ and an $\varepsilon > 0$ with the following properties (a) and (b):

a) For $n \in \mathbb{N}$ and $\theta = 1, 2$, there are closed $C_n^\theta \subset K_\theta$ and open $O_n^\theta \subset K_\theta$ with

$$C_n^{(\theta)} \subset O_n^{(\theta)}, \; O_n^{(\theta)} \cap O_{n'}^{(\theta)} = \emptyset, \text{ and } \mathrm{supp} f_n \subset C_n^{(1)} \times C_n^{(2)}$$

for $n, n' \in \mathbb{N}$ with $n \neq n'$.

b) There is a sequence $(\tilde{f}_n : n \in \mathbb{N}) \subset \mathrm{C}(K_1) \otimes \mathrm{C}(K_2)$ with

$$\| \tilde{f}_n - f_n \| < \varepsilon \text{ and } \sup_{n \in \mathbb{N}} \| \tilde{f}_n \| < \infty \text{ for } n \in \mathbb{N}.$$

Then for any weak-zero sequence $(\mu_n : n \in \mathbb{N})$ in $\mathrm{B}_1(\mathrm{M}(K_1 \times K_2))$ it follows that*

$$\limsup_{n \to \infty} |\langle \mu_n, f_n \rangle| \leq \varepsilon.$$

Proof of (4.5.4) :

To avoid ambiguities, we denote the usual norm on $C(K_1 \times K_2)$ by $\|\cdot\|_\vee$. For $n \in \mathbb{N}$ and $\theta = 1, 2$, let $C_n^{(\theta)}$ and $O_n^{(\theta)}$ be as assumed in (a) and choose $g_n^{(\theta)} \in C(K_\theta)$ with

(1) $$0 \leq g_n^{(\theta)} \leq 1\,,\ g_n^{(\theta)}|_{C_n^{(\theta)}} = 0 \text{ and } g_n^{(\theta)}|_{(O_n^{(\theta)})^C} = 1.$$

We define the following operators $P_n^{(\theta)}$, $\check{P}_n$, and $\hat{P}_n$ by

$$P_n^{(\theta)} : C(K_\theta) \to C(K_\theta)\,,\ h \mapsto g_n^{(\theta)} h\,,$$
$$\check{P}_n := P_n^{(1)} \check{\otimes} P_n^{(2)} \text{ and } \hat{P}_n := P_n^{(1)} \hat{\otimes} P_n^{(2)} \text{ for } n \in \mathbb{N} \text{ and } \theta = 1, 2$$

(c.f.(0.3)(d)) and observe that for $f_1 \in C(K_1)$ and $f_2 \in C(K_2)$ we have

$$\check{P}_n(f_1 \otimes f_2) = P_n^{(1)}(f_1) \otimes P_n^{(2)}(f_2) = (g_n^{(1)} f_1) \otimes (g_n^{(2)} f_2) = (g_n^{(1)} \otimes g_n^{(2)}) \cdot (f_1 \otimes f_2).$$

Thus, $\check{P}_n(f) = (g_n^{(1)} \otimes g_n^{(2)}) \cdot f$ for $f \in C(K_1 \times K_2)$.

By condition (b) and the definition of the projective tensor norm, we can choose for each $n \in \mathbb{N}$ finite families

$$(x(n,i) : i \leq \ell(n)) \subset B_1(C(K_1))\,,$$
$$(y(n,i) : i \leq \ell(n)) \subset B_1(C(K_2))\,, \text{ and}$$
$$(a(n,i) : i \leq \ell(n)) \subset \mathbb{R}$$

such that each $\tilde{f}_n$ defined by

$$\tilde{f}_n := \sum_{i=1}^{\ell(n)} a(n,i) x(n,i) \otimes y(n,i) \text{ for } n \in \mathbb{N}$$

satisfies $\|f_n - \tilde{f}_n\|_\vee \leq \varepsilon$ and such that

$$\sum_{i=1}^{\ell(n)} |a(n,i)| \leq 2 \sup_{n' \in \mathbb{N}} \|\tilde{f}_{n'}\|_\wedge < \infty \text{ for } n \in \mathbb{N}.$$

From assumption (a), from (1), and from the definition of $\tilde{f}_n$, we deduce for each $n \in \mathbb{N}$ that

(2) $$\check{P}_n(f_n) = (g_n^{(1)} \otimes g_n^{(2)}) f_n = f_n \text{ and}$$
$$\hat{P}_n(\tilde{f}_n) = \check{P}_n(\tilde{f}_n) = \sum_{i=1}^{\ell(n)} a(n,i)(x(n,i) g_n^{(1)}) \otimes (y(n,i) g_n^{(2)}).$$

Thus, since $\| \check{P}_n \| = 1$,

$$(3) \qquad \| \check{P}_n(\tilde{f}_n) - \check{P}_n(f_n) \|_\vee \leq \varepsilon$$

Since $C(K_\theta)$, $\theta = 1, 2$, have the Grothendieck property, we deduce from the fact that the elements of $(g_n^{(\theta)} : n \in \mathbb{N})$ have pairwise disjoint supports and from (1.1.7) that the sets

$$A^{(1)} := \{ g_n^{(1)} x(n,i) \mid i \leq \ell(n),\ n \in \mathbb{N} \} \text{ and } A^{(2)} := \{ g_n^{(2)} y(n,i) \mid i \leq \ell(n),\ n \in \mathbb{N} \}$$

are limited in $C(K_1)$ and $C(K_2)$ respectively. By (4.5.2)(b) the set $A^{(1)} \otimes A^{(2)}$, and thus the set

$$A := 2 \sup_{n' \in \mathbb{N}} \| f_{n'} \|_\wedge \cdot \operatorname{aco}(A^{(1)} \otimes A^{(2)}),$$

is limited in $C(K_1) \hat{\otimes} C(K_2)$. Since $(\hat{P}_n(\tilde{f}_n) : n \in \mathbb{N}) \subset A$ and since the identity on $C(K_1) \otimes C(K_2)$ can be extended to a linear and bounded operator $T : C(K_1) \hat{\otimes} C(K_2) \to C(K_1) \check{\otimes} C(K_2)$, we deduce that $(\check{P}_n(\tilde{f}_n) : \in \mathbb{N})$ is limited in $C(K_1) \check{\otimes} C(K_2)$.

Thus, for any normed weak*-zero sequence $(\mu_n : n \in \mathbb{N})$ in $M(K, X')$ it follows from (2) and (3) that

$$\begin{aligned} \limsup_{n \to \infty} |\langle \mu_n, f_n \rangle| &= \limsup_{n \to \infty} |\langle \mu_n, \check{P}_n(f_n) \rangle| \\ &\leq \limsup_{n \to \infty} |\langle \mu_n, \check{P}_n(\tilde{f}_n) \rangle| + \varepsilon = \varepsilon, \end{aligned}$$

which finishes the proof.

◇

Proof of Theorem (4.5.1)

(a) $\Rightarrow$(b): Theorem (4.4.5).

$\neg$(a) $\Rightarrow$ $\neg$(b):We have to show that if $A \subset C(K_1 \times K_2)$ is not limited and satisfies (b)(i), then A does not satisfy (b)(ii).

For this suppose $(f_n : n \in \mathbb{N}) \subset A$ is not limited in $C(K_1 \times K_2)$ and satisfies (b)(i).

We first observe that the conditions (b) and (c) of Lemma (4.5.3) hold by taking $K := K_1$ and $X := C(K_2)$. Moreover, (a) of (4.5.3) is satisfied, which can be seen as follows:

The sequence $(f_n : n \in \mathbb{N})$, viewed in $L_\infty(\Sigma_{K_2}, C(K_1))$, satisfies case 1 of Corollary (4.3.7)(a). This implies, by (4.3.7)(b), that for each $\xi \in K_1$ and each $N \in \mathcal{P}_\infty(\mathbb{N})$, there is an $\tilde{N} \in \mathcal{P}_\infty(N)$ for which the sequence $(f_n(\xi,\cdot) : n \in \tilde{N})$ converges pointwise on K_2. Thus, $(f_n(\xi,\cdot) : n \in \mathbb{N})$ is conditionally weakly compact in $C(K_2)$, which implies, by (1.1.7), that it is limited in $C(K_2)$.

Thus, all of the assumptions of Lemma (4.5.3) are satisfied and we can choose an $N_1 \in \mathcal{P}_\infty(\mathbb{N})$ and, for each $n \in N_1$, an open $O_n^{(1)} \subset K_1$ and a closed $C_n^{(1)} \subset K_1$ satisfying (d) and (e) of (4.5.3). We now choose, for each $n \in N_1$, a $g_n^{(1)} \in C(K_1)$ with

(1) $$0 \le g_n^{(1)} \le 1\,,\ g_n^{(1)}|_{C_n^{(1)}} = 0\,,\ \text{and } g_n^{(1)}|_{(O_n^{(1)})^C} = 1.$$

By taking $K := K_2$ and $X := C(K_1)$, we may proceed as above to show that the sequence $(g_n^{(1)} f_n : n \in N_1)$ also satisfies the assumptions of Lemma (4.5.3). Thus, we can choose an $N_2 \in \mathcal{P}_\infty(N_1)$ and, for each $n \in N_2$, an open $O_n^{(2)} \subset K_2$ and a closed $C_n^{(2)} \subset K_2$ such that (d) and (e) of (4.5.3) are satisfied and we can choose, for each $n \in N_2$, a $g_n^{(2)} \in C(K_2)$ with

(2) $$0 \le g_n^{(2)} \le 1\,,\ g_n^{(2)}|_{C_n^{(2)}} = 0\,,\ \text{and } g_n^{(1)}|_{(O_n^{(2)})^C} = 1.$$

By (e) of (4.5.3), we deduce the existence of $N_3 \in \mathcal{P}_\infty(N_2)$ and a normed w*-zero sequence $(\mu_n : n \in N_3) \subset M(K_1 \times K_2)$ with

(3) $$\varepsilon := \frac{1}{2} \inf_{n \in N_3} \langle \mu_n, (g_n^{(1)} \otimes g_n^{(2)}) f_n \rangle > 0.$$

We have shown that $((g_n^{(1)} \otimes g_n^{(2)}) f_n : n \in N_3)$ satisfies condition (a) of (4.5.4) and for the above chosen $O_n^{(\theta)}$ and $C_n^{(\theta)}$, $n \in N_3$ and $\theta = 1, 2$, but it does not satisfy the assertion of (4.5.4). This implies that (b) of (4.5.4) cannot be satisfied, and thus,

(4) any sequence $(h_n : n \in N_3) \subset C(K_1) \otimes C(K_2)$, with $\| h_n - (g_n^{(1)} \otimes g_n^{(2)}) f_n \|_\vee \le \varepsilon$ for $n \in N_3$, is unbounded in $C(K_1) \hat{\otimes} C(K_2)$.

To show that (b)(ii) is not valid, let $(\tilde{f}_n : n \in N_3) \subset C(K_1) \hat{\otimes} C(K_2)$ be arbitrary with $\| \tilde{f}_n - f_n \|_\vee < \varepsilon$ if $n \in N_3$. Since $\| (g_n^{(1)} \otimes g_n^{(2)})(f_n - \tilde{f}_n) \|_\vee \le \varepsilon$ for $n \in N_3$, (4) implies that $(\tilde{f}_n (g_n^{(1)} \otimes g_n^{(2)}) : n \in N_3)$ is unbounded in $C(K_1) \hat{\otimes} C(K_2)$ and since $\| \tilde{f}_n \|_\wedge \ge \| (g_n^{(1)} \otimes g_n^{(2)}) \tilde{f}_n \|_\wedge$ if $n \in N_3$, we deduce that $(\tilde{f}_n : n \in \mathbb{N})$ is unbounded in $C(K_1) \hat{\otimes} C(K_2)$, which finishes the proof.

◇

Finally, we want to construct three bounded subsets A, B, and C of $\ell_\infty\check{\otimes}\ell_\infty$ which demonstrate the following:

- The implication (b) $\Rightarrow$(a) of Proposition (4.4.2) is not true in general (A and B).
- Neither (b)(i) nor (b)(ii) are superfluous in Theorem (4.5.1), i.e. both are necessary to imply limitedness (C and B respectively).
- Not every limited set of $\ell_\infty\check{\otimes}\ell_\infty$ can be obtained in the way described by Proposition (4.5.2)(b), i.e. not every limited set is almost (in the sense of (1.1.3)) a finite sum of products of limited sets in ℓ_∞ (this shows the set A).

4.5.5 Examples: For each $n \in \mathbb{N}$ we define the following elements of $\ell_\infty\check{\otimes}\ell_\infty$ $(= C(\beta\mathbb{N}\times\beta\mathbb{N}) = C(\beta\mathbb{N},\ell_\infty))$:

$$x_n := \sum_{i=1}^{n} \chi_{\{i\}} \otimes \chi_{\{i\}} \qquad (= \sum_{i=1}^{n} \chi_{\{(i,i)\}}),$$

$$y_n := \sum_{i=1}^{n} \chi_{\{i\}} \otimes \sum_{j=i}^{n} \chi_{\{j\}} \qquad (= \sum_{i=1}^{n} \chi_{\{i\}\times\{i,i+1,\dots n\}}), \text{ and}$$

$$z_n := \sum_{i=1}^{2^{n-1}} \chi_{A(n,i)} \otimes \chi_{\{i\}} \qquad (= \sum_{i=1}^{2^{n-1}} \chi_{\overline{A(n,i)}} \otimes \chi_{\{i\}}),$$

where $(A(n,i) : n \in \mathbb{N}_0, i \in \{1,2,\dots,2^n\}) \subset \mathcal{P}_\infty(\mathbb{N})$ satisfies the conditions of a tree of sets.

Then for $A := \{x_n\,|\,n \in \mathbb{N}\}$, $B := \{y_n\,|\,n \in \mathbb{N}\}$, and $C := \{z_n\,|\,n \in \mathbb{N}\}$ the following hold:

a) For each $D \in \{A,B,C\}$, the sets $D^{(1)} := \{f(\xi,\cdot)\,|\,\xi \in \beta\mathbb{N},\ f \in D\}$ and $D^{(2)} := \{f(\cdot,\xi)\,|\,\xi\in\beta\mathbb{N},\ f \in D\}$ are limited in ℓ_∞.

b) A satisfies (b)(i) and (b)(ii),
B satisfies (b)(i) but not (b)(ii), and
C satisfies (b)(ii) but not (b)(i) of Theorem (4.5.5).

In particular, A is limited in $\ell_\infty\check{\otimes}\ell_\infty$ but the sets B and C are not.

Proof of (4.5.5) :

(1) $A^{(1)}$, $A^{(2)}$, $B^{(1)}$, $B^{(2)}$, $C^{(1)}$ and $C^{(2)}$ are limited in ℓ_∞:

To see this, $A^{(1)}$, $A^{(2)}$, $B^{(1)}$, $B^{(2)}$, and $C^{(1)}$ are subsets of the weakly conditionally compact set $B_1(c_o)$. Moreover,

$$C^{(2)} = \{\chi_{\overline{A(n,j)}}\,|\,n \in \mathbb{N}_0,\ j \leq 2^n\} \cup \{0\},$$

so each subsequence of $C^{(2)}$ contains either a sequence with pairwise disjoint supports or a monotone decreasing subsequence. This implies that $C^{(2)}$ is conditionally weakly compact. The assertion thus follows by (1.1.7).
(2) A and B satisfy condition (b)(i):
To see this, we observe that A and B are, for both embeddings into $L^c_\infty(\Sigma_{\beta\mathbb{N}}, \ell_\infty)$, measurable with respect to the σ-algebra generated by the countable partition

$$\pi := \{\{i\} \mid i \in \mathbb{N}\} \cup \{\beta\mathbb{N} \setminus \mathbb{N}\}.$$

(3) A and C satisfy condition (b)(ii):
For this we first make the following observation:
Let K_1 and K_2 be two compact spaces and let $g_1, g_2, \ldots, g_k \in C(K_1)$ and $h_1, h_2, \ldots, h_k \in C(K_2)$. Then

$$\begin{aligned}
2^{-k} & \sum_{A\subset\{1,\ldots,k\}} \Big(\sum_{j\in A} g_j - \sum_{j\notin A} g_j\Big) \otimes \Big(\sum_{j\in A} h_j - \sum_{j\notin A} h_j\Big) \\
&= 2^{-k} \sum_{A\subset\{1,\ldots,k\}} \sum_{i,j=1}^{k} (\chi_A(i)g_i - \chi_{A^c}(i)g_i) \otimes (\chi_A(j)h_j - \chi_{A^c}(j)h_j) \\
&= 2^{-k} \sum_{i,j=1}^{k} \sum_{A\subset\{1,\ldots,k\}} (\chi_A(i)g_i - \chi_{A^c}(i)g_i) \otimes (\chi_A(j)h_j - \chi_{A^c}(j)h_j) \\
&= 2^{-k} \sum_{i=1}^{k} \sum_{A\subset\{1,\ldots,k\}} (\chi_A(i)g_i - \chi_{A^c}(i)g_i) \otimes (\chi_A(i)h_i - \chi_{A^c}(i)h_i) \\
&\quad + 2^{-k} \sum_{i,j=1, i\neq j}^{k} \sum_{A\subset\{1,\ldots,k\}} (\chi_A(i)g_i - \chi_{A^c}(i)g_i) \otimes (\chi_A(j)h_j - \chi_{A^c}(j)h_j) \\
&= 2^{-k} \sum_{i=1}^{k} \Big(\sum_{i\in A\subset\{1,\ldots,k\}} + \sum_{i\notin A\subset\{1,\ldots,k\}} \Big)(g_i \otimes h_i) \\
&\quad + 2^{-k} \sum_{i,j=1, i\neq j}^{k} \Big(\sum_{\substack{A\subset\{1,\ldots,k\} \\ i,j\in A}} - \sum_{\substack{A\subset\{1,\ldots,k\} \\ i\in A, j\notin A}} - \sum_{\substack{A\subset\{1,\ldots,k\} \\ j\in A, i\notin A}} + \sum_{\substack{A\subset\{1,\ldots,k\} \\ i,j\notin A}} \Big)(g_i \otimes h_j) \\
&= \sum_{i=1}^{k} g_i \otimes h_i .
\end{aligned}$$

If, moreover, $(g_i : i = 1, \ldots, k)$ and $(h_j : j = 1, \ldots, k)$ are normed and have pairwise disjoint supports, we deduce that

$$\Big\|\sum_{i=1}^{k} g_i \otimes h_i\Big\|_{\wedge} \leq 2^{-k} |\mathcal{P}(\{1,2,\ldots,k\})| = 1.$$

Now we observe that, for each $n \in \mathbb{N}$, x_n and z_n can be represented in the above form with pairwise disjoint and normed $g_1, ..., g_k \in C(\beta\mathbb{N})$ and $h_1, ..., h_k \in C(\beta\mathbb{N})$ and so the assertion follows.

