STABILITY OF HEISENBERG COEFFICIENTS

A Dessertation

by

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ABSTRACT

The Heisenberg product is an associative product defined on symmetric functions which interpolates between the ordinary product and the Kronecker product. Heisenberg coefficients are Schur structure constants of the Heisenberg product and generalization of both Littlewood–Richardson coefficients and Kronecker coefficients.

In 1938, Murnaghan discovered that the Kronecker product of two Schur functions stabilizes. We prove an analogous result for the Heisenberg product of Schur functions. In 2014, Stembridge introduced the notion of stability for Kronecker triples which generalize Murnaghan's classical stability result. Sam and Snowden proved a conjecture of Stembridge concerning stable Kronecker triples, and they also showed an analogous result for Littlewood–Richardson coefficients. We show that any stable triple for Kronecker coefficients or Littlewood–Richardson coefficients also stabilizes Heisenberg coefficients, and we classify the triples stabilizing Heisenberg coefficients. We also follow Manivel and Vallejo's idea of using matrix additivity to generate Heisenberg stable triples.

DEDICATION

To my mother and my father who always believe in me and support me in pursuing what I love.

This research is dedicated to you.

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1. INTRODUCTION

Algebraic combinatorics is an area of mathematics implements techniques of abstract algebra, especially group theory and representation theory, in combinatorial problems, and conversely, applies combinatorial methods to problems in algebra.

Symmetric functions and representations of symmetric groups are two important and closely related subjects in algebraic combinatorics. Due to the Frobenius character map, these two subjects have isomorphic ring structures. On the ring of representations of symmetric groups, there are two important products, the induction product and the Kronecker product, which respectively gives structure constants Littlewood–Richardson coefficients and Kronecker coefficients. While the Littlewood–Richardson coefficient has been well understood, very little is known about the Kronecker coefficient. Although some special cases of Kronecker coefficients have been studied, it is still a challenging open problem in combinatorial representation theory to find an explicit combinatorial description for general Kronecker coefficients.

Aguiar et al. [1] and Moreira [2] introduced a (nongraded) product, Heisenberg product, on representations of symmetric groups which interpolates between the induction product and the Kronecker product, hence the structure constants of this new product, called Heisenberg coefficients, generalize both Littlewood–Richardson coefficients and Kronecker coefficients.

One remarkable property of Kronecker coefficients is the stability phenomenon discovered by Murnaghan [3] in 1938. We show that the low degree components of the Heisenberg product also have this property. Murnaghan's stability notion was later extensively generalized by Stembridge [4] by introducing a new concept called Kronecker stable triple. He and Sam and Snowden [5] characterized all Kronecker stable triples. Sam and Snowden also gave an analogous result for Littlewood–Richardson coefficients. We generalize their results to Heisenberg coefficients and using additive matrices, followed from Vallejo's idea [6], to generate Heisenberg stable triples.

2. SYMMETRIC FUNCTIONS

Symmetric functions are important in algebraic combinatorics. This chapter gives an introduction to symmetric functions. For more details about this material, see [7].

In Section 2.1, we introduce the basic notations and terminologies concerning partitions. In Section 2.2, we define several families of symmetric functions. In Section 2.3, we introduce Young tableaux and develop the connections between some symmetric functions.

2.1 Partitions

Many objects in the study of symmetric functions and combinatorial representation theory are parametrized by partitions. We begin by introducing partitions and presenting some elementary results. Throughout this thesis, $\mathbb{N}$ is the set of nonnegative integers, and for each positive integer n, we set $[n] := \{1, 2, \ldots, n\}$.

A *partition* λ is a finite weakly decreasing sequence of nonnegative integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$$

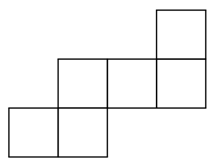
with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$, and we consider two partitions to be the same if they only differ by a string of zeroes at the end. The i-th part of λ is λ_i; the *length* of λ, denoted by $\ell(\lambda)$, is the number of nonzero parts of λ; the *size* of λ, written as $|\lambda|$, is the sum of all the parts of λ. If $|\lambda| = n$, we say λ is a partition of n and write $\lambda \vdash n$. The partition λ can be identified with its *Young diagram*, which is left-justified rows of boxes of length $\lambda_1, \lambda_2, \ldots, \lambda_r$. For example, the partition $(4, 4, 1) = (4, 4, 1, 0)$ corresponds to the diagram on the left below,

(2.1)

Sometimes we may also use another notation for partitions $\lambda = (1^{m_1}2^{m_2}\dots t^{m_t})$, where m_i is the number of times i occurs in λ. For example, $(4,4,1)$ can be written as $(1^1 4^2)$.

The *conjugate* of a partition λ, written as λ', is the partition whose Young diagram is the transpose of the diagram of λ, i.e. the diagram of λ' is obtained by the reflection of the diagram of λ over the diagonal. For example, the conjugate of $(4,4,1)$ is $(3,2,2,2)$, which corresponds to the diagram on the right in (2.1).

For partitions λ and μ, we write $\mu \subset \lambda$ to mean that the diagram of μ is contained in the diagram of λ (or equivalently, $\mu_i \leq \lambda_i$ for all i). We denote by λ/μ, called a *skew diagram*, the diagram obtained by removing the diagram of μ from the the diagram of λ. We call λ/μ the shape of the diagram. For example, let $\lambda = (4,4,2)$ and $\mu = (3,1)$, then $\mu \subset \lambda$ and the skew diagram λ/μ is the following diagram,



(2.2)

Let $\mathcal{P}_n$ be the set of all the partitions with size n. We define two orderings, reverse lexicographic ordering ($\preceq_{rl}$) and dominance ordering($\preceq_d$), on $\mathcal{P}_n$. When $\lambda, \mu \in \mathcal{P}_n$, we say $\mu \preceq_{rl} \lambda$ if either $\lambda = \mu$ or the first non-vanishing term of $\lambda_i - \mu_i$ is positive; we say $\mu \preceq_d \lambda$ if $\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i$ for all $i \geq 1$. The reverse lexicographic ordering is a total ordering. For example, it arranges the partitions in $\mathcal{P}_6$ in the following order,

$$\begin{aligned}(1,1,1,1,1,1) \preceq_{rl} (2,1,1,1,1) \preceq_{rl} (2,2,1,1) \preceq_{rl} (2,2,2) \preceq_{rl} (3,1,1,1) \preceq_{rl} (3,2,1)\\ \preceq_{rl} (3,3) \preceq_{rl} (4,1,1) \preceq_{rl} (4,2) \preceq_{rl} (5,1) \preceq_{rl} (6).\end{aligned}$$

The dominance ordering is only a partial ordering. For example, $(4,1,1)$ and $(3,3)$ in $\mathcal{P}_6$ are incomparable with respect to $\preceq_d$. Following from the definition of the two orderings, it is not hard to show

Lemma 2.1.1. *Let $\lambda, \mu \in \mathcal{P}_n$, then $\mu \preceq_d \lambda$ implies that $\mu \preceq_{rl} \lambda$.*

Another result concerning the dominance ordering is

Lemma 2.1.2. *Let $\lambda, \mu \in \mathcal{P}_n$, then $\mu \preceq_d \lambda$ if and only if $\lambda' \preceq_d \mu'$*

Proof. It is enough to show one direction. Assume $\mu \preceq_d \lambda$. If $\lambda' \npreceq_d \mu'$, then there exists a positive integer i such that

$$\lambda'_1 + \cdots + \lambda'_j \leq \mu'_1 + \cdots + \mu'_j \quad \text{for all} \quad 1 \leq j < i$$
$$\lambda'_1 + \cdots + \lambda'_i > \mu'_1 + \cdots + \mu'_i$$

In particular, we have $\lambda'_i > \mu'_i$, and

$$\lambda'_{i+1} + \lambda'_{i+2} + \cdots < \mu'_{i+1} + \mu'_{i+2} + \cdots \tag{2.3}$$

because $\lambda', \mu' \in \mathcal{P}_n$.

Let $l = \lambda'_i$ and $m = \mu'_i$, then $l > m$ and $\lambda_j \geq i$ for all $1 \leq j \leq l$. From $\mu \preceq_d \lambda$, we have

$$\mu_1 + \cdots + \mu_m \leq \lambda_1 + \cdots + \lambda_m. \tag{2.4}$$

So,

$$\begin{aligned}(\mu_1 - i) + \cdots (\mu_m - i) \leq& (\lambda_1 - i) + \cdots + (\lambda_m - i) \\ \leq& (\lambda_1 - i) + \cdots + (\lambda_m - i) + (\lambda_{m+1} - i) + \cdots + (\lambda_l - i)\end{aligned} \tag{2.5}$$

Using Young diagram, it is not hard to see that the left hand side (right hand side, respectively) of (2.5) counts the the number of boxes in the diagram of μ (λ, respectively) which are strictly to the right of the i-th column. That is,

$$(\mu_1 - i) + \cdots (\mu_m - i) = \mu'_{i+1} + \mu'_{i+2} + \cdots$$
$$(\lambda_1 - i) + \cdots + (\lambda_m - i) + (\lambda_{m+1} - i) + \cdots + (\lambda_l - i) = \lambda'_{i+1} + \lambda'_{i+2} + \cdots$$

Combining the above two equations and (2.3), we get

$$\begin{aligned}(\mu_1 - i) + \cdots (\mu_m - i) >&(\lambda_1 - i) + \cdots + (\lambda_m - i) + (\lambda_{m+1} - i) + \cdots + (\lambda_l - i)\\ \geq&(\lambda_1 - i) + \cdots + (\lambda_m - i),\end{aligned}$$

which contradicts (2.4). So $\lambda' \preceq_d \mu'$. $\square$

We define some operations on partitions. If we view partitions as vectors, we can define addition, subtraction, and scalar multiplication for partitions. Let λ and μ be partitions, and $a \in \mathbb{N}$, we define

$$\begin{aligned}\lambda + \mu &:= (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \cdots),\\ \lambda - \mu &:= (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \cdots),\\ n\lambda &:= (n\lambda_1, n\lambda_2, \cdots).\end{aligned}$$

We define $\lambda \cup \mu$ to be the partition whose parts are those of λ and μ, arranged in weakly decreasing order. For example, $(4,4,1) \cup (4,2,2) = (4,4,4,2,2,1)$. Note that the operations $+$ and $\cup$ are dual to each other in the following sense,

$$(\lambda + \mu)' = \lambda' \cup \mu'.$$

2.2 Families of Symmetric Functions

Let $X_n = (x_1, x_2, \ldots, x_n)$ be a set of n variables. A polynomial in $\mathbb{Z}[X_n]$ is said to be symmetric if it is invariant under under the action of permuting the variables. We denote the set of symmetric polynomials in n variables by Λ_n. In other words,

$$\Lambda_n := \mathbb{Z}[X_n]^{S_n}.$$

where S_n is the symmetric group of degree n and S_n acts on $\mathbb{Z}[X_n]$ by permuting the variables.

In the study of symmetric functions, we usually work with infinitely many variables, as the number of variables, as long as it is large enough, does not affect the properties we are interested in. Let $m \geq n$ be two positive integers, we have a natural (algebra) homomorphism from $\mathbb{Z}[X_m]$ to $\mathbb{Z}[X_n]$ by sending $x_{n+1} = x_{n+2} = \cdots = x_m = 0$ and other x_i's to themselves. This map induces a (surjective) homomorphism $\rho_{m,n} : \Lambda_m \twoheadrightarrow \Lambda_n$. Using this family of maps, we define the *ring of symmetric functions* Λ to be the inverse limit $\varprojlim_n \Lambda_n$ in the category of graded rings. In other words, an element f of Λ can be written as $f = (f_n)_{n\geq 0}$, where $f_n \in \Lambda_n$ with $\rho_{m,n}(f_m) = f_m(x_1, \ldots, x_n, 0, \ldots, 0) = f_n(x_1, \ldots, x_n)$ for all $m \geq n$, and the degrees of f_n's are bounded.

The ring Λ of symmetric functions is a $\mathbb{Z}$-algebra, and we can easily extend scalars to obtain a $\mathbb{Q}$-algebra,

$$\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We introduce several families of symmetric functions. Each family forms a $\mathbb{Q}$-basis for $\Lambda_{\mathbb{Q}}$, and some of them are even $\mathbb{Z}$-bases for Λ.

The first family consists of the monomial symmetric functions. Given a positive integer n, for each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, we set $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Given a partition λ with $\ell(\lambda) \leq n$, the monomial symmetric polynomials in X_n is

$$m_\lambda(X_n) := \sum_{\alpha} x^\alpha, \tag{2.6}$$

where α runs over all the distinct permutations of λ. For example,

$$m_{1,1}(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3.$$

If $\ell(\lambda) > n$, then we set $m_\lambda(X_n) = 0$. Let $m_\lambda(X)$ (or simply written as m_λ), called the monomial symmetric function indexed by λ, be the element in Λ corresponding to $(m_\lambda(X_n))_n$. For example,

$$m_{1,1} = \sum_{i<j} x_i x_j.$$

It is not hard to show that $\{m_\lambda\}_{\lambda\in\mathcal{P}}$ forms a $\mathbb{Z}$-basis for Λ.

We next introduce elementary symmetric functions. For each nonnegative integer n, we set $e_0 = 1$ and

$$e_n := \sum_{i_1<i_2<\cdots<i_n} x_{i_1}x_{i_2}\cdots x_{i_n} = m_{(1^n)} \tag{2.7}$$

For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathcal{P}$, we define the elementary symmetric function indexed by λ to be

$$e_\lambda := e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_k} \in \Lambda. \tag{2.8}$$

The generating function for the e_n's is

$$E(t) = \sum_{n\geq 0} e_n t^n = \prod_{i\geq 1}(1 + x_i t) \in \Lambda[[t]] \subset \mathbb{Z}[[X, t]]. \tag{2.9}$$

The elementary symmetric functions form a $\mathbb{Z}$-basis for Λ. For a proof, see [7].

Proposition 2.2.1. *The $\{e_n\}_{n\geq 1}$ are algebraically independent over $\mathbb{Z}$ and $\Lambda = \mathbb{Z}[e_1, e_2, \ldots]$.*

We introduce complete homogeneous symmetric functions. For each nonnegative integer n, we set $h_0 = 1$ and

$$h_n := \sum_{\lambda\in\mathcal{P}_n} m_\lambda, \tag{2.10}$$

which is the sum of all the monomial symmetric functions with degree n. For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathcal{P}$, we define the complete symmetric function indexed by λ to be

$$h_\lambda := h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_k} \in \Lambda. \tag{2.11}$$

The generating function for the h_n's is

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_{i\geq 1}(1 - x_i t)^{-1}. \tag{2.12}$$

Combining (2.9) and (2.12), we get

$$E(t)H(-t) = 1, \tag{2.13}$$

which implies that

$$\sum_{k=0}^{n} (-1)^n e_k h_{n-k} = 0 \tag{2.14}$$

for all $n \geq 1$.

Since the e_n's are algebraically independent over $\mathbb{Z}$ (or even $\mathbb{Q}$), we consider the algebra homomorphism $\omega : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ defined by

$$\omega(e_n) = h_n. \tag{2.15}$$

Using (2.14), one can show that $\omega(h_n) = e_n$ for all $n \geq 0$, which implies that ω is an involution and induces a result analogous to Proposition 2.2.1.

Proposition 2.2.2. *The $\{h_n\}_{n\geq 1}$ are algebraically independent over $\mathbb{Z}$ and $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$.*

For convenience, we usually set $e_n = h_n = 0$ for all $n < 0$.

Another natural family of symmetric functions is the power sum symmetric functions. For each integer $n \geq 1$, we define

$$p_n := \sum_{i\geq 1} x_i^n. \tag{2.16}$$

For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathcal{P}$, we define the power sum symmetric function indexed by λ to be

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k} \in \Lambda. \tag{2.17}$$

The involution ω acts on p_λ nicely,

$$w(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda. \tag{2.18}$$

The family of power sum symmetric functions is not an integral basis for Λ, however, it forms a $\mathbb{Q}$-basis for $\Lambda_{\mathbb{Q}}$. For more details, see [7].

Proposition 2.2.3. *The $\{p_n\}_{n\geq 1}$ are algebraic independent over $\mathbb{Q}$ and $\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, \ldots]$.*

We close this section by introducing the most important family of symmetric functions, Schur functions. For each partition λ, the Schur polynomial $s_\lambda[X_n]$ in the variables X_n is defined to be

$$s_\lambda[X_n] = \frac{\det(x_i^{\lambda_i+j-1})_{1\leq i,\, j\leq n}}{\det(x_i^{j-1})_{1\leq i,\, j\leq n}}, \tag{2.19}$$

where $\ell(\lambda) \leq n$. If $\ell(\lambda) > n$, we set $s_\lambda[X_n] = 0$. The Schur function s_λ indexed by λ is the one in Λ corresponding to $(s_\lambda(X_n))_n$. In particular, we have

$$s_{(1^n)} = e_n \tag{2.20}$$

$$s_n = h_n \tag{2.21}$$

The action of the involution ω on Schur functions is also understood,

$$\omega(s_\lambda) = s_{\lambda'}. \tag{2.22}$$

Proposition 2.2.4. *Schur functions form a $\mathbb{Z}$-basis for Λ.*

We can also define a Schur function using complete homogeneous symmetric functions,

$$s_\lambda = \det(h_{\lambda_j+i-j})_{i,j}. \tag{2.23}$$

This formula is called the Jacobi-Trudi determinant. We leave the last definition of Schur functions in the next section as it involves Young tableau, a combinatorial object having extensive application in algebraic combinatorics.

2.3 Young Tableaux

We begin by introducing Young tableaux. A *Young tableau* is a filling of a Young diagram which assigns a positive integer to each box of the diagram. The partition corresponding to the diagram is called the shape of the tableau. A Young tableau is called *semistandard* if the fillings is weakly increasing along the rows and strictly increasing down the columns, and such tableaux are called semistandard Young tableaux (SSYT). A Young tableaux of shape $\lambda \vdash n$ is called *standard* if it is semistandard and every number in $[n] := \{1, 2, \ldots, n\}$ is used exactly once in the filling, such tableaux are called standard Young tableaux (SYT). For example, the left diagram in (2.24) is semistandard and right one is standard.

$$\begin{array}{|c|c|c|c|c|c|}\hline 2 & 2 & 2 & 3 & 5 & 5 \\ \hline 3 & 4 & 4 & 4 \\ \cline{1-4} 5 & 5 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 11 & 12 \\ \hline 3 & 7 & 8 & 10 \\ \cline{1-4} 6 & 9 \\ \cline{1-2} \end{array} \tag{2.24}$$

Analogously, a filling of a skew diagram is called a *skew Young tableau*, and, similarly, we can define semistandard skew Young tableaux and standard skew Young tableau.

For each (skew) Young tableau T, we define the weight of T to be the sequence $w(T) = (w_1, w_2, \ldots)$ where w_i is number of occurrences of i in T. For example, the left tableau in (2.24) have the weight $(0, 3, 2, 3, 4, 0, \ldots)$.

For each partition λ, the Schur function s_λ is

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^{w(T)}, \tag{2.25}$$

where the sum is over all the SSYT of shape λ. For example, if $\lambda = (6, 4, 2)$, the two SSYT's in (2.24) contribute monomials $x_2^3 x_3^2 x_4^3 x_5^4$ and $x_1 x_2 \cdots x_{12}$, respectively, in the summation in (2.25). Note that, although it is not obvious from Equation (2.25), Schur functions are indeed symmetric.

Given two partitions μ and ν, since $s_\mu \cdot s_\nu$ is also symmetric, and Schur functions form a $\mathbb{Z}$-basis for Λ, we can consider the linear expansion of $s_\mu \cdot s_\nu$ with respect to the Schur basis,

$$s_\mu \cdot s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda, \tag{2.26}$$

where λ runs over all the partitions. The Schur structure constant $c^\lambda_{\mu,\nu}$ is called the Littlewood–Richardson coefficient. Remarkably, this coefficient is always a nonnegative integer and we have several beautiful combinatorial descriptions for it (see [7, 8, 9]). We introduce one of them below.

For a (skew) Young tableau T, we denote by $r(T)$ the reading word of T, which is obtained by reading the numbers in T from right to left in successive rows, starting with the top row. A word $a = a_1a_2\dots a_N$ ($a_i \in [n]$) is said to be a lattice permutation if for all $1 \le i \le n-1$ and $1 \le r \le N$, the number of occurrences of i in $a_1a_2\dots a_r$ is not less than the number of occurrences of $i+1$. For example, the reading word of the following tableau T is $r(T) = 11213224$, which is a lattice permutation.

$$\mathrm{T} \; = \; \begin{array}{cccc} & & \boxed{1} & \boxed{1} \\ & \boxed{1} & \boxed{2} & \\ \boxed{2} & \boxed{2} & \boxed{3} & \\ \boxed{4} & & & \end{array} \tag{2.27}$$

Proposition 2.3.1 (Littlewood–Richardson Rule)**.** *Let λ, μ, and ν be partitions. The Littlewood–Richardson coefficient $c^{\lambda}_{\mu,\nu}$ counts the number of semistandard skew Young tableaux T with shape λ/μ and weight ν such that the reading word $r(T)$ is a lattice permutation.*

We close this Chapter by showing some relations between different families of symmetric functions. We have several different expansions for the product $\prod\limits_{i,j}(1 - x_iy_j)^{-1}$

$$\prod_{i,j}(1 - x_iy_j)^{-1} = \sum_{\lambda} z_\lambda^{-1} p_\lambda(X)p_\lambda(Y) \tag{2.28}$$

$$= \sum_{\lambda} m_\lambda(X)h_\lambda(Y) = \sum_{\lambda} m_\lambda(Y)h_\lambda(X) \tag{2.29}$$

$$= \sum_{\lambda} s_\lambda(X)s_\lambda(Y) \tag{2.30}$$

where $z_\lambda = \prod\limits_i i^{m_i} m_i!$. This suggests us to construct an bilinear form on $\Lambda_{\mathbb{Q}}$ defined by

$$\langle\, s_\lambda,\ s_\mu \,\rangle = \delta_{\lambda,\mu}, \tag{2.31}$$

where $\delta_{\lambda,\mu}$ is the Kronecker delta.

Proposition 2.3.2. *Let $\{u_\lambda\}_{\lambda\in\mathcal{P}}$ and $\{v_\lambda\}_{\lambda\in\mathcal{P}}$ be $\mathbb{Q}$-bases of $\Lambda_{\mathbb{Q}}$ where u_λ and v_λ are homogeneous* *symmetric functions with degree $|\lambda|$. Then the following are equivalent:*

(1) $\langle u_\lambda,\ v_\mu \rangle = \delta_{\lambda,\mu}$ *for all λ and μ;*

(2) $\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\lambda u_\lambda(X) v_\lambda(Y)$.

Proof. Consider the Schur expansion of u_λ and v_μ

$$u_\lambda = \sum_{\alpha \vdash |\lambda|} a_{\lambda,\alpha} s_\alpha, \qquad v_\mu = \sum_{\beta \vdash |\mu|} b_{\mu,\beta} s_\beta.$$

Let $A_n = (a_{\lambda,\alpha})_{\lambda,\alpha \vdash n}$ and $B_n = (b_{\mu,\beta})_{\mu,\beta \vdash n}$ be the transition matrices, where the rows and columns are labeled by partitions and arranged in reverse lexicographic order. Then, from (2.31), we know that (1) is equivalent to

$$\sum_\alpha a_{\lambda,\alpha} b_{\mu,\alpha} = \delta_{\lambda,\mu}, \tag{2.32}$$

which means $A_n B_n^t = I$ for all $n \geq 0$.

On the other hand, from (2.30), we know that (2) is equivalent to

$$\sum_\lambda s_\lambda(X) s_\lambda(Y) = \sum_\lambda u_\lambda(X) v_\lambda(Y), \tag{2.33}$$

hence is equivalent to

$$\sum_\lambda a_{\lambda,\alpha} b_{\lambda,\beta} = \delta_{\alpha,\beta}, \tag{2.34}$$

which means $A_n^t B_n = I$ for all $n \geq 0$. Hence (1) and (2) are equivalent. □

From Proposition 2.3.2, Equations (2.28) and (2.29), we have

$$\langle m_\lambda,\ h_\mu \rangle = \langle h_\lambda,\ m_\mu \rangle = \delta_{\lambda,\mu}. \tag{2.35}$$

$$\langle p_\lambda,\ p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}. \tag{2.36}$$

From (2.31), we see that this bilinear form is symmetric and positive definite, hence an inner product on $\Lambda_{\mathbb{Q}}$. We call it the Hall inner product. Also, (2.22) (or (2.18)) implies that,

Proposition 2.3.3. *The involution ω is an isometry for the Hall inner product. That is,*

$$\langle w(f),\ w(g) \rangle = \langle f\ ,\ g \rangle$$

for all $f, g \in \Lambda_{\mathbb{Q}}$.

