Report on the MSRI semester on Topological Aspects of Real Algebraic Goemetry, Spring 2004.

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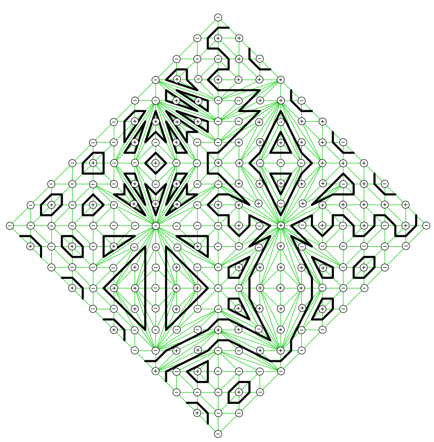
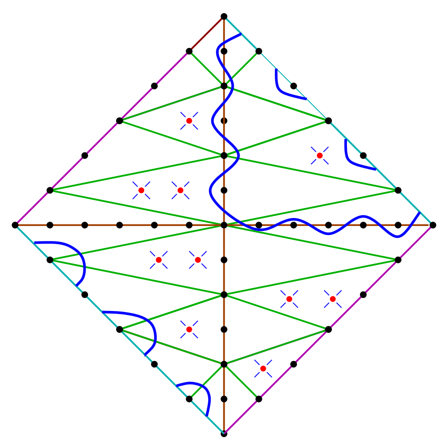
Real algebraic geometry—the geometry of varieties defined by systems of real polynomial equations—is a classical subject presently encompassing many distinct lines of inquiry. The MSRI program on Topological Aspects of Real Algebraic Geometry covered modern developments in real algebraic geometry emphasizing its topological aspects and its relations to other fields of mathematics. These relations arise as real algebraic varieties appear naturally in various mathematical contexts, in particular in applied mathematics, and there continue to be important interactions with these subjects.

Besides the traditional directions of topological classification of real algebraic varieties, the program focussed on enumerative problems and interactions with convex geometry via the theory of amoebas and tropical geometry. The main body of this report illustrates one story of an interpolation problem, its relation to real algebraic geometry and to tropical algebraic geometry, and how this story interacted with the MSRI program. It is adapted from an article written for the MSRI Autumn 2004 newsletter.

A visual demonstration of the interaction between real algebraic geometry and convex geometry is provided by these two pictures which come from constructions of real algebraic plane curves with controlled topology, and invlove convex triangulations of polytopes.

The picture on the left (due to MSRI member Ilya Itenberg) is the construction, through Viro's method of combinatorial patchworking (Viro was a program organizer) of a counterexample to a 90-year old conjecture by Ragsdale. The picture on the right (due to organizer Sottile and MSRI research professor Kharlamov) uses a modified version of the Viro construction (due to MSRI member Shustin) to construct a rational sextic with maximally many flexes and no real nodes. Both constructions predate the MSRI program.

1. Organization of the program

(1) **Introductory Workshop on Topological Aspects of Real Algebraic Geometry**
(2) **Topology and Geometry of Real Algebraic Varieties**
(3) **Real Algebraic Geometry and Geometric Modeling**
(4) **Algorithmic, Combinatorial and Applicable Real Algebraic Geometry**

The first, second and fourth workshops were traditional MSRI program workshops. (1) Introduced the topics of the semester to the members and to the mathematical public, and then (2) and (4) were devoted to particular aspects of the broad themes of the semester. The third was of a different character—it was devoted to an emerging application of toric varieties to geometric modelling, and in particular to the role that real geometry plays in these applications. It was the first workshop devoted to these applications held in North America, and has served to strengthen ties between computer scientists and mathematicians interested in these interactions.

The program also featured a weekly seminar, working seminars on tropical geometry and real algebraic algorithms, as well as a graduate course at UC Berkeley taught by Oleg Viro on the Topology of Real Algebraic Varieties.