(4) C does not satisfy (b)(i):

Let $N \in \mathcal{P}_\infty(\mathbb{N})$ and let γ and γ' be two distinct branches of $(A(n,j) : n \in N, j \leq 2^n)$, i.e. $\gamma = ((n, j(n,\gamma)) : n \in N)$ with $j(n,\gamma) \leq 2^n$ and $A(n, j(n,\gamma)) \subset A(m, j(m,\gamma))$ whenever $m < n$.

Then there exist

$$\xi \in A_\gamma := \bigcap_{n\in N} \overline{A(n, j(n,\gamma))} \quad \text{and}$$

$$\xi' \in A_{\gamma'} := \bigcap_{n\in N} \overline{A(n, j(n,\gamma'))}$$

and an $n \in N$ with $\| f_n(\xi) - f_n(\xi') \| = 2$ (take $n \in N$ with $j(n,\gamma) \neq j(n,\gamma')$).

Since there are uncountably many branches, we deduce the assertion.

(5) B does not have property (b)(ii).

We introduce the following notations:

$\|\cdot\|_p$, where $1 \leq p \leq \infty$, denotes the usual ℓ_p norm on ℓ_p and ℓ_p^n ($n \in \mathbb{N}$).

For $i, j \in \mathbb{N}$, let

$$a(i,j) := \begin{cases} 0 & \text{if } i = j \\ \frac{1}{j-i} & \text{if } i \neq j \end{cases}$$

and let M and M_n ($n \in \mathbb{N}$) be the matrices

$$M := (a(i,j) : i, j \in \mathbb{N}) \qquad \text{and} \qquad M_n := (a(i,j) : i,j \in \{1,2,...,n\}).$$

For $n \in \mathbb{N}$ and $1 \leq p \leq \infty$ define

$$P_p^{(n)} : \ell_p \to \mathbb{R}^n, \quad x \mapsto x^{(n)} := (x_1, x_2, ..., x_n) \text{ for } x = (x_i) \in \ell_p, \text{ and}$$

$$Q_p^{(n)} : \mathbb{R}^n \to \ell_p, \quad x \mapsto (x_1, x_2, ..., x_n, 0, 0, ...).$$

Now we recall the following result of Schur [29, p.212, Theorem 293]:

The mapping

$$T : \ell_2 \to \mathbb{R}^{\mathbb{N}}, \quad x = (x_i) \mapsto M \circ x = (\sum_{j=1}^{\infty} a(i,j) x_j : i \in \mathbb{N}),$$

is well defined, takes its values in ℓ_2, and is a bounded and linear operator on ℓ_2.

From this result we deduce that for each $n \in \mathbb{N}$ the operator

$$T_n := \frac{1}{n} Q_1^{(n)} \circ M^{(n)} \circ P_\infty^{(n)} : \quad \ell_\infty \to \ell_1$$

satisfies

$$\text{(6)} \qquad \begin{aligned} \|T_n\| &= \sup_{x,y\in B_1(\ell_\infty)} |\langle x, T_n(y)\rangle| \\ &= \frac{1}{n}\sup_{x,y\in B_1(\ell_\infty^n)} |\langle x, M_n \circ y\rangle| \\ &= \frac{1}{n}\sup_{x,y\in B_1(\ell_\infty^n)} |\langle \frac{x}{\|x\|_2}, M_n \circ \frac{y}{\|y\|_2}\rangle| \, \|x\|_2 \, \|y\|_2 \\ &\le \sup_{x,y\in B_1(\ell_2)} |\langle x, M \circ y\rangle| = \|T\|_2 \\ &[\|x\|_2 \le \sqrt{n}\, \|x\|_\infty \text{ for } x \in \mathbb{R}^n]. \end{aligned}$$

Finally, we need the following inequality which can be obtained by passing to integrals of minorizing functions:

$$\text{(7)} \qquad \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{1}{j-i} \ge (n-2)\ln(n-2) \text{ whenever } n \in \mathbb{N} \text{ with } n \ge 2.$$

We have, for each $n \in \mathbb{N}$ and each $\tilde{y} \in (y_n + B_{1/4}(\ell_\infty \check{\otimes} \ell_\infty)) \cap \ell_\infty \otimes \ell_\infty$, that

$$\text{(8)} \qquad \tilde{y}(i,j) := \langle \delta_{(i,j)}, \tilde{y}\rangle \begin{cases} \ge 3/4 & \text{if } j \in \{i, i+1, ..., n\} \\ \le 1/4 & \text{if } j \in \{1, 2, .., i-1\} \end{cases}$$

for any $i, j \in \{1, 2, ..., n\}$. Viewing T_n as an element of $(\ell_\infty \hat{\otimes} \ell_\infty)' = L(\ell_\infty, \ell'_\infty)$, we deduce that

$$\text{(9)} \qquad \begin{aligned} n\langle T_n, \tilde{y}\rangle &= n \sum_{i,j\in\{1,...,n\}} \tilde{y}(i,j) T_n(\chi_{\{i\}} \otimes \chi_{\{j\}}) \\ &= \sum_{i,j\in\{1,...,n\}} \tilde{y}(i,j) a(i,j) \\ &= \sum_{i=1}^{n}\sum_{j=i+1}^{n} \frac{1}{j-i}\tilde{y}(i,j) + \sum_{i=1}^{n}\sum_{j=1}^{i-1} \frac{1}{j-i}\tilde{y}(i,j) \\ &= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{1}{j-i}(\tilde{y}(i,j) - \tilde{y}(j,i)) \\ &\ge \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{1}{j-i} \\ &\ge \frac{n-2}{2}\ln(n-2). \end{aligned}$$

In order to show the assertion, let $A^{(1/4)} \subset \ell_\infty \otimes \ell_\infty$ with $\{y_n \mid n \in \mathbb{N}\} \subset A^{(1/4)} + B_{1/4}(\ell_\infty \check{\otimes} \ell_\infty)$. For each $n \in \mathbb{N}$, we can choose $\tilde{y}_n \in A^{1/4}$ with $\|\tilde{y}_n - y_n\|_\vee \leq 1/4$, which implies by (9) that

$$\limsup_{n\to\infty} |\langle T_n, y_n\rangle| \leq \frac{1}{2} \limsup_{n\to\infty} \frac{n-2}{n} \ln(n-2) = \infty .$$

By (6), $(T_n : n \in \mathbb{N})$ is bounded in $(\ell_\infty \hat{\otimes} \ell_\infty)'$, this implies that $(\tilde{y}_n : n \in \mathbb{N})$ is unbounded in $\ell_\infty \hat{\otimes} \ell_\infty$, which finishes the proof.

◇

5 Examples

Concerning examples of limited sets and Gelfand-Phillips spaces, we are still in the following unsatisfying situation:

On the one hand, all of the spaces we have found up until now which do not have the Gelfand-Phillips property, have been Grothendieck C(K)-spaces (compare (1.1.8)). On the other hand, all of the concrete examples of Gelfand-Phillips spaces which we have considered up now have been spaces X whose dual unit-ball contained X-norming and weak*-sequentially pre-compact subsets (compare section (1.2)).

It seems that the literature on this field does not present any other examples. So we want to construct the following examples in the last chapter:

Example 1 shows that the Gelfand-Phillips property is not a three space property and that there are Banach spaces which fail the Gelfand-Phillips property and are generated by weakly conditionally compact sets (by (2.1.6) and (2.3.1) non reflexive Grothendieck spaces are not generated by weakly conditionally compact subsets).

Example 2 shows that there are even spaces which do not enjoy the Gelfand-Phillips property and do not contain a copy of ℓ_1.

Example 3 is a Gelfand-Phillips space which does not admit in its dual ball a norming (up to a constant) and weak*-sequentially pre-compact subset.

Example 4 is a Gelfand-Phillips space C(K) such that K does not contain any dense and sequentially pre-compact subset.

Example 5 shows that under the continuum hypothesis one can find an infinite dimensional C(K)-space having the Gelfand-Phillips property such that every converging sequence in K is eventually stationary.

5.1 Example 1: The Gelfand-Phillips property is not a three-space-property

It is easily seen that quotients of Gelfand-Phillips spaces need not be Gelfand-Phillips spaces. In fact, the space $M(\beta\mathbb{N}) = \ell'_\infty$ is a Banach lattice which does not contain a copy of c_0, and thus, by (1.2.5)(c) it is a Gelfand-Phillips space. But ℓ'_∞ has a quotient, namely ℓ_∞, which is not a Gelfand-Phillips space (consider an isometric embedding $E : \ell_1 \to \ell_\infty$ and pass to the adjoint). In this section we want, conversely, to construct a space X, which does not enjoy the Gelfand-Phillips property, but which contains a subspace Y such that Y and X/Y are both Gelfand-Phillips spaces. By this we have proven a conjecture of L. Drewnowski in [12, p.14, Conjecture].

The space to be constructed will be a $C(K)$-space where K is the Stone compact of an algebra on $\mathbb{N}$ which is generated by $\mathcal{P}_f(\mathbb{N})$ and a well-ordered family $\mathcal{R} = (R_\alpha : \alpha < \omega) \subset \mathcal{P}_\infty(\mathbb{N})$ ($\omega \in$ Ord) with the following properties:

1) For any $0 \leq \alpha < \beta < \omega$, either $R_\beta \overset{a}{\subset} R_\alpha$ or $R_\beta \cap R_\alpha \overset{a}{=} \emptyset$.
2) For each $N \in \mathcal{P}_\infty(\mathbb{N})$, there is an $\alpha < \omega$ with $R_\alpha \subset N$.

This is a specialization of the properties which were considered in [25, 27, 31] (compare Definition (5.1.1)).

5.1.1 Definition: Let $\mathcal{R} = (R_\alpha : \alpha < \omega)$ be a well-ordered (by the ordinal ω) subset of $\mathcal{P}_\infty(\mathbb{N})$. We will say that $\mathcal{R}$ has the property (F) if

(F) for any $\alpha, \beta \in [0, \omega[$ with $\alpha < \beta$, either $R_\beta \overset{a}{\subset} R_\alpha$ or $R_\alpha \cap R_\beta \overset{a}{=} \emptyset$,

and we will say that $\mathcal{R}$ satisfies (FM) if

(FM) $\mathcal{R}$ satisfies (F) and is maximal in the following sense: for each $R_\omega \in \mathcal{P}_\infty(\mathbb{N})$, the family $\tilde{\mathcal{R}} := (R_\alpha : \alpha < \omega + 1)$ does not satisfy (F).

5.1.2 Proposition: *Let $\mathcal{R} = (R_\alpha : \alpha < \omega)$ satisfy (F).*

Then the following are equivalent:

a) $\mathcal{R}$ satisfies (FM).

b) For each $N \in \mathcal{P}_\infty(\mathbb{N})$, there is an $\alpha < \omega$ with

$$|N \cap R_\alpha| = \infty \quad \text{and} \quad |N \setminus R_\alpha| = \infty .$$

Proof of (5.1.2) :

(a) $\Rightarrow$(b): Let $\mathcal{R}$ satisfy (FM) and let $N \in \mathcal{P}_\infty(\mathbb{N})$. Then $\tilde{\mathcal{R}} := (R_\alpha : \alpha < \omega + 1)$, where $R_\omega := N$ does not satisfy (F). Since $(R_\alpha : \alpha < \omega)$ satisfies (F), there is an

$\alpha < \omega$ such that the pair (α, ω) does not satisfy the alternative in (F), i.e.

$$\begin{array}{llll} \text{either} & |N \cap R_\alpha| = \infty & \text{and} & |N \setminus R_\alpha| = \infty \\ \text{or} & |N \cap R_\alpha| < \infty & \text{and} & |N \setminus R_\alpha| < \infty . \end{array}$$

Since $|N| = \infty$, the second case cannot happen and we deduce (b).

(b) $\Rightarrow$(a): For a given $R_\omega \in \mathcal{P}_\infty(\mathbb{N})$, we have to show that $\tilde{\mathcal{R}} := (R_\alpha : \alpha < \omega + 1)$ does not satisfy (F).

By (b), we find an $\alpha < \omega$ such that

$$|R_\omega \cap R_\alpha| = \infty \qquad \text{and} \qquad |R_\omega \setminus R_\alpha| = \infty .$$

Thus, the alternative in (F) does not hold for $\beta := \omega$.

◇

5.1.3 Lemma: *Let $\mathcal{R} = (R_\alpha : \alpha < \omega) \subset \mathcal{P}_\infty(\mathbb{N})$ satisfy (F), $\mathcal{A}$ the algebra on $\mathbb{N}$ generated by $\mathcal{R}$ and $\mathcal{P}_f(\mathbb{N})$, K the Stone compact of $\mathcal{A}$ (compare (0.5)), and $\tilde{K} := K \setminus \mathbb{N}$ (compare (0.5.4)).*

a) Every regular Borel measure μ on $\tilde{K}$ has a metrizable support.

b) $B_1(M(\tilde{K}))$ is $\sigma(M(\tilde{K}), C(\tilde{K}))$-sequentially compact.

c) If $\mathcal{R}$ satisfies (FM), then $\mathbb{N}$ has no converging subsequence in K.

Proof of (5.1.3) : (compare the proof of [31, p.322, Theorem 3.7] and Remark (5.1.4))

Proof of (a): By (0.5.4)(c), the topology of K is generated by $\overline{\mathcal{A}} := \{\overline{A}^K \mid A \in \mathcal{A}\}$. Thus, we have to show that for a given $\mu \in M(\tilde{K})$ and for $C := \operatorname{supp}(\mu)$ the set $\overline{\mathcal{A}} \cap C = \{\overline{A} \cap C \mid A \in \mathcal{A}\}$ is countable. Since $\mathcal{A}$ is generated by $\mathcal{R} \cup \mathcal{P}_f(\mathbb{N})$, it is enough to show that the set

$$\mathcal{R}(C) := \{C \cap \overline{R_\alpha} \mid \alpha < \omega\}$$

is countable. For any $\alpha < \omega$, it follows from the definition of supports of measures and from the fact that $\overline{R_\alpha}$ is closed and open in K that either $|\mu|(\overline{R_\alpha} \cap C) > 0$ or $\overline{R_\alpha} \cap C = \emptyset$. Thus, it is enough to show that for a given $\varepsilon > 0$ the set

$$\mathcal{R}(C, \varepsilon) := \{C \cap \overline{R_\alpha} \mid \alpha < \omega, \quad |\mu|(C \cap \overline{R_\alpha}) \geq \varepsilon\}$$

is countable.

Let us say that a finite family $(\alpha_i : i = 1, 2, ..., \ell)$ satisfies (1) if

(1) $\overline{R_{\alpha_i}} \cap \overline{R_{\alpha_j}} \cap C = \emptyset$ and $|\mu|(\overline{R_{\alpha_i}}) \geq \varepsilon$, whenever $i, j \in \{1, 2, ..., \ell\}$, $i \neq j$.

We define

(2) $$\ell(\varepsilon) := \max\{\ell \in \mathbb{N}_0 \,|\, \exists(\alpha_i : 1 \leq i \leq \ell) \subset [0, \omega[\text{ with } (1)\}$$

(note that $\ell(\varepsilon)$ exists because μ is finite) and choose a family $(\alpha_i : i \leq \ell(\varepsilon)) \subset [0, \omega[$ satisfying (1).
From the maximality of $\ell(\varepsilon)$ it follows that any element of $\mathcal{R}(C, \varepsilon)$ lies in at least one of the sets $\mathcal{R}(C, \varepsilon, i)$, $i \leq \ell(\varepsilon)$, where

(3) $$R(C, \varepsilon, i) := \{\overline{R_\alpha} \cap C \,|\, \alpha < \omega\,,\ \overline{R_\alpha} \cap \overline{R_{\alpha_i}} \cap C \neq \emptyset\} \cap \mathcal{R}(C, \varepsilon) \text{ for } i \leq \ell(\varepsilon).$$

Thus, it is sufficient to show that $R(C, \varepsilon, i)$ is countable for a given $i \leq \ell(\varepsilon)$. For this, we first show that

$$B \prec \tilde{B} \quad :\Longleftrightarrow \quad B \supset \tilde{B} \qquad \text{for } B, \tilde{B} \in R(C, \varepsilon, i)$$

defines a well-ordering on $R(C, \varepsilon, i)$.
From (0.5.5)(a) and (F) we deduce (recall that $C \subset \tilde{K}$) that

(4) $$\text{either} \quad \overline{R_\beta} \cap C \subset \overline{R_\alpha} \cap C \quad \text{or} \quad \overline{R_\alpha} \cap \overline{R_\beta} \cap C = \emptyset$$

whenever $0 \leq \alpha < \beta < \omega$.
To show that two elements $A, B \in \mathcal{R}(C, \varepsilon, i)$ are comparable, choose $\alpha, \beta \in \omega$ with $A = \overline{R_\alpha} \cap C$ and $B = \overline{R_\beta} \cap C$. W.l.o.g. we may assume that $\alpha < \beta$ and, by (4), we have to show that $\overline{R_\alpha} \cap \overline{R_\beta} \cap C \neq \emptyset$.
This can be seen as follows: assuming that $\overline{R_\alpha} \cap \overline{R_\beta} \cap C = \emptyset$ and deducing from the definition of $\mathcal{R}(C, \varepsilon, i)$ that neither $\overline{R_\alpha} \cap \overline{R_{\alpha_i}} \cap C$ nor $\overline{R_\beta} \cap \overline{R_{\alpha_i}} \cap C$ is empty, we observe by (4) that

$$\overline{R_\alpha} \cap C \subset \overline{R_{\alpha_i}} \cap C \text{ and } \overline{R_\beta} \cap C \subset \overline{R_{\alpha_i}} \cap C$$

(note that $\overline{R_{\alpha_i}} \cap C \subset \overline{R_\alpha} \cap C$ cannot be true since $\overline{R_\beta} \cap \overline{R_{\alpha_i}} \cap C \neq \emptyset$ and $\overline{R_\alpha} \cap \overline{R_\beta} \cap C = \emptyset$ (by the assumption); in the same way we show that $\overline{R_{\alpha_i}} \cap C \subset \overline{R_\beta} \cap C$ cannot be true).