For partitions λ and μ with the same size, let $K_{\lambda,\mu}$ be the number of SSYT of shape λ and weight μ. This number is called Kostka number. It is easy to see that $K_{\lambda,\lambda} = 1$. We have the following well-known result determining when the Kostka number is positive.

Proposition 2.3.4. $K_{\lambda,\mu} > 0$ *if and only if* $\lambda \succeq_d \mu$.

Using (2.25) and the symmetry of Schur functions, it is easy to see that

$$s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu. \tag{2.37}$$

Since the monomial symmetric functions $\{m_\lambda\}$ and the complete symmetric functions $\{h_\lambda\}$ form dual bases, and the Schur functions $\{s_\lambda\}$ are self-dual with respect to the Hall inner product, we have

$$h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda. \tag{2.38}$$

Apply the involution ω to (2.38), and use (2.15) and (2.22) we have

$$e_\mu = \sum_\lambda K_{\lambda,\mu} s_{\lambda'}. \tag{2.39}$$

For complete information about the transition matrices between those different bases for $\Lambda_{\mathbb{Q}}$, see [7, Chapter 1 Section 6].

3. REPRESENTATION THEORY

Representation theory studies algebraic structures by representing the elements as linear transformations of vector spaces. In this chapter, we introduce representation theory and focus on representations of symmetric groups, which are closely related to symmetric functions. For more details about the material in this chapter, see [10, 11, 12].

In Section 3.1, we give the definition of representation theory and present some basic results about the representations of finite groups. In Section 3.2, we introduce the character theory, which is an important tool in representation theory. In Section 3.3, we describe the representations of symmetric groups. In Section 3.4, we introduce the Frobenius character map, which builds a bridge between the representations of symmetric groups and symmetric functions.

3.1 Representations of Finite Groups

Throughout this chapter, all the groups are finite unless it is otherwise stated and we let the ground field be the complex numbers $\mathbb{C}$, although all results work for any algebraic closed field with characteristic 0.

A *representation* of a finite group G is a pair (V, ρ) of a finite dimensional complex vector space V and a group homomorphism $\rho : G \to GL(V)$. When there is no confusion about the map ρ, we may simply say that V is a representation of G; we may also just write $g.\, v$ or gv for $\rho(g)(v)$, where $g \in G$ and $v \in V$.

Equivalently, representations can be understood using modules. We extend the action of G on V linearly, then V can be viewed as a $\mathbb{C}[G]$-module where $\mathbb{C}[G]$ is the group algebra associated to G defined as follows:

Definition 3.1.1. Let G be a finite group. The group algebra $\mathbb{C}[G]$ is a $\mathbb{C}$-algebra whose additive group is a complex vector space with basis

$$\{1_g | g \in G\},$$

and whose multiplication is defined by

$$1_g \cdot 1_h = 1_{gh}$$

and extended linearly using the distribution law. When there is no confusion, we may simply write g for 1_g for convenience.

For this reason, we also call a representation V of G a G-module. Note that $\mathbb{C}[G]$ can also be viewed as a G-module where the action is defined by

$$g.\, 1_h = 1_{gh} \quad \text{for all} \quad g, h \in G,$$

and this representation is called the regular representation of G.

A *subrepresentation* of a representation V is subspace a W of V which is invariant under the action of G, and this subrepresentation is called *proper* if W is a proper subspace of V. A representation is *irreducible* if it does not have a proper subrepresentation.

A map φ from a representation (V, ρ) to a representation (W, ρ') of G is called *G-linear* if it is a linear map from V to W such that the following diagram commutes.

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle \rho(g)}\big\downarrow & & \big\downarrow{\scriptstyle \rho'(g)} \\ V & \xrightarrow{\varphi} & W \end{array} \tag{3.1}$$

When there is no confusion, we simply write $\varphi g = g\varphi$. It is easy to check that Ker φ, Im φ, and Coker φ are G-modules.

Let V and W be two representations of G. We denote the set of all linear maps from V to W by $\mathrm{Hom}(V, W)$. $\mathrm{Hom}(V, W)$ is clearly a vector space and it also has a G-module structure. The action of an element $g \in G$ on a map $\varphi \in \mathrm{Hom}(V, W)$ is defined as follows:

$$(g\varphi)(v) = g\varphi(g^{-1}v) \quad \text{for all} \quad v \in V. \tag{3.2}$$

This action can be understood using the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{g\varphi} & W. \end{array} \tag{3.3}$$

In particular, if we set $W = \mathbb{C}$ and G acts on W as identity (this representation $\mathbb{C}$ is called the *trivial representation* and denoted by $\mathbf{1}_G$), we get a representation $V^* = \mathrm{Hom}(V, \mathbb{C})$ of G, and this representation (V^*, ρ^*) is called the *dual* of (V, ρ). If we fix a basis for V and take its dual basis for V^*, then, for any $g \in G$, we can present $\rho(g)$ and $\rho^*(g)$ in terms of matrices. We have

$$\rho^*(g) = (\rho(g^{-1}))^t. \tag{3.4}$$

Let V and W be representations of groups G and H respectively. Then the tensor product $V \otimes W$ is a representation of $G \times H$ with the action defined by

$$(g, h)(v \otimes w) = gv \otimes hw, \quad \text{for all} \quad g \in G, h \in H, v \in V, w \in W. \tag{3.5}$$

If $G = H$, then the tensor product $V \otimes W$ is a representation of G via the following diagonal action:

$$g(v \otimes w) = gv \otimes gw, \quad \text{for all} \quad g \in G, v \in V, w \in W. \tag{3.6}$$

We write $V \boxtimes W$ for this representation to distinguish it from the one in (3.5). With this convention, it is not hard to get the identification

$$\mathrm{Hom}(V, W) \cong V^* \boxtimes W \tag{3.7}$$

as G-modules.

We denote the set of all G-linear maps between two representations V and W by $\mathrm{Hom}_G(V, W)$. Comparing (3.1) and (3.3), we see that $\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$, which is the set of elements of $\mathrm{Hom}(V, W)$ fixed by the action of G.

The *direct sum* $V \oplus W$ of two representations V and W of G is again a G-module. We say a representation is *completely reducible* if it can be written as a direct sum of irreducibles. For our case (G is finite), any representation is completely irreducible. The key to prove this fact is the following proposition.

Proposition 3.1.2. *Let V be a representation of G, and W be a subrepresentation of V. Then there is an invariant subspace W' of V under the action of G, such that $V = W \oplus W'$.*

Following directly from this proposition, we have

Corollary 3.1.3. *Any representation of a finite group is completely reducible.*

An important result for the study of the irreducible decomposition of a representation is the following:

Proposition 3.1.4 (Schur's Lemma)**.** *Let V and W be two irreducible representations of G and $\varphi \in Hom_G(V, W)$, then*

(1) The map φ is either an isomorphism or 0.

(2) If $V = W$, then φ is a scalar multiple of the identity.

Proof. For (1), if φ is not zero, then the $\mathrm{Im}\ \varphi$ is a non-zero G-submodule of W, which implies that $\mathrm{Im}\ (\varphi) = W$ as W is irreducible. Similarly, we can show that $\mathrm{Ker}\ \varphi = 0$ using the irreducibility of V. This proves that φ is an isomorphism.

For (2), we fix a basis for V and view φ as a matrix, then φ has an eigenvalue λ as $\mathbb{C}$ is algebraically closed. Consider $\varphi - \lambda I$ (where I is the identity matrix) which also induces a G-module homomorphism from V to V. Since $\varphi - \lambda I$ has an eigenvalue 0, it is not an isomorphism. Hence, $\varphi - \lambda I = 0$ due to part (1). So $\varphi = \lambda I$. $\square$

An immediate Corollary of Schur's Lemma is

Corollary 3.1.5. *Let V and W be two representations of a group G with V being irreducible. Then dim $Hom_G(V, W) = 0$ if and only if W contains no subrepresentation isomorphic to V.*

We identify two representations if they are isomorphic. Combining Corollary 3.1.3 and Schur's Lemma, we have

Proposition 3.1.6. *For any representation V of G, there is a decomposition of V into irreducibles,*

$$V = V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}, \tag{3.8}$$

where the V_i's are pairwise nonisomorphic irreducible representations. Moreover, this decomposition is unique.

For convenience, we write this decomposition as

$$V = m_1 V_1 \oplus \cdots \oplus m_k V_k\,, \tag{3.9}$$

and we call m_i the multiplicity of V_i in V.

We introduce two operations which are important in representation theory. Let G be a group and H be a subgroup of G. A natural question is whether we can construct a representation of H from a representation of G or vice versa. One direction is straightforward, while the other one is more technical which involves the tensor of modules.

Let V be a representation of G, then we automatically have an action of H on V as H is a subgroup of G. So V is also a representation of H. We denote this representation by $\mathrm{Res}_H^G V$.

Conversely, suppose we have a representation W of H, then W is a left $\mathbb{C}[H]$-module. As $\mathbb{C}[H]$ is a subalgebra of $\mathbb{C}[G]$, we can view $\mathbb{C}[G]$ as a left $\mathbb{C}[G]$-module and right $\mathbb{C}[H]$-module. So $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ is a $\mathbb{C}[G]$-module, hence a representation of G. We denote this representation by $\mathrm{Ind}_H^G W$ and call it the *induced representation* of W from H to G. An equivalent way to understand the construction is the following

$$\mathrm{Ind}_H^G W = \bigoplus_{g \in G/H} gW. \tag{3.10}$$

where $gW = \{gw \mid w \in W\}$ is a vector space and this construction does not depend on the choice of g for the left coset gH as W is invariant under the action of H.

Both restriction and induction are transitive. Suppose $K \leq H \leq G$ are groups, and we have a representation V of G and a representation of W of K. Then

$$\mathrm{Res}_K^G V = \mathrm{Res}_K^H (\mathrm{Res}_H^G V), \tag{3.11}$$

$$\mathrm{Ind}_K^G W = \mathrm{Ind}_H^G (\mathrm{Ind}_K^H W). \tag{3.12}$$

The restriction and induction are related by the following proposition.

Proposition 3.1.7. *Let V and W be representations of G and H respectively where H is a subgroup of G, then any H-module homomorphism $\varphi : W \to \mathit{Res}_H^G V$ can be extended uniquely to a G-module homomorphism $\overline{\varphi} : \mathit{Ind}_H^G W \to V$. That is,*

$$\mathit{Hom}_H(W, \mathit{Res}_H^G V) = Hom_G(\mathit{Ind}_H^G W, V). \tag{3.13}$$

An interesting question concerning the induction and restriction is what happens when we compose them. More precisely, suppose K and H are two subgroups of G, and V is a representations of K. What do we know about $\mathrm{Res}_H^G \mathrm{Ind}_K^G V$? This problem is solved by Mackey [13]. Note that V can be viewed as a representation of gKg^{-1} for any $g \in G$ with the action defined by

$$gkg^{-1}.v = kv, \quad \text{for all} \quad k \in K, v \in V,$$

and we denote this representation by V_g. Then Mackey's Decomposition Theorem says

Proposition 3.1.8 ([13], Theorem 1)**.**

$$\mathit{Res}_H^G \mathit{Ind}_K^G V = \bigoplus_{HgK} \mathit{Ind}_{H\cap gKg^{-1}}^H \mathit{Res}_{H\cap gKg^{-1}}^{gKg^{-1}} V_g \tag{3.14}$$

3.2 Group Characters

When study the representations of a group, instead of presenting the elements of the group as matrices, it is more convenient and efficient to just consider the corresponding characters.

Definition 3.2.1. Let V be a representation of a finite group G. We fix a basis for V and present the action of $g \in G$ on V as a matrix. Then the *character* of the representation V, denoted by χ_V, is the complex-valued function on G defined by

$$\chi_V(g) = \mathrm{Tr}(g), \tag{3.15}$$

where $\mathrm{Tr}(g)$ is trace of the matrix g. It does not depend on the choice of the basis for V.

For example, the character of the regular representation $\mathbb{C}[G]$ of G is

$$\chi_{\mathbb{C}[G]}(g) = \begin{cases} |G|, & \text{if } g = e\,; \\ 0, & \text{otherwise.} \end{cases} \tag{3.16}$$

From the property of trace, we know that χ_V is a *class function* on G, i.e. χ_V is constant on the conjugacy classes of G. Let C_G be the set of all class functions on G, then C_G is a vector space (over $\mathbb{C}$) whose dimension is equal to the number of conjugacy classes of G.

We can also view $\chi_V(g)$ as the sum of the eigenvalues of g. Since $g^{|G|} = e$ where e is the identity element of G, the eigenvalues of g must be roots of unity. Hence the eigenvalues of g^{-1} are the conjugates of the eigenvalues of g. We present some basic results about characters.

Proposition 3.2.2. *Let V and W be representations of G. Then*

(1) $\chi_V(e) = \mathit{dim}\, V$.

(2) $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$.

(3) $\chi_{V \oplus W} = \chi_V + \chi_W$.

(4) $\chi_{V\boxtimes W} = \chi_V \cdot \chi_W$.

By (3.7) and Proposition 3.2.2, we have

$$\begin{aligned}\chi_{\mathrm{Hom}(V,W)} &= \chi_{V^*\boxtimes W} \\ &= \chi_{V^*}\cdot\chi_W = \overline{\chi_V}\cdot\chi_W.\end{aligned} \tag{3.17}$$

Let V and W be two representations of G. From Section 3.1, we know that $U := \mathrm{Hom}(V, W)$ is a G-module. Consider the action of $\varphi = \frac{1}{|G|}\sum_{g\in G} g \in \mathrm{End}(U)$ on U, we have

Proposition 3.2.3. *φ is a projection from U onto U^G.*

Proof. For any $h \in G$, we have

$$h\varphi = \frac{1}{|G|}h\sum_{g\in G} g = \frac{1}{|G|}\sum_{g\in G} hg = \frac{1}{|G|}\sum_{g\in G} g = \varphi.$$

So $\mathrm{Im}\,\varphi \subset U^G$.

Consider $\varphi\circ\varphi$, we have

$$\varphi\circ\varphi = \frac{1}{|G|^2}\sum_{g\in G}\sum_{h\in G} gh = \frac{1}{|G|^2}\sum_{g\in G}\sum_{h\in G} h = \frac{1}{|G|}\sum_{h\in G} h = \varphi.$$

So φ is a projection onto $\mathrm{Im}\,\varphi$. On the other hand, for any $u \in U^G$, we have

$$\varphi(u) = \frac{1}{|G|}\sum_{g\in G} gu = \frac{1}{|G|}\sum_{g\in G} u = u. \tag{3.18}$$

So $U^G \subset \mathrm{Im}\,\varphi$ and we finish the proof. $\square$

We fix a basis for U^G and extend it to a basis of U. We get

$$\begin{aligned}\dim U^G = \mathrm{Tr}(\varphi) &= \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(g) \\ &= \frac{1}{|G|}\sum_{g\in G}\chi_U(g).\end{aligned} \tag{3.19}$$

From Section 3.1, we know that $U^G = \mathrm{Hom}_G(V, W)$ is the set of all G-linear maps from V to W. When V and W are irreducible representations of G, by Schur's Lemma, we have

$$\dim U^G = \dim \mathrm{Hom}_G(V, W) = \begin{cases} 1, & \text{if } V \cong W\,; \\ 0, & \text{otherwise.} \end{cases} \tag{3.20}$$

Combining Equation (3.17), (3.19), and (3.20), we have

$$\frac{1}{|G|} \sum_{g \in G} \chi_V(g) \cdot \overline{\chi_W(g)} = \begin{cases} 1, & \text{if } V \cong W\,; \\ 0, & \text{otherwise.} \end{cases} \tag{3.21}$$

This motives us to define an (Hermitian) inner product $\langle\ ,\ \rangle_G$ on the (complex) vector space of class functions C_G:

$$\langle\, \alpha\, ,\, \beta\, \rangle_G = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \cdot \overline{\beta(g)}. \tag{3.22}$$

Then the characters of irreducible representations of G are orthonormal with respect to this inner product, that is, if V and W are two irreducible representations of G, then

$$\langle\, \chi_V\, ,\, \chi_W\, \rangle_G = \begin{cases} 1, & \text{if } V \cong W\,; \\ 0, & \text{otherwise,} \end{cases} \tag{3.23}$$

which implies that the number of nonisomorphic irreducible representations of G is less than or equal to the number of conjugacy classes of G. Using (3.23), we easily get the following results:

Corollary 3.2.4. *Let V be a representation of G, then the following are equivalent:*

(1) V is irreducible;

(2) $\langle\, \chi_V, \chi_V\, \rangle_G = 1$;

(3) V^* is irreducible.

Suppose a representation V has the irreducible decomposition as in (3.9). From (3.23), we have

$$m_i = \langle\, \chi_{V_i}\,,\, \chi_V \,\rangle_G \tag{3.24}$$

In particular, if we set $V = \mathbb{C}[G]$, the regular representation of G, and apply (3.16), then we get

$$m_i = \frac{1}{|G|}\chi_{V_i}(e)|G| = \text{dim } V_i. \tag{3.25}$$

The above equation shows

Proposition 3.2.5. *Let $\mathbb{C}[G] = \bigoplus_{i=1}^{r} m_i V_i$ be the irreducible decomposition of the regular representation $\mathbb{C}[G]$ of G, then $\{V_i\}_{i=1}^{r}$ is a complete list of irreducible representations of G, and the multiplicity of m_i of V_i in $\mathbb{C}[G]$ is equal to dim V_i.*

By computing the dimension of $\mathbb{C}[G]$ using Proposition 3.2.5, we get an identity

$$\text{dim } \mathbb{C}[G] = \sum_{i=1}^{r} (\text{dim } V_i)^2. \tag{3.26}$$

The orthonormality of the irreducibles also shows that any representation is determined by its character. When there is no confusion, we may simply write the representation for its character for convenience. For example, we write (3.23) as follows

$$\langle\, V\,,\, W \,\rangle_G = \begin{cases} 1, & \text{if } V \cong W\,; \\ 0, & \text{otherwise}, \end{cases} \tag{3.27}$$

With this notation, Proposition 3.1.7 can be rephrased as

Proposition 3.2.6 (Frobenius Reciprocity)**.** *Let V and W be representations of G and H respectively where H is a subgroup of G, then*

$$\langle\, V\,,\, \mathit{Ind}_H^G\, W \,\rangle_G = \langle\, \mathit{Res}_H^G\, V\,,\, W \,\rangle_H. \tag{3.28}$$

Finally, we show that the number of irreducible representations of G is equal to the number of conjugacy classes of G. It is enough to prove

Lemma 3.2.7. *If $\alpha \in C_G$ is a class function and $\langle\, \alpha\, ,\, \chi_V\, \rangle_G = 0$ for all irreducible representation V of G, then $\alpha = 0$.*

Proof. Consider $\varphi_{\alpha,V} := \sum\limits_{g\in G} \alpha(g)g \in \operatorname{End}(V)$. The map $\varphi_{\alpha,V}$ is G-linear as α is a class function. Due to Schur's Lemma, we have $\varphi_{\alpha,V} = \lambda I$ for some $\lambda \in \mathbb{C}$. So $\operatorname{Tr}(\varphi_{\alpha,V}) = \lambda \dim V$. On the other hand,

$$\begin{aligned}
\operatorname{Tr}(\varphi_{\alpha,V}) &= \sum_{g\in G} \alpha(g)\operatorname{Tr}(g) \\
&= \sum_{g\in G} \alpha(g)\chi_V(g) \\
&= |G|\langle\, \alpha\, ,\, \chi_{V^*}\, \rangle_G \\
&= 0.
\end{aligned} \tag{3.29}$$

So $\lambda = 0$. Hence $\varphi_{\alpha,V} = 0$ for any irreducible representation V, which implies that $\sum\limits_{g\in G} \alpha(g)g$ is zero on any representation of G. In particular, we consider the regular representation $\mathbb{C}[G]$. Since $\sum\limits_{g\in G} \alpha(g)g = 0$ on $\mathbb{C}[G]$, we have

$$0 = \sum_{g\in G} \alpha(g)g.\, 1_e = \sum_{g\in G} \alpha(g)1_g,$$

which means that $\alpha(g) = 0$ for all $g \in G$ as $\{1_g\}_{g\in G}$ is a basis for $\mathbb{C}[G]$. So $\alpha = 0$. $\square$

Proposition 3.2.8. *The number of irreducible representations of G is equal to the number of conjugacy classes of G. Equivalently, the characters $\{\chi_V\}$ given by the irreducible representations form an orthonormal basis for C_G.*

3.3 Representations of Symmetric Groups

Given a set X, let S_X be the group whose elements are bijections from X to itself. An element of S_X is called a permutation of X. In particular, the symmetric group of degree n is $S_n := S_{[n]}$. It is known that two permutations in S_n are in the same conjugacy class if and only if they have the same cycle type. For each cycle type, if we arrange the length of the cycles in weakly decreasing order, we get a partition of n. So we can associate each conjugacy class with a partition of n. On the other hand, from Proposition 3.2.8, we know that the number of irreducible representations of S_n is equal to the number of conjugacy classes of S_n, so we can index the irreducible representations by partitions of n. We construct an irreducible representation V_λ, called a *Specht module*, of S_n for each $\lambda \vdash n$.

Definition 3.3.1. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$ be a (weak) composition of n, written as $\alpha \vDash n$, then the associated *Young subgroup* of S_n is,

$$S_\alpha = S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_l}, \tag{3.30}$$

where S_{α_i} permutes the set $\{\alpha_1 + \cdots + \alpha_{i-1} + 1, \alpha_1 + \cdots + \alpha_{i-1} + 2, \dots, \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i\}$. The number of elements in S_α is $\alpha! := \alpha_1!\, \alpha_2! \,\cdots\, \alpha_l!$.

We consider the induced representation M_λ ($\lambda \vdash n$) of the trivial representation $\mathbf{1}_{S_\lambda}$ from S_λ to S_n. That is,

$$M_\lambda := \mathrm{Ind}_{S_\lambda}^{S_n} \mathbf{1}_{S_\lambda}. \tag{3.31}$$

Using (3.10), we can write this induced representation as

$$M_\lambda = \bigoplus_{i=1}^{k} \sigma_i \mathbf{1}_{S_\lambda}, \tag{3.32}$$

where $\{\sigma_1, \sigma_2, \dots, \sigma_k\}$ is transversal for the left cosets of S_λ in S_n.

We introduce Young tabloids, which are equivalence classes of Young tableaux, to help us understand the induced representation M_λ. In this section, each Young tableau has the filling set $[n]$ for some n and each number in $[n]$ is used exactly once in the filling.

Definition 3.3.2. Two tableaux T_1 and T_2 with the same shape λ are *row equivalent* (denoted by $T_1 \sim T_2$) if corresponding rows of the two tableaux have the same elements. An equivalence class of such tableaux is called a *tabloid* with shape λ. That is, for a Young tableaux T with shape λ, the tabloid it represents, denoted by $[T]$, is

$$[T] := \{S \in \mathbf{YT}(\lambda) | S \sim T\}. \tag{3.33}$$

For example, the following is a tabloid with shape $(2, 1)$.

$$\left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\right] = \left\{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \quad , \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\right\} \tag{3.34}$$

For each partition $\lambda \vdash n$, we have a natural (well-defined) action of S_n on the set of all tabloids with shape λ:

$$\sigma[T] = [\sigma T], \quad \sigma \in S_n \tag{3.35}$$

where σT is the Young tableau obtained by applying σ to the entries of T. For example, take $\sigma = (132) \in S_3$, we have

$$\sigma \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline \sigma(2) & \sigma(1) \\ \hline \sigma(3) & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$$

Let T_λ be the Young tableau with shape λ whose entries in the i-th row, reading from left to right, are $\lambda_1 + \cdots + \lambda_{i-1} + 1, \lambda_1 + \cdots + \lambda_{i-1} + 2, \ldots, \lambda_1 + \cdots + \lambda_{i-1} + \lambda_i$. For example,

$$T_{(4,2,1)} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & \multicolumn{2}{|c}{} \\ \cline{1-2} 7 & \multicolumn{3}{|c}{} \\ \cline{1-1}\end{array}.$$

By identifying $\sigma_i \mathbf{1}_{S_\lambda}$ with $T_i := [\sigma_i T_\lambda]$, we have

$$M_\lambda = \mathbb{C}\{[T_1], [T_2], \ldots, [T_k]\}, \tag{3.36}$$

as S_n-module, where $\{[T_1], [T_2], \ldots, [T_k]\}$ is a complete list of tabloids with shape λ. In particular,

$$M_{(n)} = \mathbb{C}\{[T_{(n)}]\}, \tag{3.37}$$

which is the trivial representation of S_n, and

$$M_{(1^n)} \cong \mathbb{C}[S_n], \tag{3.38}$$

which is the regular representation of S_n. From our definition of tabloids, it is not hard to show that

Proposition 3.3.3. *The representation M_λ of S_n is generated by any tabloid with shape λ. That is, for any tabloid $[T]$ with shape λ, we have*

$$M_\lambda = \mathbb{C}S_n[T] \tag{3.39}$$

Moreover, dim $M_\lambda = \frac{n!}{\lambda!}$, the number of tabloids with shape λ.