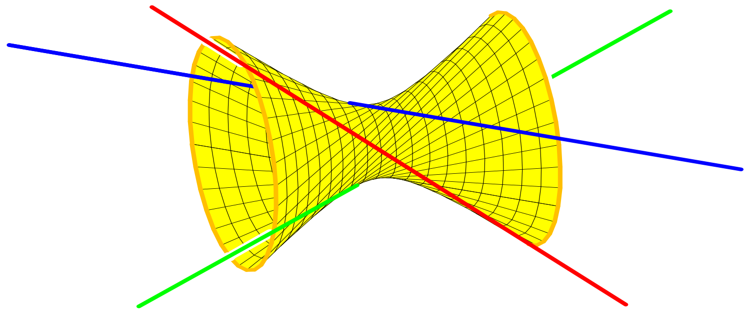
2. Tropical Interpolation

Everyone knows that two points determine a line, and many people who have studied geometry know that five points on the plane determine a conic. In general, if you have m random points in the plane and you want to pass a rational curve of degree d through all of them, there may be no solution to this interpolation problem (if m is too big), or an infinite number of solutions (if m is too small), or a finite number of solutions (if m is just right). It turns out that "m just right" means $m = 3d-1$ ($m = 2$ for lines and $m = 5$ for conics).

A harder question is, if $m = 3d-1$, *how many* rational curves of degree d interpolate the points? Let's call this number N_d, so that $N_1 = 1$ and $N_2 = 1$ because the line and conic of the previous paragraph are unique. It has long been known that $N_3 = 12$, and in 1873 Zeuthen [10] showed that $N_4 = 620$. That was where matters stood until 1989, when Ran [6] gave a recursion for these numbers. About ten years ago, Kontsevich and Manin [4] used associativity in quantum cohomology of $\mathbb{P}^2$ to give the elegant recursion

$$N_d \;=\; \sum_{a+b=d} N_a N_b \left(a^2b^2\binom{3d-4}{3a-2} - a^3b\binom{3d-4}{3a-1}\right) \;.$$

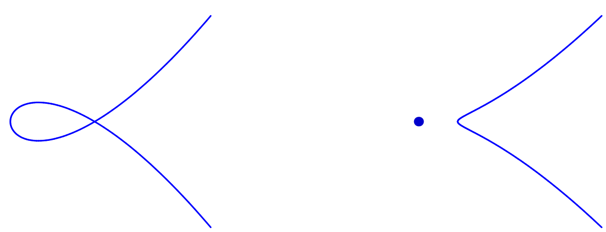
The research themes in the MSRI Winter 2004 semester on Topological Aspects of Real Algebraic Geometry included enumerative real algebraic geometry, tropical geometry, real plane curves, and applications of real algebraic geometry. All are woven together in the unfolding story of this interpolation problem, a prototypical problem of *enumerative geometry*, which is the art of counting geometric figures determined by given incidence conditions. Here is another problem: how many lines in space meet four given lines? To answer this, note that three lines lie on a unique doubly-ruled hyperboloid.



The three lines lie in one ruling, and the second ruling consists of the lines meeting the given three lines. Since the hyperboloid is defined by a quadratic equation, a fourth line will meet it in two points. Through each of these two points there is a line in the second ruling, and these are the two lines meeting our four given lines.

Enumerative geometry works best over the *complex* numbers, as the number of real figures depends rather subtly on the configuration of the figures giving the incidence conditions. For example, the fourth line may meet the hyperboloid in two real points, or in two complex conjugate points, and so there are either two or no real lines meeting all four. Based on many examples, we have come to expect that any enumerative problem may have all of its solutions be real [8].

Another such problem is the 12 rational curves interpolating 8 points in the plane. Most mathematicians are familiar with the nodal (rational) cubic shown on the left below. There is another type of real rational cubic, shown on the right.



In the second curve, two complex conjugate branches meet at the isolated point. If we let $N(\tau)$ be the number of real curves of type τ interpolating 8 given points, then Kharlamov and Degtyarev [1] showed that

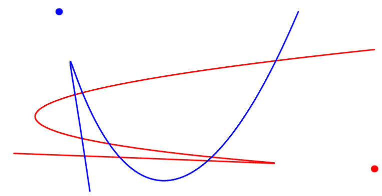
$$N(\propto) \;-\; N(\cdot\!<) \;=\; 8\,.$$

Their elementary topological methods are described on the web[†].