But this would imply that the family $(\tilde{\alpha}_j : 1 \leq j \leq \ell(\varepsilon)+1)$, where

$$\tilde{\alpha}_j := \begin{cases} \alpha_j & \text{if } j \leq \ell(\varepsilon),\, j \neq i \\ \alpha & \text{if } j = i \\ \beta & \text{if } j = \ell(\varepsilon)+1 \end{cases} \qquad \text{for } j \in \{1,2,\ldots,\ell(\varepsilon)+1\},$$

would satisfy (1), which is a contradiction of the maximality of $\ell(\varepsilon)$. Thus, we have shown that $\mathcal{R}(C,\varepsilon,i)$ is linearly ordered.

To show that $\mathcal{R}(C,\varepsilon,i)$ is well-ordered, let $\tilde{\mathcal{R}} \subset \mathcal{R}(C,\varepsilon,i)$ be non empty. Then there exists $\tilde{\alpha} := \min\{\alpha < \omega \,|\, C \cap \overline{R_\alpha} \in \tilde{\mathcal{R}}\}$. Since for each $B = \overline{R_\alpha} \cap C \in \tilde{\mathcal{R}}$ it follows that $\alpha \geq \tilde{\alpha}$, we deduce from (4) that $\overline{R_\alpha} \cap C \subset \overline{R_{\tilde{\alpha}}} \cap C$; thus, $\overline{R_{\tilde{\alpha}}} \cap C$ is minimal in $\tilde{\mathcal{R}}$.

Now the set $\{B \setminus \mathrm{succ}(B) \,|\, B \in \mathcal{R}(C,\varepsilon,i)\}$, where $\mathrm{succ}(B)$ is the successor of B (with respect to $\prec$) if it exists and $\emptyset$ if not, consists of pairwise disjoint, clopen (with respect to the topology on C), and non empty subsets of C with strictly positive $|\mu|$-measure. Since $|\mu|$ is finite, we deduce that $\mathcal{R}(C,\varepsilon,i)$ is countable, and thus, the assertion.

Proof of (b): Let $(\mu_n : n \in \mathbb{N}) \subset B_1(M(\tilde{K}))$ and set $\mu := \sum_{n\in\mathbb{N}} 2^{-n}|\mu_n|$. Then $C := \mathrm{supp}(\mu)$ contains the support of each μ_n. Since C is metrizable, $M(C)$ is weak*-sequentially compact. Since the inclusion $E : M(C) \to M(\tilde{K})$ is weak*-continuous (it is the adjoint of the restriction-map $R : C(\tilde{K}) \to C(C)$) and since E maps each μ_n to itself, we deduce the assertion.

Proof of (c): Assume that $\mathcal{R}$ satisfies (FM) and let $N \in \mathcal{P}_\infty(\mathbb{N})$. By (5.1.2), there is an $\alpha < \omega$ with

$$\liminf_{n\in N, n\to\infty} \delta_n(R_\alpha) = 0 \quad \text{and} \quad \limsup_{n\in N, n\to\infty} \delta_n(R_\alpha) = 1\,,$$

which implies that N does not converge in K since $\overline{R_\alpha}$ is closed and open in K.

◇

5.1.4 Remark: The idea to consider Stone compacts K over algebras generated by systems $\mathcal{R} \subset \mathcal{P}_\infty(\mathbb{N})$, for wich

$$R \overset{a}{\subset} \tilde{R} \quad \text{or} \quad R \overset{a}{\supset} \tilde{R} \quad \text{or} \quad R \cap \tilde{R} \overset{a}{=} \emptyset \qquad \text{whenever } R, \tilde{R} \in \mathcal{R},$$

originates from D. H. Fremlin (compare [31, p.322, line 8]). It was used to construct counter-examples for the weak*-sequentially compactness of dual balls:

a) R. Haydon showed in [31, p.322, Theorem 3.2] that $C(K)$ does not contain $\ell_1(\Gamma)$ for an uncountable set Γ and that the dual ball of $C(K)$ is not weak*-sequentially compact. In the proof of (5.1.3), we use essentially his ideas.

b) J. Hagler and F. Sullivan [27, p.501] showed in a direct way (without using (5.1.3)(a)) that $C(K)/c_o$ has a weak*-sequentially compact dual ball (recall that $C(K)/c_o \equiv C(\tilde{K})$ by (0.5.5)), and thus, they showed that the property of Banach spaces having weak*-sequentially compact dual balls is not a three-space property.

c) J. Hagler and E. Odell [25] constructed a non-separable James tree-space Y and showed that there is a subspace of $C(K) \oplus_2 Y$ which does not contain a copy of ℓ_1 and does not admit a weak*-sequentially compact dual ball; thus, they sharpened the result cited in (a).

We want to show now that a family $\mathcal{R} = (R_\alpha : \alpha < \omega) \subset \mathcal{P}_\infty(\mathbb{N})$ satisfying (FM) can be chosen with the property that the corresponding $C(K)$, where K is defined as in (5.1.3), is not a Gelfand-Phillips space. Therefore, we need first the following result about the cardinality of systems $\mathcal{R}$ which satisfy (FM):

5.1.5 Lemma: *Let $\mathcal{R} = (R_\alpha : \alpha < \omega) \subset \mathcal{P}_\infty(\mathbb{N})$ satisfy (FM).*

Then $|\mathcal{R}| = |\omega_c|$

(note that this assertion follows trivially from (5.1.3)(c) if we assume the continuum hypothesis).

Proof of (5.1.5) :

Let $\mathcal{R} = (R_\alpha : \alpha < \omega)$ satisfy (FM) and hence, condition (b) of (5.1.2).

First we observe that

(1) if $\alpha, \beta \in [0, \omega[$, $|R_\alpha \cap R_\beta| = \infty$, and $|R_\alpha \setminus R_\beta| = \infty$,
then $\alpha < \beta$ and $R_\beta \overset{a}{\subset} R_\alpha$.

This can be seen as follows:

From $|R_\alpha \cap R_\beta| = \infty$ and (F) it follows, that

$$R_\alpha \overset{a}{\subset} R_\beta \text{ and } \alpha \leq \beta \qquad \text{or} \qquad R_\beta \overset{a}{\subset} R_\alpha \text{ and } \beta \leq \alpha.$$

Since $|R_\alpha \setminus R_\beta| = \infty$ the first possibility cannot be true and α and β cannot be equal.

Secondly, we show that

(2) for each $\alpha < \omega$, there are $\alpha_1, \alpha_2 \in]\alpha, \omega[$ with

$$R_{\alpha_1} \cap R_{\alpha_2} \overset{a}{=} \emptyset \qquad \text{and} \qquad R_{\alpha_1} \cup R_{\alpha_1} \overset{a}{\subset} R_\alpha .$$

For this let $\alpha < \omega$. By (5.1.2)(b) (set $N := R_\alpha$), there is a $\beta < \omega$ with

$$|R_\alpha \cap R_\beta| = \infty \quad \text{and} \quad |R_\alpha \setminus R_\beta| = \infty, \tag{3}$$

which implies by (1) that $\alpha < \beta$ and $R_\beta \overset{a}{\subset} R_\alpha$. Thus,

$$\alpha_1 := \min\{\beta \in]\alpha, \omega[\mid R_\beta \text{ satisfies (3) and } R_\beta \overset{a}{\subset} R_\alpha\}$$

exists.

Then we apply (5.1.2)(b) to $N := R_\alpha \setminus R_{\alpha_1}$ and we find an $\alpha_2 < \omega$ such that

$$|R_\alpha \cap R^c_{\alpha_1} \cap R_{\alpha_2}| = \infty \quad \text{and} \quad |R_\alpha \cap R^c_{\alpha_1} \setminus R_{\alpha_2}| = \infty, \tag{4}$$

wich implies by (1) that

$$\alpha < \alpha_2, \quad R_{\alpha_2} \overset{a}{\subset} R_\alpha, \text{ and with } \beta := \alpha_2 \text{ (3) is satisfied.} \tag{5}$$

From the minimality of α_1 we deduce that $\alpha_1 \leq \alpha_2$ and from (4) we have $\alpha_1 < \alpha_2$. Since by (4) it is not possible that $R_{\alpha_2} \overset{a}{\subset} R_{\alpha_1}$, we deduce from (F) that $R_{\alpha_1} \cap R_{\alpha_2} \overset{a}{=} \emptyset$. Thus, (2) is shown.

Using (2), we can inductively choose a family $(\alpha(n,j) : n \in \mathbb{N}_0, j \in \{1, ..., 2^n\})$ with

$$R_{\alpha(n,j)} \cap R_{\alpha(n,i)} \overset{a}{=} \emptyset \text{ and } R_{\alpha(n+1,2j-1)} \cup R_{\alpha(n+1,2j)} \overset{a}{\subset} R_{\alpha(n,j)}, \tag{6}$$

whenever $n \in \mathbb{N}$ and $i, j \in \{1, ..., 2^n\}$ with $i \neq j$.

Let Br be the set of all branches of $((n,j) : n \in \mathbb{N}_0, j \in \{1, ..., 2^n\})$, i.e. the set of all $\gamma = (j(n,\gamma) : n \in \mathbb{N}_0) \subset \mathbb{N}$ with $j(0,\gamma) = 1$ and $j(n+1,\gamma) \in \{2j(n,\gamma), 2j(n,\gamma)-1\}$. For every $\gamma \in$Br, we choose an $N_\gamma \in \mathcal{P}_\infty(\mathbb{N})$ such that

$$N_\gamma \overset{a}{\subset} R_{\alpha(n,j(n,\gamma))} \text{ for } n \in \mathbb{N} \tag{7}$$

(note that this is possible since, by (6), the sets $\bigcap_{m \leq n} R_{\alpha(m,j(m,\gamma))}$, $n \in \mathbb{N}_0$, are of infinite cardinality).

Applying (5.1.2)(b) for each $\gamma \in$Br, we find $\alpha(\gamma) < \omega$ with

$$|N_\gamma \cap R_{\alpha(\gamma)}| = \infty \text{ and } |N_\gamma \setminus R_{\alpha(\gamma)}| = \infty.$$

Using (7), we deduce therefore that

$$|R_{\alpha(n,\gamma(n))} \cap R_{\alpha(\gamma)}| = \infty \text{ and } |R_{\alpha(n,\gamma(n))} \setminus R_{\alpha(\gamma)}| = \infty,$$

and thus, by (1), $R_{\alpha(\gamma)}\overset{a}{\subset}R_{\alpha(n,j(n,\gamma))}$ for all $n \in \mathbb{N}$ and $\gamma \in$Br.
Since for two distinct $\gamma, \gamma' \in$Br there is an $n \in \mathbb{N}$ with $j(n,\gamma) \neq j(n,\gamma')$, it follows from (6) that

$$R_{\alpha(\gamma)} \cap R_{\alpha(\gamma')} \overset{a}{=} \emptyset \quad \text{whenever } \gamma, \gamma' \in \text{Br with } \gamma \neq \gamma' .$$

In particular, we deduce that $|\mathcal{R}| \geq |\omega_c|$ (note that $|\text{Br}| = |\omega_c|$) and thus the assertion.

◇

5.1.6 Theorem: *(Example (1))*

There exists a family $\mathcal{R} = (R_\alpha : \alpha < \omega_c) \subset \mathcal{P}_\infty(\mathbb{N})$ which satisfies (F) and, moreover:
(5.1.6.1.) for each $N \in \mathcal{P}_\infty(\mathbb{N})$ there is an $\alpha < \omega_c$ with $R_\alpha \subset N$.

Let K be the Stone compact corresponding to the algebra generated by $\mathcal{R} \cup \mathcal{P}_f(\mathbb{N})$. Then the sequence $(\chi_{\{n\}} : n \in \mathbb{N})$ is limited in $C(K)$ (we consider $\mathbb{N}$ as a subset of K as in Proposition (0.5.4)).

Using Lemma (5.1.3)(b) and (1.2.2), we deduce therefore that $C(K)/c_0$ ($\equiv C(K \setminus \mathbb{N})$ by (0.5.5)) and c_0 (viewed as a subspace of $C(K)$ by (0.5.4)(d)) have the Gelfand-Phillips property but $C(K)$ does not.

Proof of (5.1.6) :
Let $(N_\alpha : \alpha < \omega_c)$ be a well-ordering of $\mathcal{P}_\infty(\mathbb{N})$. In order to show the existence of $(R_\alpha : \alpha < \omega_c)$, we choose by transfinite induction, for each $\beta < \omega_c$, an $R_\beta \in \mathcal{P}_\infty(\mathbb{N})$ such that
(1)(β) $R_\beta \subset N_\beta$, and
(2)(β) for any $\alpha < \beta$, either $R_\beta \cap R_\alpha \overset{a}{=} \emptyset$ or $R_\beta \overset{a}{\subset} R_\alpha$.

We suppose that $(R_\alpha : \alpha < \beta)$ has been chosen for a $\beta < \omega_c$ and we set

$$I := \{\alpha < \beta \,|\, |R_\alpha \cap N_\beta| = \infty\}$$

and

$$\tilde{\mathcal{R}} := (\tilde{R}_\alpha : \alpha \in I), \text{ where } \tilde{R}_\alpha := R_\alpha \cap N_\beta \text{ for } \alpha \in I .$$

Then the cardinality of $\tilde{\mathcal{R}}$ is strictly less then the cardinality of ω_c ($|\tilde{\mathcal{R}}| \leq |I| \leq |\beta| < |\omega_c|$). Thus, I is order isomorphic to an ordinal $\tilde{\omega}$ less then ω and $\tilde{\mathcal{R}}$ satisfies condition (F) as a subset of $\mathcal{P}_\infty(N_\beta)$. From Lemma (5.1.4) we deduce that $\tilde{\mathcal{R}}$

cannot be maximal; thus, there is an $\tilde{R}_\beta \in \mathcal{P}_\infty(N_\beta)$ such that $(\tilde{R}_\alpha : \alpha \in I \cup \{\beta\})$ satisfies (F).
Taking $R_\beta := \tilde{R}_\beta$, (1)($\beta$) is satisfied. In order to show (2)(β), let $\alpha < \beta$.
If $\alpha \notin I$, we deduce from the definition of I and R_β that $R_\beta \cap R_\alpha \overset{a}{=} \emptyset$.
If $\alpha \in I$ we deduce from the fact that $(\tilde{R}_\alpha : \alpha \in I \cup \{\beta\})$ satisfies (F) that

$$\text{either } (R_\beta \cap N_\beta) \cap (R_\alpha \cap N_\beta) \overset{a}{=} \emptyset \text{ or } R_\beta \cap N_\beta \overset{a}{\subset} R_\alpha \cap N_\beta .$$

Since $R_\beta \subset N_\beta$, we deduce the assertion, which finishes the induction step.

Since each $\overline{R_\alpha}$ is open and closed in K, we deduce that for each $N \in \mathcal{P}_\infty(\mathbb{N})$ the sequence $(\chi_{\{n\}} : n \in N)$ contains a subsequence (choose an α with $R_\alpha \subset N$) which admits a supremum in $C(K)$ (namly $\chi_{\overline{R_\alpha}}$). Thus, we deduce from (3.3.2) that the sequence $(\chi_{\{n\}} : n \in \mathbb{N})$ is limited in $C(K)$.

◇

5.1.7 Proposition: *Let $\mathcal{R} = (R_\alpha : \alpha < \omega) \subset \mathcal{P}_\infty(\mathbb{N})$ satisfy (F) and let K be the Stone compact corresponding to the algebra generated by $\mathcal{R}$ and $\mathcal{P}_f(\mathbb{N})$. Then the set*

$$G := \{\chi_{\overline{R_\alpha}} \mid \alpha < \omega_c\} \cup \{\chi_{\{n\}} \mid n \in \mathbb{N}\} \cup \{1\}$$

is conditionally $\sigma(C(K), M(K))$-compact and generates $C(K)$.

In particular, the space $C(K)$ of Theorem (5.1.6) is not a Gelfand-Phillips space, but it is conditionally weakly compactly generated and, by Corollary (2.3.3), every in $C(K)$ limited set is relatively weakly compact.

Proof of (5.1.7) :

(1) G is conditionally weakly compact.
Let $(f_n : n \in \mathbb{N}) \subset G = \{\chi_{\overline{R_\alpha}} \mid \alpha < \omega_c\} \cup \{\chi_{\{n\}} \mid n \in \mathbb{N}\} \cup \{1\}$. From (F) and (0.5.5) we deduce that for $\tilde{K} := K \setminus \mathbb{N}$ the sequence $(f_n|_{\tilde{K}} : n \in \mathbb{N})$ contains a subsequence $(f_n|_{\tilde{K}} : n \in N)$ which is either decreasing or consists only of elements with pairwise disjoint support. Thus, it is $\sigma(C(\tilde{K}), M(\tilde{K}))$-Cauchy. Since $M(K)$ is the complemented sum of $M(\tilde{K})$ and $\overline{\operatorname{span}(\{\delta_n \mid n \in \mathbb{N}\})} = \ell_1$, a separable space, we deduce the assertion (1).