For each Young tableau T with shape $\lambda \vdash n$, we associate it with two groups, the *row-stabilizer*,

$$R_T = S_{R_1} \times S_{R_2} \times \cdots \times S_{R_l}, \tag{3.40}$$

and the *column-stabilizer*,

$$C_T = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_k}, \tag{3.41}$$

where $R_1, R_2, \ldots, R_l$ are the rows and $C_1, C_2, \ldots, C_k$ are the columns of T.

For example, if

$$T = \begin{array}{|c|c|c|c|} \hline 2 & 6 & 5 & 7 \\ \hline 4 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}\ , \tag{3.42}$$

then

$$R_T = S_{\{2,5,6,7\}} \times S_{\{1,4\}} \times S_{\{3\}},$$
$$C_T = S_{\{2,3,4\}} \times S_{\{1,6\}} \times S_{\{5\}} \times S_{\{7\}}.$$

Let $r_T = \sum_{\sigma \in R_T} \sigma$, $c_T = \sum_{\sigma \in C_T} \mathrm{sgn}(\sigma)\sigma \in \mathbb{C}[S_n]$, and $v_T = c_T[T] \in M_\lambda$. For example, if T is the tableau in (3.42), we have

$$c_T = (e - (23) - (24) - (34) + (234) + (243))(e - (16)),$$
$$\begin{aligned} v_T = {} & \left[\ytableaushort{2657,41,3}\right] - \left[\ytableaushort{3657,41,2}\right] - \left[\ytableaushort{4657,21,3}\right] - \left[\ytableaushort{2657,31,4}\right] \\ & + \left[\ytableaushort{3657,21,4}\right] + \left[\ytableaushort{4657,31,2}\right] - \left[\ytableaushort{2157,46,3}\right] + \left[\ytableaushort{3157,46,2}\right] \\ & + \left[\ytableaushort{4157,26,3}\right] + \left[\ytableaushort{2157,36,4}\right] - \left[\ytableaushort{3157,26,4}\right] - \left[\ytableaushort{4157,36,2}\right]. \end{aligned}$$

We have the following results.

Lemma 3.3.4. *Let T be a tableau with shape $\lambda \vdash n$ and $\sigma \in S_n$, then*

(1) $R_{\sigma T} = \sigma R_T \sigma^{-1}$,

(2) $C_{\sigma T} = \sigma C_T \sigma^{-1}$,

(3) $c_{\sigma T} = \sigma c_T \sigma^{-1}$,

(4) $v_{\sigma T} = \sigma v_T$.

Definition 3.3.5. For each partition $\lambda \vdash n$, the associated S_n-module V_λ, called the *Specht module*, is the submodule of M_λ spanned by $\{v_T \mid T \text{ is a Young tableau with shape } \lambda\}$.

From Lemma 3.3.4 (4), we know that the Specht module V_λ is generated by v_T for any tableau T with shape λ. The following lemma is important for the proof of the irreducibility of the Specht module V_λ. For a proof, see [12].

Lemma 3.3.6. *Let S and T be Young tableaux of the shapes λ and μ respectively. If $\lambda \not\succ_d \mu$, then exactly one of the following occurs:*

(1) There are two distinct integers that occur in the same column of S and in the same row of T;

(2) $\lambda = \mu$ and there is some $\alpha \in C_S$ and $\beta \in R_T$ such that $\alpha S = \beta T$.

We define a total ordering on the set of all Young tableaux with n boxes. The *column word* of of a tableau S, denoted by $\mathrm{col}(S)$, is obtained by reading the numbers in S from bottom to top in successive columns. Let S and T be Young tableaux of the shapes λ and μ, then we say $S \succ T$ if either $\lambda \succ_{rl} \mu$, or $\lambda = \mu$ and the largest entry that is in a different box in the two tableaux occurs earlier in the column word $\mathrm{col}(S)$ than in $\mathrm{col}(T)$. For example, this ordering arranges the Young tableau with shape $(2, 1)$ in the following order:

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} > \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} > \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} > \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} > \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} > \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{3.43}$$

From the definition of this ordering, we can easily get that

Lemma 3.3.7. *Let T be a SYT, then for any $\sigma \in R_T$ and $\tau \in C_T$, we have*

$$\sigma T \succeq T \succeq \tau T. \tag{3.44}$$

Combining Lemmas 2.1.1, 3.3.6, and 3.3.7, we get

Corollary 3.3.8. *If S and T are standard tableaux with $T \succ S$, then there are two distinct integers in the same column of S and same row of T.*

Using Lemma 3.3.6, we can show

Lemma 3.3.9. *Let S and T be Young tableaux of the shapes λ and μ respectively, and $\lambda \not\succ_d \mu$. If there is a pair of integers in the same column of S and in the same row of T, then $c_S[T] = 0$. If there is no such a pair, then $c_S[T] = \pm v_S$.*

Note that Lemma 3.3.9 shows

Corollary 3.3.10. *Let λ and μ be partitions of n with $\mu \succ_{rl} \lambda$, and T be a Young tableau with shape λ, then we have*

$$c_T M_\lambda = c_T V_\lambda = \mathbb{C} \cdot v_T \neq 0 \tag{3.45}$$

$$c_T M_\mu = c_T V_\mu = 0 \tag{3.46}$$

From Corollary 3.3.10, we know that V_λ and V_μ are not isomorphic as S_n-modules. It also implies that V_λ is irreducible. Indeed, if $V_\lambda = U \oplus W$, where U and W are S_n-modules, then from (3.45), we have

$$\mathbb{C} \cdot v_T = c_T V_\lambda = c_T U \oplus c_T W \subset U \oplus W\ . \tag{3.47}$$

So either U or W contains v_T. Without loss of generality, assume that $v_T \in U$, then

$$V_\lambda = \mathbb{C}[S_n] v_T \subset U \subset V_\lambda. \tag{3.48}$$

Hence $V_\lambda = U$, and this proves the irreducibility of V_λ. For each partition λ of n, we construct an irreducible representation V_λ of S_n. Since the number of conjugacy classes of S_n is equal to the number of partitions of n and also to the number of irreducible representations of S_n, we have

Proposition 3.3.11. *The set of Specht modules $\{V_\lambda \mid \lambda \vdash n\}$ is a complete list of irreducible representations of S_n.*

Using Lemma 3.3.7, it is not hard to see that $\{\, v_T \,|\, T \in SYT(\lambda) \,\}$ is linear independent in V_λ. Applying a "straightening algorithm" (see [12, Section 7.4]), we can show that these v_T's also span V_λ. Hence,

Proposition 3.3.12. *The set $\{\, v_T \mid T \in SYT(\lambda) \,\}$ is a basis for V_λ. So dim $V_\lambda = |$SYT$(\lambda)|$.*

Consider two special examples, $V_{(n)}$ and $V_{(1^n)}$. They are both one dimensional representations of S_n as there is only one SYT with each of shapes (n) and (1^n). By the definition of actions of S_n

on Specht modules, it is not hard to see that $V_{(n)}$ is the trivial representation and $V_{(1^n)}$ is the *sign representation* $\rho : S_n \to GL(\mathbb{C})$, where $\rho(\sigma) = \mathrm{sgn}(\sigma)$ for all $\sigma \in S_n$.

Applying Proposition 3.3.12 to the identity (3.26), we get

$$n! = \sum_{T \in \mathrm{SYT}(\lambda)} |\mathrm{SYT}(\lambda)|^2. \tag{3.49}$$

This identity can also be proved combinatorially using the Robinson–Schensted–Knuth (RSK) correspondence, see [10, 12].

3.4 Frobenius Character Map

Recall that a representation of a group G is determined by its character, and the character is a class functions on G. We consider the graded $\mathbb{Z}$-module

$$R = \bigoplus_{n \geq 0} R_n, \tag{3.50}$$

where R_n is the $\mathbb{Z}$-module generated by the irreducible characters of S_n and with the convention that $R_0 = \mathbb{Z}$. We can endow R a ring structure by defining the (external) multiplication as follows. Let V and W be representations of S_n and S_m respectively, then $V \otimes W$ is a representation of $S_n \times S_m$. Observe that $S_n \times S_m$ can be naturally embedded into S_{n+m}, hence we get an induced representation $U := \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} V \otimes W$. This product is called the *induction product*, and we will study this product in more detail in Chapter 4. We define the multiplication (denoted by $\bullet$) to be

$$\chi_V \bullet \chi_W := \chi_U \ . \tag{3.51}$$

It is not hard to verify that this product is well-defined, commutative and associative. Hence, R is a commutative, associative, graded ring with unit.

Recall that in Section 3.2, we define an inner product on the vector space of class functions which makes the irreducible characters form an orthonormal basis. We use this inner product to define a scalar product on R. Let $f, g \in R$ where $f = \sum f_n$, $g = \sum g_n$ and $f_n, g_n \in R_n$, then we define

$$\langle\, f \,,\, g \,\rangle_R := \sum_{n\geq 0} \langle\, f_n \,,\, g_n \,\rangle_{S_n} . \tag{3.52}$$

We define a $\mathbb{Z}$-linear map $\mathcal{F} : R \to \Lambda_{\mathbb{C}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ as follows

$$\mathcal{F}(f) = \sum_{n\geq 0} \mathcal{F}(f_n) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\sigma\in S_n} f(\sigma) p_{c(\sigma)}, \tag{3.53}$$

where $f = \sum_{n\geq 0} f_n \in R$, $f_n \in R_n$, and $c(\sigma)$ denotes the cycle type of σ. This map is called the *Frobenius character map*, and it builds a connection between the representations of symmetric groups and symmetric functions by the following result

Proposition 3.4.1. *The Frobenius character map $\mathcal{F}$ is an isometric (ring) isomorphism of R onto Λ, where the metrics on R and Λ are defined by $\langle\ ,\ \rangle_R$ and the Hall inner product respectively. In particular, $\mathcal{F}(V_\lambda) = s_\lambda$ and $\mathcal{F}(M_\lambda) = h_\lambda$.*

For this reason, we also call $\langle\ ,\ \rangle_R$ the Hall inner product, and simply write $\langle\ ,\ \rangle$ for it. For a proof of this proposition, see [7].

Recall that $V_{(1^n)}$ is the sign representation of S_n. Let $\lambda \vdash n$, we consider the representation $V_\lambda \boxtimes V_{1^n}$ of S_n. From Proposition 3.2.2(4) and Equation (3.53), we have

$$\mathcal{F}(V_\lambda \boxtimes V_{(1^n)}) = \frac{1}{n!} \sum_{\sigma\in S_n} \chi_{V_\lambda}(\sigma)\mathrm{sgn}(\sigma) p_{c(\sigma)} \tag{3.54}$$

Note that $\mathrm{sgn}(\sigma) = (-1)^{c_1+c_2+\cdots+c_l-l} = (-1)^{n-\ell(c(\sigma))}$, where $(c_1, c_2, \ldots, c_l) = c(\sigma)$ is the cycle type of σ. Applying (2.18) and (2.22), and Proposition 3.4.1 to (3.54), we have

$$\begin{aligned} \mathcal{F}(V_\lambda \boxtimes V_{(1^n)}) &= \frac{1}{n!} \sum_{\sigma\in S_n} \chi_{V_\lambda}(\sigma)\omega(p_{c(\sigma)}) = \omega(\mathcal{F}(V_\lambda)) \\ &= \omega(s_\lambda) = s_{\lambda'} = \mathcal{F}(V_{\lambda'}). \end{aligned} \tag{3.55}$$

By Proposition 3.4.1, we have

$$V_\lambda \boxtimes V_{(1^n)} = V_{\lambda'}. \tag{3.56}$$

Hence, applying the involution ω to a symmetric function is equivalent to tensoring the corresponding representation with the sign representation.

4. PRODUCTS ON REPRESENTATIONS OF SYMMETRIC GROUPS/SYMMETRIC FUNCTIONS

There are several interesting products on representations of symmetric groups (also on symmetric functions due to the Frobenius character map).

In Section 4.1, we recall the induction product which we introduced in Section 3.4. In Section 4.2, we introduce the Kronecker product and present some related results. In Section 4.3, we define the Heisenberg product which interpolates between the induction product and the Kronecker product and show some results about this product.

4.1 Induction product

Let V and W be representations of S_n and S_m respectively. Recall that the induction product of V and W is $\text{Ind}_{S_n \times S_m}^{S_{n+m}}(V \otimes W)$. Let λ, μ, and ν be partitions of $n+m$, n, and m respectively. Recall that the Littlewood-Richardson coefficient $c_{\mu,\nu}^{\lambda}$ is the coefficient of s_λ in the Schur expansion of the ordinary product $s_\mu \cdot s_\nu$ of Schur functions. From Proposition 3.4.1, we can can rewrite Equation (2.26) as

$$\text{Ind}_{S_n \times S_m}^{S_{n+m}}(V_\mu \otimes V_\nu) = \bigoplus_{\nu \vdash n+m} c_{\mu,\nu}^{\lambda} V_\lambda. \tag{4.1}$$

So the Littlewood-Richardson coefficient $c_{\mu,\nu}^{\lambda}$ can also be considered as the multiplicity of V_λ in the decomposition of $\text{Ind}_{S_n \times S_m}^{S_{n+m}}(V_\mu \otimes V_\nu)$ into irreducibles, hence

$$c_{\mu,\nu}^{\lambda} = \langle\ \text{Ind}_{S_n \times S_m}^{S_{n+m}}(V_\mu \otimes V_\nu)\ ,\ V_\lambda\ \rangle_{S_{n+m}} \tag{4.2}$$

Applying Proposition 3.2.6 to (4.2), we have

$$\begin{aligned} c_{\lambda,\mu}^{\nu} &= \langle\ \text{Ind}_{S_n \times S_m}^{S_{n+m}}(V_\mu \otimes V_\nu)\ ,\ V_\lambda\ \rangle_{S_{n+m}} \\ &= \langle\ V_\mu \otimes V_\nu\ ,\ \text{Res}_{S_n \times S_m}^{S_{n+m}} V_\lambda\ \rangle_{S_n \times S_m}. \end{aligned}$$

4.2 Kronecker product

Let V and W be representations of S_n. The *Kronecker product* of V and W is the tensor product $V \boxtimes W = \mathrm{Res}_{S_n}^{S_n \times S_n}(V \otimes W)$, where S_n is viewed as a subgroup of $S_n \times S_n$ through the diagonal map. Let λ, μ, and ν be partitions of n. The *Kronecker coefficient* $g^{\lambda}_{\mu,\nu}$ is the multiplicity of V_λ in the decomposition of $\mathrm{Res}_{S_n}^{S_n \times S_n}(V_\mu \otimes V_\nu)$ into irreducibles. That is,

$$\mathrm{Res}_{S_n}^{S_n \times S_n}(V_\mu \otimes V_\nu) = \bigoplus_{\lambda \vdash n} g^{\lambda}_{\mu,\nu} V_\lambda.$$

Using the above formula, we can define the Kronecker product (denoted by $*$) for symmetric functions:

$$s_\mu * s_\nu = \sum_{\lambda \vdash n} g^{\lambda}_{\mu,\nu} s_\lambda.$$

While the Littlewood–Richardson coefficients are well-studied and have several beautiful combinatorial interpretations, an explicit combinatorial or geometric description for the Kronecker coefficients is still unknown. In 1938, Murnaghan [3] discovered a remarkable stability property for the Kronecker coefficients. He stated without proof that for any partitions λ, μ, and ν with the same size, the sequence $\left\{g^{\lambda+(n)}_{\mu+(n),\nu+(n)}\right\}$ is eventually constant. We discuss this in more detail in Chapter 5.

Although an explicit method to compute the Kronecker product of Schur functions is still unknown, we have combinatorial ways to describe the Kronecker products of two power sum symmetric functions and two complete homogeneous symmetric functions. The formula for the power sum symmetric functions is the following:

Proposition 4.2.1. *Let λ and μ be partitions with the same size, then*

$$p_\lambda * p_\mu = z_\lambda \, \delta_{\lambda,\mu} \, p_\lambda. \tag{4.3}$$

The formula for the complete homogeneous symmetric functions is more complicated. Given a matrix A, we arrange its entries in weakly decreasing order. The result sequence is called the

π-*sequence* of A and denoted by $\pi(A)$. Let $\beta, \gamma \vDash n$ and $\alpha \vdash n$. We denote by $\mathcal{M}(\beta,\gamma)$ the set of matrices with nonnegative integer entries, row-sum vector β and column-sum vector γ, and $\mathcal{M}(\beta,\gamma)_\alpha$ the subset of $\mathcal{M}(\beta,\gamma)$ whose elements have π-sequence α. Then the Kronecker product of two complete homogeneous symmetric functions can be computed using the following formula:

Proposition 4.2.2. *Let β and γ be (weak) compositions of n, then the Kronecker product of h_β and h_γ is*

$$h_\beta * h_\gamma = \sum_{A\in\mathcal{M}(\beta,\gamma)} h_{\pi(A)} = \sum_{\alpha\vdash n} \sum_{A\in\mathcal{M}(\beta,\gamma)_\alpha} h_\alpha.$$

4.3 Heisenberg product

Aguiar et al. [1] and Moreira [2] introduced a new (nongraded) product which interpolates between the induction product and the Kronecker product.

Definition 4.3.1. (Heisenberg product) Let V and W be representations of S_n and S_m respectively. Fix an integer $l \in [\max\{m,n\}, m+n]$, and let $p = l-m$, $q = n+m-l$, and $r = l-n$. We have the (commutative) diagram of inclusions (solid arrows):

$$\begin{tikzcd}
S_p \times S_q \times S_q \times S_r \arrow[r, hook] \arrow[d, dashed, bend right, "\mathrm{Res}"'] & S_{p+q} \times S_{q+r} = S_n \times S_m & \color{red}{V \otimes W} \\
S_p \times S_q \times S_r \arrow[r, hook] \arrow[u, hook, "id_{S_p}\times\Delta_{S_q}\times id_{S_r}"] \arrow[ur, hook] \arrow[r, dashed, bend left, "\mathrm{Ind}"] & S_{p+q+r} = S_l & \color{red}{(V\#W)_l}
\end{tikzcd} \tag{4.4}$$

The Heisenberg product (denoted by $\#$) of V and W is

$$V\#W = \bigoplus_{l=\max(n,m)}^{n+m} (V\#W)_l, \tag{4.5}$$

where the degree l component is defined using the dashed arrows in the diagram:

$$(V\#W)_l = \mathrm{Ind}_{S_p\times S_q\times S_r}^{S_l} \mathrm{Res}_{S_p\times S_q\times S_r}^{S_n\times S_m} (V \otimes W). \tag{4.6}$$

When $l = m + n$, $(V\#W)_l = \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(V \otimes W)$, which is the induction product of representations; when $l = n = m$, $(V\#W)_l = \mathrm{Res}_{S_l}^{S_l \times S_l}(V \otimes W)$, which is the Kronecker product of representations. The Heisenberg product connects the induction product and the Kronecker product. Remarkably, this product is associative [1, Theorem 2.3, Theorem 2.4, Theorem 2.6]. They first construct the Heisenberg product on the category of species where the associativity is straightforward, then they build an isomorphism from the category of species to the category of representations of symmetric groups which preserves the Heisenberg product, hence proving the associativity of the Heisenberg product on the representations of symmetric groups. We give a direct proof of associativity in Appendix A. The *Heisenberg coefficient* $h_{\mu,\nu}^{\lambda}$ is the multiplicity of V_λ in the decomposition of $V_\mu \# V_\nu$ into irreducibles, i.e.

$$V_\mu \# V_\nu = \bigoplus_{l=\max\{n,m\}}^{n+m} \bigoplus_{\lambda \vdash l} h_{\mu,\nu}^{\lambda} V_\lambda,$$

and we set $h_{\mu,\nu}^{\lambda} = 0$ if λ, μ, or ν is not a partition. Similar to the Kronecker product, we can use the above formula to define the Heisenberg product (also denoted by $\#$) for symmetric functions:

$$s_\mu \# s_\nu = \sum_{l=\max\{n,m\}}^{n+m} \sum_{\lambda \vdash l} h_{\mu,\nu}^{\lambda} s_\lambda.$$

As the Kronecker product, the Heisenberg product of two Schur functions is not well understood, but there are combinatorial ways to compute the Heisenberg products of two power sum symmetric functions and two complete homogeneous symmetric functions which generalize Proposition 4.2.1 and 4.2.2.

Proposition 4.3.2 ([1] Theorem 3.4)**.** *Let λ and μ be partitions, then*

$$p_\lambda \# p_\mu = \sum_{\substack{\alpha \cup \beta = \lambda \\ \beta \cup \gamma = \mu}} z_\beta \, p_{\alpha \cup \beta \cup \gamma}. \tag{4.7}$$

To describe the Heisenberg product of two complete homogeneous symmetric functions, we introduce some notation. Given three finite sequences of real numbers α, β, and γ. Let $\mathcal{F}(\alpha,\beta)$ be the set of matrices with real entries, zero at the top left corner, row-sum vector (ignoring the first row) α and column-sum vector (ignoring the first column) β. We denote by $\mathcal{H}(\alpha,\beta)$ the set of matrices in $\mathcal{F}(\alpha,\beta)$ with integer entries, and $\mathcal{H}(\alpha,\beta)_\gamma$ the subset of $\mathcal{H}(\alpha,\beta)$ whose elements have π-sequence γ.