Since there are at most 12 such curves, $N(\propto) + N(\cdot\!<) \leq 12$, and so there are 8, 10, or 12 real rational cubics interpolating 8 real points in the plane, depending upon the number (0, 1, or 2) of cubics with an isolated point. Thus there will be 12 real rational cubics

[†] `http://www.math.tamu.edu/~sottile/stories/real_cubics.html`

interpolating any 8 of the 9 points of intersection of the two cubics below.



Welschinger [9], who was an MSRI postdoc in the program, developed this example into a theory. In general, the singularities of a real rational plane curve C are nodes or isolated points. The parity of the number of nodes is its *sign* $\sigma(C) \in \{\pm 1\}$. Given $3d-1$ real points in the plane, Welschinger considered the quantity

$$\left| \sum \sigma(C) \right| ,$$

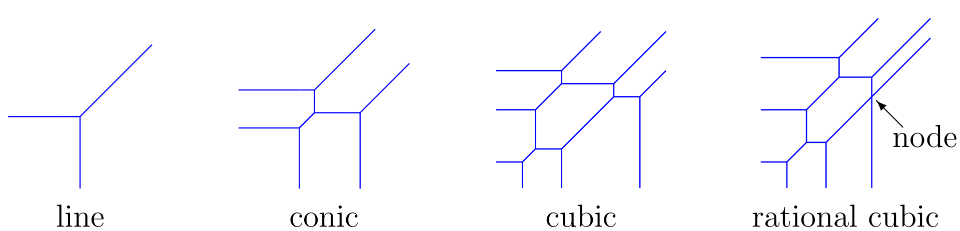
the sum over all real rational curves C of degree d that interpolate the points. He showed that this weighted sum does not depend upon the choice of points. Write W_d for this invariant of Welschinger. For example, we just saw that $W_3 = 8$.

This was a breakthrough, as W_d was (almost) the first truly non-trivial invariant in enumerative real algebraic geometry. Note that W_d is a lower bound for the number of real rational curves through $3d-1$ real points in the plane, and $W_d \leq N_d$.

Mikhalkin, who was an organizer of the semester, provided the key to computing W_d using tropical algebraic geometry [5]. This is the geometry of the tropical semiring, where the operations of max and $+$ on real numbers replace the usual operations of $+$ and $\cdot$. A tropical polynomial is a piecewise linear function of the form

$$T(x, y) \;=\; \max_{(i,j)\in\Delta} \{x \cdot i + y \cdot j + c_{i,j}\} ,$$

where $\Delta \subset \mathbb{Z}^2$ is the finite set of exponents of T and $c_{i,j} \in \mathbb{R}$ are its coefficients. When Δ is the set of all nonnegative (i, j) with $i + j \leq d$, then the tropical polynomial T defines a tropical (plane) curve of degree d, which is the set of points (x, y) where $T(x, y)$ is not differentiable. Here are some tropical curves.



The degree of a tropical curve is the number of rays tending to infinity in either of the three directions West, South, or North East. A tropical curve is rational if it is a piecewise-linear immersion of a tree. Nodes have valence 4.

Mikhalkin showed that there are only finitely many rational tropical curves of degree d interpolating $3d-1$ generic points. While the number of such curves does depend upon the choice of points, Mikhalkin attached positive multiplicities to each tropical curve so that the weighted sum does not, and is in fact equal to N_d. He also reduced these multiplicities and the enumeration of tropical curves to the combinatorics of lattice paths within a triangle of side length d.

Mikhalkin used a correspondence involving the map $\mathrm{Log}\colon (\mathbb{C}^*)^2 \to \mathbb{R}^2$ defined by $(x,y) \mapsto (\log|x|, \log|y|)$, and a certain 'large complex limit' of the complex structure on $(\mathbb{C}^*)^2$. Under this large complex limit, rational curves of degree d interpolating $3d-1$ points in $(\mathbb{C}^*)^2$ deform to 'complex tropical curves', whose images under Log are ordinary tropical curves interpolating the images of the points. The multiplicity of a tropical curve T is the number of complex tropical curves which project to T.

