US Participation in workshop:
Enumeration and bounds in real algebraic geometry

CONTENTS

1. Summary

From January to June 2008, the Bernoulli Center at the École Polytechnique Fédéral de Lausanne in Lausanne, Switzerland will host a program on "Real and Tropical Algebraic Geometry". Embedded within this program are several week-long workshops, including one on "Enumeration and bounds in real algebraic geometry" (April 21–25 2008, organized by Frank Sottile of Texas A&M University and Boris Shapiro of the University of Stockholm). This proposal asks for $12,000 to help cover the expenses of approximately 10 participants in this workshop, to ensure that US researchers, and especially younger scientists, will be able to attend. While the program "Real and Tropical Algebraic Geometry" at the Bernoulli centre may be able to provide some support for local accommodation, most of its (scant) resources are earmarked for European participants.

A priori information on the real solutions to polynomial equations is important for the applications of mathematics. It is notoriously difficult to extract any information about real solutions without first solving them completely by finding all complex solutions. Recently, breakthroughs have led to new insight about bounds, both upper and lower, for the numbers of real solutions to some structured polynomial equations. At the same time, a better understand has arisen about average-case behavior and other scientists have begun to apply these results. The workshop is intended to be small (20 participants) and very focused on current developments and future prospects on real solutions to polynomial equations.

Intellectual Merits. The intellectual merits of this proposal are the importance of further improvement in our knowledge about real solutions to systems of equations. Scientific meetings of this level and size are necessary for the timely development of this field.

Broader Impacts. The broader impacts are the development of human resources from the further training of younger scientists in the United States, and the dissemination of these new results and ideas.

2. Scientific Description

A priori information on the real solutions to polynomial equations is important for the applications of mathematics. It is notoriously difficult to extract any information about real solutions without first solving them completely by finding all complex solutions. The most notable exception to this rule is due to Descartes [8], who showed that the number of positive zeroes of a univariate polynomial is at most the number of changes in the signs of the coefficients when read from lowest to highest power in the variable. This bounds the number of positive zeroes by the number of coefficients of the polynomial, minus 1.

A multivariate version of this bound was given by Khovanskii in 1980 [15]. He showed that a system of n polynomials in n variables having a total of $n+k+1$ distinct monomial terms has at most $2^{\binom{n+k}{2}}(n+1)^k$ positive real solutions. Thus the number of real solutions to a system of polynomial equations has a completely different character than the number of complex solutions, which depends on the degrees of the polynomials. The dependence of this bound on the number of monomials shows that it is most relevant for polynomials involving few monomials (fewnomials), and is thus called the *fewnomial bound*. This fundamental finiteness result implies many similar bounds on the topology of real algebraic varieties defined by fewnomials.

It has found recent applications in work of Hampton and Moeckel [12] on relative equilibria in the four-body problem and of Albouy [1] on Euler configurations, and in Gabrielov, Novikov, and Shapiro [10] on equilibria of a potential arising from point charges. It also led to the theory of *o*-minimality—a model-theoretic approach to finiteness of definable sets. A related theory of Pfaffian functions was initiated by Khovanskii with subsequent development by Gabrielov and his coauthors, including Zell [11].

Recently, Bertrand, Bihan, and Sottile developed a completely different approach to fewnomial bounds [4, 5], which led to a much smaller fewnomial bound of $\frac{e^2+3}{4}2^{\binom{k}{2}}n^k$. Their approach is based on a duality between polynomial systems and master functions of hyperplane arrangements and it has led to a new bound of $\frac{e^4+3}{4}2^{\binom{k}{2}}n^k$ for all (not just positive) real solutions [3]. Like Khovanskii's bound it implies new bounds on the topology of some real varieties. It is also sharp in a sense; there is a construction of systems with $k^{-k}n^k$ solutions [6]. Thus the term n^k is the best possible.

Many open questions remain, which include improving the factor $2^{\binom{k}{2}}$, applying these new bounds more comprehensively to the topology of real varieties, and finding applications for them outside of real algebraic geometry. One purpose of this workshop is to disseminate these new ideas and incubate work in these new directions by bringing together some of the most active researchers in this area, including Albouy, Basu, Bates, Bihan, Gabrielov, Khovanskii, Moeckel, Rojas, and Sottile.

Another recent trend in this area is the discovery of lower bounds for the number of real solutions to systems of polynomials. This was first found by Eremenko and Gabrielov in the Schubert calculus [9] who showed that some problems in the Schubert calculus were forced to have many real solutions. This inspired work of Sottile and Soprunova [19], who developed a theory for lower bounds for the number of solutions to sparse polynomial systems. This was based on toric geometry and geometric combinatorics, and it was extended by Joswig and Witte [14]. Together with the Itenberg-Kharlamov-Shustin proof of the non-triviality of the Welschinger invariant [13], this work suggests that we can expect such lower bounds to be ubiquitous. Extending these results to cover a wider variety of polynomial systems would have significant implications in some applications of mathematics. Another goal of this workshop is to discuss the existing theory of lower bounds for real solutions and how to extend it, especially to mixed systems of sparse polynomials and other structured systems arising in applications.