Example 4.3.3. The following matrix is in $\mathcal{H}((18,10),(12,18,3))_{(7,6,5,5,4,4,3,2,2,1)}$

$$\begin{pmatrix} 0 & 4 & 6 & 1 \\ 4 & 5 & 7 & 2 \\ 2 & 3 & 5 & 0 \end{pmatrix}.$$

Proposition 4.3.4 ([1] Theorem 3.1)**.** *Let β and γ be two (weak) compositions, then the Heisenberg product of h_β and h_γ is*

$$h_\beta \# h_\gamma = \bigoplus_{A\in\mathcal{H}(\beta,\gamma)} h_{\pi(A)}.$$

Briand et al. [14] showed that four families of coefficients (Kronecker coefficients, plethysm coefficients, Littlewood–Richardson coefficients, and the Kostka–Foulkes polynomials) share symmetries related to the operations of taking complements with respect to rectangles. We follow the notations that are used in [14], and prove an analogous result for Heisenberg coefficients. We use "bialternants" formula (2.19) of the Schur polynomial. Recall that $X_n = \{x_1, x_2, \ldots, x_n\}$, and we define ${X_n}^\vee := \{x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$ to be the set of the inverses of variables in X_n. The partition (k^n) has n parts all equal to k. Given a partition λ, let $\square_{k,n}(\lambda) = (k-\lambda_n,\, k-\lambda_{n-1},\, \ldots,\, k-\lambda_1)$. When $k \geq \lambda_1$ and $n \geq \ell(\lambda)$, the partition $\square_{k,n}(\lambda)$ is the complement of λ in the $n\times k$ rectangle (k^n). With this convention and using (2.19), it is not hard to show

$$s_{\lambda+(k^n)}(X_n) = (x_1x_2\cdots x_n)^k s_\lambda(X_n), \tag{4.8}$$

$$s_\lambda({X_n}^\vee) = s_{\square_{0,n}(\lambda)}(X_n). \tag{4.9}$$

From [1, Theorem 12.1], we have

$$s_\lambda(XY+X+Y)=\sum_{\mu,\nu} h^\lambda_{\mu,\nu} s_\mu(X)s_\nu(Y), \tag{4.10}$$

where $X=\{x_1,x_2,\ldots\}$, $Y=\{y_1,y_2,\ldots\}$, and $XY+X+Y=\{x_iy_j,x_i,y_j \mid i,j \geq 1\}$. Restricting variables to X_m and Y_n gives us

$$s_\lambda(X_mY_n+X_m+Y_n)=\sum_{\mu,\nu} h^\lambda_{\mu,\nu} s_\mu(X_m)s_\nu(Y_n). \tag{4.11}$$

Taking the inverses of variables in (4.11), we get

$$s_\lambda({X_m}^\vee{Y_n}^\vee+{X_m}^\vee+{Y_n}^\vee)=\sum_{\mu,\nu} h^\lambda_{\mu,\nu} s_\mu({X_m}^\vee)s_\nu({Y_n}^\vee). \tag{4.12}$$

Multiplying both sides by $(\prod_{i,j} x_iy_j)^k(\prod_i x_i)^k(\prod_j y_j)^k=(\prod_i x_i)^{kn+k}(\prod_j y_j)^{km+k}$, for k sufficiently large, and using (4.8) and (4.9), we get

$$s_{\square_{k,\,mn+m+n}(\lambda)}(X_mY_n+X_m+Y_n)=\sum_{\mu,\nu} h^\lambda_{\mu,\nu} s_{\square_{kn+k,\,m}(\mu)}(X_m)s_{\square_{km+k,\,n}(\nu)}(Y_n). \tag{4.13}$$

Using (4.10) and the fact that the family of Schur polynomials $\{s_\alpha(X_n) \mid \ell(\alpha) \leq n\}$ forms a $\mathbb{Z}$-basis for Λ_n, we get a rectangle symmetry for Heisenberg coefficients:

Theorem 4.3.5. *Let m, n, and k be nonnegative integers and λ, μ, and ν be three partitions such that $\lambda \subset ((k)^{mn+m+n})$, $\mu \subset ((kn+k)^m)$, and $\nu \subset ((km+k)^n)$, then*

$$h^\lambda_{\mu,\nu}=h^{\square_{k,\,mn+m+n}(\lambda)}_{\square_{kn+k,\,m}(\mu),\,\square_{km+k,\,n}(\nu)}. \tag{4.14}$$

5. STABILITY OF SCHUR STRUCTURE CONSTANTS

As we mentioned in Section 4.2, the Kronecker coefficient is not well understood, but Murnaghan [3] discovered a stability phenomenon for the Kronecker product of two Schur functions. There are many proofs with different flavours for this fact, see [15, 16, 17]. In 2014, Stembridge [4] vastly generalized Murnaghan's notion of the stability of Kronecker coefficients by introducing a new concept, Kronecker stable triple. We generalize those results to Heisenberg coefficients.

In Section 5.1, we introduce Murnaghan's stability result and prove an analogous result for the Heisenberg product of Schur functions. In Section 5.2, we define the stable Heisenberg coefficient, and show how to recover the usual Heisenberg coefficients from the stable ones which generalizes an analogue formula for the Kronecker coefficients in [15]. In Section 5.3, we introduce Stembridge's generalized stability of Kronecker coefficients and prove analogous result for Heisenberg coefficients. In Section 5.4, We follow Vallejo's idea [6] of using matrix additivity to generate stable triples for Heisenberg coefficients.

The results presented in this chapter are from [18, 19]. Table 5.1 and 5.2 are from [18].

5.1 Classic Stability Result

There is some interesting general work on representation stability by Church, Ellenberg, and Farb [20, 21, 22], and Sam and Snowden [23]. Church et al. use the FI-module to study the stability pattern, and Sam and Snowden use the theory of twisted commutative algebras [24] to study the stability phenomenon. In this section, we focus on the stability phenomenon of the Kronecker product discovered by Murnaghan [3]. We introduce some notations which will be used throughout this section. Let α be a finite integer sequence. Define α^+ to be the sequence obtained from α by adding 1 to the first part, $\alpha^+ := (\alpha_1 + 1, \alpha_2, \alpha_3, \dots)$; similarly, set $\alpha^- := (\alpha_1 - 1, \alpha_2, \alpha_3, \dots)$. Let $\overline{\alpha} = (\alpha_2, \alpha_3, \dots)$ be the sequence obtained from α by removing the first part.

Given an eventually constant sequence $\{a_n\}_{n\geq 0}$ with stable value L, we denote by SStab$(\{a_n\})$ the smallest integer n_0, such that for all $n \geq n_0$, $a_n = L$. We say that this sequence stabilizes when

$n \geq M$ as long as $M \geq n_0$, and the stabilization begins at $n = n_0$. For a sequence of symmetric functions $\{F_n\}_{n\geq 0}$, where F_n has the Schur expansion $F_n = \sum_\alpha a_n^\alpha s_\alpha$ (we set $a_n^\alpha = 0$ if α is not a partition), we say the sequence $\{F_n\}$ stabilizes if for any α (not necessarily a partition), the sequence $\left\{a_n^{\alpha+(n)}\right\}_{n\geq 0}$ is eventually constant, and there exist N, such that $\mathrm{SStab}(\left\{a_n^{\alpha+(n)}\right\}) \leq N$ for all α. Let n_1 be the smallest N having this property, and we denote it by $\mathrm{FStab}(\{F_n\})$. From the definition, we have $n_1 = \mathrm{FStab}(\{F_n\}) = \max_\alpha \left\{\mathrm{SStab}(\left\{a_n^{\alpha+(n)}\right\})\right\}$. We say the sequence of symmetric functions $\{F_n\}$ stabilizes when $n \geq M$ as long as $M \geq n_1$, and the stabilization begins at $n = n_1$.

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and a positive integer n, let $\lambda[n]$ be the sequence $(n - |\lambda|, \lambda_1, \lambda_2, \dots)$. When $n \geq |\lambda| + \lambda_1$, $\lambda[n]$ is a partition of n. The stability of the Kronecker product means that for any partitions λ and μ, the sequence of symmetric functions $\left\{s_{\lambda[n]} * s_{\mu[n]}\right\}_{n\geq 0}$ stabilizes when n is large enough. This phenomenon is best shown on an example. Let $\lambda = (2)$ and $\mu = (1,1)$, we compute the Kronecker product $s_{n-2,2} * s_{n-2,1,1}$ for $n \geq 4$:

$$
\begin{aligned}
s_{2,2} * s_{2,1,1} &= s_{3,1} + s_{2,1,1} \\
s_{3,2} * s_{3,1,1} &= s_{4,1} + s_{3,2} + 2s_{3,1,1} + s_{2,2,1} + s_{2,1,1,1} \\
s_{4,2} * s_{4,1,1} &= s_{5,1} + s_{4,2} + 2s_{4,1,1} + s_{3,3} + 2s_{3,2,1} + s_{3,1,1,1} + s_{2,2,1,1} \\
s_{5,2} * s_{5,1,1} &= s_{6,1} + s_{5,2} + 2s_{5,1,1} + s_{4,3} + 2s_{4,2,1} + s_{4,1,1,1} + s_{3,3,1} + s_{3,2,1,1} \\
s_{6,2} * s_{6,1,1} &= s_{7,1} + s_{6,2} + 2s_{6,1,1} + s_{5,3} + 2s_{5,2,1} + s_{5,1,1,1} + s_{4,3,1} + s_{4,2,1,1}.
\end{aligned}
$$

Observe that the last two equations are only different in the first part of the indexing partitions. Indeed, for $n \geq 7$, we have

$$
\begin{aligned}
s_{n-2,2} * s_{n-2,1,1} = {} & s_{n-1,1} + s_{n-2,2} + 2s_{n-2,1,1} + s_{n-3,3} + 2s_{n-3,2,1} \\
& + s_{n-3,1,1,1} + s_{n-4,3,1} + s_{n-4,2,1,1}.
\end{aligned}
$$

In this example, the stabilization of the sequence $\left\{g^{(n-3,2,1)}_{(n,2),(n-2,1,1)}\right\}_{n\geq 0}$ (the coefficients of the red colored terms) begins at $n = 6$. The sequence of symmetric functions $\{s_{n-2,2} * s_{n-2,1,1}\}_{n\geq 0}$ stabilizes when $n \geq N$ as long as $N \geq 7$, and the stabilization begins at $n = 7$.

In the above example, one can also observe that, for fixed partition ν, the sequence of coefficients of $s_{\nu[n]}$ in the expansion is weakly increasing as n increases. This was shown by Brion [25] and Manivel [26]:

Proposition 5.1.1. *Let λ, μ, and ν be partitions. The sequence $\left\{g^{\lambda[n]}_{\mu[n],\nu[n]}\right\}_n$ is weakly increasing.*

The sequence $\left\{g^{\lambda[n]}_{\mu[n],\nu[n]}\right\}$ is eventually constant according to the stability of Kronecker coefficients. Write $\overline{g}^{\lambda}_{\mu,\nu}$ for the stable value of this sequence and call it a *reduced Kronecker coefficient*. In our example, we see that $\overline{g}^{(2,1)}_{(2),(1,1)} = 2$ and $\overline{g}^{(1,1,1)}_{(2),(1,1)} = 1$. Moreover, Murnaghan [3] claimed that $\overline{g}^{\lambda}_{\mu,\nu}$ vanishes unless

$$|\lambda| \leq |\mu| + |\nu|, \quad |\mu| \leq |\lambda| + |\nu|, \quad |\nu| \leq |\lambda| + |\mu|,$$

which are triangle inequalities. When $|\lambda| = |\mu| + |\nu|$, $\overline{g}^{\lambda}_{\mu,\nu}$ is equal to the Littlewood-Richardson coefficient $c^{\lambda}_{\mu,\nu}$ [3].

Briand et al. [15] determined when the Kronecker product stabilizes and provided another condition for the reduced Kronecker coefficient to be nonzero.

Proposition 5.1.2 ([15] Theorem 1.2)**.** *Let λ and μ be partitions. The sequence of symmetric functions $\left\{s_{\lambda[n]} * s_{\mu[n]}\right\}_{n\geq 0}$ stabilizes, and the stabilization begins at $n = |\lambda| + |\mu| + \lambda_1 + \mu_1$.*

Proposition 5.1.3 ([15] Theorem 3.2)**.** *Let λ and μ be partitions, then*

$$\max\{|\nu| + \nu_1 | \, \nu \text{ partition}, \, \overline{g}^{\nu}_{\lambda,\mu} > 0\} = |\lambda| + |\mu| + \lambda_1 + \mu_1.$$

Proposition 5.1.3 will be used later in the proof of Theorem 5.1.4.

By the definition of the Heisenberg product (see Diagram (4.4)), when b is much greater than a and c, the right hand side of Equation (4.6) behaves like the Kronecker product. A natural question is whether we can develop a stability result for this degree component.

Theorem 5.1.4. *Given nonnegative integers r and t and two partitions λ and μ, the sequence $\left\{(s_{\lambda[n]}\#s_{\mu[n-r]})_{n+t}\right\}_{n\geq 0}$ of symmetric functions stabilizes, and the stabilization begins at $n = |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$.*

We give an example of the stabilization of the Heisenberg product.

Let us take $\lambda = (1,1)$, $\mu = (1)$. We check the stability of the two lowest degree components of $s_{(1,1)[n]}\#s_{(1)[n-1]}$:

$$s_{1,1,1}\#s_{1,1} = (s_{2,1,1,1}+s_{1,1,1,1,1}) + (s_{3,1}+s_{2,2}+2s_{2,1,1}+s_{1,1,1,1}) + (s_3+s_{2,1}),$$

$$\begin{aligned} s_{2,1,1}\#s_{2,1} = {} & (s_{4,2,1}+s_{4,1,1,1}+s_{3,3,1}+s_{3,2,2}+2s_{3,2,1,1}+s_{3,1,1,1,1}+s_{2,2,2,1}+s_{2,2,1,1,1}) \\ & + (s_{5,1}+3s_{4,2}+4s_{4,1,1}+2s_{3,3}+8s_{3,2,1}+6s_{3,1,1,1}+3s_{2,2,2}+6s_{2,2,1,1}+4s_{2,1,1,1,1}+s_{1,1,1,1,1,1}) \\ & + (s_5+5s_{4,1} + 7s_{3,2}+9s_{3,1,1}+8s_{2,2,1}+7s_{2,1,1,1}+2s_{1,1,1,1,1}) \\ & + (s_4+3s_{3,1}+2s_{2,2}+3s_{2,1,1} + s_{1,1,1,1}). \end{aligned}$$

The lowest degree component $(s_{(1,1)[n]}\#s_{(1)[n-1]})_n$ for $n \geq 5$:

$$\begin{aligned} (s_{3,1,1}\#s_{3,1})_5 &= s_5 + 3s_{4,1} + 4s_{3,2} + 4s_{3,1,1} + 4s_{2,2,1} + 3s_{2,1,1,1} + s_{1,1,1,1,1}, \\ (s_{4,1,1}\#s_{4,1})_6 &= s_6 + 3s_{5,1} + 4s_{4,2} + 4s_{4,1,1} + 2s_{3,3} + 5s_{3,2,1} + 3s_{3,1,1,1} + s_{2,2,2} + 2s_{2,2,1,1} \\ &\quad + s_{2,1,1,1,1}, \\ (s_{5,1,1}\#s_{5,1})_7 &= s_7 + 3s_{6,1} + 4s_{5,2} + 4s_{5,1,1} + 2s_{4,3} + 5s_{4,2,1} + 3s_{4,1,1,1} + s_{3,3,1} + s_{3,2,2} + 2s_{3,2,1,1} \\ &\quad + s_{3,1,1,1,1}, \\ (s_{6,1,1}\#s_{6,1})_8 &= s_8 + 3s_{7,1} + 4s_{6,2} + 4s_{6,1,1} + 2s_{5,3} + 5s_{5,2,1} + 3s_{5,1,1,1} + s_{4,3,1} + s_{4,2,2} + 2s_{4,2,1,1} \\ &\quad + s_{4,1,1,1,1}, \\ &\vdots \end{aligned}$$

To ease comparison, we create a table for this. The coefficients are the coefficients in the expansion in the Schur basis, of, respectively (in this order):

$$s_n, s_{(n-1,1)}, s_{(n-2,2)}, s_{(n-2,1,1)}, s_{(n-3,3)}, s_{(n-3,2,1)}, s_{(n-3,1,1,1)},$$
$$s_{(n-4,3,1)}, s_{(n-4,2,2)}, s_{(n-4,2,1,1)}, s_{(n-4,1,1,1,1)}.$$

n	coefficients in $(s_{n-2,1,1} \# s_{n-2,1})_n$										
3	1	1									
4	1	3	2	3			1				
5	1	3	4	4	2	5	3				1
6	1	3	4	4	2	5	3		1	1	1
$n \geq 7$	1	3	4	4	2	5	3	1	1	2	1

Table 5.1: Schur expansion of $(s_{n-2,1,1} \# s_{n-2,1})_n$ for $n \geq 3$.

We color the number red if it reaches the stable value and the circled numbers are where we estimate the corresponding sequence of Heisenberg coefficients will stabilize using Corollary 5.2.2. We can see that when $n \geq 7$, the Schur expansion of this degree component always has the same Heisenberg coefficients in the Schur expansion, and the only difference is the first part of the indexing partitions. The stabilization of the sequence of the lowest degree components of $s_{n-2,1,1} \# s_{n-2,1}$ happens at $n = 7$ (using Theorem 5.1.4 with $r = 1$ and $t = 0$, the stabilization begins at $n = 2 + 1 + 1 + 1 + 2 = 7$). When $n \geq 7$, we have

$$\begin{aligned}(s_{n-2,1,1} \# s_{n-2,1})_n = s_n + 3s_{n-1,1} + 4s_{n-2,2} + 4s_{n-2,1,1} + 2s_{n-3,3} + 5s_{n-3,2,1} \\ + 3s_{n-3,1,1,1} + s_{n-4,3,1} + s_{n-4,2,2} + 2s_{n-4,2,1,1} + s_{n-4,1,1,1,1}.\end{aligned} \tag{5.1}$$

From Table 5.1, we can also see that different columns (i.e. sequences $\left\{h^{\nu[n]}_{(n-2,1,1),(n-2,1)}\right\}$ for different ν) stabilize at different steps, we give an estimate for this in Corollary 5.2.2.

We also compute the second lowest degree component $(s_{(1,1)[n]} \# s_{(1)[n-1]})_{n+1}$ for $n \geq 5$, and create a table (see Table 5.2 on the next page) for the result, where the coefficients are the coefficients in the expansion in the Schur basis, of, respectively (in this order):

$s_{n+1}, s_{(n,1)}, s_{(n-1,2)}, s_{(n-1,1,1)}, s_{(n-2,3)}, s_{(n-2,2,1)}, s_{(n-2,1,1,1)}, s_{(n-3,4)}, s_{(n-3,3,1)}, s_{(n-3,2,2)},$

$s_{(n-3,2,1,1)}, s_{(n-3,1,1,1,1)}, s_{(n-4,5)}, s_{(n-4,4,1)}, s_{(n-4,3,2)}, s_{(n-4,3,1,1)}, s_{(n-4,2,2,1)}, s_{(n-4,2,1,1,1)},$

$s_{(n-4,1,1,1,1,1)}, s_{(n-5,5,1)}, s_{(n-5,4,2)}, s_{(n-5,4,1,1)}, s_{(n-5,3,3)}, 4s_{(n-5,3,2,1)}, s_{(n-5,3,1,1,1)}, s_{(n-5,2,2,2)},$

$s_{(n-5,2,2,1,1)}, s_{(n-5,2,1,1,1,1)}.$

n	coefficients in $\left(s_{n-2,1,1}\#s_{n-2,1}\right)_{n+1}$																										
5	1	7	13	16	7	24	16			7	13	7							1									
6	1	7	15	17	13	33	19		17	16	26	10					8	7	2									
7	1	7	15	17	15	34	19	6	26	18	29	10			10	13	12	10	2							1	2	1
8	1	7	15	17	15	34	19	8	27	18	29	10		9	12	16	12	10	2				1	4	3	1	2	1
9	1	7	15	17	15	34	19	8	27	18	29	10	2	10	12	16	12	10	2		2	3	1	4	3	1	2	1
$n \geq 10$	1	7	15	17	15	34	19	8	27	18	29	10	2	10	12	16	12	10	2	1	2	3	1	4	3	1	2	1

Table 5.2: Schur expansion of $(s_{n-2,1,1}\#s_{n-2,1})_{n+1}$ for $n \geq 5$.

This computation shows that the sequence of the second lowest degree components of $s_{n-2,1,1}\#s_{n-2,1}$ stabilizes at $n = 10$ (using Theorem 5.1.4 with $r = 1$ and $t = 1$, the stabilization begins at $n = 2+1+1+1+3+2 = 10$). When $n \geq 10$, we have

$$\begin{aligned}(s_{n-2,1,1}\#s_{n-2,1})_{n+1} = {} & s_{n+1} + 7s_{n,1} + 15s_{n-1,2} + 17s_{n-1,1,1} + 15s_{n-2,3} + 34s_{n-2,2,1} \\ & + 19s_{n-2,1,1,1} + 8s_{n-3,4} + 27s_{n-3,3,1} + 18s_{n-3,2,2} + 29s_{n-3,2,1,1} \\ & + 10s_{n-3,1,1,1,1} + 2s_{n-4,5} + 10s_{n-4,4,1} + 12s_{n-4,3,2} + 16s_{n-4,3,1,1} \\ & + 12s_{n-4,2,2,1} + 10s_{n-4,2,1,1,1} + 2s_{n-4,1,1,1,1,1} + s_{n-5,5,1} + 2s_{n-5,4,2} \\ & + 3s_{n-5,4,1,1} + s_{n-5,3,3} + 4s_{n-5,3,2,1} + 3s_{n-5,3,1,1,1} + s_{n-5,2,2,2} \\ & + 2s_{n-5,2,2,1,1} + s_{n-5,2,1,1,1,1}.\end{aligned} \tag{5.2}$$

To prove Theorem 5.1.4, we first prove a stability property of the Littlewood–Richardson coefficient.

Lemma 5.1.5. *Let λ, μ and ν be partitions with $|\nu| = |\lambda| + |\mu|$,*

(1) If $\nu_1 - \nu_2 \geq |\lambda|$, then $c^{\nu}_{\lambda,\mu} = c^{\nu^+}_{\lambda,\mu^+}$.

(2) If $\mu_1 - \mu_2 \geq |\lambda|$, then $c^{\nu}_{\lambda,\mu} = c^{\nu^+}_{\lambda,\mu^+}$.

Proof. By Proposition 2.3.1, $c^{\gamma}_{\alpha,\beta}$ (α, β, and γ are partitions) counts the number of semi-standard skew tableaux of shape γ/β and weight α whose row reading word is a lattice permutation. Let $T^{\gamma}_{\alpha,\beta}$ be the set of these tableaux. We show that $|T^{\nu}_{\lambda,\mu}| = |T^{\nu^+}_{\lambda,\mu^+}|$.

Note that $T^{\nu}_{\lambda,\mu} = \emptyset$ unless $\mu \subset \nu$, and $\mu \subset \nu$ if and only if $\mu^+ \subset \nu^+$, hence it is enough to consider the case $\mu \subset \nu$. The skew diagrams ν/μ and ν^+/μ^+ differ only by a shift of the first row. Since $\nu_1 - \nu_2 \geq |\lambda|$, the first row (may be empty) of ν/μ is disconnected from the rest of the skew diagram, and similarly for ν^+/μ^+. This gives us a natural bijection between $T^{\nu}_{\lambda,\mu}$ and $T^{\nu^+}_{\lambda,\mu^+}$. Hence $|T^{\nu}_{\lambda,\mu}| = |T^{\nu^+}_{\lambda,\mu^+}|$, and (1) is proved.

The proof of (2) is the same, as $\mu_1 - \mu_2 \geq |\lambda|$ also implies that the first row of ν/μ is disconnected from the rest of it. $\square$

Remark 5.1.6. When λ, μ, and ν do not satisfy the conditions in Lemma 5.1.5, the one unit shift of the first row may fail to be a bijection between $T^{\nu}_{\lambda,\mu}$ and $T^{\nu^+}_{\lambda,\mu^+}$. However, it is still a well-defined injection from $T^{\nu}_{\lambda,\mu}$ to $T^{\nu^+}_{\lambda,\mu^+}$, which means $c^{\nu}_{\lambda,\mu} \leq c^{\nu^+}_{\lambda,\mu^+}$. In other words, the sequence $\left\{c^{\nu+(n)}_{\lambda,\mu+(n)}\right\}$ is weakly increasing and is constant when n is large.

Theorem 5.1.4 states that FStab$(\{(s_{\lambda[n]}\#s_{\mu[n-r]})_{n+t}\}_n) = |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$. We first show that FStab$(\{(s_{\lambda[n]}\#s_{\mu[n-r]})_{n+t}\}) \leq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$, i.e.

$$h^{\nu^-}_{\lambda[n],\mu[n-r]} = h^{\nu}_{\lambda[n+1],\mu[n-r+1]} \tag{5.3}$$

for all $\nu \vdash n + t + 1$ when $n \geq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$.

To prove (5.3), we express the Heisenberg coefficient in terms of the Littlewood-Richardson coefficients and the Kronecker coefficients.

Lemma 5.1.7. *For each* $\nu \vdash l$,

$$h^{\nu}_{\lambda,\mu} = \sum_{\substack{\alpha \vdash a,\, \rho \vdash c,\, \tau \vdash n \\ \beta,\eta,\delta \vdash b}} c^{\lambda}_{\alpha,\beta}\; c^{\mu}_{\eta,\rho}\; g^{\delta}_{\beta,\eta}\; c^{\tau}_{\alpha,\delta}\; c^{\nu}_{\tau,\rho} \tag{5.4}$$

where $\max(n,m) \leq l \leq n+m$, $a = l - m$, $b = m + n - l$, *and* $c = l - n$.