What about real curves? Following this correspondence, Mikhalkin attached a real multiplicity to each tropical curve and showed that if the tropical curves interpolating a given $3d-1$ points have total real multiplicity N, then there are $3d-1$ real points which are interpolated by N real rational curves of degree d. This real multiplicity is again expressed in terms of lattice paths.

What about Welschinger's invariant? In the same way, Mikhalkin attached a signed weight to each tropical curve (a tropical version of Welschinger's sign) and showed that the corresponding weighted sum equals Welschinger's invariant. As before, this tropical signed weight may be expressed in terms of lattice paths.

During the semester at MSRI, Itenberg, Kharlamov, and Shustin [3] used Mikhalkin's results to estimate Welschinger's invariant. They showed that $W_d \geq \frac{1}{3}d!$, and also

$$\log W_d \;=\; \log N_d + O(d), \qquad \text{and} \qquad \log N_d \;=\; 3d\log d + O(d)\,.$$

Thus at least logarithmically, *most* rational curves of degree d interpolating $3d-1$ real points in the plane are real.

There are two other instances of this phenomenon of lower bounds, the first of which predates Welschinger's work. Suppose that d is even and let $W(s)$ be a real polynomial of degree $k(d-k+1)$. Then MSRI program members Eremenko and Gabrielov [2] showed that there exist real polynomials $f_1(s),\ldots,f_k(s)$ of degree d whose Wronski determinant is $W(s)$. In fact, they proved a lower bound on the number of k-tuples of polynomials, up to an equivalence. Similarly, while at MSRI, Soprunova and organizer Sottile [7] studied sparse polynomial systems associated to posets, showing that the number of real solutions is bounded below by the sign-imbalance of the poset. Such lower bounds to enumerative problems, which imply the existence of real solutions, are important for applications.

For example, this story was recounted at dinner one evening at the MSRI Workshop on Geometric Modeling and Real Algebraic Geometry in April 2004. A participant, Schicho, realized that the result $W_3 = 8$ for cubics explained why a method he had developed always seemed to work. This was an algorithm to compute an approximate parametrization of an arc of a curve, via a real rational cubic interpolating 8 points on the arc. It remained to find conditions that guaranteed the existence of a solution which is close to the arc. This was just solved by Fiedler-Le Touzé, an MSRI postdoc who had studied cubics (not necessarily rational) interpolating 8 points to help classify real plane curves of degree 9.

References

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[6] Z. Ran, *Enumerative geometry of singular plane curves*, Invent. math. **97** (1989), 447–465.
[7] E. Soprunova and F. Sottile, *Lower Bounds for Real Solutions to Sparse Polynomial Systems*, `arXiv:math.AG/0409504`.
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[10] H. G. Zeuthen, *Almindelige Egenskaber ved Systemer af plane Kurver*, Danske Videnskabernes Selskabs Skrifter, Naturvidenskabelig og Mathematisk, Afd. **10** Bd. IV (1873), 286–393.

3. Participation

There were 83 members in residence at MSRI for two weeks or more during the program. Of these, 72 received some form of support: 17 senior researchers, 8 postdoctoral fellows and 44 general members. 39 participants were in residence for 2 months or more, with all postdocs and 6 senior members (including 3 organizers) in residence for the entire semester. 35 of the funded participants came from US institutions, and of those 6 are from Group I institutions (defined by the AMS classification). There were 12 funded women in the program. Of the eight postdocs, 2 were women, 1 Latino, 4 non-US, and 2 from Group 1 institutions. All postdocs went from MSRI to academic positions, 2 at Group 1 US universities, 2 at other US institutions, and 4 at foreign institutions.