(2) G generates $C(K)$.
For this we remark that

$$\tilde{D} := \mathcal{P}_f(\mathbb{N}) \cup \{\mathbb{N} \setminus A \mid A \in \mathcal{P}_f(\mathbb{N})\} \cup \{(R_\alpha \setminus A) \cup B \mid \alpha < \omega,\ A, B \in \mathcal{P}_f(\mathbb{N})\}$$

is closed under taking intersections
(remark that for $\alpha < \beta < \omega$ and $A, B, \tilde{A}, \tilde{B} \in \mathcal{P}_f(\mathbb{N})$ we deduce from (F) that the set $((R_\alpha \setminus A) \cup B) \cap ((R_\beta \setminus \tilde{A}) \cup \tilde{B})$ is either almost empty or almost equal to R_β) and we deduce the assertion from (0.5.3).

◇

5.1.8 Remark: From Proposition (5.1.2) we deduce that the system $(R_\alpha : \alpha < \omega_c)$ constructed in Theorem (5.1.6) satisfies (FM). In (5.4) we will construct, under the continuum hypothesis, another system $\hat{\mathcal{R}} = (\hat{R}_\alpha : \alpha < \omega_c)$ which satisfies (FM) but has the property that the corresponding $C(\hat{K})$ is a Gelfand-Phillips space.

5.2 Example 2: A Banach space which does not contain ℓ_1 and does not have the Gelfand-Phillips property

In this section we want to construct a Banach space which does not contain ℓ_1 and which does not have the Gelfand-Phillips property. This example shows that the result of J. Bourgain and J. Diestel (c.f. Corollary (2.3.3)(a)) cannot be sharpened in the following way: from the assumption that $\ell_1 \not\subset X$, it does not follow that all limited sets of X are relatively compact. Secondly, this example strengthens the result of J. Hagler and E. Odell [25] cited in (5.1.4)(c) (note that the weak*-sequential compactness of the dual ball of the space X implies the Gelfand-Phillips property of X).

In order to construct the desired example, we could follow the construction in [25] using the $C(K)$ constructed in Theorem (5.1.6). Since the main ideas of the proof in [25] are usable, but not the results themselves, we estimate that it is shorter to use other methods, namely the factorization theorem of W. J. Davis, T. Figiel , W. B. Johnson and A. Pełczyński, [7]. This follows an idea of C. Stegall (cited from [32]), who proposed to use this method to find a space which does not contain ℓ_1 and does not admit a w*-sequentially compact dual ball.

We recall same notations from [7]:

5.2.1 Definition: Let W be a bounded, closed, and absolutely convex subset of X.

For $n \in \mathbb{N}$, let $\|\cdot\|_n$ be the Minkowski functional of $U_n := 2^n W + 2^{-n}\mathrm{B}_1(X)$, i.e.

$$\|x\|_n := \inf\{r > 0 \mid x \in r(2^n W + 2^{-n}\mathrm{B}_1(X))\} \quad \text{for } n \in \mathbb{N},\ x \in X.$$

Let $Y := Y(X, W)$ be the space

$$Y := \{x \in X \mid \sum_{n\in\mathbb{N}} \|x\|_n^2 < \infty\}$$

endowed with the norm

$$|\!|\!|\cdot|\!|\!| : Y \to \mathbb{R}, \quad y \mapsto \Big(\sum_{n\in\mathbb{N}} \|y\|_n^2\Big)^{1/2}$$

Denote the inclusion of Y into X by $j = j(X, W)$ and the ℓ_2 sum of the spaces $(X, \|\cdot\|_n)$ by $Z := Z(X, W)$, i.e.

$$Z := \{(x_n : n \in \mathbb{N}) \subset X \mid \sum_{n\in\mathbb{N}} \|x_n\|_n^2 < \infty\}.$$

We denote the norm on Z by $|\!|\!|\cdot|\!|\!|_2$.

5.2.2 Lemma: *Let $W \subset B_1(X)$ be bounded, closed and absolutely convex and let $\|\cdot\|_n$ for $n \in \mathbb{N}$, $(Y, |\!|\!| \cdot |\!|\!|)$, j, and Z be as in (5.2.1). Then*

a) *for $n \in \mathbb{N}$ and $x \in X$,* $2^{-(n+1)} \|x\| \le \|x\|_n \le 2^n \|x\|$ *and* $\|x\|_n \le 2^{-n}$ *whenever $x \in W$,*

b) *$W \subset B_1(Y, |\!|\!| \cdot |\!|\!|)$ and $(Y, |\!|\!| \cdot |\!|\!|)$ is a Banach space,*

c) *$E : Y \to Z$, $y \mapsto (y : n \in \mathbb{N})$, is an isometric embedding, and*

d) *Y does not contain ℓ_1 if and only if W is weakly conditionally compact.*

Proof of (5.2.2) :
Proof of (a): obvious.
Proof of (b): [7, p.313, Lemma 1(i) and (ii)].
Proof of (c): obvious.
Proof of (d)($\Rightarrow$): If Y does not contain ℓ_1, then $B_1(Y, |\!|\!| \cdot |\!|\!|)$ is weakly conditionally compact by Rosenthal's ℓ_1 theorem. Since j is continuous (which follows from (a)), $B_1(Y, |\!|\!| \cdot |\!|\!|)$ is also weakly conditionally compact when viewed as a subset of X and we deduce the assertion from the first part of (b).
Proof of (d)($\Leftarrow$): Let W be conditionally $\sigma(X, X')$-compact. For each $n \in \mathbb{N}$, we deduce from the definition of $|\!|\!| \cdot |\!|\!|$ that for each $n \in \mathbb{N}$

$$B_1(Y, |\!|\!| \cdot |\!|\!|) \subset 2^n W + 2^{-n} B_1(X),$$

and thus,

$$B_1(Y, |\!|\!| \cdot |\!|\!|) \subset \bigcap_{n \in \mathbb{N}} 2^n W + 2^{-n} B_1(X),$$

which implies that $B_1(Y, |\!|\!| \cdot |\!|\!|)$ is conditionally $\sigma(X, X')$-compact.
In order to show that $B_1(Y, |\!|\!| \cdot |\!|\!|)$ is $\sigma(Y, Y')$ conditionally compact (which implies that Y does not contain ℓ_1), let $(y_n : n \in \mathbb{N}) \subset B_1(Y, |\!|\!| \cdot |\!|\!|)$ be arbitrary. We first deduce from the observations above that there is a subsequence $(y_n : n \in N)$, $N \in \mathcal{P}_\infty(\mathbb{N})$, such that $(\langle x', y_n \rangle : n \in \mathbb{N})$ converges for each $x' \in X'$. Secondly, we remark that in Z', namely the ℓ_2-sum of $((X', \|\cdot\|_n) : n \in \mathbb{N})$, the subspace V,

$$V := \{(x'_n : n \in \mathbb{N}) \subset X' \mid |\{n \in \mathbb{N} \mid x'_n \neq 0\}| < \infty\},$$

is dense. For each $v = (x'_1, x'_2, \ldots, x'_m, 0, 0 \ldots) \in V$ it follows that

$$(\langle v, E(y_n) \rangle : n \in N) = (\sum_{i=1}^{n} \langle x'_m, y_n \rangle : n \in N)$$

is convergent. Since $(E(y_n) : n \in N)$ is bounded in Z, we deduce that it is $\sigma(Z, Z')$-Cauchy, which implies by (c) that $(y_n : n \in N)$ is $\sigma(Y, Y')$-Cauchy and this finishes the proof.

◇

5.2.3 Lemma: *Let $\mathcal{R} := (R_\alpha : \alpha < \omega_c)$ and K be as in Theorem (5.1.6). Then for each bounded sequence $(\mu_n : n \in \mathbb{N}) \subset \mathrm{M}(K)$ satisfying:*

$$r := \limsup_{n\to\infty} \langle \mu_n, \chi_{\{n\}} \rangle > 0$$

there is an $\alpha < \omega_c$ such that

$$\limsup_{n\to\infty} \langle \mu_n, \chi_{R_\alpha} \rangle > \frac{r}{2}\,.$$

Proof of (5.2.3) :

Let $(\mu_n : n \in \mathbb{N}) \subset \mathrm{M}(K)$ and $r > 0$ satisfy the assumption. Then there is an $N \in \mathcal{P}_\infty(\mathbb{N})$ with

$$(1) \qquad \mu_n(\{n\}) = \langle \mu_n, \chi_{\{n\}} \rangle \geq \frac{3r}{4}, \quad (n \in N).$$

Applying the lemma of Rosenthal [9, p.82, Rosenthal's Lemma], we find an $M \in \mathcal{P}_\infty(N)$ such that

$$(2) \qquad |\mu_n|(M \setminus \{n\}) < \frac{r}{4} \quad \text{for } n \in M\,.$$

We now choose an uncountable family $(M_i : i \in I) \subset \mathcal{P}_\infty(M)$ such that M_i and M_j are almost disjoint for $i \neq j$ and we deduce from (0.5.5)(a) that $\overline{M_i} \setminus M_i$ and $\overline{M_j} \setminus M_j$ are disjoint if $i \neq j$. Since I is uncountable, we find $L := M_{i_0}$ such that

$$(3) \qquad |\mu_n|(\overline{L} \setminus L) = 0\,, \text{ for } n \in L\,.$$

By condition (5.1.6.1), we find $\alpha < \omega_c$ with $R_\alpha \subset L$ and we deduce from (1), (2), and (3) that for each $n \in R_\alpha$

$$\begin{aligned} \mu_n(\overline{R_\alpha}) &= \mu_n(\{n\}) + \mu_n(R_\alpha \setminus \{n\}) + \mu_n(\overline{R_\alpha} \setminus R_\alpha) \\ &\geq \mu_n(\{n\}) - |\mu_n|(R_\alpha \setminus \{n\}) - |\mu_n|(\overline{R_\alpha} \setminus R_\alpha) \\ &\geq \frac{3r}{4} - \frac{r}{4} = \frac{r}{2}, \end{aligned}$$

which implies the assertion.

◇

With these preparations we can formulate:

5.2.4 Theorem: *(Example 2)*

Let $\mathcal{R} := (R_\alpha : \alpha < \omega_c)$ and $C(K)$ be as in Theorem (5.1.6) and let Y be the space introduced in (5.2.1) for $W := \overline{\mathrm{aco}(\{\chi_{\{n\}} \mid n \in \mathbb{N}\} \cup \{\chi_{\overline{R_\alpha}} \mid \alpha < \omega_c\})}$.

Then Y does not contain ℓ_1 and $(\chi_{\{n\}} : n \in \mathbb{N})$ is not relatively compact but it is limited in Y.

Proof of (5.2.4) :

By Proposition (5.1.7) and (1.1.9), W is conditionally weakly compact and we deduce from Lemma (5.2.3)(d) that Y does not contain ℓ_1.

From (5.2.2)(a) we deduce for $i \neq j$ in $\mathbb{N}$ that

$$|||\chi_{\{i\}} - \chi_{\{j\}}||| \geq \| \chi_{\{i\}} - \chi_{\{j\}} \|_1 \geq 2^{-2},$$

which implies that $(\chi_{\{n\}} : n \in \mathbb{N})$ is not relatively compact in Y.

In order to show that $(\chi_{\{n\}} : n \in \mathbb{N})$ is limited in Y, let $(y'_n : n \in \mathbb{N}) \subset B_1(Y')$ with

$$(1) \qquad r := \limsup_{n\to\infty} \langle y'_n, \chi_{\{n\}} \rangle > 0 .$$

We have to show that $(y'_n : n \in \mathbb{N})$ is not weak*-zero convergent.

By Lemma (5.2.2)(c), there is, for each $n \in \mathbb{N}$, a sequence $(\mu(n,m) : m \in \mathbb{N}) \subset M(K)$ with $\sum_{m\in\mathbb{N}} \|\mu(n,m)\|_n^2 \leq 1$ such that

$$\langle y'_n, y \rangle = \sum_{m\in\mathbb{N}} \langle \mu(n,m), y \rangle , \quad \text{for each } y \in Y \text{, and } n \in \mathbb{N}.$$

Choosing an $m_0 \in \mathbb{N}$ with $2^{-m_0} < \frac{r}{4}$ and defining $\mu_n := \sum_{m \leq m_0} \mu(n,m)$, for $n \in \mathbb{N}$, we deduce from the fact that $\|\cdot\|$ and $\|\cdot\|_n$ are equivalent norms on $C(K)$ that $(\mu_n : n \in \mathbb{N})$ is bounded in $M(K)$ and that for each $n \in \mathbb{N}$

$$\begin{aligned} \langle \mu_n, \chi_{\{n\}} \rangle &= \langle y'_n, \chi_{\{n\}} \rangle - \sum_{m>m_0} \langle \mu(n,m), \chi_{\{n\}} \rangle \\ &\geq \langle y'_n, \chi_{\{n\}} \rangle - \sum_{m>m_0} \|\mu(n,m)\|_m \, \|\chi_{\{n\}}\|_m \\ &\geq \langle y'_n, \chi_{\{n\}} \rangle - \sum_{m>m_0} 2^{-m} \\ &\quad [\text{since } \|\mu(n,m)\|_m \leq 1 \text{ and using (5.2.2)(a)}] \\ &\geq \langle y'_n, \chi_{\{n\}} \rangle - \frac{r}{4} . \end{aligned}$$

This implies, together with (1), that $\limsup_{n\to\infty}\langle\mu_n,\chi_{\{n\}}\rangle \geq 3/4r$, and thus, by (5.2.2), there exists an $\alpha < \omega_c$ with

$$\limsup_{n\to\infty}\langle\mu_n,\chi_{\overline{R_\alpha}}\rangle \geq \frac{3r}{8}\,.$$

Since $\chi_{\overline{R_\alpha}} \in W \subset B_1(Y, \|\!|\cdot|\!\|)$ (by Lemma (5.2.2)(b)), we deduce that

$$\begin{aligned}
\limsup_{n\to\infty}|\langle y_n',\chi_{\overline{R_\alpha}}\rangle| &= \limsup_{n\to\infty}\Big|\langle\mu_n,\chi_{\overline{R_\alpha}}\rangle + \sum_{m>m_0}\langle\mu(n,m),\chi_{\overline{R_\alpha}}\rangle\Big| \\
&\geq \frac{3r}{8} - \sum_{m>m_0}\|\mu(n,m)\|_m\,\|\chi_{\overline{R_\alpha}}\|_m \\
&\geq \frac{3r}{8} - 2^{-m_0} \quad \geq \quad \frac{r}{8} \\
&\text{[since } \|\mu(n,m)\|_m \leq 1 \text{ and using (5.2.2)(a)]},
\end{aligned}$$

which implies that $(y_n' : n \in \mathbb{N})$ is not $\sigma(Y',Y)$-zero convergent and finishes the proof.

◇

5.3 Example 3 and 4: The Gelfand-Phillips property does not imply (w*-spcnc) and the Gelfand-Phillips property of C(K) does not imply that K contains a sequentially pre-compact dense subset.

Using the compact space K of Theorem (5.1.6), we will construct for each $k \in \mathbb{N}$ a closed subspace X_k of $C([0,\omega_c k] \times K)$, where $[0,\eta]$ is endowed with the order topology for $\eta \in \mathrm{Ord}$, such that X_k is a Gelfand-Phillips space and such that every $c > 0$, for which there is a w*-sequentially pre-compact $D \subset B_1(X_k')$ with the property

$$\| x \| \leq c \sup\{|\langle x', x\rangle| \mid x' \in D\} \quad \text{for all } x \in X_k ,$$

is not greater than $\frac{1}{k+1}$.

We deduce that the ℓ_1 sum $(\oplus_{k\in\mathbb{N}} X_k)_{\ell_1}$ still has the Gelfand-Phillips property but not the property (w*-spcnc) (see Proposition (1.2.2) and (1.2.3)). We will show that the space X_1 can be isometrically represented as C(K)-space such that K does not have a dense sequentially precompact subset; this answers a question posed by L. Drewnowski in [13, p.408, Remarks 3.3.(3)].

We begin with the well known result about $M([0,\eta], X)$, for the sake of completeness, we include a proof.

5.3.1 Lemma: *If* $\eta \in \mathrm{Ord}$, *then*

$$M([0,\eta], X) = \left\{ \sum_{n\in\mathbb{N}} x_n \delta_{\alpha_n} \;\middle|\; \begin{array}{c} \alpha_n \in [0,\eta] \text{ and } x_n \in X \text{ for } n \in \mathbb{N}, \\ \sum_{n\in\mathbb{N}} \| x_n \| < \infty \end{array} \right\}.$$

Proof of (5.3.1) :

Since $\mu \in M([0,\eta], X)$ is a sum of Dirac measures if and only if the variation has this property, it is enough to show that each positive $\mu \in M([0,\eta])$ is a sum of Dirac-measures.

We will show this by transfinite induction.

Assuming that the assertion is true for all $\tilde{\eta} < \eta$, we distinguish the following three cases:

Case 1: $\eta = \tilde{\eta} + 1$ and thus

$$\mu = \mu|_{[0,\tilde{\eta}]} + \mu(\{\eta\}) \cdot \delta_\eta$$

which, together with the assumption, implies the assertion.