Proof. Consider the diagram (4.4) we used to define the Heisenberg product. Given partitions $\lambda \vdash n$ and $\mu \vdash m$, $V_\lambda \otimes V_\mu$ is a representation of $S_n \times S_m (= S_{a+b} \times S_{b+c})$. We compute the Heisenberg product of V_λ and V_μ in three steps.

$$\begin{tikzcd}
S_a\times S_b\times S_b\times S_c \arrow[rr, hook] \arrow[d, bend right, "(2)"'] & & S_{a+b}\times S_{b+c} = S_n\times S_m \arrow[ll, bend right, "(1)"'] \\
S_a \times S_b \times S_c \arrow[u, hook'] \arrow[urr, hook] \arrow[rr, hook, "(3)"'] \arrow[dr, "(3.1)"'] & & S_{a+b+c} = S_l \\
& S_{a+b} \times S_c = S_n \times S_c \arrow[ur, hook, "(3.2)"'] &
\end{tikzcd} \tag{5.5}$$

First, we restrict the representation from $S_n \times S_m$ to $S_a \times S_b \times S_b \times S_c$,

$$\mathrm{Res}^{S_n\times S_m}_{S_a\times S_b\times S_b\times S_c}(V_\lambda\otimes V_\mu)=\bigoplus_{\substack{\alpha\vdash a\\ \beta\vdash b}}\bigoplus_{\substack{\eta\vdash b\\ \rho\vdash c}}c^{\lambda}_{\alpha,\beta}\ c^{\mu}_{\eta,\rho}\,V_\alpha\otimes V_\beta\otimes V_\eta\otimes V_\rho. \tag{1}$$

Second, pull back to $S_a\times S_b\times S_c$ along the diagonal map of S_b. For $\alpha\vdash a$, $\rho\vdash c$, and $\beta,\eta\vdash b$ we have,

$$\mathrm{Res}^{S_a\times S_b\times S_b\times S_c}_{S_a\times S_b\times S_c}(V_\alpha\otimes V_\beta\otimes V_\eta\otimes V_\rho)=\bigoplus_{\delta\vdash b}g^{\delta}_{\beta,\eta}\,V_\alpha\otimes V_\delta\otimes V_\rho. \tag{2}$$

The final step is the induction from $S_a\times S_b\times S_c$ to $S_{a+b+c}(=S_l)$. Break this step into two substeps as in (5.5). Given $\alpha\vdash a$, $\delta\vdash b$, and $\rho\vdash c$, we have:

$$\begin{aligned}\mathrm{Ind}^{S_l}_{S_a\times S_b\times S_c}(V_\alpha\otimes V_\delta\otimes V_\rho)&=\mathrm{Ind}^{S_l}_{S_n\times S_c}\mathrm{Ind}^{S_n\times S_c}_{S_a\times S_b\times S_c}(V_\alpha\otimes V_\delta\otimes V_\rho) \qquad (3)\\ &=\bigoplus_{\substack{\tau\vdash n\\ \nu\vdash l}}c^{\tau}_{\alpha,\delta}\ c^{\nu}_{\tau,\rho}\ V_\nu.\end{aligned}$$

Combining (1), (2), and (3) together, gives

$$\begin{aligned}(V_\lambda\otimes V_\mu)_l&=\mathrm{Ind}^{S_l}_{S_a\times S_b\times S_c}\mathrm{Res}^{S_a\times S_b\times S_b\times S_c}_{S_a\times S_b\times S_c}\mathrm{Res}^{S_n\times S_m}_{S_a\times S_b\times S_b\times S_c}(V_\lambda\otimes V_\mu)\\ &=\bigoplus_{\substack{\alpha\vdash a,\rho\vdash c,\tau\vdash n\\ \beta,\eta,\delta\vdash b,\nu\vdash l}}c^{\lambda}_{\alpha,\beta}\ c^{\mu}_{\eta,\rho}\ g^{\delta}_{\beta,\eta}\ c^{\tau}_{\alpha,\delta}\ c^{\nu}_{\tau,\rho}\ V_\nu\end{aligned}$$

So for $\nu\vdash l$,

$$h^{\nu}_{\lambda,\mu}=\sum_{\substack{\alpha\vdash a,\rho\vdash c,\tau\vdash n\\ \beta,\eta,\delta\vdash b}}c^{\lambda}_{\alpha,\beta}\ c^{\mu}_{\eta,\rho}\ g^{\delta}_{\beta,\eta}\ c^{\tau}_{\alpha,\delta}\ c^{\nu}_{\tau,\rho},$$

as claimed. □

We set $c^{\nu}_{\lambda,\mu}=0$ when λ, μ, or ν is not a partition. Then (5.4) holds for all sequences ν with sum

l. Applying (5.4), to prove (5.3), it is enough to show that, when $n \geq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$,

$$\sum_{(\alpha,\beta,\eta,\rho,\delta,\tau)\in T} c^{\lambda[n]}_{\alpha,\beta}\ c^{\mu[n-r]}_{\eta,\rho}\ g^{\delta}_{\beta,\eta}\ c^{\tau}_{\alpha,\delta}\ c^{\nu^-}_{\tau,\rho} = \sum_{(\alpha^*,\beta^*,\eta^*,\rho^*,\delta^*,\tau^*)\in T^*} c^{\lambda[n+1]}_{\alpha^*,\beta^*}\ c^{\mu[n+1-r]}_{\eta^*,\rho^*}\ g^{\delta^*}_{\beta^*,\eta^*}\ c^{\tau^*}_{\alpha^*,\delta^*}\ c^{\nu}_{\tau^*,\rho^*} \tag{5.6}$$

for all $\nu \vdash n + t + 1$, where

$$\begin{aligned} T &= \{(\alpha,\beta,\eta,\rho,\delta,\tau) \mid \alpha \vdash r+t, \rho \vdash t, \tau \vdash n, \beta,\eta,\delta \vdash n-r-t\}; \\ T^* &= \{(\alpha^*,\beta^*,\eta^*,\rho^*,\delta^*,\tau^*) \mid \alpha^* \vdash r+t, \rho^* \vdash t, \tau^* \vdash n+1, \\ &\qquad \beta^*,\eta^*,\delta^* \vdash n-r-t+1\}. \end{aligned}$$

Define $f : T \longmapsto \mathbb{Z}_{\geq 0}$ and $f^* : T^* \longmapsto \mathbb{Z}_{\geq 0}$ as follows:

$$f(\alpha,\beta,\eta,\rho,\delta,\tau) = c^{\lambda[n]}_{\alpha,\beta}\ c^{\mu[n-r]}_{\eta,\rho}\ g^{\delta}_{\beta,\eta}\ c^{\tau}_{\alpha,\delta}\ c^{\nu^-}_{\tau,\rho},$$

$$f^*(\alpha^*,\beta^*,\eta^*,\rho^*,\delta^*,\tau^*) = c^{\lambda[n+1]}_{\alpha^*,\beta^*}\ c^{\mu[n+1-r]}_{\eta^*,\rho^*}\ g^{\delta^*}_{\beta^*,\eta^*}\ c^{\tau^*}_{\alpha^*,\delta^*}\ c^{\nu}_{\tau^*,\rho^*}.$$

Then Equation (5.6) becomes:

$$\sum_{u\in T} f(u) = \sum_{u^*\in T^*} f^*(u^*). \tag{5.7}$$

Some terms in the sums of (5.7) vanish. Let us consider only the nonvanishing terms.

Let $T_0 = T \setminus f^{-1}(0)$ and $T_0^* = T^* \setminus f^{*-1}(0)$, then (5.7) is equivalent to

$$\sum_{u\in T_0} f(u) = \sum_{u^*\in T_0^*} f^*(u^*). \tag{5.8}$$

Lemma 5.1.8. *When $n \geq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$, the embedding φ from T to T^*:*

$$\varphi(\alpha,\beta,\eta,\rho,\delta,\tau) = (\alpha,\beta^+,\eta^+,\rho,\delta^+,\tau^+)$$

induces a map $\varphi|_{T_0}$ from T_0 to T_0^*. Moreover, $f|_{T_0} = f^* \circ \varphi|_{T_0}$.

Proof. For all $u = (\alpha, \beta, \eta, \rho, \delta, \tau) \in T_0$, we show that β, η, δ, and τ have large enough first parts so that we can apply Proposition 5.1.2 and Lemma 5.1.5 to the Kronecker coefficients and the Littlewood-Richardson coefficients appearing in the definition of f.

Since $n \geq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$, we have

$$\lambda[n]_1 - \lambda[n]_2 = n - |\lambda| - \lambda_1 \geq |\mu| + \mu_1 + 3t + 2r \geq r + t \ \ (= |\alpha|)$$

and

$$\mu[n-r]_1 - \mu[n-r]_2 = n - r - |\mu| - \mu_1 \geq |\lambda| + \lambda_1 + 3t + r \geq t \ \ (= |\rho|).$$

Using Lemma 5.1.5 (1), we get

$$c^{\lambda[n]}_{\alpha,\beta} = c^{\lambda[n+1]}_{\alpha,\beta^+} \quad \text{and} \quad c^{\mu[n-r]}_{\eta,\rho} = c^{\mu[n+1-r]}_{\eta^+,\rho}.$$

As $\beta \subset \lambda[n]$, $|\overline{\beta}| \leq |\lambda| < n - r - t$ and $(\overline{\beta})_1 \leq \lambda_1$. Similarly, we have $|\overline{\eta}| \leq |\mu| < n - r - t$ and $(\overline{\eta})_1 \leq \mu_1$. Since β and η are both partitions of $n - r - t$, they can be written as $\beta = \overline{\beta}[n - r - t]$ and $\eta = \overline{\eta}[n - r - t]$ respectively. They both have large first parts. More specifically, we have

$$n - r - t \geq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 2t + r \geq |\overline{\beta}| + |\overline{\eta}| + (\overline{\beta})_1 + (\overline{\eta})_1.$$

By Proposition 5.1.2, we have

$$g^{\delta}_{\beta,\eta} = g^{\delta^+}_{\beta^+,\eta^+} = \overline{g}^{\overline{\delta}}_{\overline{\beta},\overline{\eta}}.$$

From Proposition 5.1.3,

$$|\overline{\delta}| + (\overline{\delta})_1 \leq |\overline{\beta}| + |\overline{\eta}| + (\overline{\beta})_1 + (\overline{\eta})_1$$

for otherwise $g^{\delta}_{\beta,\eta} = 0$, which implies that $f(u) = 0$, a contradiction. Hence,

$$|\delta| - \delta_1 + \delta_2 \leq |\lambda| + |\mu| + \lambda_1 + \mu_1,$$

which gives us

$$\delta_1 - \delta_2 \geq n - r - t - |\lambda| - |\mu| - \lambda_1 - \mu_1 \geq 2t + r \geq |\alpha|.$$

Applying Lemma 5.1.5 (2), we get

$$c^{\tau}_{\alpha,\delta} = c^{\tau^+}_{\alpha,\delta^+}.$$

Since $c^{\tau}_{\alpha,\delta} \neq 0$, after Proposition 2.3.1, we have

$$\tau_2 \leq \delta_2 + |\alpha| \quad \text{and} \quad \tau_1 \geq \delta_1.$$

So

$$\tau_1 - \tau_2 \geq \delta_1 - (\delta_2 + |\alpha|) \geq 2t + r - (r + t) = t = |\rho|.$$

Hence, by Lemma 5.1.5 (2), we get

$$c^{\nu^-}_{\tau,\rho} = c^{\nu}_{\tau^+,\rho}.$$

So

$$f(\alpha, \beta, \eta, \rho, \delta, \tau) = f^*(\varphi(\alpha, \beta, \eta, \rho, \delta, \tau))(\neq 0), \tag{5.9}$$

which means $\varphi(T_0) \subset T_0^*$ and $f|_{T_0} = f^* \circ \varphi|_{T_0}$. $\square$

To show that φ is a bijection between T_0 and T_0^*, we construct a reverse map.

Lemma 5.1.9. *When $n \geq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$, the map $\phi : (\alpha, \beta, \eta, \rho, \delta, \tau) \longrightarrow (\alpha, \beta^-, \eta^-, \rho, \delta^-, \tau^-)$ is well-defined from T_0^* to T_0. Moreover, $f^*|_{T_0^*} = f \circ \phi$.*

Proof. Take $u = (\alpha, \beta, \eta, \rho, \delta, \tau) \in T_0^*$, we first show that $\beta^-, \eta^-, \delta^-$, and τ^- are partitions.

Since $f^*(u) \neq 0$, we get $c_{\alpha,\beta}^{\lambda[n+1]} \neq 0$. Applying Proposition 2.3.1, we must have $\overline{\beta} \subset \lambda$ and $\lambda[n+1] - \beta_1 \leq |\alpha|$. Hence,

$$\beta_1 - \beta_2 \geq (n + 1 - |\lambda| - |\alpha|) - \lambda_1 \geq |\mu| + \mu_1 + 2t + r + 1 \geq 1.$$

So β^- is a partition. Similarly, we can show that η^- is a partition. Using Proposition 5.1.2 and Proposition 5.1.3 as we did in the proof of Lemma 5.1.8, we see that δ^- is a partition for

$$\delta_1 - \delta_2 \geq 2t + r + 1 \geq 1.$$

As $c_{\alpha,\delta}^{\tau} \neq 0$, we have $\tau_1 \geq \delta_1$ and $\tau_2 \leq \delta_2 + |\alpha|$. This shows that τ^- is a partition because

$$\tau_1 - \tau_2 \geq \delta_1 - (\delta_2 + |\alpha|) \geq t + 1 \geq 1.$$

Then by the same argument as in the proof of Lemma 5.1.8, we can show that $f^*|_{T_0^*} = f \circ \phi$, which implies that $\phi(T_0^*) \subset T_0$. $\square$

Proof of Theorem 5.1.4. Combining Lemma 5.1.8 and Lemma 5.1.9, we know φ is a bijection between T_0 and T_0^*. With this and (5.9), we prove (5.8), and hence

$$\text{FStab}((s_{\lambda[n]} \# s_{\mu[n-r]})_{n+t}) \leq |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r.$$

To prove that stabilization begins at $|\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$, it is enough to show that there exists $\nu \vdash n + t$ with $\nu_1 = \nu_2$ (then ν^- is not a partition) such that $h_{\lambda[n],\mu[n-r]}^{\nu} \neq 0$ when $n = |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$. We use the Formula (5.4) for $h_{\lambda[n],\mu[n-r]}^{\nu} \neq 0$ (replace λ and ν by $\lambda[n]$ and $\mu[n-r]$ respectively, and set $l = n + t$), and take

$$\alpha = (a) = (r + t), \rho = (c) = (t),$$

$$\beta = \lambda[n] - \alpha = (n - |\lambda| - r - t, \lambda_1, \lambda_2, \dots) = (|\mu| + \lambda_1 + \mu_1 + 2t + r, \lambda_1, \dots),$$

$$\eta = \mu[n-r] - \rho = (n - r - |\mu| - t, \mu_1, \mu_2, \dots) = (|\lambda| + \lambda_1 + \mu_1 + 2t + r, \mu_1, \dots),$$

$$\delta = (\overline{\beta}+\overline{\eta})[n-r-t] = (n-r-t-|\overline{\beta}|-|\overline{\eta}|, \beta_2+\eta_2, \beta_3+\eta_3, \dots) = (\lambda_1+\mu_1+2t+r, \lambda_1+\mu_1, \dots),$$

$$\tau = (\delta_1, \delta_2 + |\alpha|, \delta_3, \dots) = (\lambda_1 + \mu_1 + 2t + r, \lambda_1 + \mu_1 + r + t, \lambda_2 + \mu_2, \dots),$$

$$\nu = (\tau_1, \tau_2 + |\rho|, \dots) = (\lambda_1 + \mu_1 + 2t + r, \lambda_1 + \mu_1 + 2t + r, \lambda_2 + \mu_2, \dots).$$

By the Pieri Rule, $1 = c^{\lambda[n]}_{\alpha,\beta} = c^{\mu[n-d]}_{\eta,\rho} = c^{\tau}_{\alpha,\delta} = c^{\nu}_{\tau,\rho}$, as α and ρ have only one part each. Since $|\overline{\delta}| = |\overline{\beta}| + |\overline{\eta}|$, we have $g^{\delta}_{\beta,\eta} = \overline{g}^{\overline{\delta}}_{\overline{\beta},\overline{\eta}} = c^{\overline{\delta}}_{\overline{\beta},\overline{\eta}}$ (note that $\overline{\delta} = \overline{\beta} + \overline{\eta}$) which is also nonzero according to Proposition 2.3.1.

So $h^{\nu}_{\lambda[n],\mu[n-r]} \neq 0$ and $\nu_1 = \nu_2 = \lambda_1+\lambda_2+2t+r$, this proves that $n = |\lambda|+|\mu|+\lambda_1+\mu_1+3t+2r$ is where the stabilization begins. $\square$

When $n < |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r$, following the same arguments as in the proof of Lemma 5.1.8 (except for using Proposition 5.1.1 and Remark 5.1.6 instead of Proposition 5.1.2 and Lemma 5.1.5), we can show that the map φ in Lemma 5.1.8 induces an injection from T_0 to T_0^* with $f|_{T_0} \leq f^* \circ \varphi|_{T_0}$. This gives us the following corollary:

Corollary 5.1.10. *Given three partitions λ, μ, and ν and two nonnegative integers r and t, the sequence $\left\{h^{\nu[n+t]}_{\lambda[n],\mu[n-r]}\right\}_n$ is weakly increasing.*

5.2 Stable Heisenberg Coefficients

Given partitions λ, μ, and ν, Theorem 5.1.4 tells us that the sequence $\left\{h^{\nu+(n)}_{\lambda+(n),\mu+(n)}\right\}_{n=0}^{\infty}$ is eventually constant. We write $\overline{h}^{\nu}_{\lambda,\mu}$ for that constant value, and call it a *stable Heisenberg coefficient*. Stable Heisenberg coefficients generalize reduced Kronecker coefficients. By the way we define a stable Heisenberg coefficient, we have

$$\overline{h}^{\nu}_{\lambda,\mu} = \overline{h}^{\nu+(n)}_{\lambda+(n),\mu+(n)}, \quad \text{for all nonnegative integers } n.$$

The reason we restrict n to be a nonnegative integer is that $\lambda + (n)$, $\mu + (n)$, and $\nu + (n)$ need to be partitions. But we can drop this restriction and extend the definition by setting

$$\overline{h}^{\nu-(n)}_{\lambda-(n),\mu-(n)} = \overline{h}^{\nu}_{\lambda,\mu}, \quad \text{for all nonnegative integers } n.$$

We call a finite integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ an h-partition if $\alpha_2 \geq \alpha_3 \geq \cdots \geq \alpha_k > 0$. A stable Heisenberg coefficient, in the new definition, is indexed by three h-partitions, and

$$\overline{h}^{\nu}_{\lambda,\mu} = \overline{h}^{\nu+(n)}_{\lambda+(n),\mu+(n)}, \quad \text{for all integers } n.$$

where λ, μ, and ν are h-partitions.

Murnaghan [3] pointed out that the reduced Kronecker coefficients determine the Kronecker product. Briand et al. [15, Theorem 1.1] gave an exact formula to recover the Kronecker coefficients from the reduced ones, and Bowman et al. [27] interpreted this formula in terms of the representation theory of the partition algebra. Analogously, the stable Heisenberg coefficients also determine the Heisenberg product, even for small values of n. This can be proved using vertex operators on symmetric functions, and the idea of the proof is the same as the proof of the stability of the Kronecker product in [17]. We prove this in Appendix B.

Consider the lowest degree component of $s_{2,1,1} \# s_{2,1}$ as an example. Let $n = 4$, then (5.1) gives

$$\begin{aligned}(s_{2,1,1} \# s_{2,1})_4 = s_4 &+ 3s_{3,1} + 4s_{2,2} + 4s_{2,1,1} + 2s_{1,3} + 5s_{1,2,1} \\ &+ 3s_{1,1,1,1} + s_{0,3,1} + s_{0,2,2} + 2s_{0,2,1,1} + s_{0,1,1,1,1}.\end{aligned} \tag{5.10}$$

We use the Jacobi-Trudi determinant (2.23) as the definition of Schur functions. We no longer require λ to be a partition, and λ can be any finite integer sequence. Then (2.23) gives us 0 or ± 1

times some Schur function. Applying (2.23) to the right hand side of (5.10), we have

$$s_{1,3} = -s_{2,2}, \quad s_{0,3,1} = -s_{2,1,1}, \quad s_{0,2,1,1} = -s_{1,1,1,1}, \quad \text{and}$$
$$s_{1,2,1} = s_{0,2,2} = s_{0,1,1,1,1} = 0.$$

So (5.10) gives us

$$(s_{2,1,1}\#s_{2,1})_4 = s_4 + 3s_{3,1} + 2s_{2,2} + 3s_{2,1,1} + s_{1,1,1,1},$$

which coincides with the result we had in Section 5.1. This example shows the process to recover the Heisenberg coefficients from the stable ones. The following theorem generalizes the formula in [15, Theorem 1.1], and recovers the Kronecker coefficient as a special case.

Theorem 5.2.1. *Let λ, μ, and ν be partitions with $|\nu| \geq |\lambda| \geq |\mu|$, then*

$$h^{\nu}_{\lambda,\mu} = \sum_{i=1}^{4|\nu|-|\lambda|-|\mu|} (-1)^{i-1} \overline{h}^{\nu^{\dagger i}}_{\lambda,\mu}, \tag{5.11}$$

where $\nu^{\dagger i} = (\nu_i - i + 1, \nu_1 + 1, \nu_2 + 1, \ldots, \nu_{i-1} + 1, \nu_{i+1}, \nu_{i+2}, \ldots)$.

Consider an example. From the example we computed in Section 5.1, we know that $h^{(2,2)}_{(2,1,1),(2,1)} = 2$. On the other hand, using the Formula (5.11), we have

$$\begin{aligned} h^{(2,2)}_{(2,1,1),(2,1)} = \overline{h}^{(2,2)}_{(2,1,1),(2,1)} - \overline{h}^{(1,3)}_{(2,1,1),(2,1)} + \overline{h}^{(-2,3,3)}_{(2,1,1),(2,1)} \\ - \overline{h}^{(-3,3,3,1)}_{(2,1,1),(2,1)} + \cdots . \end{aligned} \tag{5.12}$$

From (5.1), we have

$$\overline{h}^{(2,2)}_{(2,1,1),(2,1)} = 4, \quad \overline{h}^{(1,3)}_{(2,1,1),(2,1)} = 2,$$

and

$$\overline{h}^{(2,2)^{\dagger i}}_{(2,1,1),(2,1)} = 0, \text{ when } i \geq 3.$$

So (5.12) gives us

$$h^{(2,2)}_{(2,1,1),(2,1)} = 4 - 2 = 2.$$

Proof of Theorem 5.2.1. From Theorem 5.1.4, we know that when $n \geq |\overline{\lambda}| + |\overline{\mu}| + \lambda_2 + \mu_2 + 3(|\nu| - |\lambda|) + 2(|\lambda| - |\mu|) - |\lambda|$, the Heisenberg coefficients of $(s_{\lambda+(n)} \# s_{\mu+(n)})_{n+|\nu|}$ stabilize, i.e.

$$\begin{aligned}
(s_{\lambda+(n)} \# s_{\mu+(n)})_{n+|\nu|} &= \sum_{\tau \vdash n+|\nu|} h^{\tau}_{\lambda+(n),\mu+(n)} s_\tau \\
&= \sum_{\tau \vdash n+|\nu|} \overline{h}^{\tau}_{\lambda+(n),\mu+(n)} s_\tau \\
&= \sum_{\tau \vdash n+|\nu|} \overline{h}^{\tau-(n)}_{\lambda,\mu} s_\tau.
\end{aligned}$$

So

$$(s_\lambda \# s_\mu)_{|\nu|} = \sum_{\tau \vdash n+|\nu|} \overline{h}^{\tau-(n)}_{\lambda,\mu} s_{\tau-(n)}. \tag{5.13}$$

To get $h^{\nu}_{\lambda,\mu}$ from (5.13), we determine which $s_{\tau-(n)}$'s would give us $\pm s_\nu$. Suppose the length of τ is l. From the Jacobi-Trudi formula, we know that $s_{\tau-(n)} = \pm s_\nu$ if and only if the length of ν is at most l and $(\tau_1 - n, \tau_2, \tau_3, \ldots, \tau_l) + (l-1, l-2, \ldots, 0)$ is a permutation of $(\nu_1, \nu_2, \ldots, \nu_l) + (l-1, l-2, \ldots, 0)$. This happens when there is an i ($1 \leq i \leq l$) such that

$$\tau_1 - n + (l-1) = \nu_i + l - i,$$

$$\tau_j + (l-j) = \nu_{j-1} + (l-j+1), \;\; j = 2, 3, 4, \ldots, i,$$

$$\tau_j + (l-j) = \nu_j + (l-j), \;\; j = i+1, i+2, \ldots, l,$$

which is equivalent to

$$\tau - (n) = \nu^{\dagger i},$$

and when this happens,

$$s_{\tau-(n)} = (-1)^{i-1} s_\nu.$$

So the coefficient of s_ν in $(s_\lambda \# s_\mu)_{|\nu|}$ is

$$h_{\lambda,\mu}^{\nu} = \sum_{i=1}^{l} (-1)^{i-1} \overline{h}_{\lambda,\mu}^{\nu^{\dagger i}}. \tag{5.14}$$

Take $n = 3|\nu| - |\lambda| - |\mu| \geq |\overline{\lambda}| + |\overline{\mu}| + \lambda_2 + \mu_2 + 3(|\nu| - |\lambda|) + 2(|\lambda| - |\mu|) - |\lambda|$, since $l \leq |\tau| = n + |\nu|$, (5.4) can be written as

$$h_{\lambda,\mu}^{\nu} = \sum_{i=1}^{4|\nu|-|\lambda|-|\mu|} (-1)^{i-1} \overline{h}_{\lambda,\mu}^{\nu^{\dagger i}}.$$

$\square$

Now we use Theorem 5.1 to estimate when $\left\{ h_{\lambda[n],\mu[n-r]}^{\nu[n+t]} \right\}_n$ stabilizes for given partitions λ, μ, and ν and nonnegative integers r and t.