4. Partial list of publications resulting from the semester

Another perspective on the semester is provided by the following partial list of publications that resulted or benefited from visits to MSRI during the semester. The semester on Topological Aspects of Real Algebraic Geometry enjoyed scientific links with the other two programs running at MSRI in the year 2003-2004, Discrete and Computational Geometry and Symplectic Geometry. In particular, any works on moduli spaces or tropical geometry (11, 19, 25, 27, 28, 33–36, 40, 44, 48-50), were of interest to or also involved members from the other two programs, and the publications 5, 9, 16, 20, 24, 32, 42, and 43 involved either participants or topics from one of the other programs.

(1) M. Aguiar and F. Sottile, *Cocommutative Hopf algebras of permutations and trees* `math.QA/0403101`, Journal of Algebraic Combinatorics, to appear.
(2) M. Aguiar and F. Sottile, *Structure of the Loday-Ronco Hopf algebra of trees*, J. Algebra, to appear.
(3) S. Akbulut and S. Durusoy, *An involution acting nontrivially on Heegard-Floer homology*, Fields Institute Proceedings, to appear.
(4) S. Akbulut and H. King, *Transcendental submanifolds of* $\mathbb{RP}^n$, Commentarii Math. Helv., to appear.

(5) S. Akbulut and S. Salur, *Calibrated manifolds and gauge theory*, submitted.
(6) C. Andradas, T. Recio, and J.R. Sendra, *La variedad de Weil para variedades unirracionales* , in "Contribuciones Matemáticas. Homenaje al profesor Enrique Outerelo" Revista Matemática Complutense. 2004. 33–53.
(7) B. Bertrand, F. Bihan, and F. Sottile, *Polynomial systems with few real zeroes*, submitted.
(8) F. Bihan, M. Franz, J. van Hamel, and C. McCrory, *Is every toric variety an M-variety?*.
(9) R. Brandenberg and T. Theobald, *Radii minimal projections of polytopes and constrained optimization of symmetric polynomials*. `math.MG/0311017`.
(10) F. Catanese, S. Hoşten, A. Khetan, and B. Sturmfels, *The Maximum Likelihood Degree*, submitted, `math.AG/0406533`.
(11) O. Ceyhan, *On moduli of pointed real curves of genus* 0, *math.AG/0207058*, IMRN, to appear.
(12) A. Dickenstein, L. Matusevich, T. Sadykov, *Bivariate hypergeometric D-modules*, Advances in Mathematics, to appear.
(13) A. Eremenko, A. Gabrielov, M. Shapiro, and A. Vainshtein, *Rational functions and real Schubert calculus*, Proceedings of the AMS, to appear, `math.AG/0407408`
(14) A. Gabrielov, D. Novikov, and B. Shapiro, *Mystery of point charges*, `math-ph/0409009`.
(15) R. Goldman and R. Krasauskas, *Control point structures for entire toric surfaces.*
(16) J.E. Goodman, A. Holmsen, R. Pollack, K. Ranestad and F. Sottile, *Cremona Convexity, Frame Convexity, and a Theorem of Santaló*, submitted. `math.MG/0409219`.
(17) D. Grigoriev and D. Pasechnik, *Polynomial-time computing over quadratic maps I. Sampling in real algebraic sets*, submitted. `cs.SC/0403008`.
(18) D. Grigoriev and D. Pasechnik, *Polynomial-time computing over quadratic maps II. Topological invariants*,
(19) C. Haase and I. Zharkov, *Integral affine structures on spheres III: complete intersections*, `math.AG/0504181`.
(20) M. Harada, T. Holm, *The equivariant cohomology of hypertoric varieties and their real loci*, `math.DG/0405422`.
(21) J. Huisman, M. Lattarulo, *Anisotropic real curves and bordered line arrangements*, submitted.
(22) J. Huisman, F. Mangolte, *Every connected sum of lens spaces is a real component of a uniruled algebraic variety*, submitted, `math.AG/0412159`.
(23) E. De Klerk, M. Laurent, and P. Parrilo, *A PTAS for the minimization of polynomials of fixed degree over the simplex*, 2004.
(24) E.de Klerk, J.Maharry, D. Pasechnik, R.B.Richter, and G.Salazar, *Improved bounds for the crossing numbers of $K_{m,n}$ and K_n*, submitted, `math.CO/0404142`.
(25) I. Itenberg,V. Kharlamov, and E. Shustin, *Logarithmic equivalence of Welschinger and Gromov-Witten invariants* Russian Math. Surveys 59 (2004), no. 6, 1093–1116.
(26) R. Krasauskas, *Bezier patches on toric surfaces*, Computer Aided Geometric Design, to appear.
(27) S. Kwon, *Intersection theoretic properties on the moduli space of genus 0 stable maps to D-convex symplectic 4-manifold*, `math.SG/0502297`.