Perhaps more important than upper or lower bounds is the average, or expected number of real solutions. This has a long history beginning with work of Bloch and Polya in 1932. For us, the relevant thread begins with work of Shub and Smale [18], who showed that the expected number of real solutions to a system of polynomials was the square root of the Bézout number of complex solutions (they used Haar measure from an action of the orthogonal group). Similar results were later obtained by Rojas [16] for some systems of sparse polynomials. Azaïs and Wschebor [2] gave a new proof of the result of Shub and Smale based on the Rice formula, and this was greatly extended by Bürgisser [7] to give the expected Euler characteristic of a random real algebraic variety. He extended and unified much recent work in this area, and his formulae look like characteristic class calculations—suggesting a deeper algebraic theory behind these average-case results. Also related is work of Shiffman-Zelditch [17] and Zrebiec [20] on the related question of the distribution of zeroes of random polynomial systems. This workshop will also address the average-case behaviour of the number and distribution of the zeroes of systems of polynomial equations.

3. Description of workshop

Our plan is to run a small workshop focused on the topic of "Enumeration and bounds in real algebraic geometry" embedded within the program at the Bernoulli center on "Real and tropical geometry". (`http://bernoulli.epfl.ch/tropical/`) Some key personnel plan to be in Lausanne in the middle of April, including Gabrielov, Itenberg, Kharlamov, Khovanskii, Shapiro, Shustin, and Sottile. The program has resources to support the visits of a few additional scientists, primarily from Europe. We hope with this proposal to enable about 8-10 people to attend from the United States, including several younger scientists.

Our intention is to gather the leading researchers in these related topics, together with young scientists interested and working on aspects of real solutions to systems of equations. We will exploit the small size of the workshop, scheduling survey talks, discussions on trends and open problems, as well as more traditional research talks. We hope to map out important trends in these areas and foster collaboration among the participants.

Sottile (Texas A&M University) and Shapiro (University of Stockholm) are the organizers of this workshop. This workshop, with its focus on real solutions and choice of topics is unique; While real solutions to systems of equations is often a topic covered in some meetings in real algebraic geometry (most notably in April 2004 at the MSRI, where one day out of five was devoted to this topic), it is only with the recent surge of results and interest that it makes sense to devote an entire workshop to this topic. As the subject is now moving fast, we have a desire to bring together those with the most interest to give this line of research an additional impetus.

The meeting has been advertised through the Bernoulli center website and via European research networks. We have also been contacting potential participants directly.

Potential participants We are beginning to invite people to attend the workshop. We plan to approach at least the following people, as well as some graduate students.

Visitors to Bernoulli Centre and local participants

A. Bernig (Fribourg), B. Bertrand (Geneva), E. Brugalle (Paris) A. Gabrielov (Purdue), I. Itenberg (Strasbourg), V. Kharlamov (Strasbourg), A. Khovanskii (Toronto), B. Shapiro (Stockholm), E. Shustin (Tel Aviv), F. Sottile (Texas A&M).

US Participants

S. Basu (Georgia Tech), D. Bates (IMA Minneapolis), C. Hillar (Texas A&M), R. Moeckel (Minnesota), M. Rojas (Texas A&M), E. Soprunova (Kent State), S. Zelditch (John Hopkins), T. Zell (Georgia State), S. Zrebic (Texas A&M).

European Participants

F. Bihan (Savoie), D. Novikov (Weizmann), N. Witte (Darmstadt), P. Bürgisser (Paderborn), A. Albouy (Paris 6), S. Yakovenko (Weizmann), M. Joswig (Darmstadt).

4. Facilities

The Bernoulli centre is located on the campus of EPFL (École Polytechnique Fédéral de Lausanne) in Lausanne, Switzerland. It provides offices to long-term visitors, a seminar and a conference room, and its staff provide assistance with lodging to all visitors.

Budget Justification

This proposal is requesting \$12,000 to provide on the average of \$1,200 support for each of 10 participants from the United States to attend the workshop "Enumeration and bounds in real algebraic geometry" (21–25 April 2008) at the Bernoulli center at EPFL in Lausanne, Switzerland. The estimated cost of airfare to Geneva is about \$1,000 in April, and the Bernoulli center projects lodging and food in Laussanne for a 5-day period to be 900 Swiss francs, or \$760.

We expect to give most of this support for younger participants, but also to ensure the participation of the most important US scientists.

The Bernoulli Centre, through its program on real and tropical geometry will provide logistical support for the workshop and also largely fund the visits of Brugalle, Gabrielov, Itenberg, Kharlamov, Khovanskii, Shapiro, Shustin, Sottile. In addition, the workshop currently has a budget of 4,000 Swiss francs, which will be used to bring at least three additional European researchers to the workshop (The budget may increase due to possible cancellations of other long-term visitors to the program.)

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