Ideals and Varieties Exercises

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Exercise 1. Let $\mathbb{A}_K^{n^2}$ be identified with the set of $M_{n \times n}$ matrices.

1. Show that the general linear group $GL_n(K) \subset \mathbb{A}_K^{n^2}$ of invertible matrices is not algebraic.
2. How can it be made algebraic?
3. Repeat both parts for the algebraic torus $(\mathbb{C}^*)^n \subset \mathbb{C}^n$.

Exercise 2. Let $f(x) = x^4 + x^3 - x^2 + x - 2$, and $g(x) = x^3 + x^2 + x + 1$. Use the Sylvester matrix of f and g to investigate whether or not the polynomials share a common root in $\mathbb{Q}[x]$. Then, do this with the Euclidean algorithm.

Exercise 3. How would you describe lines in $\mathbb{A}_K^2$? What is the algebraic interpretation of your description?

Exercise 4. A map $\phi : \mathbb{A}_k^2 \to \mathbb{A}_K^2$ is an *affine transformation* if and only if there exists a vector $v \in \mathbb{K}^2$ and a matrix $A \in GL_2(K)$ such that $\phi(x) = Ax + v$ for all $x \in \mathbb{A}_K^2$. Show that the set of all affine transformations, denoted by $\mathrm{Aff}(\mathbb{A}_K^2)$, forms a group.

Exercise 5. Show that the Zariski topology on $\mathbb{A}_k^n$ is not Hausdorff when K is an infinite field. What happens when K is finite?

Exercise 6. The equation $x^2 + y^2 = z^2$ has many solutions over $\mathbb{Z}$. If (a, b, c) is a solution and $n \in \mathbb{N}$, then $(n \cdot a, n \cdot b, n \cdot c)$ is also a solution. Find infinitely many solutions where (a, b, c) have no common factor greater than 1.

Exercise 7. Show that $V(x + y, x^2) = V(x + y, y^2)$.

Exercise 8. Show that the set of matrices in $\mathbb{A}_k^{n^2}$ with a repeated eigenvalue is an algebraic set.