(28) S. Kwon, *Transversality properties on the moduli space of genus 0 stable maps to a smooth rational projective surface and their real enumerative implications*, `math.AG/0410379`.

(29) C. Lenart, S. Robinson, and F. Sottile, *Grothendieck Polynomials via Permutation Patterns and chains in the Bruhat Order*, submitted. `math.CO/0405539`.

(30) C. Lenart and F. Sottile, *A Pieri-type formula for the K-theory of a flag manifold*, Trans. AMS, to appear.

(31) S.Yu.Orevkov, *Maximal number of real roots of $f(x) - g(x)$ given the multiplicities of roots of polynimials f and g*. (In preparation).

(32) Y. Ozan, *Triviality of symplectic $SU(2)$-action on homology*, `math.SG/0404121`.

(33) G. Mikhalkin, *Amoebas and tropical geometry*, book in progress.

(34) G. Mikhalkin, *Amoebas of algebraic varieties and tropical geometry*, for the the volume "Different faces in Geometry", `math.AG/0403015`.

(35) G. Mikhalkin, *Enumerative tropical algebraic geometry in $\mathbb{R}^2$*, J. Amer. Math. Soc. 18 (2005), 313-377.

(36) T. Nishinou and B. Siebert, *Toric degenerations of toric varieties and tropical curves*, `math.AG/0409060`.

(37) V. Powers, B. Reznick, C. Scheiderer, and F. Sottile, *A New Approach to Hilbert's Theorem on Ternary Quartics*, Comptes Rendus Mathematique (Paris), **339**, Issue 9, Pages 617-620.

(38) J. Ruffo, Y. Sivan, E. Soprunonva and F. Sottile, *Conjectures and experimentation in the real enumerative geometry of flag manifolds*, `math.AG/0502040`

(39) I. Scherbak, *Intersections of Schubert varieties and highest weight vectors in tensor products of sl_{N+1}-representations*, submitted, `math.RT/0409329`.

(40) E. Shustin, *A tropical calculation of the Welschinger invariants of real toric Del Pezzo surfaces*, submitted, `math.AG/0406099`.

(41) E. Soprunova and F. Sottile, *Lower Bounds for Real Solutions to Sparse Polynomial Systems*, submitted, `math.AG/0409504`.

(42) B. Sturmfels and S. Sullivant, *Toric ideals of phylogenetic invariants*, submitted, `q-bio.PE/0402015`.

(43) B. Sturmfels and J. Yu, *Classification of Six-Point Metrics*, `math.MG/0403147`, submitted.

(44) T. Theobald, *On the frontiers of polynomial computations in tropical geometry*, `math.CO/0411012`.

(45) D. Weinberg, N. Willis, *Singular Points of Real Quartic Curves Via Computer Algebra*, submitted.

(46) D. Weinberg, N. Willis, *Singular Points of Real Sextic Curves I*, in progress.

(47) D. Weinberg, N. Willis, *Singular Points of Real Sextic Curves II*, in progress.

(48) J-Y. Welschinger, *Invariants of real symplectic 4-manifolds out of reducible and cuspidal curves*, `math.SG/0502355`.

(49) J-Y. Welschinger, *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Invent. Math., to appear.

(50) J-Y. Welschinger, *Towards relative invariants of real symplectic 4-manifolds*, `math.SG/0502358`.