Case 2: There is an increasing sequence $(\eta_n : n \in \mathbb{N}) \subset [0,\eta[$ with $\eta = \sup_{n\in\mathbb{N}} \eta_n$, and thus,

$$\mu = \mu_{[0,\eta_1]} + \mu(\{\eta\})\delta_\eta + \sum_{n\in\mathbb{N}} \mu|_{[\eta_n+1,\eta_{n+1}]},$$

which, together with the assumption, implies the assertion.
Case 3: $\eta = \sup_{\alpha<\eta}$ and the supremum of each sequence in $[0,\eta[$ lies in $[0,\eta[$.
Since μ is regular, there is an increasing sequence $(\alpha_n : n \in \mathbb{N}) \subset [0,\eta[$ with

$$\mu([0,\alpha_n[) \geq \mu([0,\eta[) - \frac{1}{n} \text{ for } n \in \mathbb{N}.$$

Thus, for $\alpha := \sup_{n\in\mathbb{N}} \alpha_n \ (< \eta)$,

$$\mu = \mu|_{[0,\alpha]} + \mu(\{\eta\})\delta_\eta,$$

which, together with the assumption, implies the assertion.

◇

5.3.2 Lemma: *Let $\mathcal{R} := (R_\alpha : \alpha < \omega_c)$ and $C(K)$ be as in Theorem (5.1.6). For each $\beta < \omega_c$ we define,*

$$Y_\beta := \overline{\operatorname{span}(c_\circ \cup \{\chi_{\overline{R_\alpha}} \mid \alpha \leq \beta\})} \quad \text{and} \quad Y^\beta := \overline{\operatorname{span}(c_\circ \cup \{\chi_{\overline{R_\alpha}} \mid \beta < \alpha < \omega_c\})}$$

Then

a) *for each $N \in \mathcal{P}_\infty(\mathbb{N})$, the sequence $(\delta_n : n \in N)$ has a $\sigma(Y'_\beta, Y_\beta)$ converging subsequence, and*
b) *$(\chi_{\{n\}} : n \in \mathbb{N})$ is limited in Y^β.*

Proof of (5.3.2) :

Proof of (a): Let $\beta < \omega_c$ and $N \in \mathcal{P}_\infty(\mathbb{N})$ and set $I := \{\alpha \leq \beta \mid |R_\alpha \cap N| = \infty\}$. The system $\tilde{R} := (R_\alpha \cap N : \alpha \in I)$ satisfies (F) as subset of $\mathcal{P}_\infty(N)$ (I is well-ordered and thus can be identified with an ordinal $\tilde{\beta} \leq \beta$) and from Lemma (5.1.2) we deduce that it cannot be maximal. Thus, by (5.1.5), there is an $M \in \mathcal{P}_\infty(N)$ such that for all $\alpha \in I$

$$|R_\alpha \cap M| < \infty \quad \text{or} \quad |M \setminus R_\alpha| < \infty.$$

Together with the definition of I, we deduce that $\lim_{n\in M} \delta_n(\overline{R_\alpha})$ exists for each $\alpha \leq \beta$ and since $(\delta_n : n \in \mathbb{N})$ converges also with respect to $\sigma(c'_\circ, c_\circ)$, we deduce the assertion.

Proof of (b): It is enough to show that the system $\tilde{\mathcal{R}} := (R_\alpha : \beta < \alpha < \omega_c)$ satisfies condition (5.1.6.1) of Theorem (5.1.6), since then it follows from Lemma (5.2.3) that $(\chi_{\{n\}} : n \in \mathbb{N})$ is limited in Y^β.
For this, let $N \in \mathcal{P}_\infty(\mathbb{N})$ be arbitrary. We choose a family $(N_\alpha : \alpha < \omega_c)$ of pairwise almost disjoint infinite subsets of N (see (0.5.6)(c)). Since $\mathcal{R}$ satisfies (5.1.6.1), we find for each $\alpha < \omega_c$ an $\hat{\alpha} < \omega_c$ with $R_{\hat{\alpha}} \subset N_\alpha$. This implies that $\hat{\alpha}_1 \neq \hat{\alpha}_2$ if $\alpha_1 \neq \alpha_2$, and, since $|\beta| < \omega_c$, we find an $\alpha < \omega_c$ for which $\beta < \hat{\alpha} < \omega_c$; this implies the assertion.

◇

5.3.3 Theorem: *(Example 3)*
Let $\mathcal{R} := (R_\alpha : \alpha < \omega_c)$ and $C(K)$ be as in Theorem (5.1.6).

a) Let $k \in \mathbb{N}$. For each $j \in \{1, ..., k\}$ and $\alpha \in [0, \omega_c[$, we consider the following element of $C([0, \omega_c k] \times K)$:

$$f_{(\alpha,j)} := \chi_{(\{0\} \cup]\omega_c \cdot (j-1)+\alpha, \omega_c \cdot j]) \times \overline{R_\alpha}} = \left(\chi_{\{0\}} + \chi_{]\omega_c \cdot (j-1)+\alpha, \omega_c \cdot j]}\right) \otimes \chi_{\overline{R_\alpha}}$$

(note that $\{0\}$ and $]\alpha, \beta]$, $0 < \alpha < \beta < \omega_c$, are open and closed in $[0, \omega_c]$, thus $f_{(\alpha,j)} \in C([0, \omega_c k] \times K)$).
We define

$$X_k := \overline{\operatorname{span}(\{g \otimes h \mid g \in C([0, \omega_c k]),\, h \in c_0\} \cup \{f_{(\alpha,j)} \mid 0 \le \alpha < \omega_c,\, j \le k\})}.$$

Then it follows:

i) X_k has the property (w-spcnc) (see (1.2.3)); in particular, it is a Gelfand-Phillips space.*

ii) The supremum of all $c > 0$ for which there is a $\sigma(X_k', X_k)$-sequentially pre-compact $D \subset B_1(X_k')$ with

$$\| x \| \ge c \sup_{x' \in D} |\langle x', x \rangle| \quad \text{for all } x \in X$$

is not greater than $1/(k+1)$. In particular, X_k does not satisfy (w-spcn).*

b) The space $Y := (\oplus_{k \in \mathbb{N}} X_k)_{\ell_1}$ has the Gelfand-Phillips property, but not the property (w-spcnc).*

Proof of (5.3.3) :
Proof of (a): Since X_k is a subspace of $C([0, \omega_c k] \times K)$, we can extend each $x' \in X_k'$ to an element $\tilde{x}' \in C([0, \omega_c k] \times K)' = M([0, \omega_c k], M(K))$ with $\| x' \| = \| \tilde{x}' \|$.

Thus, we view each $x' \in X_k'$ as an element of $M([0,\omega_c k], M(K))$ and to avoid ambiguities we must always be precise about which w* topology we consider on $M([0,\omega_c], M(K))$ and we distinguish between the following norm and semi-norm on $M([0,\omega_c k], M(K))$:

$\|\cdot\|$ is the variation-norm on $M([0,\omega_c k], M(K))$ (thus the dual norm of the usual norm on $C([0,\omega_c k]\times K)$) and $|\!|\!|\cdot|\!|\!|$ is the semi-norm on $M([0,\omega_c k], M(K))$ generated by X_k, i.e.:

$$|\!|\!|\mu|\!|\!| := \sup_{x \in B_1(X_k)} |\langle \mu, x\rangle| \quad \text{for } \mu \in M([0,\omega_c k], M(K)).$$

Proof of (a)(i): We first show that

(1) the sequence $(\mu_n : n \in \mathbb{N})$, where

$$\mu_n := \frac{1}{k+1}(\delta_0 \otimes \delta_n - \sum_{j=1}^{k} \delta_{\omega_c \cdot j} \otimes \delta_n) \text{ for } n \in \mathbb{N},$$

converges in $\sigma(X_k', X_k)$ to zero.

For this, we observe that for $\alpha < \omega$, $j \in \{1, 2..., k\}$ and $n \in \mathbb{N}$ we have

$$\langle \mu_n, f_{(\alpha,j)}\rangle = \frac{1}{k+1}(\chi_{\overline{R_\alpha}}(n) - \chi_{\overline{R_\alpha}}(n)) = 0$$

and, for $g \in C([0,\omega_c k])$ and $h \in c_o$,

$$\langle \mu_n, g \otimes h\rangle = \frac{1}{k+1}(g(0) - \sum_{j=1}^{k} g(\omega_c \cdot j))h(n) \xrightarrow[n\to\infty]{} 0,$$

which implies (1).

Secondly, we prove that

(2) for each $j \in \{1, 2..., k\}$, the set

$$D_j := \{\delta_{\omega_c(j-1)+\alpha} \otimes \delta_n \,|\, n \in \mathbb{N},\ 0 < \alpha < \omega_c\}$$

is $\sigma(X_k', X_k)$-sequentially pre-compact.

For this let $(\alpha_n : n \in \mathbb{N}) \subset]0, \omega_c[$, $(m_n : n \in \mathbb{N}) \subset \mathbb{N}$, and $j \in \{1, 2, ..., k\}$. By taking subsequences if necessary, we may assume that $(\alpha_n : n \in \mathbb{N})$ and $(m_n : n \in \mathbb{N})$ are both (not necessarily strictly) increasing. For each $\alpha < \omega_c$, $i \in \{1, 2..., k\}$, and $n \in \mathbb{N}$, it follows that

$$\langle \delta_{\omega_c(j-1)+\alpha_n} \otimes \delta_{m_n}, f_{(\alpha,i)}\rangle = \begin{cases} 0 & \text{if } i \neq j \\ 0 & \text{if } i = j \text{ and } \alpha_n \leq \alpha \\ \chi_{\overline{R_\alpha}}(m_n) & \text{if } i = j \text{ and } \alpha_n > \alpha. \end{cases}$$

By Lemma (5.3.2)(a), there is an $M \in \mathcal{P}_\infty(\mathbb{N})$ for which $(\chi_{\overline{R_\alpha}}(m_n) : n \in M)$ converges whenever $\alpha \leq \sup_{n\in\mathbb{N}} \alpha_n$ $(< \omega_c)$. Thus

$$\lim_{n\in M}\langle \delta_{\omega_c(j-1)+\alpha_n}\otimes\delta_{m_n}, f_{(\alpha,i)}\rangle = \begin{cases} 0 & \text{if } i\neq j \\ 0 & \text{if } i=j \text{ and } \alpha_n\leq\alpha \text{ for each } n\in\mathbb{N} \\ \lim_{n\in M}\chi_{\overline{R_\alpha}}(m_n) & \text{if } i=j \text{ and if there is an } n\in\mathbb{N} \\ & \quad\text{with } \alpha_n>\alpha \end{cases}$$

and, for $g\in C([0,\omega_c k])$ and $h\in c_o$,

$$\lim_{n\in M}\langle \delta_{\omega_c(j-1)+\alpha_n}\otimes\delta_{m_n}, g\otimes h\rangle = \begin{cases} 0 & \text{if } m_n \underset{n\to\infty}{\longrightarrow}\infty \\ g(\sup_{n\in\mathbb{N}}\omega_c(j-1)+\alpha_n)h(m) & \text{if } m_n=m, \text{ for all but finitely many } n\in\mathbb{N}, \end{cases}$$

which implies that $(\delta_{\omega_c(j-1)+\alpha_n}\otimes\delta_{m_n} : n\in M)$ converges in $\sigma(X_k',X_k)$ and finishes the proof of (2).

In order to show (a)(i), it remains to prove that

(3) the set $D:=\{\mu_n\,|\,n\in\mathbb{N}\}\cup\bigcup_{j\leq k}D_j$ (μ_n as in (1)) norms X up to the constant $1/(k+1)^2$.

For this, let $f\in X_k$ and $\varepsilon>0$ be arbitrary. We distinguish two cases:

Case 1: $\|f\|\leq (k+1)\sup_{0<\alpha\leq\omega_c k;\,\xi\in K}|f(\alpha,\xi)|$.

Then there are $m\in\mathbb{N}$, $j\in\{1,\ldots,k\}$, and $\alpha\in]0,\omega_c[$ with

$$|\langle\delta_{\omega_c(j-1)+\alpha}\otimes\delta_m, f\rangle|\geq \sup_{0<\beta\leq\omega_c k;\,\xi\in K}|f(\beta,\xi)|-\varepsilon\geq\frac{1}{k+1}\|f\|-\varepsilon$$

(note that $\mathbb{N}$ is dense in K and that $\bigcup_{j=1}^k]\omega_c(j-1),\omega_c j[$ is dense in $]0,\omega_c k]$). Thus, we deduce in this case that f is normed up to the factor $1/(k+1)$ by the elements of $\bigcup_{1\leq j\leq k}D_j$.

Case 2: $\|f\|>(k+1)\sup_{0<\alpha\leq\omega_c k;\,\xi\in K}|f(\alpha,\xi)|$. Thus, $|f(\cdot,\cdot)|$ takes its maximum in the set $\{0\}\times K$ and, since $\mathbb{N}$ is dense in K, we find an $m\in\mathbb{N}$ with

$$\begin{aligned}\|f\|-\varepsilon &\leq |f(0,m)| \\ &\leq |f(0,m)-\sum_{j=1}^k f(\omega_c\cdot j,m)|+|\sum_{j=1}^k f(\omega_c\cdot j,m)| \\ &\leq |\langle\delta_0\otimes\delta_m-\sum_{j=1}^k\delta_{\omega_c j}\otimes\delta_m, f\rangle|+\sum_{j=1}^k\frac{1}{k+1}\|f\|.\end{aligned}$$

Hence,

$$\begin{aligned}\left|\frac{1}{k+1}\langle\delta_0\otimes\delta_m-\sum_{j=1}^k\delta_{\omega_c j}\otimes\delta_m, f\rangle\right| &\geq \frac{1}{k+1}\left(\|f\|-\varepsilon-\frac{k}{k+1}\|f\|\right) \\ &> \frac{1}{(k+1)^2}\|f\|-\varepsilon.\end{aligned}$$

Thus, in this case, f is normed up to the factor $1/(k+1)^2$ by the elements of $(\mu_n : n \in \mathbb{N})$.

Proof of (ii): We suppose that $\tilde{D} \subset \mathrm{B}_1(x'_k)$ is $\sigma(X'_k, X_k)$-sequentially pre-compact and that $c > 0$ is such that for each $n \in \mathbb{N}$ there is a $x'_n \in \tilde{D}$ with

$$(4) \qquad \liminf_{n\to\infty} \langle \chi_{\{0\}\times\{n\}}, x'_n \rangle \geq c\,.$$

We have to show that $c \leq 1/(k+1)$.

From Lemma (5.3.1) we deduce that for each $n \in \mathbb{N}$ and $j \in \{1, ..., k\}$ there is a sequence $(\alpha(n,j,m) : m \in \mathbb{N}) \subset]0, \omega_c[$, a sequence $(\nu(n,j,m) : m \in \mathbb{N}) \subset \mathrm{B}_1(\mathrm{M}(K))$ and $(\nu(n,j) : j = 0, 1, ..., k) \subset \mathrm{B}_1(\mathrm{M}(K))$ such that

$$(5) \qquad \sum_{j=1}^{k} \sum_{m\in\mathbb{N}} \| \nu(n,j,m) \| + \sum_{j=0}^{k} \| \nu(n,j) \| \leq 1$$

and

$$(6) \quad x'_n = \delta_0 \otimes \nu(n,0) + \sum_{j=1}^{k} \delta_{\omega_c j} \otimes \nu(n,j) + \sum_{j=1}^{k} \sum_{m\in\mathbb{N}} \delta_{\omega_c(j-1)+\alpha(n,j,m)} \otimes \nu(n,j,m)\,.$$

We define

$$(7) \qquad \beta := \sup_{j\leq k, n, m\in\mathbb{N}} \alpha(n,j,m) \qquad (< \omega_c)$$

and

$$(8) \qquad \mu(n,j) := \nu(n,0) + \nu(n,j) \qquad \text{for } n \in \mathbb{N}, j \leq k$$

and choose $N \in \mathcal{P}_\infty(\mathbb{N})$ for which $(x'_n : n \in \mathbb{N})$ converges in $\sigma(X'_k, X_k)$ to an x'_0 (assumption on $\tilde{D}$). We deduce from (6) and (7), for $j \in \{1, ..., k\}$, $\alpha \in]\beta, \omega_c[$, and $n \in N$ that

$$(9) \quad \mu(n,j)(\overline{R_\alpha}) = \langle \nu(n,0), \chi_{\overline{R_\alpha}} \rangle + \langle \nu(n,j), \chi_{\overline{R_\alpha}} \rangle = \langle x'_n, f_{(\alpha,j)} \rangle \underset{n\in N}{\longrightarrow} \langle x'_0, f_{(\alpha,j)} \rangle$$

and, for $n \in N$ and $m \in \mathbb{N}$, that

$$\begin{aligned} \mu(n,j)(\{m\}) &= \langle \nu(n,0), \chi_{\{m\}} \rangle + \langle \nu(n,j), \chi_{\{m\}} \rangle \\ &= \langle x'_n, \chi_{\{0\}\times\{m\}} + \chi_{]\omega_c(j-1)+\beta, \omega_c j]\times\{m\}} \rangle \\ &\underset{n\in N}{\longrightarrow} \langle x'_0, \chi_{\{0\}\times\{m\}} + \chi_{]\omega_c(j-1)+\beta, \omega_c j]\times\{m\}} \rangle \end{aligned}$$

Thus, we have shown that $(\mu(n,j) : n \in N)$ converges in $\sigma(Y^{\beta'}, Y^{\beta})$ and, since $(\chi_{\{n\}} : n \in N)$ is limited in Y^{β} by Lemma (5.3.2), we deduce that

$$\limsup_{n\in N} |\mu(n,j)(\{n\})| = 0 \qquad \text{for each } j \leq k.$$

Together with (8), (6), and (4) this implies that

$$\begin{aligned}(10) \qquad \limsup_{n\in N} \nu(n,j)(\{n\}) &= \limsup_{n\in N}(-\nu(n,0)(\{n\}) + \mu(n,j)(\{n\})) \\ &= \limsup_{n\in N} -\nu(n,0)(\{n\}) \\ &= -\liminf_{n\in N}\langle x'_n, \chi_{\{0\}\times\{n\}}\rangle \leq -c,\end{aligned}$$

and thus,

$$\begin{aligned} 1 &\geq \liminf_{n\in N} \|x'_n\| \\ &\geq \liminf_{n\in N}\langle x'_n, \chi_{\{0\}\times\{n\}} - \sum_{j=1}^{k} \chi_{]\omega_c(j-1)+\beta,\omega_c j]\times\{n\}}\rangle \\ &\geq c - \limsup_{n\in N} \sum_{j=1}^{k} \nu(j,n)(\{n\}) \\ &\text{[by (4), (6) and (7)]} \\ &\geq c - k(-c) = (k+1)c \\ &\text{[by (10)]},\end{aligned}$$

which implies that $c \leq 1/(k+1)$ and finishes the proof of (a)(ii).