Corollary 5.2.2. *The sequence of Heisenberg coefficients* $\left\{ h_{\lambda[n],\mu[n-r]}^{\nu[n+t]} \right\}_{n\geq 0}$ *stabilizes when* $n \geq \frac{1}{2}(|\lambda| + |\mu| + |\nu| + \lambda_1 + \mu_1 + \nu_1 - 1) + r + t$.

Proof. Formula (5.11) gives us

$$h_{\lambda[n],\mu[n-r]}^{\nu[n+t]} = \sum_{i=1}^{2n+4t+r} (-1)^{i-1} \overline{h}_{\lambda[n],\mu[n-r]}^{\nu[n+t]^{\dagger i}}, \tag{5.15}$$

So $h_{\lambda[n],\mu[n-r]}^{\nu[n+t]}$ reaches the stable value when $\overline{h}_{\lambda[n],\mu[n-r]}^{\nu[n+t]^{\dagger i}} = 0$ for all $i \geq 2$. By Theorem 5.1.4, $\left\{ (s_{\lambda[n]} \# s_{\mu[n-r]})_{n+t} \right\}_{n\geq 0}$ stabilizes at $n = |\lambda| + |\mu| + \lambda_1 + \mu_1 + 3t + 2r =: m$, so

$$\overline{h}_{\lambda[n],\mu[n-r]}^{\nu[n+t]^{\dagger i}} = \overline{h}_{\lambda[m],\mu[m-r]}^{\nu[n+t]^{\dagger i}+(m-n)} = h_{\lambda[m],\mu[m-r]}^{\nu[n+t]^{\dagger i}+(m-n)}.$$

Since for $i \geq 2$, we have

$$\begin{aligned}\nu[n+t]^{\dagger i} + (m-n) =&(\nu_{i-1} - i + 1, n + t - |\nu| + 1, \nu_1 + 1, \ldots, \nu_{i-2} + 1, \nu_i, \nu_{i+1}, \ldots) \\ &+ (m-n) \\ =&(\nu_{i-1} - i + 1 + m - n, n + t - |\nu| + 1, \nu_1 + 1, \ldots, \nu_{i-2} + 1, \nu_i, \ldots).\end{aligned}$$

When $n \geq \frac{1}{2}(|\lambda| + |\mu| + |\nu| + \lambda_1 + \mu_1 + \nu_1 - 1) + t + r$, we have

$$n + t - |\nu| + 1 > \nu_{i-1} - i + 1 + m - n, \quad \text{for all} \quad i \geq 2.$$

So $h_{\lambda[m],\mu[m-r]}^{\nu[n+t]^{\dagger i}+(m-n)} = 0$ for all $i \geq 2$, which proves the corollary. $\square$

We go back to Table 1 and compute the lower bound for the stabilization of each column using Corollary 5.2.2. We circle the number corresponding to those lower bounds. We can see that, in this case, the lower bounds are the places where the stabilizations of the Heisenberg coefficients begin, except for $h_{(n-2,1,1),(n-2,1)}^{(n-3,3)}$, $h_{(n-2,1,1),(n-2,1)}^{(n-3,2,1)}$, and $h_{(n-2,1,1),(n-2,1)}^{(n-4,1,1,1,1)}$.

5.3 Generalized Stability

Stembridge [4] vastly generalized Murnaghan's stability notion by introducing the concept of a stable triple.

Definition 5.3.1. A triple (α, β, γ) of partitions of the same size with $g_{\beta,\gamma}^{\alpha} > 0$ is a *K-triple*. It is *K-stable* if, for any other triple of partitions (λ, μ, ν) with $|\lambda| = |\mu| = |\nu|$, the sequence $\left\{g_{\mu+n\beta,\nu+n\gamma}^{\lambda+n\alpha}\right\}_{n\geq 0}$ is eventually constant.

Thus, Murnaghan showed that $((1), (1), (1))$ is K-stable. Stembridge conjectured a characterization for K-stability and he proved its necessity. Sam and Snowden [5] proved the sufficiency.

Proposition 5.3.2. *A K-triple (α, β, γ) is K-stable if and only if $g_{n\beta,n\gamma}^{n\alpha} = 1$ for all $n > 0$.*

Sam and Snowden [5] also proved an analogous result for Littlewood–Richardson coefficients, which can also be deduced from some earlier work (see [5, Remark 4.7]).

Definition 5.3.3. A triple (α, β, γ) of partitions with $|\alpha| = |\beta| + |\gamma|$ and $c^{\alpha}_{\beta,\gamma} > 0$ is an *LR-triple*. It is *LR-stable* if, for any other triple of partitions (λ, μ, ν) with $|\lambda| = |\mu| + |\nu|$, the sequence $\left\{c^{\lambda+n\alpha}_{\mu+n\beta,\nu+n\gamma}\right\}_{n\geq 0}$ is eventually constant.

Proposition 5.3.4 ([5] Theorem 4.6)**.** *The following are equivalent for an LR-triple* (α, β, γ)*,*

(a) (α, β, γ) *is LR-stable.*

(b) $c^{\alpha}_{\beta,\gamma} = 1$.

(c) $c^{n\alpha}_{n\beta,n\gamma} = 1$ *for all* $n > 0$.

Remark 5.3.5. Sam and Snowden [5] did not require $c^{\alpha}_{\beta,\gamma} > 0$, which should be added. For example, when β is not contained in α, we have that $c^{\alpha}_{\beta,\gamma} = 0$ and $\left\{c^{\lambda+n\alpha}_{\mu+n\beta,\nu+n\gamma}\right\}$ is eventually zero.

In Section 5.1, we showed that Heisenberg coefficients stabilize in low degrees, which is analogous to Murnaghan's stability result. It is worthwhile trying to also generalize stability for Heisenberg coefficients.

Definition 5.3.6. A triple (α, β, γ) of partitions with $\max\{|\beta|, |\gamma|\} \leq |\alpha| \leq |\beta| + |\gamma|$ and $h^{\alpha}_{\beta,\gamma} > 0$ is an *H-triple*. It is *H-stable* if, for any other triple of partitions (λ, μ, ν) with $\max\{|\mu|, |\nu|\} \leq |\lambda| \leq |\mu| + |\nu|$, the sequence $\left\{h^{\lambda+n\alpha}_{\mu+n\beta,\nu+n\gamma}\right\}_{n\geq 0}$ is eventually constant.

Our result in Section 5.1 is that $((1), (1), (1))$ is an H-stable triple.

We show that the K-stable triples and LR-stable triples are H-stable. As in Section 5.1, we begin with a stability result for Littlewood–Richardson coefficients.

Lemma 5.3.7. *Given partitions* λ*,* μ*,* ν*,* α*, and a positive integer* $n \geq |\mu|$ *with* $|\lambda + n\alpha| = |\mu| + |\nu|$ *and* $\nu \subset \lambda + n\alpha$*, then*

(1) $\nu - (n - |\mu|)\alpha$ *is a partition.*

(2) When n *is large, we have* $c^{\lambda+n\alpha}_{\mu,\nu} = c^{\lambda+(n+1)\alpha}_{\mu,\nu+\alpha}$.

Proof. For part (1), we first show that $(n-|\mu|)\alpha \subset \nu$. It is enough to prove that $((n-|\mu|)\alpha)_i \leq \nu_i$ for all i. When $\mu = (0)$ (empty partition) or $\alpha_i = 0$, this is trivially true. We consider the nontrivial case. Since

$$|\mu| = |\lambda + n\alpha| - |\nu| \geq (\lambda + n\alpha)_i - \nu_i,$$

we have

$$\nu_i \geq (\lambda + n\alpha)_i - |\mu| \geq n\alpha_i - |\mu| = (n - |\mu|)\alpha_i + |\mu|(\alpha_i - 1) \geq ((n - |\mu|)\alpha)_i.$$

We then show that $\nu - (n - |\mu|)\alpha$ is a partition. To see this, it suffices to show that

$$(\nu - (n - |\mu|)\alpha)_i \geq (\nu - (n - |\mu|)\alpha)_{i+1}, \quad \text{for all } \ i,$$

that is,

$$\nu_i - \nu_{i+1} \geq (n - |\mu|)(\alpha_i - \alpha_{i+1}). \tag{5.16}$$

This is obviously true when $\alpha_i = \alpha_{i+1}$. If $\alpha_i > \alpha_{i+1}$, note that $\nu_i \geq \lambda_i + n\alpha_i - |\mu|$ and $\nu_{i+1} \leq \lambda_{i+1} + n\alpha_{i+1}$, we have

$$\begin{aligned}\nu_i - \nu_{i+1} &\geq \lambda_i + n\alpha_i - |\mu| - (\lambda_{i+1} + n\alpha_{i+1}) \geq n(\alpha_i - \alpha_{i+1}) - |\mu| \\ &\geq (n - |\mu|)(\alpha_i - \alpha_{i+1}).\end{aligned}$$

So (5.16) holds, and we have proved part (1).

Part (2) follows from part (1) and Proposition 5.3.4, as $(\,\alpha, (0), \alpha\,)$ is LR-stable. □

Similarly, we have the following result:

Lemma 5.3.8. *Given partitions λ, μ, ν, α, and an positive integer $n \geq |\mu|$, with $|\lambda| = |\mu| + |\nu + n\alpha|$ and $\nu + n\alpha \subset \lambda$, then*

(1) $\lambda - (n - |\mu|)\alpha$ is a partition.

(2) When n is large, we have $c^{\lambda}_{\mu,\nu+n\alpha} = c^{\lambda+\alpha}_{\mu,\nu+(n+1)\alpha}$.

Remark 5.3.9. Proposition 5.3.4 does not give a lower bound for what large means for n in part (2) of Lemma 5.3.7. However, it is not hard to see that when $n \geq 2|\mu|$, the connected components of the skew shapes $(\lambda + n\alpha)/\nu$ and $(\lambda + (n+1)\alpha)/(\nu + \alpha)$ are the same except for some horizontal shifts. The Littlewood–Richardson rule then implies that $c^{\lambda+n\alpha}_{\mu,\nu} = c^{\lambda+(n+1)\alpha}_{\mu,\nu+\alpha}$. Similarly, for Lemma 5.3.8 (2), $n \geq 2|\mu|$ is enough to guarantee the stability.

Theorem 5.3.10. *A K-stable triple is H-stable.*

Let (α, β, γ) with $\alpha, \beta, \gamma \vdash s > 0$ be K-stable. Suppose λ, μ, and ν are partitions with $\lambda \vdash p$, $\mu \vdash q$, and $\nu \vdash r$ and $\max\{q, r\} \leq p \leq q + r$. Theorem 5.3.10 states that the sequence $\left\{h^{\lambda+n\alpha}_{\mu+n\beta,\nu+n\gamma}\right\}$ is eventually constant. According to Proposition 5.1.7, we have

$$h^{\lambda+n\alpha}_{\mu+n\beta,\nu+n\gamma} = \sum_{K_n} c^{\mu+n\beta}_{\xi,\theta}\ c^{\nu+n\gamma}_{\eta,\rho}\ g^{\delta}_{\theta,\eta}\ c^{\tau}_{\xi,\delta}\ c^{\lambda+n\alpha}_{\tau,\rho}, \tag{5.17}$$

where

$$K_n = \{(\xi, \theta, \eta, \rho, \delta, \tau) \mid \theta, \eta, \delta \vdash (q + r - p) + ns,\ \xi \in p - r,\ \rho \in p - q,\ \tau \vdash q + ns\}.$$

Define $f_n : K_n \longmapsto \mathbb{Z}_{\geq 0}$ as follows:

$$f_n(\xi, \theta, \eta, \rho, \delta, \tau) = c^{\mu+n\beta}_{\xi,\theta}\ c^{\nu+n\gamma}_{\eta,\rho}\ g^{\delta}_{\theta,\eta}\ c^{\tau}_{\xi,\delta}\ c^{\lambda+n\alpha}_{\tau,\rho} \quad \text{(the summands in (5.17))}. \tag{5.18}$$

Some terms in the sum of (5.17) vanish. Let us consider only the nonvanishing terms. Let $K^0_n = K_n \smallsetminus f_n^{-1}(0)$. To prove Theorem 5.3.10, it is enough to prove

$$\sum_{u \in K^0_n} f_n(u) = \sum_{u \in K^0_{n+1}} f_{n+1}(u) \tag{5.19}$$

for n sufficiently large.

We have a natural embedding $\varphi_n : K_n \hookrightarrow K_{n+1}$,

$$\varphi_n(\xi, \theta, \eta, \rho, \delta, \tau) = (\xi, \theta + \beta, \eta + \gamma, \rho, \delta + \alpha, \tau + \alpha).$$

We show that when n is large, φ_n induces a bijection between K_n^0 and K_{n+1}^0 with $f_n = f_{n+1} \circ \varphi_n$. From the definition of K-stability, we know that there exists a positive integer N, such that for all $n \geq N$, we have

$$g_{\zeta+n\beta,\pi+n\gamma}^{\epsilon+n\alpha} = g_{\zeta+(n+1)\beta,\pi+(n+1)\gamma}^{\epsilon+(n+1)\alpha}, \tag{5.20}$$

for all $\epsilon, \zeta, \pi \vdash q + r - p + (2p - q - r)s$.

Lemma 5.3.11. *When $n \geq N + 3p - q - 2r$, $\varphi_n|_{K_n^0}$: $K_n^0 \longrightarrow K_{n+1}^0$ is a well-defined bijection. Moreover, $f_n|_{K_n^0} = f_{n+1} \circ \varphi_n|_{K_n^0}$.*

Proof. Take any $u = (\xi, \theta, \eta, \rho, \delta, \tau) \in K_n^0$. Since $f(u) \neq 0$, we have $c_{\xi,\theta}^{\mu+n\beta} \neq 0$. So $\theta \subset \mu + n\beta$. According to Lemma 5.3.7 and Remark 5.3.9, we know that $\theta - (n - p + r)\beta$ is a partition of $(q + r - p) + (p - r)s$ and $c_{\xi,\theta}^{\mu+n\beta} = c_{\xi,\theta+\beta}^{\mu+(n+1)\beta}$. Similarly, we can show that $\eta - (n - p + q)\gamma$, $\tau - (n - p + q)\alpha$, and $\delta - (n - 2p + q + r)\alpha$ are partitions, and

$$c_{\eta,\rho}^{\nu+n\gamma} = c_{\eta,\rho+\gamma}^{\nu+(n+1)\gamma}, \quad c_{\tau,\rho}^{\lambda+n\alpha} = c_{\tau+\alpha,\rho}^{\lambda+(n+1)\alpha}, \quad c_{\xi,\delta}^{\tau} = c_{\xi,\delta+\alpha}^{\tau+\alpha}.$$

Since δ, θ, and η can be written as

$$\delta = \delta' + (n - 2p + q + r)\alpha, \ \theta = \theta' + (n - 2p + q + r)\beta, \ \eta = \eta' + (n - 2p + q + r)\gamma$$

for some partitions δ', θ', and η' of $(q + r - p) + (2p - q - r)s$. From (5.20), we have

$$g_{\eta,\rho}^{\delta} = g_{\eta+\beta,\rho+\gamma}^{\delta+\alpha}.$$

Hence, $f_{n+1}(\varphi_n(u)) = f_n(u)(\neq 0)$. So $\varphi_n|_{K_n^0}$ is a well-defined embedding from K_n^0 into K_{n+1}^0. To construct the inverse map, we consider $\psi_{n+1}(\xi, \theta, \eta, \rho, \delta, \tau) = (\xi, \theta - \beta, \eta - \gamma, \rho, \delta - \alpha, \tau - \alpha)$.

Nearly the same arguments show that the inverse map induces an injection from K_{n+1}^0 to K_n^0. So $\varphi_n|_{K_n^0}$ is a bijection. $\square$

Proof of Theorem 5.3.10. Applying Lemma 5.3.11, we prove (5.19), hence Theorem 5.3.10. $\square$

Theorem 5.3.10 shows that some Heisenberg coefficients in low degree components stabilize. Our next result gives a stability result for the relatively high degree components.

Theorem 5.3.12. *LR-stable triples are H-stable.*

The idea of the proof is the same (without using stability of Kronecker coefficients) as the proof for Theorem 5.3.10. Given an LR-stable triple (α, β, γ) with $\alpha \vdash a+b$, $\beta \vdash a$, and $\gamma \vdash b$, and partitions $\lambda \vdash p$, $\mu \vdash q$, and $\nu \vdash r$ with $\max\{q, r\} \leq p \leq q+r$. We define $f'_n : LR_n \longmapsto \mathbb{Z}_{\geq 0}$ as follows:

$$f'_n(\xi, \theta, \eta, \rho, \delta, \tau) = c_{\xi,\theta}^{\mu+n\beta}\ c_{\eta,\rho}^{\nu+n\gamma}\ g_{\theta,\eta}^{\delta}\ c_{\xi,\delta}^{\tau}\ c_{\tau,\rho}^{\lambda+n\alpha}. \tag{5.21}$$

where

$$LR_n = \{(\xi, \theta, \eta, \rho, \delta, \tau) \mid \theta, \eta, \delta \vdash (q+r-p),\ \xi \vdash p-r+na,\ \rho \vdash p-q+nb,\ \tau \vdash q+na\}.$$

Applying Proposition 5.1.7, Theorem 5.3.12 states that

$$\sum_{u \in LR_n^0} f'_n(u) = \sum_{u \in LR_{n+1}^0} f'_{n+1}(u) \tag{5.22}$$

for all large n, where $LR_n^0 = LR_n \setminus {f'}_n^{-1}(0)$.

Proof of Theorem 5.3.12. Consider the map $\phi_n : LR_n \hookrightarrow LR_{n+1}$,

$$\phi_n(\xi, \theta, \eta, \rho, \delta, \tau) = (\xi+\beta, \theta, \eta, \rho+\gamma, \delta, \tau+\beta).$$

Using Lemma 5.3.7, Lemma 5.3.8, and the same idea in the proof of Lemma 5.3.11, it follows that $f'_n = f'_{n+1} \circ \phi_n$ on LR_n^0 when n is large, and it is not hard to see that ϕ_n is a bijection between

LR_n^0 and LR_{n+1}^0. So (5.22) is true, and hence we prove the theorem. $\square$

Proposition 5.3.2 and Proposition 5.3.4 (3) have a similar form. A natural question is whether the necessary and sufficient condition for being an H-stable triple has the same form. The answer is yes, and Pelletier [28, Theorem 3.6] proved the following direction.

Proposition 5.3.13. *An H-triple (α, β, γ) is H-stable if $h^{n\alpha}_{n\beta,n\gamma} = 1$ for all $n > 0$.*

Remark 5.3.14. By Propositions 5.3.2 and 5.3.4, Proposition 5.3.13 also shows that K-stable triples and LR-stable triples are H-stable.

We prove the reverse direction and complete the characterization of H-stability using the monotonicity of Heisenberg coefficients. This is deduced from the monotonicity of Littlewood–Richardson coefficients and Kronecker coefficients. We start with the monotonicity of Kronecker coefficients. Stembridge [17] proved the following for Kronecker coefficients.

Proposition 5.3.15. *Let (α, β, γ) be a K-triple. Then*

(1) the sequence $\left\{g^{\lambda+n\alpha}_{\mu+n\beta,\nu+n\gamma}\right\}$ is weakly increasing for any partitions λ, μ, and ν with the same size.

(2) if $g^{\alpha}_{\beta,\gamma} \geq 2$, then $g^{n\alpha}_{n\beta,n\gamma} \geq n+1$.

Using the hive model of Littlewood-Richardson coefficients (see [8, 9]), we prove an analogous result for Littlewood-Richardson coefficients.

Proposition 5.3.16. *Let (α, β, γ) be an LR-triple. Then*

(1) the sequence $\left\{c^{\lambda+n\alpha}_{\mu+n\beta,\nu+n\gamma}\right\}$ is weakly increasing for any partitions λ, μ, and ν with $|\lambda| = |\mu| + |\nu|$.

(2) if $c^{\alpha}_{\beta,\gamma} \geq 2$, then $c^{n\alpha}_{n\beta,n\gamma} \geq n+1$.

Proof. We follow the notation used for hives in [9, Section 4]. Let k be a positive integer larger than the lengths of λ, μ, ν, α, β, and γ. We define (coordinatewise) addition and scalar multiplication on hives (as what we do for vectors and matrices).

For (1), it suffices to show $c^{\lambda}_{\mu,\nu} \leq c^{\lambda+\alpha}_{\mu+\beta,\nu+\gamma}$. Since $c^{\alpha}_{\beta,\gamma} \geq 1$, there exists a hive $\Delta \in H_k(\alpha, \beta, \gamma)$. Then the map: $\iota : H_k(\lambda, \mu, \nu) \hookrightarrow H_k(\lambda + \alpha, \mu + \beta, \nu + \gamma)$

$$\iota(\Theta) = \Theta + \Delta$$

where $\Theta \in H_k(\lambda, \mu, \nu)$, gives a well-defined injection. So (1) is proved.

For (2), we have two different hives Δ_1 and Δ_2 in $H_k(\alpha, \beta, \gamma)$ as $c^{\alpha}_{\beta,\gamma} \geq 2$. Then $i\Delta_1 + (n-i)\Delta_2$ $(0 \leq i \leq n)$ give $n+1$ different hives in $H_k(n\alpha, n\beta, n\gamma)$, so $c^{n\alpha}_{n\beta,n\gamma} \geq n+1$. □

Propositions 5.1.7, 5.3.15, and 5.3.16 together imply the following:

Proposition 5.3.17. *Let (α, β, γ) be a H-triple. Then*

(1) the sequence $\left\{h^{\lambda+n\alpha}_{\mu+n\beta,\nu+n\gamma}\right\}$ is weakly increasing for any partitions λ, μ, and ν with $\max\{|\mu|, |\nu|\} \leq |\lambda| \leq |\mu| + |\nu|$.

(2) if $h^{\alpha}_{\beta,\gamma} \geq 2$, then $h^{n\alpha}_{n\beta,n\gamma} \geq n+1$.

Proof. For (1), it is enough to show $h^{\lambda}_{\mu,\nu} \leq h^{\lambda+\alpha}_{\mu+\beta,\nu+\gamma}$. Since $h^{\alpha}_{\beta,\gamma} > 0$, by Formula (5.4), there exists a sextuple $(\xi, \theta, \eta, \rho, \delta, \tau)$ of partitions with appropriate sizes such that $c^{\beta}_{\xi,\theta}\, c^{\gamma}_{\eta,\rho}\, g^{\delta}_{\theta,\eta}\, c^{\tau}_{\xi,\delta}\, c^{\alpha}_{\tau,\rho} > 0$. The triples appearing in the coefficients on the left hand side are LR-triples or K-triples. As in the proof of Theorems 5.3.10 and 5.3.12, applying Proposition 5.1.7, we write

$$h^{\lambda}_{\mu,\nu} = \sum_{u \in \Lambda} f_u \quad \text{and} \quad h^{\lambda+\alpha}_{\mu+\beta,\nu+\gamma} = \sum_{u' \in \Lambda'} f'_{u'},$$

where Λ and Λ' are sets of sextuples of partitions with appropriate sizes, f_u and $f'_{u'}$ are the summands given by the sextuples u and u'. We view the sextuples as vectors whose coordinates are partitions, so we may define addition and scalar multiplication for them. The map $u \longrightarrow u + (\xi, \theta, \eta, \rho, \delta, \tau) =: u'$ embeds Λ into Λ'. From Proposition 5.3.15 and Proposition 5.3.16, we know that $f_u \leq f'_{u'}$, so (1) is proved.

