Tensors: From Data to Statistics
First Session with Bernd Sturmfels at Ibadan

Every participant is asked to answer the following three binary questions:
Are you from Ibadan or from elsewhere? Are you male or female? Do you wear glasses or not?

The responses will be summarized in a table of format $2 \times 2 \times 2$, to be called the *Ibadan tensor*:

$$u \quad = \quad \left(\begin{pmatrix} u_{000} & u_{001} \\ u_{010} & u_{011} \end{pmatrix}, \begin{pmatrix} u_{100} & u_{101} \\ u_{110} & u_{111} \end{pmatrix} \right)$$

Question 1: A tensor p is called *independent* (or *rank one*) if its entries can be written in the form $p_{ijk} = a_i b_j c_k$. Show that the independent $2 \times 2 \times 2$-tensors form an irreducible algebraic variety $\mathscr{I}$.

Question 2: Compute a minimal generating set for the prime ideal of the variety $\mathscr{I}$.

Question 3: Evaluate each ideal generator at the Ibadan tensor u, and record the sign (positive, zero, or negative). What does this sign mean in statistics? (Hint: conditional independence)

Question 4: Fix a term order that refines the weights given by the Ibadan tensor u. Compute the reduced Gröbner bases for this ideal, and find all minimal primes of the initial monomial ideal.

Question 5: How many triangulations does the 3-dimensional cube have? Is this related to $\mathscr{I}$?

Question 6: Find a tensor in $\mathscr{I}$ that minimizes the Euclidean distance to the Ibadan tensor u.
Is that tensor unique? Does it have nonnegative entries? If not, find a closest nonnegative tensor.

Question 7: Find a nonnegative tensor in $\mathscr{I}$ that minimizes the likelihood function $\prod p_{ijk}^{u_{ijk}}$.

Question 8: Compare your answers for #6 and #7. Find and interpret $(a_0, a_1), (b_0, b_1), (c_0, c_1)$.

Question 9: Design a hypothesis test for independence and apply it to the Ibadan tensor u.