Proof of (b): For $k \in \mathbb{N}$, let $E_k : X_k \to (\oplus_{k'\in\mathbb{N}} X_{k'})_{\ell_1}$ be the canonical embedding. For a w*-sequentially pre-compact $D \subset B_1((\oplus_{k'\in\mathbb{N}} X_{k'})'_{\ell_1}))$, we deduce from (a)(ii) that

$$\inf_{x\in X_k\setminus\{0\}} \sup_{x'\in D} \frac{|\langle x, E'_k(x')\rangle|}{\|x\|} \leq \frac{1}{k+1}.$$

which implies that $(\oplus_{k'\in\mathbb{N}} X_{k'})_{\ell_1}$ does not have the property (w*-spcnc). However, (4.2.6) implies that it is a Gelfand-Phillips space.

◇

5.3.4 Theorem: *(Example 4)*
Let $\mathcal{R}$ and K be as in Theorem (3.1.6). We assume that $R_0 = \mathbb{N}$, (otherwise we pass to $\tilde{\mathcal{R}} = (\tilde{R}_\alpha : \alpha < \omega_c)$, where $\tilde{R}_0 := \mathbb{N}$, $\tilde{R}_\alpha := R_{\alpha+1}$, if $\alpha < \omega_0$, and $\tilde{R}_\alpha := R_\alpha$ if $\omega_0 \leq \alpha < \omega_c$).

Then the space X_1 constructed in Theorem (3.5.3)(a) for $k := 1$ is isometrically isomorphic to $C(K_1)$ for a compact K_1.

From (5.5.3)(a)(ii) we deduce that K_1 cannot contain a dense pre-sequentially compact set D, because otherwise it would follow that the set of all Dirac-measures on D would norm the elements of $C(K_1)$ and is $\sigma(M(K_1), C(K_2))$-sequentially precompact.

Proof of (5.3.4) :

It is enough to show that X_1 is closed under multiplication in $C([0,\omega_c]\times K)$. If we have shown this, we observe that $1 = \chi_{[0,\omega_c]\times K} = f_{(1,0)}$ and deduce the assertion from [39, p.65,Theorem 9].

In order to do this, we have to show that for any $f_1, f_2 \in G$, with

$$G := \{f_{(1,\alpha)} \mid \alpha < \omega_c\} \cup \{g \otimes h \mid g \in C([0,\omega_c], h \in c_o)\},$$

it follows that $f_1 \cdot f_2 \in \text{span}(G)$.

We distinguish the following cases (which are the only ones up to a permutation):

Case 1: $f_1 = g_1 \otimes h_1$ and $f_2 = g_2 \otimes h_2$ with $g_1, g_2 \in C([0,\omega_c])$ and $h_1, h_2 \in C([0,\omega_c])$; then

$$f_1 \cdot f_2 = (g_1 \cdot g_2) \otimes (h_1 \cdot h_2) \in G.$$

Case 2: $f_1 = g_1 \otimes h_1$ and $f_2 = f_{(1,\alpha)}$, for an $\alpha \in [0,\omega_c[$; then

$$f_1 \cdot f_2 = (g_1 \cdot \chi_{\{0\}\cup]\alpha,\omega_c]}) \otimes (h_1 \cdot \chi_{\overline{R_\alpha}}) \in G.$$

Case 3: $f_1 = f_{(1,\beta)}$ and $f_2 = f_{(1,\alpha)}$ with $0 \le \alpha \le \beta < \omega_c$; by (F) two cases are possible:

Case 3(a): $R_\alpha \cap R_\beta \overset{a}{=} \emptyset$, and thus,

$$f_1 \cdot f_2 = \chi_{(\{0\}\cup]\beta,\omega_c])\times(R_\alpha \cap R_\beta)} \in G.$$

Case 3(b): $R_\beta \overset{a}{\subset} R_\alpha$, and thus,

$$\begin{aligned} f_1 \cdot f_2 &= \chi_{(\{0\}\cup]\beta,\omega_c])\times(\overline{R_\alpha} \cap \overline{R_\beta})} \\ &= \chi_{(\{0\}\cup]\beta,\omega_c])\times\overline{R_\beta}} - \chi_{(\{0\}\cup]\beta,\omega_c])\times(R_\beta \setminus R_\alpha)} \in \text{span}(G). \end{aligned}$$

This verifies the asssertion and finishes the proof.

◇

5.4 (CH) Example 5 and 6: Two Gelfand Phillips C(K)-spaces with interesting additional properties

Using the continuum hypothesis, we will construct two compact spaces K_1 and K_2, both of infinite cardinality, with the following properties:

C(K_1) and C(K_2) both have the Gelfand-Phillips property. Moreover, both compacts have a dense subset D such that every sequence $(\xi_n : n \in \mathbb{N})$ in D contains a subsequence $(\xi_{n_k} : k \in \mathbb{N})$ such that $(\delta_{\xi_{n(2k)}} - \delta_{\xi_{n(2k-1)}} : k \in \mathbb{N})$ is w*-zero convergent (we will easily deduce from this property that C(K_1) and C(K_2) are Gelfand-Phillips spaces).

Every convergent sequence in K_1 is eventually stationary. Roughly speaking, this means that on the one hand sequences of Dirac-measures on K_1 does only converge in the trivial case, but on the other hand there are enough convergent differences of Dirac- measures to insure the Gelfand-Phillips property for C(K_2).

K_2 is a Stone compact of an algebra on $\mathbb{N}$ which is generated by $\mathcal{P}_f(\mathbb{N})$ and a system $(R_\alpha : \alpha < \omega_c) \subset \mathcal{P}_\infty(\mathbb{N})$ satisfying (F) and (FM) of Definition (5.1.1).

The construction of K_1 was pointed out to the author by D. Fremlin [20], who we whish to thank in this place for the permission to use it. The space K_2 can be constructed using similar ideas. Since some of the technical arguments for both constructions are the same, we will formulate them in the following lemmas. We begin by introducing the following notations:

5.4.1 Definition:

a) Let $\mathcal{F}$ be the set of all strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$, (so $\mathcal{F}$ is a representation of $\mathcal{P}_\infty(\mathbb{N})$).

b) Let $\tilde{\mathcal{F}} \subset \mathcal{F}$, $f \in \mathcal{F}$ and $N \in \mathcal{P}(\mathbb{N})$.
N is called *f-admissible* if there is an $i_0 \in \mathbb{N}$ such that

$$f(2i) \in N \iff f(2i-1) \in N \qquad \text{for all } i \in \mathbb{N}, i \geq i_0 ,$$

N is called *strictly f-admissible* if

$$f(2i) \in N \iff f(2i-1) \in N \qquad \text{for all } i \in \mathbb{N};$$

and N will be called (*strictly*) *$\tilde{\mathcal{F}}$-admissible* if, for each $f \in \tilde{\mathcal{F}}$, N is (strictly) f-admissible.

c) We will say that $\tilde{\mathcal{F}} \subset \mathcal{F}$ satisfyes condition (E) if
(E) for all $J \in \mathcal{P}_f(\tilde{\mathcal{F}})$ and $m \in \mathbb{N}$ there is a $K \in \mathcal{P}_f(\mathbb{N})$ with

i) $m \in K$, and

ii) K is strictly J-admissible.

5.4.2 Proposition: *Let $\tilde{\mathcal{F}} \subset \mathcal{F}$.*

a) *The set of all (strictly) $\tilde{\mathcal{F}}$-admissible subsets of $\mathbb{N}$ is a (σ-)algebra on $\mathbb{N}$.*

b) *Suppose that $\tilde{\mathcal{F}}$ satisfies (E). For $m \in \mathbb{N}$ and $J \in \mathcal{P}_f(\tilde{\mathcal{F}})$, we define recursively the following sequence $(L(J,m,n): n \in \mathbb{N}_0) \subset \mathcal{P}_f(\mathbb{N})$:*
$L(J,m,0) := \{m\}$ and, assuming that $L(J,m,n)$ has been chosen for $n \in \mathbb{N}_0$, we set

$$L(J,m,n+1) :=$$
$$L(J,m,n) \cup \left\{ f(2i), f(2i-1) \;\middle|\; \begin{array}{c} i \in \mathbb{N} \text{ and } f \in J, \text{ with} \\ \{f(2i), f(2i-1)\} \cap L(J,m,n) \neq \emptyset \end{array} \right\}.$$

Then $(L(J,m,n): n \in \mathbb{N}_0)$ is eventually stationary in $\mathcal{P}_f(\mathbb{N})$ and the set

$$K(J,m) := \cup_{n \in \mathbb{N}_0} L(J,m,n)$$

satisfies the conditions (i) and (ii) of (E), and is contained in each strictly J-admissible set which owns m.

Proof of (5.4.2) :
Proof of (a): obvious.
Proof of (b): Let $m \in \mathbb{N}$ and $J \subset \mathcal{P}_f(\tilde{\mathcal{F}})$ and let $K \in \mathcal{P}(\mathbb{N})$ satisfy the conditions (i) and (ii) of (E).
By induction we show, for each $n \in \mathbb{N}_0$, that

$$(1)(n) \qquad L(J,m,n) \subset K.$$

For $n = 0$ the assertion follows from the definition of $L(J,m,0)$ and from (E)(i). We suppose that $(1)(n)$ is satisfied for $n \in \mathbb{N}_0$ and that $k \in L(J,m,n+1)$ is arbitrary. W.l.o.g. we may assume that $k \notin L(J,m,n)$. From the definition of $L(J,m,n+1)$ it follows that there are $f \in J$, $\theta \in \{0,1\}$, and $i \in \mathbb{N}$ with $k = f(2i-\theta)$ and $f(2i+\theta-1) \in L(J,m,n)$. This implies, by (E)(ii), that $k \in K$, which finishes the induction step.

Since $(L(J,m,n): n \in \mathbb{N}_0)$ is monotone and since K can be chosen to be finite, we deduce that $(L(J,m,n): n \in \mathbb{N}_0)$ is eventually stationary and that $K(J,m)$ is finite and satisfies (E)(i).

Moreover, we deduce from the defintion of $(L(J,m,n):n\in\mathbb{N}_0)$ that

$$f(2i)\in L(J,m,n)\Rightarrow f(2i-1)\in L(J,m,n+1)\Rightarrow f(2i)\in L(J,m,n+2)$$

for $n,i\in\mathbb{N}$ and $f\in J$,

which implies that $K(J,m)$ satisfies (E)(ii).

◇

5.4.3 Lemma: *Let $\tilde{\mathcal{F}}\subset\mathcal{F}$ satisfy (E), $R\in\mathcal{P}_\infty(\mathbb{N})$ be $\tilde{\mathcal{F}}$-admissible, and $N\in\mathcal{P}_\infty(\mathbb{N})$ have an infinite intersection with R.*

Then for each $J\in\mathcal{P}_\infty(\tilde{\mathcal{F}})$ there is a sequence $(K_n:n\in\mathbb{N})\in\mathcal{P}_f(\mathbb{N})$ with

a) $K_n\cap K_m=\emptyset$ for $n,m\in\mathbb{N}$, with $n\neq m$,

b) $K_n\subset R$ and $N\cap K_n\neq\emptyset$ for $n\in\mathbb{N}$, *and*

c) *K_n is strictly J-admissible for $n\in\mathbb{N}$.*

Proof of (5.4.3) :

Let $J\in\mathcal{P}_f(\tilde{\mathcal{F}})$. Recursively, we choose a strictly increasing sequence $(m_n:n\in\mathbb{N})\subset N\cap R$ such that the sequence $(K(J,m_n):n\in\mathbb{N})$ (defined as in (5.4.2)(b)) is pairwise disjoint.

For $n=1$ we set $m_1:=\min(N\cap R)$ and, assuming that $m_1<m_2<...m_{n-1}$ are already chosen, we note that $\bigcup_{j<n}K(J,m_j)$ is strictly J-admissible and finite. Thus, there exists

$$m_n:=\min(N\cap R)\setminus\{1,2,...,\max\big(\bigcup_{j<n}K(J,m_j)\big)\},$$

and since the set $\mathbb{N}\setminus\bigcup_{j<n}K(J,m_j)$ is strictly J-admissible (by (5.4.2)(a)) and owns m_n it contains $K(J,m_n)$ (by (5.4.2)(b)).

We now want to show that there is an n_0 such that

(1) $K(J,m_n)\subset R$ for any $n\in\mathbb{N}$ with $n\geq n_0$.

Assuming that this were not true, we find an $M\in\mathcal{P}_\infty(\mathbb{N})$ such that

$$K(J,m_n)\setminus R\neq\emptyset\quad\text{whenever } n\in M.$$

Since $L(J,m_n,0)=\{m_n\}\subset R$, there exists, for each $n\in M$, the number

$$\ell_n:=\max\{\tilde{\ell}\in\mathbb{N}_0\,|\,L(J,m_n,\tilde{\ell})\subset R\}.$$

By the definition of $L(J,m_n,\ell_n+1)$, we find for each $n\in M$ an $f_n\in J$, an $i_n\in\mathbb{N}$, and a $\theta_n\in\{0,1\}$ such that for each $n\in M$

$$f_n(2i_n-\theta_n)\in L(J,m_n,\ell_n)\subset R\quad\text{and}\quad f_n(2i_n-1+\theta_n)\in L(J,m_n,\ell_n+1)\setminus R.$$

Since J is finite, we find an $\tilde{M} \in \mathcal{P}_\infty(M)$ and an $f \in J$ with $f_n = f$ for each $n \in \tilde{M}$. By the assumptions on $(m_n : n \in \mathbb{N})$, the elements of $(f(2i_n - \theta_n) : n \in \tilde{M})$ are pairwise distinct and lie in R, while $(f(2i_n - 1 + \theta_n) : n \in \tilde{M})$ lies in $\mathbb{N} \setminus R$. But this is a contradiction of the assumption that R is f-admissible; thus, we have shown (1).

Now taking $K_n := K(J, m_{n_0+n})$ for $n \in \mathbb{N}$, we deduce the assertion.

◇

5.4.4 Lemma: *Let $\tilde{\mathcal{F}} \subset \mathcal{F}$ satisfy (E) and be countable, let $(R_n : n \in \mathbb{N})$ be a sequence of $\tilde{\mathcal{F}}$-admissible sets with $R_{n+1} \overset{a}{\subset} R_n$ for each $n \in \mathbb{N}$, and let $N \in \mathcal{P}_\infty(\mathbb{N})$ satisfy $|N \cap R_n| = \infty$ for $n \in \mathbb{N}$.*

Then there exists an $\tilde{\mathcal{F}}$-admissible R with the following properties:

a) $R \overset{a}{\subset} R_n$ for each $n \in \mathbb{N}$, and

b) $|R \cap N| = \infty$ and $|N \setminus R| = \infty$.

Proof of (5.4.4) :

Let $\tilde{\mathcal{F}} = (f_n : n \in \mathbb{N})$.

By induction we choose, for each $n \in \mathbb{N}$, a $K_n \in \mathcal{P}_f(\mathbb{N})$ with

(1)(n) $K_n \subset \bigcap_{j \le n} R_j$,

(2)(n) $K_n \cap N \neq \emptyset$,

(3)(n) $K_n \cap K_m = \emptyset$ for $m < n$, and

(4)(n) K_n is strictly $\{f_1, f_2, ..., f_n\}$-admissible.

If $n = 1$, we apply Lemma (5.4.3) to $\tilde{N} =: N$, $\tilde{R} := R_1$ and, $\tilde{J} := \{f_1\}$ to find a sequence $(\tilde{K}_n^{(1)} : n \in \mathbb{N}) \subset \mathcal{P}_f(\mathbb{N})$ satisfying (a), (b), and (c) of (5.4.3). Choosing $K_1 := \tilde{K}_n^{(1)}$, we observe that (1)(1), (2)(1), (3)(1), and (4)(1) are satisfied.

Assuming that $K_1, K_2, ..., K_{n-1}$ have already been chosen, we apply Lemma (5.4.3) to $\tilde{N} := N$, $\tilde{R} := \bigcap_{j \le n} R_j$ (note that $\tilde{R} \cap N \overset{a}{=} R_n \cap N \in \mathcal{P}_\infty(\mathbb{N})$), and $\tilde{J} := \{f_1, f_2 ..., f_n\}$ to get pairwise disjoint $\tilde{K}_m^{(n)} \in \mathcal{P}_f(\mathbb{N})$, $m \in \mathbb{N}$, each of them satisfying (1)(n),(2)(n), and (4)(n). Since $\bigcup_{j<n} K_j$ is finite, we find an $m \in \mathbb{N}$ such that $K_n := \tilde{K}_m^{(n)}$ satisfies (3)(n) also. Thus, we have finished the induction step.

We now choose $R := \bigcup_{n \in \mathbb{N}} K_{2n}$ and deduce from (1)(n) that for each $m \in \mathbb{N}$

$$R \setminus R_m = \bigcup_{n \in \mathbb{N}} K_{2n} \setminus R_m \subset \bigcup_{2j<m} K_{2j} \in \mathcal{P}_f(\mathbb{N}),$$

which implies (a). From (2)(n) and (3)(n) we deduce that

$$R \cap N = \bigcup_{n \in \mathbb{N}} K_{2n} \cap N \in \mathcal{P}_\infty(\mathbb{N}) \text{ and}$$

$$N \setminus R = N \setminus \bigcup_{n \in \mathbb{N}} K_{2n} \cap N \supset \bigcup_{n \in \mathbb{N}} K_{2n-1} \cap N \in \mathcal{P}_\infty(\mathbb{N}).$$

Finally, we deduce for $n \in \mathbb{N}$ and each $i \in \mathbb{N}$ with $i \geq i_0 := 1 + \max(\{i \in \mathbb{N} \mid f_n(2i) \in \bigcup_{2j<n} K_{2j}\})$ from (3) (n) and (4)(n), that

$$f_n(2i) \in R \iff f_n(2i) \in \bigcup_{2j \geq n} K_{2j} \iff f_n(2i-1) \in \bigcup_{2j \geq n} K_{2j} \iff f_n(2i-1) \in R,$$

which implies that R is $(f_n : n \in \mathbb{N})$-admissible.