For (2), if $h^{\alpha}_{\beta,\gamma} \geq 2$, then there are two possibilities.

Case 1. There exists a sextuple $(\xi, \theta, \eta, \rho, \delta, \tau)$ of partitions with appropriate sizes such that $c^{\beta}_{\xi,\theta}\ c^{\gamma}_{\eta,\rho}\ g^{\delta}_{\theta,\eta}\ c^{\tau}_{\xi,\delta}\ c^{\alpha}_{\tau,\rho} \geq 2$. So all the five coefficients on the left hand side are positive and at least one of them is at least 2. From Formula (5.4), Propositions 5.3.15 and 5.3.16, we have

$$h^{n\alpha}_{n\beta,n\gamma} \geq c^{n\beta}_{n\xi,n\theta}\ c^{n\gamma}_{n\eta,n\rho}\ g^{n\delta}_{n\theta,n\eta}\ c^{n\tau}_{n\xi,n\delta}\ c^{n\alpha}_{n\tau,n\rho} \geq n+1.$$

Case 2. Two distinct sextuples $u = (\xi, \theta, \eta, \rho, \delta, \tau)$ and $u' = (\xi', \theta', \eta', \rho', \delta', \tau')$ give positive summands for $h^{\alpha}_{\beta,\gamma}$. Then $iu + (n-i)u'$ $(1 \leq i \leq n)$ gives $n+1$ different sextuples, and due to Propositions 5.3.15 and 5.3.16, they all give positive summands for $h^{n\alpha}_{n\beta,n\gamma}$, so $h^{n\alpha}_{n\beta,n\gamma} \geq n+1$. Hence, we prove the proposition. □

Combining Proposition 5.3.13 and 5.3.17, we achieve the main theorem of this thesis.

Theorem 5.3.18. *An H-triple* (α, β, γ) *is H-stable if and only if* $h^{n\alpha}_{n\beta,n\gamma} = 1$ *for all* $n > 0$.

5.4 Additive Matrices

Manivel [29] and Vallejo [6] used additive matrices to produce examples of K-stable triples. We first recall some definitions and results concerning additive matrices, then we give an analogous result for H-stable triples.

Definition 5.4.1. A $p \times q$ matrix $A = (a_{i,j})$ with nonnegative integer entries is called *K-additive* if there exist real numbers $x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q$, such that

$$a_{i,j} > a_{k,l} \Longrightarrow x_i + y_j > x_k + y_l$$

for all $i, k \in [p]$ and $j, l \in [q]$.

Recall that in Section 4.2, we define $\mathcal{M}(\beta, \gamma)_{\alpha}$ to be the set of nonnegative integer matrices with row-sum (resp. column-sum) vector β (resp. γ) and π-sequence α, where β and γ are (weak) compositions of some n and α is a partition of n.

Proposition 5.4.2 ([6] Theorem 1.1)**.** *Let α, β, and γ be partitions of the same size. If there is a matrix $A \in \mathcal{M}(\beta, \gamma)_\alpha$ which is K-additive, then (α, β, γ) is K-stable.*

Moreover, Manivel [29, Section 5.3] showed that each K-additive matrix defines a regular face of the corresponding Kronecker polyhedron, of minimal dimension.

One of the most important steps in Vallejo's proof of Proposition 5.4.2 is is the Proposition 4.2.2. We introduce H-additive matrices and use Proposition 4.3.4 to show that each H-additive matrix gives an H-stable triple.

Definition 5.4.3. A $(p+1) \times (q+1)$ matrix $A = (a_{i,j})$ with nonnegative integer entries and $a_{1,1} = 0$ is called *H-additive* if there exist real numbers $x_1 = 0, x_2, \dots, x_{p+1}, y_1 = 0, y_2, \dots, y_{q+1}$, such that

$$a_{i,j} > a_{k,l} \Longrightarrow x_i + y_j > x_k + y_l$$

for all $(i,j),(k,l) \in [p+1] \times [q+1] \smallsetminus \{(1,1)\}$.

With this definition, the matrix in Example 4.3.3 is H-additive (consider setting $x_0 = y_0 = 0, x_1 = 1, x_2 = -1, y_1 = 1, y_2 = 3, y_3 = -2$). Recall that in Section 4.3, for (weak) compositions α and β, and a partition γ, we define $\mathcal{H}(\alpha, \beta)_\gamma$ to be the set of nonnegative integer matrices with zero at the top left corner, row-sum vector (ignoring the first row) α, column-sum vector (ignoring the first column) β, and $\pi-$sequence γ.

Theorem 5.4.4. *Let α, β, and γ be partitions with* $\max\{|\beta|, |\gamma|\} \leq |\alpha| \leq |\beta| + |\gamma|$*. If there is a matrix $A \in \mathcal{H}(\beta, \gamma)_\alpha$ which is H-additive, then (α, β, γ) is H-stable.*

Remark 5.4.5. Theorem 5.4.4 is equivalent to Proposition 5.4.2 if $|\alpha| = |\beta| = |\gamma|$. The only LR-stable triples it can produce are in the form $(\beta \cup \gamma, \beta, \gamma)$. It is not hard to see $c_{\beta,\gamma}^{\beta\cup\gamma} = 1$.

The proof for Theorem 5.4.4 is similar to Onn and Vallejo's proof [6, 30] for Proposition 5.4.2 with some changes, as we are looking at slightly different matrices. Consequently, we only give a sketch.

We first recall some basic notions, many introduced in [6, 30], which will be used in our proof. We move away from integers for a while and work with real numbers. For a vector $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$, we denote by $\pi(\mathbf{a})$ the vector formed by the entries of $\mathbf{a}$ arranged in weakly decreasing order. We say that $\mathbf{a}$ is dominated by $\mathbf{b}$ (both are vectors in $\mathbb{R}^m$), written as $\mathbf{a} \preccurlyeq \mathbf{b}$, if

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i \quad \text{and} \quad \sum_{i=1}^{k} \pi(\mathbf{a})_i \leq \sum_{i=1}^{k} \pi(\mathbf{b})_i, \ \text{ for all } \ k \in [m].$$

If $\mathbf{a} \preccurlyeq \mathbf{b}$ and $\pi(\mathbf{a}) \neq \pi(\mathbf{b})$, then we write $\mathbf{a} \prec \mathbf{b}$. In particular, when $\mathbf{a}$ and $\mathbf{b}$ are partitions of some integer n, then $\preccurlyeq$ coincides with the dominance ordering $\preceq_d$ for partitions.

For a permutation $\rho \in S_m$, we set $\mathbf{a}_\rho := (a_{\rho(1)}, a_{\rho(2)}, \ldots, a_{\rho(m)})$. The *permutohedron* determined by $\mathbf{a}$ is the convex hull of the vectors of the form a_ρ:

$$P(\mathbf{a}) = \mathrm{conv}\{\mathbf{a}_\rho \mid \rho \in S_m\}.$$

Proposition 5.4.6 (Rado [31])**.** $\quad P(\boldsymbol{a}) = \{\boldsymbol{x} \in \mathbb{R}^m \mid \boldsymbol{x} \preccurlyeq \boldsymbol{a}\}.$

Suppose α and β are two finite sequences of real numbers whose lengths are p and q respectively. Recall that in Section 4.3, for (weak) compositions α and β, we define $\mathcal{F}(\alpha, \beta)$ to be the set of matrices with real entries, zero at the top left corner, row-sum vector (ignoring the first row) α and column-sum vector (ignoring the first column) β. We consider the linear map $\Phi : \mathcal{F}(\alpha, \beta) \longrightarrow \mathbb{R}^{pq+p+q}$,

$$\Phi(A) = (a_{1,2}, a_{1,3}, \ldots, a_{1,q+1}, a_{2,1}, a_{2,2}, \ldots, a_{2,q+1}, \ldots, a_{p+1,1}, a_{p+1,2}, a_{p+1,q+1}),$$

where $A = (a_{i,j}) \in \mathcal{F}(\alpha, \beta)$. A matrix $A \in \mathcal{F}(\alpha, \beta)$ is real-minimal if there is no other matrix $B \in \mathcal{F}(\alpha, \beta)$ such that $\pi(B) \prec \pi(A)$. Real-minimality has the following equivalent interpretations.

Proposition 5.4.7. *Let $A \in \mathcal{F}(\alpha, \beta)$. The following are equivalent:*

(1) A is real-minimal.

(2) $P(\Phi(A)) \cap \Phi(\mathcal{F}(\alpha, \beta)) = \{\Phi(A)\}$.

(3) there exists a hyperplane $H \subset \mathbb{R}^{pq+p+q}$ *containing* $\Phi(\mathcal{F}(\alpha, \beta))$ *such that*

$$P(\Phi(A)) \cap H = \{\Phi(A)\}.$$

See [30, Section 5] for the proof of this proposition. Although the matrices we are working with are different, the proof still applies.

Proposition 5.4.8. *Let* $A \in \mathcal{F}(\alpha, \beta)$, $\boldsymbol{a} = \Phi(A)$. *Then* A *is real-minimal if and only if there is some vector* $\boldsymbol{n} \in \mathbb{R}^{pq+p+q}$ *such that*

(1) $\boldsymbol{n}$ *is orthogonal to* $\Phi(\mathcal{F}(\alpha, \beta))$.

(2) For each transposition $\sigma = (s\ s{+}1) \in S_{pq+p+q}$ *such that* $a_s \neq a_{s+1}$, *one has* $\langle\, \boldsymbol{n}\,, \sigma\boldsymbol{a} - \boldsymbol{a}\,\rangle > 0$.

Remark 5.4.9. The second condition is equivalent to $\langle\, \mathbf{n}\,, \mathbf{x} - \mathbf{a}\,\rangle > 0$ for all $\mathbf{x} \in P(\mathbf{a}), \mathbf{x} \neq \mathbf{a}$.

Again, one can use the proof in [30, Proposition 6.1] to prove this. The definition of H-additivity can be extended naturally to matrices with real entries, and we next show that real-minimality is equivalent to H-additivity for real matrices.

Theorem 5.4.10. *Let* $A \in \mathcal{F}(\alpha, \beta)$. *Then* A *is real-minimal if and only if* A *is H-additive.*

Following Onn and Vallejo [30], we first construct a matrix. Let $M = (m_{i,j})$ be a $(p+q) \times (pq + p + q)$ matrix with

$$m_{i,j} = \begin{cases} 1, & \text{if } 1 \leq i \leq p \text{ and } i(q+1) \leq j \leq q + i(q+1); \\ 1, & \text{if } p+1 \leq i \leq p+q \text{ and } j = s - p + k(q+1), \text{ for some } 0 \leq k \leq p; \\ 0, & \text{otherwise.} \end{cases}$$

For example, if $p = 2$ and $q = 3$, then

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{p+q}$ be the rows of M. From the definition of M, it is obvious that these $p+q$ vectors are linearly independent. The set $\Phi(\mathcal{F}(\alpha, \beta))$ is exactly the set of (transpose of) solutions of the following matrix equation:

$$M\mathbf{x} = \mathbf{y}$$

where $\mathbf{y} = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)'$. For a vector $\mathbf{z} = (x_2, \ldots x_{p+1}, y_2, \ldots, y_{q+1})$, we have

$$\mathbf{z}M = \Phi((x_i + y_j)_{(i,j)\in[p+1]\times[q+1]}),$$

where we set $x_1 = y_1 = 0$.

Proof of Theorem 5.4.10. Suppose $A \in \mathcal{F}(\alpha, \beta)$ is real-minimal, then there exists a vector $\mathbf{n} \in \mathbb{R}^{pq+p+q}$ satisfying the two conditions in Proposition 5.4.8. Since $\mathbf{n}$ is orthogonal to $\Phi(\mathcal{F}(\alpha, \beta))$, $\mathbf{n}$ must be in the row space of M. So there are unique numbers $x_2, \ldots, x_{p+1}, y_2, \ldots, y_{q+1}$ such that

$$-\mathbf{n} = x_2\mathbf{r}_1 + \cdots + x_{p+1}\mathbf{r}_p + y_2\mathbf{r}_{p+1} + \cdots + y_{q+1}\mathbf{r}_{p+q}.$$

Let $\mathbf{z} = (x_2, \ldots x_{p+1}, y_1, \ldots, y_{q+1})$, then $-\mathbf{n} = \mathbf{z}M = -\Phi((x_i + y_j))$. Following the arguments in [30, Theorem 6.2] proves this theorem. $\square$

From Proposition 5.4.6, 5.4.7, and Theorem 5.4.10, we have

Corollary 5.4.11. *Let $A \in \mathcal{F}(\alpha, \beta)$. Then A is H-additive if and only if*

$$P(\pi(A)) \cap \Phi(\mathcal{F}(\alpha, \beta)) = \{\Phi(A)\}.$$

Vallejo showed that the Kronecker coefficient indexed by the K-triple produced by a K-additive matrix is 1.

Lemma 5.4.12 ([32] Corollary 4.2)**.** *Let $A \in \mathcal{M}(\beta, \gamma)_\alpha$ be K-additive where α, β, and γ are partitions with the same size, then $g^{\alpha}_{\beta,\gamma} = 1$.*

The same is true for Heisenberg coefficients and H-additive matrices.

Lemma 5.4.13. *Let $A \in \mathcal{H}(\beta, \gamma)_\alpha$ be H-additive, where α, β, and γ are partitions with $\max\{|\beta|, |\gamma|\} \leq |\lambda| \leq |\beta| + |\gamma|$, then $h^{\alpha}_{\beta,\gamma} = 1$.*

Proof. We first show that $h^{\alpha}_{\beta,\gamma} \geq 1$. Suppose $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_q)$. Since $A = (a_{ij})$ is additive, then there exists real numbers x_i's and y_j's ($i \in [p+1]$ and $j \in [q+1]$) satisfy the condition in Definition 5.4.3. After permuting rows and columns if necessary, we may assume $x_2 \geq x_3 \geq \cdots \geq x_{p+1}$ and $y_2 \geq y_3 \geq \cdots \geq y_{q+1}$. By H-additivity, this assumption implies that $a_{i,j} \geq a_{i,j+1}$ for all $1 \leq i \leq p+1, 2 \leq j \leq q$, and $a_{i,j} \geq a_{i+1,j}$ for all $2 \leq i \leq p, 1 \leq j \leq q+1$. Set $\beta^{(1)} = (a_{2,1}, a_{3,1}, \ldots, a_{p+1,1})$, $\gamma^{(1)} = (a_{1,2}, a_{1,3}, \ldots, a_{1,q+1})$, $\beta^{(2)} = \beta - \beta^{(1)}$, and $\gamma^{(2)} = \gamma - \gamma^{(1)}$, then these four are all partitions and, by the Littlewood–Richardson rule, we have

$$c^{\beta}_{\beta^{(1)},\beta^{(2)}} = c^{\gamma}_{\gamma^{(1)},\gamma^{(2)}} = 1. \tag{5.23}$$

Let $A^{(1)}$ be the the submatrix of A obtained by removing the first row, and $A^{(2)}$ be the submatrix of $A^{(1)}$ obtained by removing the first column. We set $\alpha^{(1)} = \pi(A^{(1)})$ and $\alpha^{(2)} = \pi(A^{(2)})$. From Remark 5.4.5, we have

$$c^{\alpha}_{\gamma^{(1)},\alpha^{(1)}} = c^{\alpha^{(1)}}_{\beta^{(1)},\alpha^{(2)}} = 1, \tag{5.24}$$

as $\alpha = \gamma^{(1)} \cup \alpha^{(1)}$ and $\alpha^{(1)} = \beta^{(1)} \cup \alpha^{(2)}$. Note that $A^{(2)} \in \mathcal{M}(\beta^{(2)}, \gamma^{(2)})_{\alpha^{(2)}}$ is K-additive, so, due to Lemma 5.4.12, we have

$$g^{\alpha^{(2)}}_{\beta^{(2)},\gamma^{(2)}} = 1. \tag{5.25}$$

Using Proposition 5.1.7 and Equations (5.23), (5.24), and (5.25), we have $h^{\alpha}_{\beta,\gamma} \geq 1$.

On the other hand, using Equation (2.38) and properties of Kostka numbers, we have

$$\begin{aligned} h^{\alpha}_{\beta,\gamma} &= \langle\, s_\beta \# s_\gamma \,,\, s_\alpha \,\rangle \leq \langle\, h_\beta \# h_\gamma \,,\, s_\alpha \,\rangle = \langle \sum_{A \in \mathcal{H}(\beta,\gamma)} h_{\pi(A)} \,,\, s_\alpha \,\rangle \\ &= \langle \sum_{\delta} |\mathcal{H}(\beta,\gamma)_\delta| \, h_\delta \,,\, s_\alpha \,\rangle = \langle \sum_{\delta} \sum_{\epsilon \succcurlyeq \delta} |\mathcal{H}(\beta,\gamma)_\delta| K_{\epsilon,\delta} \, s_\epsilon \,,\, s_\alpha \,\rangle \\ &= \sum_{\delta \preccurlyeq \alpha} |\mathcal{H}(\beta,\gamma)_\delta| K_{\alpha,\delta} \end{aligned} \tag{5.26}$$

Since $A \in \mathcal{H}(\beta,\gamma)_\alpha$ is H-additive, according to Corollary 5.4.11, we have

$$P(\alpha) \cap \Phi(\mathcal{H}(\beta,\gamma)) = \{\Phi(A)\}.$$

Hence, it follows that $|\mathcal{H}(\beta,\gamma)_\delta| = 0$ for all $\delta \prec \alpha$, and $|\mathcal{H}(\beta,\gamma)_\alpha| = 1$. Equation (5.26) shows that $h^{\alpha}_{\beta,\gamma} \leq 1$, and proves the lemma. $\square$

Proof of Theorem 5.4.4. If a matrix A is H-additive, then nA is H-additive. Consequently, by Lemma 5.4.13, $h^{n\alpha}_{n\beta,n\gamma} = 1$ for all $n > 0$. By Proposition 5.3.13, (α,β,γ) is H-stable. $\square$

Remark 5.4.14. One may prove Theorem 5.4.4 without using Proposition 5.3.13. See [6, Section 5], and the proof there applies here. Also, as in [30, Theorem 7.1], given a rational matrix A with zero at the top left corner, it can be decided in polynomial time whether A is H-additive.

6. CONCLUSION

In general, we expect that Heisenberg coefficients share some properties which Littlewood–Richardson coefficients and Kronecker coefficients have. The rectangular symmetry showed in Chapter 4 and the classic/generalized stability results in Chapter 5 are examples of this. There are some other directions we can try.

There are also combinatorial formulas for the Kronecker product of two Schur functions when one of the index partitions is a hook shape partition or is a two row partition. Can we find combinatorial formulas for the Heisenberg product $s_\mu \# s_\nu$ when μ is a hook shape partition or a two-rows partition? Formula (5.4) suggests that this is possible.

People have studied the Littlewood–Richardson cone and the Kronecker cone, and they showed that the Littlewood–Richardson coefficient and the Kronecker coefficient have polynomiality [33, 34] and quasi-polynomiality [29] respectively. We can try to verify that whether the Heisenberg coefficient have quasi-polynomiality.

The triples of partitions of nonvanishing Heisenberg coefficients form a semigroup, which suggests us considering the cone, called Heisenberg cone, generated those triples. People already studied and had some results about the Littlewood–Richardson cone and the Kronecker cone, and these two cones sit naturally inside the Heisenberg cone. It would be interesting to explore relations among the three cones.

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APPENDIX A

PROOF OF THE ASSOCIATIVITY OF THE HEISENBERG PRODUCT

In this appendix, we prove the associativity of the Heisenberg product using representation theory.

We study three types of double cosets. We start with the double cosets of Young subgroups. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ be two (weak) compositions of n. For each $\sigma \in S_n$, we get an $l \times m$ matrix $A = (a_{ij})$ of nonnegative integers, where

$$a_{ij} = |\{\alpha_1 + \cdots + \alpha_{i-1} + 1, \ldots, \alpha_1 + \cdots + \alpha_i\} \cap \sigma\{\beta_1 + \cdots + \beta_{i-1} + 1, \ldots, \beta_1 + \cdots + \beta_i\}|. \quad \text{(A.1)}$$

It is not hard to check that $A \in \mathcal{M}(\alpha, \beta)$.

Proposition A.0.1. *(Young subgroup double coset theorem) Let α and β be two compositions of n. The map $\sigma \to (a_{ij})$ given by* (A.1) *induces a bijection from the set of double cosets $S_\alpha \sigma S_\beta$ to $\mathcal{M}(\alpha, \beta)$.*

For a proof, see [35]. In particular, when $l = m = 2$, the Young subgroups S_α and S_β can be written as $S_\alpha = S_i \times S_{n-i}$ and $S_\beta = S_j \times S_{n-j}$, and the element in $\mathcal{M}(\alpha, \beta)$ has the form

$$\begin{pmatrix} a & i-a \\ j-a & n-i-j+a \end{pmatrix}, \quad \text{(A.2)}$$

where $\max\{0, i+j-n\} \le a \le \min\{i, j\}$. Let $\sigma_a \in S_n$ be one of the elements (for this element see [36, Chapter 5]) corresponding to the matrix in (A.2), such that

$$S_\beta \cap \sigma_a^{-1} S_\alpha \sigma_a = S_a \times S_{j-a} \times S_{i-a} \times S_{n-i-j+a} \quad \text{(A.3)}$$

$$\sigma_a S_\beta \sigma_a^{-1} \cap S_\alpha = S_a \times S_{i-a} \times S_{j-a} \times S_{n-i-j+a} \quad \text{(A.4)}$$

Let $T_\alpha := S_a \times S_{i-a} \times S_{j-a} \times S_{n-i-j+a}$, then T_α is a subgroup of S_α, and it can also be viewed as a subgroup of S_β by switching the middle two factors S_{i-a} and S_{j-a}. Combining (A.3) and (A.4) and applying Mackey's Decomposition Theorem, we get

$$\operatorname{Res}^{S_n}_{S_\alpha} \operatorname{Ind}^{S_n}_{S_\beta} = \bigoplus_a \operatorname{Ind}^{S_\alpha}_{T_a} \operatorname{Res}^{S_\beta}_{T_a} . \tag{A.5}$$

To visualize this, we consider the diagram

$$\begin{tikzcd}
 & S_n \arrow[dr, bend left, "\text{Res}"] & \\
S_j \times S_{n-j} \arrow[ur, hook] \arrow[ur, bend left, "\textbf{Ind}"] \arrow[dr, bend left, "\textbf{Res}"] & & S_i \times S_{n-i} \arrow[ul, hook'] \\
\bigoplus_a & S_a \times S_{i-a} \times S_{j-a} \times S_{n-i-j+a} \arrow[ul, hook] \arrow[ur, hook'] \arrow[ur, bend left, "\text{Ind}"] &
\end{tikzcd} \tag{A.6}$$

The left hand side of (A.5) follows the blue arrows and the right hand side follows the red arrows.

Another type of double coset is the following. Let $\alpha \vDash n$, then $S_\alpha \times S_n$ is a subgroup of $S_n \times S_n$. Note that S_n can also be viewed as a subgroup of $S_n \times S_n$ along the diagonal map, and it is not hard to show that $S_n \cdot (S_\alpha \times S_n) = S_n \times S_n$. So $S_n \setminus S_n \times S_n \ / \ S_\alpha \times S_n$ has only one double coset. Choosing the identity element to be the representative of the double coset and applying Mackey's Decomposition Theorem, we get

$$\operatorname{Res}^{S_n \times S_n}_{S_n} \operatorname{Ind}^{S_n \times S_n}_{S_\alpha \times S_n} = \operatorname{Ind}^{S_n}_{S_\alpha} \operatorname{Res}^{S_\alpha \times S_n}_{S_\alpha} . \tag{A.7}$$

where the two restrictions in the above equations are the pull backs of the diagonal maps. Similarly, we can show that

$$\operatorname{Res}^{S_{a+b} \times S_c \times S_{a+b+c}}_{S_{a+b} \times S_c} \operatorname{Ind}^{S_{a+b} \times S_c \times S_{a+b+c}}_{S_a \times S_b \times S_c \times S_{a+b+c}} = \operatorname{Ind}^{S_{a+b} \times S_c}_{S_a \times S_b \times S_c} \operatorname{Res}^{S_a \times S_b \times S_c \times S_{a+b+c}}_{S_a \times S_b \times S_c} . \tag{A.8}$$

The last type of double coset has the form $(G \times H') \setminus (G \times H) \,/\, (G' \times H)$ where G' and H' are subgroups of finite groups G and H respectively. It is easy to show that there is only one double coset. We choose the identiy element to be the representative of the double coset and apply the Mackey's Decomposition Theorem,

$$\mathrm{Res}^{G\times H}_{G\times H'}\,\mathrm{Ind}^{G\times H}_{G'\times H} \;=\mathrm{Ind}^{G\times H'}_{G'\times H'}\,\mathrm{Res}^{G'\times H}_{G'\times H'}\;. \tag{A.9}$$

Let U, V, and W be representations of S_l, S_m, and S_n respectively. To show that the Heisenberg product is associative, we need to show that

$$(U\#V)\#W = U\#(V\#W) \tag{A.10}$$

We first compute the left hand side of (A.10). By the definition of the Heisenberg product, we have

$$(U\#V)\#W = \bigoplus_{i,j} \mathrm{Ind}^{S_{l+m-i-j}}_{S_{(l+m-i-j,j,n-j)}}\,\mathrm{Res}^{S_{(l+m-i,n)}}_{S_{(l+m-i-j,j,n-j)}}\,\mathrm{Ind}^{S_{(l+m-i,n)}}_{S_{(l-i,i,m-i,n)}}\,\mathrm{Res}^{S_{(l,m,n)}}_{S_{(l-i,i,m-i,n)}}\,U\otimes V\otimes W. \tag{A.11}$$

To visualize this, see the (solid) blue arrows in Diagram (A.20). Since all the groups in Diagram (A.20) are Young subgroups, we just write their index compositions for convenience. Note that the arrows for (A.11) in Diagram (A.20) has the pattern " $\searrow\nearrow\searrow\nearrow$ ". Our goal is straightening this pattern by applying Mackey's Decomposition Theorem to get a pattern of " $\searrow\nearrow$ ". To achieve our goal, we need the following straightening steps. Applying (A.5), we have

$$\mathrm{Res}^{S_{(l+m-i,n)}}_{S_{(l+m-i-j,j,n)}}\mathrm{Ind}^{S_{(l+m-i,n)}}_{S_{(l,m-i,n)}} = \bigoplus_k \mathrm{Ind}^{S_{(l+m-i-j,j,n)}}_{S_{(k,l-k,l+m-i-j-k,k+j-l,n)}}\mathrm{Res}^{S_{(l,m-i,n)}}_{S_{(k,l-k,l+m-i-j-k,k+j-l,n)}}. \tag{A.12}$$

See the "block" in (A.20) labeled by (A.12). The left hand side follows the dashed blue arrows and the right hand side follows the dashed red arrows, and we illustrate the following steps in the diagram in the same way. Applying (A.5) again, we have

$$\mathrm{Res}^{S_{(l,m-i,n)}}_{S_{(k,l-k,m-i,n)}}\mathrm{Ind}^{S_{(l,m-i,n)}}_{S_{(l-i,i,m-i,n)}} = \bigoplus_t \mathrm{Ind}^{S_{(k,l-k,m-i,n)}}_{S_{(t,l-i-t,k-t,i+t-k,m-i,n)}}\mathrm{Res}^{S_{(l-i,i,m-i,n)}}_{S_{(t,l-i-t,k-t,i+t-k,m-i,n)}}. \tag{A.13}$$

Applying (A.9), we have

$$\operatorname{Res}^{S_{(l+m-i-j,j,n)}}_{S_{(l+m-i-j,j,j,n-j)}} \operatorname{Ind}^{S_{(l+m-i-j,j,n)}}_{S_{(k,l-k,l+m-i-j-k,k+j-l,n)}} = \operatorname{Ind}^{S_{(l+m-i-j,j,j,n-j)}}_{S_{(k,l-k,l+m-i-j-k,k+j-l,j,n-j)}} \operatorname{Res}^{S_{(k,l-k,l+m-i-j-k,k+j-l,n)}}_{S_{(k,l-k,l+m-i-j-k,k+j-l,j,n-j)}}. \tag{A.14}$$

Applying (A.7), we have

$$\operatorname{Res}^{S_{(l+m-i-j,j,j,n-j)}}_{S_{(l+m-i-j,j,n-j)}} \operatorname{Ind}^{S_{(l+m-i-j,j,j,n)}}_{S_{(l+m-i-j,l-k,k+j-l,j,n-j)}} = \operatorname{Ind}^{S_{(l+m-i-j,j,n-j)}}_{S_{(l+m-i-j,l-k,k+j-l,n-j)}} \operatorname{Res}^{S_{(l+m-i-j,l-k,k+j-l,j,n-j)}}_{S_{(l+m-i-j,l-k,k+j-l,n-j)}}. \tag{A.15}$$

Applying (A.9), we have

$$\operatorname{Res}^{S_{(l+m-i-j,l-k,k+j-l,j,n-j)}}_{S_{(l+m-i-j,l-k,j+k-l,n-j)}} \operatorname{Ind}^{S_{(l+m-i-j,l-k,k+j-l,j,n-j)}}_{S_{(k,l-k,l+m-i-j-k,k+j-l,j,n-j)}} = \operatorname{Ind}^{S_{(l+m-i-j,l-k,j+k-l,n-j)}}_{S_{(k,l-k,l+m-i-j-k,k+j-l,n-j)}} \operatorname{Res}^{S_{(l+m-i-j,l-k,k+j-l,n-j)}}_{S_{(k,l-k,l+m-i-j-k,k+j-l,n-j)}}. \tag{A.16}$$

Applying (A.9), we have

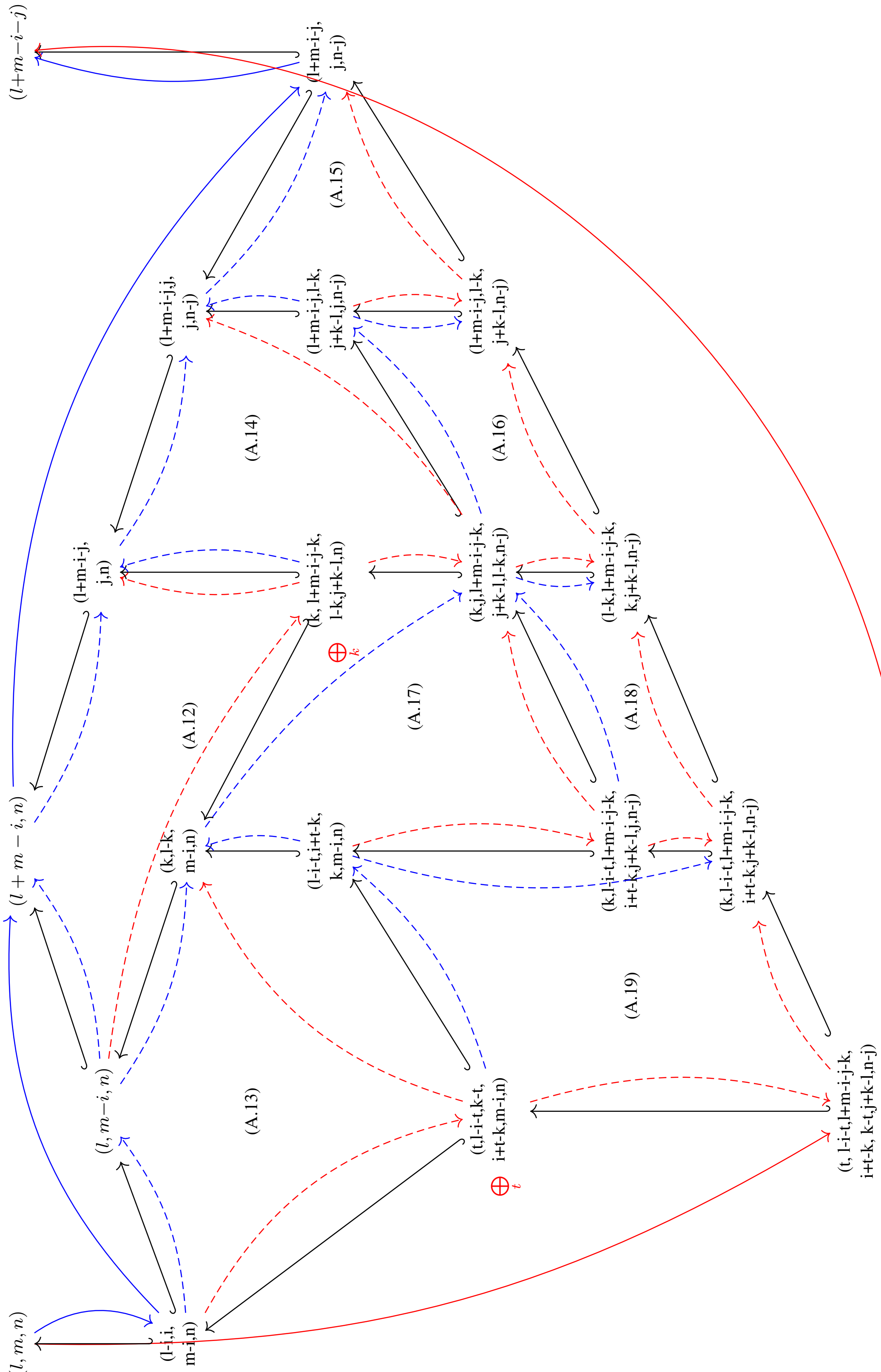
$$\operatorname{Res}^{S_{(k,l-k,m-i,n)}}_{S_{(k,l-k,l+m-i-j-k,j+k-l,j,n-j)}} \operatorname{Ind}^{S_{(k,l-k,m-i,n)}}_{S_{(k,l-i-t,i+t-k,m-i,n)}} = \operatorname{Ind}^{S_{(k,l-k,l+m-i-j-k,j+k-l,j,n-j)}}_{S_{(k,l-i-t,i+t-k,l+m-i-j-k,j+k-l,j,n-j)}} \operatorname{Res}^{S_{(k,l-i-t,i+t-k,m-i,n)}}_{S_{(k,l-i-t,i+t-k,l+m-i-j-k,j+k-l,j,n-j)}}. \tag{A.17}$$

Applying (A.8), we have

$$\operatorname{Res}^{S_{(k,l-k,l+m-i-j-k,j+k-l,j,n-j)}}_{S_{(k,l-k,l+m-i-j-k,j+k-l,n-j)}} \operatorname{Ind}^{S_{(k,l-k,l+m-i-j-k,j+k-l,j,n-j)}}_{S_{(k,l-i-t,i+t-k,l+m-i-j-k,j+k-l,j,n-j)}} = \operatorname{Ind}^{S_{(k,l-k,l+m-i-j-k,j+k-l,n-j)}}_{S_{(k,l-i-t,i+t-k,l+m-i-j-k,j+k-l,n-j)}} \operatorname{Res}^{S_{(k,l-i-t,i+t-k,l+m-i-j-k,j+k-l,j,n-j)}}_{S_{(k,l-i-t,i+t-k,l+m-i-j-k,j+k-l,n-j)}}. \tag{A.18}$$

Applying (A.9), we have

$$\operatorname{Res}^{S_{(k,l-i-t,i+t-k,m-i,n)}}_{S_{(k,l-i-t,i+t-k,l+m-i-j-k,j+k-l,n-j)}} \operatorname{Ind}^{S_{(k,l-i-t,i+t-k,m-i,n)}}_{S_{(t,l-i-t,k-t,i+t-k,m-i,n)}} = \operatorname{Ind}^{S_{(k,l-i-t,i+t-k,l+m-i-j-k,j+k-l,n-j)}}_{S_{(t,l-i-t,,k-t,i+t-k,l+m-i-j-k,j+k-l,n-j)}} \operatorname{Res}^{S_{(t,l-i-t,k-t,i+t-k,m-i,n)}}_{S_{(t,l-i-t,k-t,i+t-k,l+m-i-j-k,j+k-l,n-j)}}. \tag{A.19}$$


(l+m−i−j)
(l+m-i-j, j,n-j)
(l+m-i-j,j, j,n-j)
(l+m-i-j,l-k, j+k-l,j,n-j)
(l+m-i-j,l-k, j+k-l,n-j)
(A.15)
(A.14)
(A.16)
(l+m-i-j, j,n)
(k, l+m-i-j-k, l-k,j+k-l,n)
(k,j,l+m-i-j-k, j+k-l,l-k,n-j)
(l-k,l+m-i-j-k, k,j+k-l,n-j)
⊕k
(A.12)
(A.17)
(A.18)
(l + m − i, n)
(k,l-k, m-i,n)
(l-i-t,i+t-k, k,m-i,n)
(k,l-i-t,l+m-i-j-k, i+t-k,j+k-l,j,n-j)
(k,l-i-t,l+m-i-j-k, i+t-k,j+k-l,n-j)
(A.19)
(l, m−i, n)
(A.13)
(t,l-i-t,k-t, i+t-k,m-i,n)
(t, l-i-t,l+m-i-j-k, i+t-k, k-t,j+k-l,n-j)
⊕t
(l, m, n)
(l-i,i, m-i,n)

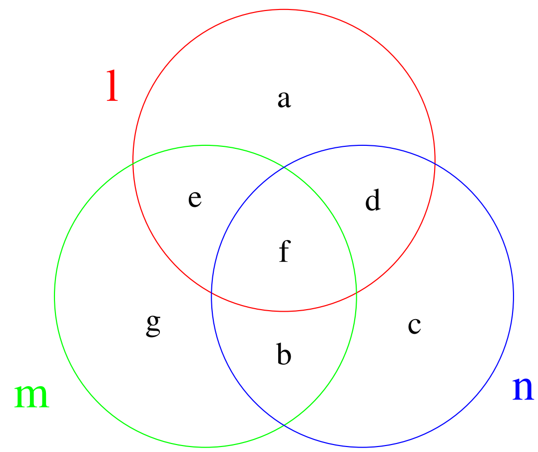
(A.20)

To better understand what we get after all the straightening steps, we parametrize our indices i, j, k, and t. Let

$$a = t,\ b = j + k - l,\ c = n - j,\ d = l - i - t,$$

$$e = k - t,\ f = i + t - k,\ g = l + m - i - j - k.$$

These new indices can be visualized using the following Venn diagram



(A.21)

where all the numbers in the red (green, blue respectively) circle add up to l (m,n respectively). Then Diagram (A.20) shows that

$$(U\#V)\#W = \bigoplus \mathrm{Ind}_{S_{(a,b,c,d,e,f,g)}}^{S_{(a+b+c+d+e+f+g)}} \mathrm{Res}_{S_{(a,b,c,d,e,f,g)}}^{S_{(l,m,n)}} U \otimes V \otimes W, \tag{A.22}$$

where the direct sum are taken over all the possible a, b, c, d, e, f, g fit the Venn diagram (A.21). By the symmetry of the right hand side of (A.22), it is not hard to see that $(U\#V)\#W = U\#(V\#W)$, which shows the associativity of the Heisenberg product.

APPENDIX B

ANOTHER PROOF OF THE STABILITY OF HEISENBERG COEFFICIENTS

We give another proof of the stability of Heisenberg coefficients using vertex operators. Note that some of the notations we use here are different from those in [17].

Recall that we defined a scalar product (2.31) on symmetric functions. The adjoint, with respect to this scalar product, of the operation of multiplying by a symmetric function $f \in \Lambda$ is denoted by $D_f : \Lambda \longrightarrow \Lambda$. That is,

$$\langle\, D_f(g),\ h\,\rangle = \langle\, g,\ fh\,\rangle \qquad \text{for all} \quad g, h \in \Lambda \tag{B.1}$$

Let $\Lambda^\wedge$ be the algebra of symmetric formal series, and $\sigma_1, \sigma_{-1} \in \Lambda^\wedge$, where $\sigma_1 := h_0 + h_1 + \cdots$, and $\sigma_{-1} = e_0 - e_1 + e_2 - \cdots$. We can naturally extend the scalar product and the adjoint operator to $\Lambda^\wedge$. In Section 5.2, we extend the indices of Schur functions from partitions to finite integer sequences.

Definition B.0.1. The linear map $\mathit{\Gamma} : \Lambda \to \Lambda^\wedge$ is defined by

$$\mathit{\Gamma}(s_\alpha) = \sum_{n \in \mathbb{Z}} s_{\alpha[n]} \tag{B.2}$$

and extending to Λ linearly. Here, $\alpha = (\alpha_1, \ldots, \alpha_l)$ is a finite integer sequence and $\alpha[n] = (n - \alpha_1, \alpha_2, \ldots, \alpha_l)$.

This map can be interpreted using the operator we introduced.

Lemma B.0.2 ([17] Lemma 3.2)**.** *For any $f \in \Lambda$, we have $\mathit{\Gamma}(f) = \sigma_1\, D_{\sigma_{-1}}(f)$.*

Lemma B.0.3 ([17] Lemma 3.3)**.** *D_{σ_1} and $D_{\sigma_{-1}}$ are ring automorphisms of Λ, with $(D_{\sigma_1})^{-1} = D_{\sigma_{-1}}$. Moreover, $D_{\sigma_{-1}} f = f(X-1)$ and $D_{\sigma_1}(f) = f(X+1)$.*

Let λ be a partition, then it is not hard to see

$$s(X+1) = \sum_{\mu} a_{\mu}^{\lambda}\, s_{\mu}(X) \tag{B.3}$$

where $a_{\mu}^{\lambda} = 1$ if λ/μ is a horizontal strip (no two boxes in the same column); $a_{\mu}^{\lambda} = 0$ otherwise. In particular, $a_{\mu}^{\lambda} = 0$ unless $\mu \subset \lambda$.

Lemma B.0.4. *Let λ be a partition and $f \in \Lambda$, then*

$$D_{s_\lambda}(\sigma_1\, f) = \sigma_1 \sum_{\mu\subset\lambda} a_{\mu}^{\lambda}\, D_{s_\mu}(f). \tag{B.4}$$

Proof. For any $g \in \Lambda$, we have

$$\begin{aligned}
\langle\, D_{s_\lambda}(\sigma_1\, f)\,,g\,\rangle &= \langle\, \sigma_1\, f\,, s_\lambda\, g\,\rangle = \langle\, f\,, D_{\sigma_1}(s_\lambda\, g)\,\rangle \\
&= \langle\, f\,, (s_\lambda\, g)\,(X+1)\,\rangle \quad \text{(Lemma B.0.3)} \\
&= \langle\, f, s_\lambda(X+1)\, g(X+1)\,\rangle \\
&= \sum_{\mu\subset\lambda} a_{\mu}^{\lambda}\,\langle\, f\,, s_\mu(X)\, g(X+1)\,\rangle \quad \text{(Equation (B.3))} \\
&= \sum_{\mu\subset\lambda} a_{\mu}^{\lambda}\,\langle\, D_{s_\mu}(f)\,, D_{\sigma_1}(g)\,\rangle \quad \text{(Lemma B.0.3)} \\
&= \sum_{\mu\subset\lambda} a_{\mu}^{\lambda}\,\langle\, \sigma_1\, D_{s_\mu}(f)\,, g\,\rangle,
\end{aligned}$$

which proves the lemma. □

Lemma B.0.5 ([17] Theorem 2.1)**.** *Let $\{U_\lambda\}$ and $\{V_\lambda\}$ form dual bases of Λ, and $f, g \in \Lambda$. Then,*

$$(\sigma_1\, f) * (\sigma_1\, g) = \sigma_1 \sum_{\alpha,\beta} (D_{U_\alpha}(f))\,(D_{V_\beta}(g))(U_\alpha * V_\beta), \tag{B.5}$$

where the sum is taken over all partitions α and β and $*$ is the Kronecker product.

Let r, t, λ, and μ be as stated in Theorem 5.1.4, then $\sum\limits_{n\in\mathbb{Z}}(s_{\lambda[n]}\#s_{\mu[n-r]})_{n+t}$ can be written as

$$\sum_{\substack{\alpha\,\vdash\, t+r\\ \beta\,\vdash\, t}} \left(D_{s_\alpha}(\mathit{\Gamma}(s_\lambda)) * D_{s_\beta}(\mathit{\Gamma}(s_\mu))\right) s_\alpha\, s_\beta. \tag{B.6}$$

By Lemma B.0.2 and B.0.4, we have

$$\begin{aligned} D_{s_\alpha}(\mathit{\Gamma}(s_\lambda)) &= D_{s_\alpha}(\sigma_1\, D_{\sigma_{-1}}(s_\lambda)) \\ &= \sigma_1 \sum_{\gamma} a^{\alpha}_{\gamma}\, D_{s_\gamma}(D_{\sigma_{-1}}(s_\lambda)) \end{aligned} \tag{B.7}$$

Similarly, we have

$$\begin{aligned} D_{s_\beta}(\mathit{\Gamma}(s_\mu)) &= D_{s_\beta}(\sigma_1\, D_{\sigma_{-1}}(s_\mu)) \\ &= \sigma_1 \sum_{\rho} a^{\beta}_{\rho}\, D_{s_\rho}(D_{\sigma_{-1}}(s_\mu)) \end{aligned} \tag{B.8}$$

Using Lemma B.0.5, we have

$$\begin{aligned} \text{(B.6)} &= \sum_{\substack{\alpha\,\vdash\, t+r\\ \beta\,\vdash\, t}} \sum_{\gamma,\rho} a^{\alpha}_{\gamma}\, a^{\beta}_{\rho} \big((\sigma_1\, D_{s_\gamma}(D_{\sigma_{-1}}(s_\lambda))) * (\sigma_1\, D_{s_\gamma}(D_{\sigma_{-1}}(s_\lambda)))\big)\, s_\alpha\, s_\beta \\ &= \sum_{\alpha,\beta,\gamma,\rho} a^{\alpha}_{\gamma}\, a^{\beta}_{\rho}\, \sigma_1 \sum_{\eta,\tau} \big(D_{s_\eta}(D_{s_\gamma}(D_{\sigma_{-1}}(s_\lambda)))\, (D_{s_\tau}(D_{s_\rho}(D_{\sigma_{-1}}(s_\mu)))\big)\, (s_\eta * s_\tau)\, s_\alpha\, s_\beta \\ &= \sum_{\alpha,\beta,\gamma,\rho} a^{\alpha}_{\gamma}\, a^{\beta}_{\rho}\, \sigma_1\, D_{\sigma_{-1}}\big(\sum_{\eta,\tau} (D_{s_\eta}(D_{s_\gamma}(s_\lambda)))\, (D_{s_\tau}(D_{s_\rho}(s_\mu)))\, D_{\sigma_1}(s_\eta * s_\tau)\, D_{\sigma_1}(s_\alpha)\, D_{\sigma_1}(s_\beta)\big) \\ &= \sigma_1\, D_{\sigma_{-1}}\big(\sum_{\substack{\alpha,\beta,\gamma\\ \rho,\eta,\tau}} a^{\alpha}_{\gamma}\, a^{\beta}_{\rho}\, (D_{s_\eta}(D_{s_\gamma}(s_\lambda)))\, (D_{s_\tau}(D_{s_\rho}(s_\mu)))\, (s_\eta * s_\tau)(X{+}1) s_\alpha(X{+}1) s_\beta(X{+}1)\big) \end{aligned} \tag{B.9}$$

where $\alpha \vdash t{+}r, \beta \vdash t, \gamma, \rho, \eta$, and τ are all partitions. Note that a summand in the last expression of (B.9) is zero unless

$$\eta \subset \lambda, \gamma \subset \lambda, \tau \subset \mu, \text{ and } \rho \subset \mu \tag{B.10}$$

The conditions in (B.10) shows that the degrees of the Schur functions (in variables set X) appearing in the Schur expansion of the summation of the last expression of (B.9) are bounded above by $d := |\lambda| + |\mu| + 2t + r$. Suppose the summation is equal to $\sum_{|\nu|\le d} a_\nu\, s_\nu$. Then by Lemma B.0.2 and Equation (B.9), we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} (s_{\lambda[n]} \# s_{\mu[n-r]})_{n+t} &= \sigma_1\, D_{\sigma_{-1}}\Big(\sum_{|\nu|\le d} a_\nu\, s_\nu\Big) = \sum_{|\nu|\le d} \sigma_1\, D_{\sigma_{-1}}(a_\nu\, s_\nu) \\ &= \sum_{n\in\mathbb{Z}} \sum_{|\nu|\le d} a_\nu\, s_{\nu[n]}.\end{aligned} \tag{B.11}$$

This shows the stability of the degree component $(s_{\lambda[n]} \# s_{\mu[n-r]})_{n+t}$, and it is not hard to see that this degree component stabilizes when $n \ge 2d = 2|\lambda| + 2|\mu| + 4t + 2r$. Equation (B.11) also proves that the stable formula for $(s_{\lambda[n]} \# s_{\mu[n-r]})_{n+t}$ applies for even small n's, which is why we have (5.13).