◇

5.4.5 Lemma: *Let $\tilde{\mathcal{F}}$ be a countable subset of $\mathcal{F}$ satisfying (E) and let $\mathcal{R} \subset \mathcal{P}_\infty(\mathbb{N})$ be a countable subset of $\tilde{\mathcal{F}}$-admissible sets, and let $N \in \mathcal{P}_\infty(\mathbb{N})$.*

Then there exists an $f \in \mathcal{F}$ with the following properties:

a) *$\{f\} \cup \tilde{\mathcal{F}}$ satisfies (E),*

b) *each $R \in \mathcal{R}$ is $\{f\} \cup \tilde{\mathcal{F}}$-admissible, and*

c) *$f(\mathbb{N}) \subset N$.*

Proof of (5.4.5) :

Let $\tilde{\mathcal{F}} = (f_n : n \in \mathbb{N})$ and $\mathcal{R} = (R_n : n \in \mathbb{N})$ and choose a non-principal ultra-filter $\mathcal{U}$ on $\mathbb{N}$ with $N \in \mathcal{U}$.

By induction we choose, for each $k \in \mathbb{N}$, $f(2k-1)$, $f(2k) \in \mathbb{N}$, and $K_k \in \mathcal{P}_f(\mathbb{N})$ such that

(1)(k) $f(i) \in K_k$ for all $1 \leq i < 2k-1$ and $\{1, 2, \ldots, k\} \subset K_k$,

(2)(k) K_k is strictly $\{f_1, ..., f_k\}$-admissible,

(3)(k) $f(2k-1) < f(2k)$ and, if $k > 1$, then $f(2k-2) < f(2k-1)$, and

(4)(k) $f(2k-1)$, $f(2k) \in N_k$, where

$$N_k := \Big(N \cap \bigcap\{R_j \mid j \leq k,\ R_j \in \mathcal{U}\} \cap \bigcap\{R_j^c \mid j \leq k,\ R_j \notin \mathcal{U}\}\Big) \setminus \bigcup_{j \leq k} K_j$$

(note that N_k is a finite intersection of elements of $\mathcal{U}$, and thus, $N_k \in \mathcal{U}$).

For $k = 1$, we take $K_1 := K(\{f_1\}, 1)$ (defined as in (5.4.2)) and

$$f(1) := \min(N_1) \text{ and } f(2) := \min(N_1 \setminus \{f(1)\})$$

and note that (1)(1)-(4)(1) are satisfied.
If, for a $k \in \mathbb{N}$, $k > 1$, and each $i < k$, $f(2i)$, $f(2i-1)$, and K_i are already chosen, we set

$$K_k := \bigcup_{m \leq f(2k-2)} K(\{f_1, ..., f_k\}), m).$$

Thus, K_k is finite, by (5.4.2)(b), and satisfies $(1)(k)$ and $(2)(k)$ (note that $f(2k-2) \geq 2k-2 \geq k$ and that $f(2k-2) \geq f(i)$ whenever $i \leq 2k-2$). Taking

$$f(2k-1) := \min(N_k) \quad \text{and} \quad f(2k) := \min(N_k \setminus \{f(2k-1)\}),$$

$(4)(k)$ and the first part of $(3)(k)$ follow. The second part of $(3)(k)$ follows from the fact that $N_k \subset K_k^{\complement} \subset \{1, 2, ..., f(2k-2)\}^{\complement}$. Thus, we have finished the induction step.

For this choice of f, we now have to verify (a), (b), and (c):
Proof of (a): In order to verify (E), let $J \in \mathcal{P}_f(\tilde{\mathcal{F}}) \cup \{f\}$ and $m \in \mathbb{N}$ be arbitrary. We set $k := \max(\{m\} \cup \{n \mid f_n \in J\})$ and show that K_k satisfies (E)(i) and (ii). Since $k \geq m$ (E)(i) follows from $(1)(k)$. For each $\tilde{f} \in J \setminus \{f\}$, it follows from $(2)(k)$ that K_k is strictly $\tilde{f}$-admissible, while for $\tilde{f} = f$ we deduce for each $i \in \mathbb{N}$ that

$$\begin{aligned} f(2i) \in K_k \Rightarrow \quad & i < k \qquad \Rightarrow 2i-1 < 2k-1 \\ & [\text{By } (4)(i),\ i \geq k \text{ would imply that } f(2i) \in N_i \subset K_k^{\complement}] \\ & \Rightarrow f(2i-1) \in K_k \\ & [(1)(k)] \\ & \Rightarrow \quad i < k \qquad \Rightarrow 2i < 2k-1 \Rightarrow f(2i) \in K_k \\ & [\text{as above}], \end{aligned}$$

which implies that K_k is strictly f-admissible.
Proof of (b). By the assumption, it is enough to show that each $R = R_j \in \mathcal{R}$ is f-admissible. This follows from $(4)(i)$ since, for $i \geq j$,

$$f(2i) \in R_j \iff R_j \in \mathcal{U} \iff f(2i-1) \in R_j.$$

Proof of (c): $(4)(k)$ $(k \in \mathbb{N})$.

◇

5.4.6 Lemma: *Let $\tilde{\mathcal{F}}$ be a countable subset of $\mathcal{F}$ satisfying (E) and let $(R_n: n \in \mathbb{N}) \subset \mathcal{P}_\infty(\mathbb{N})$ be a pairwise disjoint sequence of $\tilde{\mathcal{F}}$-admissible sets.*

Then there exists $R \in \mathcal{P}_\infty(\mathbb{N})$ such that

a) R is $\tilde{\mathcal{F}}$-admissible, and

b) $R \subset \bigcup_{n\in\mathbb{N}} R_{2n-1}$ and $R_{2n-1} \overset{a}{\subset} R < \infty$ for $n \in \mathbb{N}$.

Proof of (5.4.6) :

Let $\tilde{\mathcal{F}} = (f_n: n \in \mathbb{N})$. For each $k \in \mathbb{N}$, we set

$$(1)\ L_k := \bigcup_{m,\ell \le k} \{f_m(2i), f_m(2i-1) \mid i \in \mathbb{N} \text{ and } |\{f_m(2i), f_m(2i-1)\} \cap R_\ell| = 1\}.$$

Since R_ℓ is $\tilde{\mathcal{F}}$-admissible, each L_k is finite, and thus, by the assumption that $\tilde{\mathcal{F}}$ satisfies (E),

$$K_k := \bigcup_{\ell \in L_k} K(\{f_1, ..., f_k\}, \ell)$$

is finite also (by (5.4.2)) for each $k \in \mathbb{N}$.

Moreover, $R_k \setminus K_k$ is strictly $\{f_1, ..., f_k\}$-admissible. Indeed, for each $i \in \mathbb{N}$, $j \le k$, and $\theta \in \{0, 1\}$ we have

$$\begin{aligned} f_j(2i-\theta) \in R_k \setminus K_k &\Rightarrow f_j(2i-1+\theta) \in R_k \text{ and } f_j(2i-\theta) \notin K_k \\ &\quad [\text{Otherwise, } |\{f_j(2i), f_j(2i-1)\} \cap R_k| = 1, \\ &\quad \text{and thus, by (1), } f_j(2i-\theta) \in L_k \subset K_k] \\ &\Rightarrow f_j(2i-1+\theta) \in R_k \setminus K_k \\ &\quad [K_k \text{ is strictly } f_j-\text{admissible}]. \end{aligned}$$

We deduce that the set

$$R := \bigcup_{k\in\mathbb{N}} R_{2k-1} \setminus K_{2k-1} = \bigcup_{2k-1<j} R_{2k-1} \setminus K_{2k-1} \cup \bigcup_{2k-1\ge j} R_{2k-1} \setminus K_{2k-1} \quad (j \in \mathbb{N})$$

is f_j-admissible for each $j \in \mathbb{N}$ (by (5.4.2)(a)); hence, R is $\tilde{\mathcal{F}}$-admissible. Since each K_k is finite and since $(R_n: n \in \mathbb{N})$ is pairwise disjoint, we deduce that R satisfies condition (b).

◇

5.4.7 Theorem: *(Example 5)*

We assume the continuum hypothesis ($\omega_1 = \omega_c$).

Then there exists an algebra $\mathcal{A}$ on $\mathbb{N}$ which contains $\mathcal{P}_f(\mathbb{N})$ such that the Stone compact K of $\mathcal{A}$ has the following properties:

a) *Every convergent sequence in K is eventually stationary.*

b) *Every strictly increasing sequence $(k_n : n \in \mathbb{N}) \subset \mathbb{N}$ contains a subsequence $(k_{n(m)}: m \in \mathbb{N})$ for which $(\delta_{k_{n(2m)}} - \delta_{k_{n(2m-1)}}: m \in \mathbb{N})$ is $\sigma(\mathrm{M}(K), \mathrm{C}(K))$-zero convergent.*

c) *$\mathrm{C}(K)$ has the Gelfand-Phillips property.*

Proof of (5.4.7) :

We first well-order $\mathcal{P}_\infty(\mathbb{N})$ by $(N_\alpha : \alpha < \omega_1)$ and the set of all sequences $(A_n : n \in \mathbb{N}) \subset \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}$ having pairwise disjoint elements by

$$((A(\alpha, n) : n \in \mathbb{N}) : \alpha < \omega_1).$$

By transfinite induction we choose, for each $\alpha < \omega_1$, $f_\alpha \in \mathcal{F}$ and $R_\alpha \in \mathcal{P}_\infty(\mathbb{N})$ such that

(1)(α) $f_\alpha(\mathbb{N}) \subset N_\alpha$,

(2)(α) $(f_\beta : \beta \le \alpha)$ satisfies (E),

(3)(α) i) for each $\beta < \alpha$ the set R_β is f_α-admissible, and

ii) R_α is $(f_\beta : \beta \le \alpha)$-admissible,

(4)(α) i) if $|A(\alpha, n)| = 1$ for each $n \in \mathbb{N}$, then

$$|\{n \in \mathbb{N} \mid A(\alpha, n) \subset R_\alpha\}| = \infty \quad \text{and} \quad |\{n \in \mathbb{N} \mid A(\alpha, n) \setminus R_\alpha \neq \emptyset\}| = \infty,$$

ii) if $|A(\alpha, n)| = \infty$ and if $A(\alpha, n)$ is $(f_\beta : \beta \le \alpha)$-admissible for each $n \in \mathbb{N}$, then

$$R_\alpha \subset \bigcup_{n \in \mathbb{N}} A(\alpha, 2n-1) \quad \text{and, for each } n \in \mathbb{N},\ A(\alpha, 2n-1) \overset{a}{\subset} R_\alpha.$$

We assume that for $\alpha < \omega_1$, $\tilde{\mathcal{F}} := (f_\beta : \beta < \alpha)$ and $\tilde{\mathcal{R}} := (R_\beta : \beta : \beta < \alpha)$ have been chosen.

Applying Lemma (5.4.5) to $\tilde{\mathcal{F}}$, $\tilde{\mathcal{R}}$, and $N := N_\alpha$, we get an $f_\alpha \in \mathcal{F}$ for which (1)(α), (2)(α), and (3)(α)(i) are satisfied (note that $\tilde{\mathcal{F}}$ satisfies (E) because $(f_\beta : \beta < \tilde{\alpha})$ satisfies (E) for each $\tilde{\alpha} < \alpha$).

If $(A(\alpha,n) : n \in \mathbb{N})$ satisfies neither of the two cases in (4)(α), we set $R_\alpha := \mathbb{N}$ and note that in this case (3)(α)(ii) is satisfied and that (4)(α) is empty.
If $|A(\alpha,n)| = 1$ for each $n \in \mathbb{N}$, we apply Lemma (5.4.4), to $\tilde{\mathcal{F}} \cup \{f_\alpha\}$, $R_n := \mathbb{N}$ for $n \in \mathbb{N}$, and $N := \bigcup_{n\in\mathbb{N}} A(\alpha,n)$ to get an $\tilde{\mathcal{F}} \cup \{f_\alpha\}$-admissible set R_α (which implies (3)(α)(ii)) satisfying (b) of (5.4.4), which implies (4)(α).
If $A(\alpha,n)$ is of infinite cardinality and is $\tilde{\mathcal{F}} \cup \{f_\alpha\}$-admissible, for each $n \in \mathbb{N}$, we apply Lemma(5.4.6) to $\tilde{\mathcal{F}} \cup \{f_\alpha\}$ and $R_n := A(\alpha,n)$ ($n \in \mathbb{N}$) to get an $\tilde{\mathcal{F}} \cup \{f_\alpha\}$-admissible set $R_\alpha \in \mathcal{P}_\infty(\mathbb{N})$ (thus, (3)(α)(ii) is satisfied) which satisfies (b) of (5.4.6) and hence implies (4)(α).
Thus, we have finished the induction step.

Now taking

$$\mathcal{A} := \{R \in \mathcal{P}(\mathbb{N}) \,|\, R \text{ is } (f_\alpha : \alpha < \omega_1) - \text{admissible}\},$$

(thus $\mathcal{P}_f(\mathbb{N}) \subset \mathcal{A}$), we have to verify that the Stone compact K of $\mathcal{A}$ satisfies (a), (b), and (c) of the assertion.

We first note that $(R_\alpha : \alpha < \omega_1) \subset \mathcal{A}$ (for $\alpha, \beta \in [0,\omega_1[$, it follows from (3)($\alpha$)(i) that R_β is f_α-admissible if $\beta < \alpha$ while, if $\beta \geq \alpha$, then we deduce from (3)(β)(ii) that R_β is f_α-admissible).
Proof of (a): We have to show that a given family $(\xi_n : n \in \mathbb{N}) \subset K$ of pairwise distinct elements does not converge. W.l.o.g. we can assume that one of the following cases happens:
Case 1: $(\xi_n : n \in \mathbb{N}) \subset \mathbb{N}$.
Then there is an $\alpha < \omega_1$ with $A(\alpha,n) = \{\xi_n\}$ for $n \in \mathbb{N}$. From (4)(α)(ii) we deduce that

$$\{n \in \mathbb{N} \,|\, \xi_n \in R_\alpha\} \cap R_\alpha \quad \text{and} \quad \{n \in \mathbb{N} \,|\, \xi_n \in R_\alpha\} \setminus R_\alpha$$

are both of infinite cardinality, which implies that $(\xi_n : n \in \mathbb{N})$ does not converge.
Case 2: $(\xi_n : n \in \mathbb{N}) \subset K \setminus \mathbb{N}$.
By passing to a subsequence, we may assume that there are pairwise disjoint $A_n \in \mathcal{A}$ ($n \in \mathbb{N}$) with $\xi_n \in \overline{A_n}$. Thus, there is an $\alpha < \omega_1$ with $A(\alpha,n) = A_n$, for $n \in \mathbb{N}$ and, by (4)(α) (note that $|A_n| = \infty$ because $\xi_n \in K \setminus \mathbb{N}$), we deduce that

$$R_\alpha \subset \bigcup_{n\in\mathbb{N}} A_{2n-1}\,, \text{ and for each, } n \in \mathbb{N} \quad A_{2n-1} \overset{a}{\subset} R_\alpha < \infty.$$

Thus, $\xi_{2n-1} \in \overline{R_\alpha}$ and $\xi_{2n} \notin \overline{R_\alpha}$ for each $n \in \mathbb{N}$, which implies that $(\xi_n : n \in \mathbb{N})$ does not converge.

Proof of (b): Let $(k_n : n \in \mathbb{N})$ be strictly increasing in $\mathbb{N}$. Then there is an $\alpha < \omega_1$ with $N_\alpha = \{k_n \mid n \in \mathbb{N}\}$ and from (1)(α) we deduce that $f_\alpha(\mathbb{N}) \subset N_\alpha$. From the definition of $\mathcal{A}$ and the definition of admissibility we deduce, for each $A \in \mathcal{A}$, that

$$\lim_{i\to\infty} \langle \delta_{f_\alpha(2i)} - \delta_{f_\alpha(2i-1)}, \chi_{\overline{A}} \rangle = 0,$$

which implies the assertion.

Proof of (c): By Theorem (3.1.3), it is enough to prove that a given normed sequence $(g_n : n \in \mathbb{N}) \subset C(K)$ with elements having pairwise disjoint supports is not limited in $C(K)$. Since $\mathbb{N}$ is dense in K, we find for such a sequence an increasing $(k_n : n \in \mathbb{N}) \subset \mathbb{N}$ such that $|g_n(k(n))| \geq \frac{1}{2}$, and thus, $g_n(k(m)) = 0$ for $n, m \in \mathbb{N}$ with $n \neq m$. By (b), there is a subsequence $(k(n(m)) : m \in \mathbb{N})$ for which $(\delta_{k(n(2m))} - \delta_{k(n(2m-1))} : m \in \mathbb{N})$ is weak*-zero convergent. Since

$$\langle g_{n(2m)}, \delta_{k(n(2m))} - \delta_{k(n(2m-1))} \rangle \geq \frac{1}{2} \text{ for each } m \in \mathbb{N},$$

we deduce the assertion.

5.4.8 Theorem: *(Example 6)*

We assume the continuum hypothesis.

Then there exists a family $\mathcal{R} = (R_\alpha : \alpha < \omega_1) \subset \mathcal{P}_\infty(\mathbb{N})$ satisfying (F) and (FM) of Definition (5.1.1) such that $C(K)$ is a Gelfand-Phillips space, were K is the Stone compact of the algebra generated by $\mathcal{P}_f(\mathbb{N})$ and $\mathcal{R}$.

Proof of (5.4.8) :

Let $(N_\alpha : \alpha < \omega_1)$ be a well-ordering of $\mathcal{P}_\infty(\mathbb{N})$.

By transfinite induction we choose, for each $\alpha < \omega_1$, $R_\alpha \in \mathcal{P}_\infty(\mathbb{N})$ and $f_\alpha \in \mathcal{F}$ such that

(1)(α) either $R_\alpha \overset{a}{\subset} R_\beta$ or $R_\alpha \cap R_\beta \overset{a}{=} \emptyset$ for each $\beta < \alpha$,

(2)(α) $|N_\alpha \cap R_\alpha| = \infty$ and $|N_\alpha \setminus R_\alpha| = \infty$,

(3)(α) $(f_\beta : \beta \leq \alpha)$ satisfies (E),

(4)(α) $f_\alpha(\mathbb{N}) \subset N_\alpha$, and

(5)(α) i) R_β is f_α-admissible for each $\beta < \alpha$,

ii) R_α is $(f_\beta : \beta \leq \alpha)$-admissible.

Assuming that for $\alpha < \omega_1$ the families $\tilde{\mathcal{F}} := (f_\beta : \beta < \alpha)$ and $\tilde{\mathcal{R}} := (R_\beta : \beta < \alpha)$ are already chosen, we note that $(f_\beta : \beta < \alpha)$ has property (E) because, for each

$\tilde{\alpha} < \alpha$, $(f_\beta : \beta < \tilde{\alpha})$ satisfies (E) and $(R_\beta : \beta < \alpha)$ satisfies (F) because of (1)($\tilde{\alpha}$) ($\tilde{\alpha} < \alpha$).

Applying Lemma (5.4.5) to $\tilde{\mathcal{F}}$, $\tilde{\mathcal{R}}$, and $N := N_\alpha$ we find an $f_\alpha \in \mathcal{F}$ such that (3)(α), (4)(α) and (5)(α)(i) are satisfied.

By the Zorn's lemma, we find an $I \subset [0, \alpha[$ satisfying the following properties (6), (7) and (8):

(6) $|R_\beta \cap N_\alpha| = \infty$ for each $\beta \in I$,

(7) $R_{\tilde{\beta}} \overset{a}{\subset} R_\beta$ whenever β, $\tilde{\beta} \in I$ with $\beta < \tilde{\beta}$, and

(8) I is maximal in the following sense:

For each $\beta \in [0, \alpha[\setminus I$, $I \cup \{\beta\}$ does not satsify (6) or (7).

(Note that $\emptyset$ satisfies (6) and (7) and that for each linearly ordered (by inclusion) $\mathcal{I} \subset \mathcal{P}([0, \alpha[)$ consisting of elements satisfying (6) and (7), the set $\bigcup \mathcal{I}$ satisfies (6) and (7) also.)

If $I \neq \emptyset$, we choose a non decreasing sequence $(\alpha_n : n \in \mathbb{N}) \subset I$ with

$$\sup_{n \in \mathbb{N}} \alpha_n = \sup I \tag{9}$$

(we recall that α is countable)
and, if $\hat{I} := [0, \alpha[\setminus I$ is not empty, there exists a sequence $(\beta_n : n \in \mathbb{N})$ such that $\hat{I} = \{\beta_n \mid n \in \mathbb{N}\}$.

For each $n \in \mathbb{N}$, we define

$$A_n := \begin{cases} \mathbb{N} & \text{if } I = \hat{I} = \emptyset \\ \mathbb{N} \setminus \bigcup_{j \le n} R_{\beta_n} & \text{if } I = \emptyset \text{ and } \hat{I} \neq \emptyset \\ R_{\alpha_n} & \text{if } I \neq \emptyset \text{ and } \hat{I} = \emptyset \\ R_{\alpha_n} \setminus \bigcup_{j \le n} R_{\beta_n} & \text{if } I \neq \emptyset \text{ and } \hat{I} \neq \emptyset. \end{cases} \tag{10}$$

By (7), the sequence $(A_n : n \in \mathbb{N})$ is decreasing with respect to "$\overset{a}{\subset}$". We want to show that for each $n \in \mathbb{N}$

$$|A_n \cap N_\alpha| = \infty. \tag{11}$$

If $\hat{I} = \emptyset$, (11) follows from (6).

We suppose that $\hat{I} \neq \emptyset$ and that (11) is not true; hence, we find an $n \in \mathbb{N}$ for which (if $I = \emptyset$, set $R_{\alpha_n} := \mathbb{N}$)

$$R_{\alpha_n} \cap N_\alpha \overset{a}{\subset} \bigcup_{j \le n} R_{\beta_j} \cap N_\alpha ,$$

and thus, by (7),

$$R_{\alpha_m} \cap N_\alpha \overset{a}{\subset} \bigcup_{j \leq n} R_{\beta_j} \cap N_\alpha \quad \text{whenever } m \geq n\,.$$

We deduce from (6) and (7) that there is a $j_0 \leq n$ such that

$$|R_{\alpha_m} \cap R_{\beta_{j_0}}| = \infty \text{ for each } m \in \mathbb{N} \quad \text{and } |R_{\beta_{j_0}} \cap N_\alpha| = \infty\,.$$

By (1)(α), this implies that for each $m \in \mathbb{N}$

$$\text{either} \qquad \beta_{j_0} \leq \alpha_m \text{ and } R_{\alpha_m} \overset{a}{\subset} R_{\beta_{j_0}} \qquad \text{or} \qquad \beta_{j_0} > \alpha_m \text{ and } R_{\beta_{j_0}} \overset{a}{\subset} R_{\alpha_m}\,.$$

From (7) and (9) we conclude for each $\tilde{\alpha} \in I$:

$$\text{either} \qquad \beta_{j_0} \leq \tilde{\alpha} \text{ and } R_{\tilde{\alpha}} \overset{a}{\subset} R_{\beta_{j_0}} \qquad \text{or} \qquad \beta_{j_0} > \tilde{\alpha} \text{ and } R_{\beta_{j_0}} \overset{a}{\subset} R_{\tilde{\alpha}}\,.$$

This implies that $I \cup \{\beta_{j_0}\}$ satisfies (6) and (7), which contradicts (8). Thus, we have proven (11).

Now we are in a position to apply Lemma (5.4.4) to $\tilde{\mathcal{F}} \cup \{f_\alpha\}$, $\tilde{R}_n := A_n$ for $n \in \mathbb{N}$ and $\tilde{N} := N_\alpha$, to get an $\tilde{\mathcal{F}} \cup \{f_\alpha\}$-admissible $R_\alpha \in \mathcal{P}_\infty(\mathbb{N})$ (thus, (5)(α)(ii) is satisfied) which satisfies (a) and (b) of (5.4.4). Thus, we deduce 2(α). In order to show (1)(α) let $\beta < \alpha$.

If $\beta \in \hat{I}$, then we deduce that $R_\alpha \cap R_\beta \overset{a}{=} \emptyset$ from (5.4.4)(a) and (10).

If $\beta \in I$, then there is an $n \in \mathbb{N}$ with $\beta \leq \alpha_n$ and we deduce from (7) and (5.4.4) that

$$R_\alpha \overset{a}{\subset} R_{\alpha_n} \overset{a}{\subset} R_\beta\,,$$

which implies the assertion and finishes the induction step.

From (1)(α), (2)(α), and Proposition (5.1.2) we deduce that $(R_\alpha : \alpha < \omega_1)$ satisfies (FM). To show that K is a Gelfand-Phillips space we proceed as in the proof of Theorem (5.4.7): first we deduce from (4)(α) and (5)(α) that each $(k_n : n \in \mathbb{N})$ contains a subsequence $(k_{n(m)} : m \in \mathbb{N})$ such that

$$(\delta_{k_{n(2m)}} - \delta_{k_{n(2m-1)}} : m \in \mathbb{N})$$

is a w*-zero sequence; and then we observe that this, together with Theorem (3.1.3), implies the Gelfand-Phillips property for C(K).

◇

Bibliography

[1] D. Amir and J. Lindenstrauss, "The structure of weakly compact sets in Banach spaces", *Ann. of Math. (2)* **88**, (1968), 34–46.

[2] S. A. Argyros, "On non separable Banach spaces", *Trans. Amer. Math. Soc.* **270/1**, (1982), 193–216.

[3] S. A. Argyros, J. Bourgain and T. Zachariades, "A result on the isomorphic embeddability of $\ell_1(\Gamma)$", *Stud. Math.* **78**, (1984), 77–92.

[4] J. Bourgain and J. Diestel, "Limited operators and strict cosingularity", *Math. Nachrichten* **119** (1984), 55–58.

[5] R. E. Brackebusch, "James spaces on general trees", Dissertation, Ohio State (1985).

[6] P. Cembranos, "$C(K,E)$ contains a complemented copy of c_o", *Proc. Amer. Math. Soc.* **91**, (1984), 556–558.

[7] W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński, "Factoring weakly compact operators", *J. Funct. Anal.*, **17**, 311–327. (1974

[8] J. Diestel and J. J. Uhl Jr., "Progress in vector measures 1977–83", in: Measure theory and its applications, Proc. Conf. Sherbrooke/Canada 1982, Lecture Notes 1033, Springer Verlag (1983), 144–192.

[9] J. Diestel, "Sequences and series in Banach spaces", Graduate Texts in Mathematics 92, Springer Verlag (1984).

[10] J. Diestel and J. J. Uhl Jr., "Vector measures", AMS Math. Surveys 15, Providence (R.I.), (1977).

[11] N. Dinculeanu, "Vector measures", VEB Deutscher Verlag der Wissenschaften, Berlin, (1966).

[12] L. Drewnowski, "On Banach spaces with the Gelfand-Phillips property", unpublished.

[13] L. Drewnowski, "On Banach spaces with the Gelfand-Phillips property", *Math. Zeitschrift* **193**, (1986), 405–411.

[14] L. Drewnowski and G. Emmanuele, "On Banach spaces with the Gelfand-Phillips property. II ", preprint, (1986).

[15] N. Dunford and J. T. Schwartz, "Linear operators", Part I, Interscience, New York and London, (1958).

[16] G. A. Edgar, "A long James space", in: Measure Theory, Oberwolfach (ed.: D. Kölzow), Lecture Notes 794, Springer Verlag (1980), 31–37.

[17] G. Emmanuele, "Gelfand-Phillips property in a Banach space of vector valued

measures", *Math. Nachrichten* **127**, (1986), 21–23.

[18] G. Emmanuele and F. Räbiger, "On the Gelfand-Phillips property in $\varepsilon-$tensor products", *Math. Zeitschrift* **191**, (1986), 485–488.

[19] C. Foiaş and I. Singer, "Some remarks on the representation of linear operators in spaces of vector valued continuous functions", *Rev. Roumaine Math. Pures Appl.* **5**, (1960), 729–752.

[20] D. Fremlin, personal communication.

[21] I. Gelfand, "Abstrakte Funktionen und lineare Operatoren", *Mat. Sb.* **46 (NS 4)**, (1938), 235–284.

[22] A. Grothendieck, "Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$", *Canadian J. Math.* **5**, (1953), 129–173.

[23] A. Grothendieck, "Résumé de la théorie métrique des produits tensoriels topologique", *Bol. Soc. math. Sao Paolo* **8**, (1956), 1–79.

[24] A. Grothendieck, "Topological vector spaces", Notes on Mathematics and its Applications, Gordon and Breach (1973).

[25] J. Hagler and E. Odell, "A Banach space not containing ℓ_1 whose dual ball is not weak* sequentially compact", *Israel J. Math.* **22**, (1975), 290–294.

[26] J. Hagler and W. B. Johnson,"On Banach spaces whose dual balls are not weak* sequentially compact", *Israel J. Math.* **28/4**, (1977), 325–330.

[27] J. Hagler and F. Sullivan, "Smoothness and weak* sequential compactness", *Proc. Amer. Math. Soc.* **78/4**, (1980), 497–503.

[28] P. R. Halmos, "Lectures on Boolean algebras", D. van Nostrand Reinhold Mathematical Studies 1, (1963)

[29] G. H. Hardy, J. E. Littlewood and G. Pólya, "Inequalities", Cambridge at the University Press, (1967).

[30] J. Harksen, "Tensortopologien", Dissertation, Kiel, (1979).

[31] R. Haydon, "On Banach spaces which contain $\ell_1(\tau)$ and types of measures on compact spaces", *Israel J. Math.* **28/4**, (1977), 313–324.

[32] R. Haydon, Review, *Math. rev.* **58,1**, (1979), 58#2174.

[33] R. Haydon, "A non-reflexive Grothendieck space that does not contain ℓ_∞", *Israel J. Math.* **40/1**, (1981), 65–73.

[34] R. Haydon, "An unconditional result about Grothendieck spaces", preprint (1986).

[35] R. Haydon, M. Levy and E. Odell, "On sequences without weak* convergent convex block subsequences", *Proc. Amer. Soc.* **101**, (1987), 94–98.

[36] R. C. James, "Weak compactness and reflexivity", *Israel J. Math.* **2**, (1964), 101–119.

[37] G. Köthe, "Lösbarkeitsbedingungen für Gleichungen mit unendlich vielen Unbekannten", *J. reine und angew. Math.*, (1938), 193–213

[38] G. Köthe, "Stanislaw Mazur's contributions to functional analysis", Math. Ann. **277**, (1987), 489–528.

[39] H. E. Lacey, "The isometric theory of classsical Banach spaces", Die Grundlehren der math. Wissenschaften 208, Springer Verlag (1974).

[40] J. Lindenstrauss and L. Tzafriri, "Classical Banach spaces I. Sequence spaces", Ergebnisse der Mathematik und ihrer Grenzgebiete Vol. 92, Springer Verlag, (1977).

[41] J. Lindenstrauss and L. Tzafriri, "Classical Banach spaces II. Function spaces", Ergebnisse der Mathematik und ihrer Grenzgebiete Vol. 97, Springer Verlag, (1979).

[42] H. P. Lotz and H. P. Rosenthal, "Embeddings of C(Δ) in Banach lattices", *Israel J. Math.* **31/2**, (1978), 169–179.

[43] S. Negrepontis, "Banach spaces and topology", in: Handbook of set theoretic topology (ed.: K. Kunen and J. E. Vaughan), North-Holland (1984).

[44] E. Odell, "Applications of Ramsey theorems to Banach space theory", in: Notes in Banach spaces (ed.:H. E. Lacey), University Press, Austin and London (1980), 379–404.

[45] A. Pełczyński, "Banach spaces on which every unconditionally converging operator is weakly compact", *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **10**, (1962), 641–648.

[46] R. S. Phillips, "On linear transformations", *Trans. Amer. Math. Soc.* 48, (1940), 516–541.

[47] B.v.Querenburg, "Mengentheoretische Topologie", Hochschultext, Springer Verlag, (1973).

[48] F. Räbiger, "Beiträge zur Strukturtheorie der Grothendieck-Räume", Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Springer Verlag, (1973).

[49] H. P. Rosenthal, "On the structure of weak*-compact subsets of a dual Banach space", preprint.

[50] E. Saab, personal communication.

[51] H. H. Schäfer, "Banach lattices and positive operators", Springer Verlag,

(1974).

[52] L. Schwartz, "Théorie des distribution à valeur vectorielles (I)", *Ann. Inst. Fourier (Grenoble)* **7**, (1957),1–137.

[53] Z. Semadeni, "Banach Spaces of Continuous Functions 1", Monografie Matematyczne, PWN-Polish Scientific Publishers 55, Warschau (1971).

[54] I. Singer, "Bases in Banach spaces II", Springer Verlag, Berlin, (1981).

[55] P. Suppes, "Axiomatic set theory", D. van Nostrand, (1960).

[56] M. Talagrand, "Un nouveau C(K) qui possède la propriété de Grothendieck", *Israel J. Math.* **37/1–2**, (1980), 181–191.

Lebenslauf

Geburtsdatum:	Am 19.10.1954 bin ich
Geburtsort:	in München geboren.
Schulbildung:	September 1961, Eintritt in die deutsche Volksschule in Barcelona/Spanien.
	September 1963, Volksschule Etterschlag/Wörthsee bei München;
	September 1965, Übertritt in das Max-Planck-Gymnasium, Pasing-München,
	November 1966, deutsch-peruanische Schule Max-Uhle in Arequipa/Peru;
	April 1969, Internatsgymnasium Schloß Plön/Schleswig-Holstein;
	Mai 1970, Luitpold-Gymnasium, München, an dem ich am 22.06.1974 das Abitur bestand.
Studienzeit:	01.11.1974, Beginn des Studiums der Mathematik und Physik an der Ludwig-Maximilians-Universität, Münche
	1.11.1975 bis 1.2.1977, Unterbrechung durch Einberufung zum Zivildienst. Ich arbeitete in der Soziotherapiestation des Bezirkskrankenhauses Haar.
	1.5.77, Wiederaufnahme des Studiums.
	Oktober 1979 bis Juli 1980, Studium am "Laboratoire de Probabilité" der Universität Paris-6 mit Hilfe eines Stipendiums des Deutschen Akademischen Austauschdienstes.
Abschlußprüfung:	Im Juli 1982 legte ich die Hauptdiplomprüfung im Fach Mathematik ab.
Berufliche Tätigkeiten:	Vom 1.9.1982 bis 31.10.1984 war ich bei der "Winterthur"-Lebensversicherung beschäftigt.
	Vom 1.1.1985 bis 31.12.1986 erhielt ich ein Stipendium der Bayerischen Graduiertenförderung und begann mit der Anfertigung einer Dissertation.
	Seit 1.1.87 arbeite ich am Mathematischen Institut der Ludwig-Maximilians Universität, München als wissenschaftliche Hilfskraft.

München, den 20.12
Thomas Schlumpre