Manuscript of 21 February 2005.

LINE TANGENTS TO FOUR TRIANGLES IN THREE-DIMENSIONAL SPACE

HERVÉ BRÖNNIMANN, OLIVIER DEVILLERS, SYLVAIN LAZARD, AND FRANK SOTTILE

ABSTRACT. We investigate the lines tangent to four triangles in $\mathbb{R}^3$. By a construction, there can be as many as 62 tangents. We show that there are at most 162 connected components of tangents, and at most 156 if the triangles are disjoint. In addition, if the triangles are in (algebraic) general position, then the number of tangents is finite and it is always even.

INTRODUCTION

Motivated by visibility problems, we investigate lines tangent to four triangles in $\mathbb{R}^3$. In computer graphics and robotics, scenes are often represented as unions of not necessarily disjoint polygonal or polyhedral objects. The objects that can be seen in a particular direction from a moving viewpoint may change when the line of sight becomes tangent to one or more objects in the scene. Since this line of sight is tangent to a subset of the edges of the polygons and polyhedra representing the scene, we are also led to questions about lines tangent to segments and to polygons. Four polygons will typically have finitely many common tangents, while 5 or more will have none and 3 or fewer will have either none or infinitely many.

This paper is the third in a series of papers by the authors and their collaborators investigating such questions. The paper [2] investigated the lines of sight tangent to four convex polyhedra in a scene of k convex but not necessarily disjoint polyhedral objects, and proved that there could be up to but no more than $\Theta(n^2k^2)$ connected components of such lines. (The earlier version [1] only proved this bound for the considerably easier case of disjoint convex polyhedra in algebraic general position.) We would like, however, to investigate how high the constants hidden in the $O()$ notation are. The paper [3] offers a detailed study of transversals to n line segments in $\mathbb{R}^3$ and proved that although there are at most 2 such transversals for four segments in (algebraic) general position, there are always at most n such connected components of transversals in any case. In this paper, we consider the slightly more complicated (but more relevant) case of four triangles in $\mathbb{R}^3$, and establish lower and upper bounds on the number of tangent lines.

Research of Brönnimann supported in part by NSF CAREER Grant CCR-0133599.

Research of O. Devillers supported by IST Program of the EU as a Shared-cost RTD (FET Open) Project under Contract No IST-2000-26473 (ECG - Effective Computational Geometry for Curves and Surfaces).

Research of Sottile is supported in part by NSF CAREER grant DMS-0134860.

Research initiated at the Second McGill-INRIA Workshop on Computational Geometry in Computer Graphics, February 7–14 2003 co-organized by H. Everett, S. Lazard, and S. Whitesides, and held at the Bellairs Research Institute of McGill University in Holetown, St. James, Barbados, West Indies. A preliminary version appeared in Proc. Canad. Conf. Comput. Geom., Montreal, August 2004.

A *triangle* in $\mathbb{R}^3$ is the convex hull of three distinct (and non-collinear) points in $\mathbb{R}^3$. A line is *tangent* to a triangle if it meets an edge of the triangle. Note that a line tangent to each of four triangles forming a scene corresponds to an unoccluded line of sight in that scene. If there are $k > 4$ triangles, then the bound $\Theta(k^2)$ of [2] stands (as the total number of edges is $n = 3k$ and one of the lower bound example is made of triangles). We thus investigate the case of four triangles. Let $n(t_1, t_2, t_3, t_4)$ be number of lines tangent to four triangles t_1, t_2, t_3, and t_4 in $\mathbb{R}^3$. This number may be infinite if the lines supporting the edges of the different triangles are not in general position.

Our first step is to consider the algebraic relaxation of this geometric problem in which we replace each edge of a triangle by the line in $\mathbb{CP}^3$ supporting it, and then ask for the set of lines in $\mathbb{CP}^3$ which meet one supporting line from each triangle. Since there are $3^4 = 81$ such quadruples of supporting lines, this is the disjunction of 81 instances of the classical problem of transversals to four given lines in $\mathbb{CP}^3$. As there are two such transversals to four given lines in general position, we expect that this algebraic relaxation has 162 solutions.

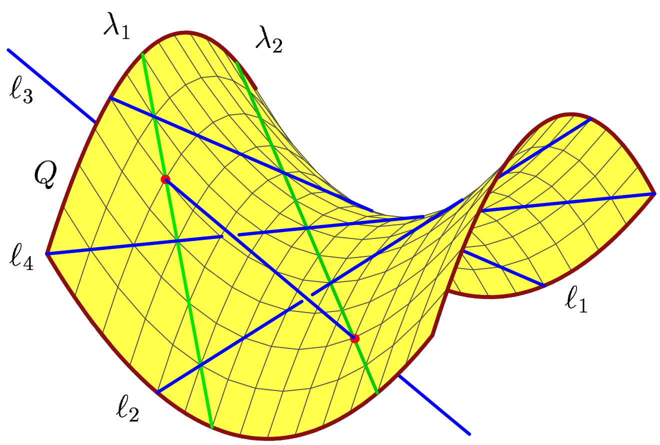

FIGURE 1. The lines ℓ_1, ℓ_2 and ℓ_3 span a hyperbolic paraboloid Q which meets line ℓ_4 in two points. The two lines λ_1 and λ_2 are the transversals to the four lines ℓ_1 ℓ_2, ℓ_3, and ℓ_4.

We say that four triangles t_1, t_2, t_3, t_4 are in (algebraic) *general position* if each of the 81 quadruples of supporting lines have two transversals and all 162 transversals are distinct. Let $\mathcal{T}$ be the configuration space of all quadruples of triangles in $\mathbb{R}^3$ and $T \subset \mathcal{T}$ consist of those quadruples which are in general position. Thus if $(t_1, t_2, t_3, t_4) \in T$, the number $n(t_1, t_2, t_3, t_4)$ is finite and is at most 162.

Our first result is a congruence.

Theorem 1. *If* $(t_1, t_2, t_3, t_4) \in T$*, then* $n(t_1, t_2, t_3, t_4)$ *is even.*

Our primary interest is the number

$$N \ := \ \max\{n(t_1, t_2, t_3, t_4) \mid (t_1, t_2, t_3, t_4) \in T\}\,.$$

Our results about this number N are two-fold. First, we show that $N \geqslant 62$.

Theorem 2. *There are four disjoint triangles in* T *with* 62 *common tangent lines.*

The idea is to perturb a configuration of four lines in $\mathbb{R}^3$ with two real transversals, such as in Figure 1. The triangles in our construction are very 'thin'—the smallest angle measures about 10^{-11} degrees. We ran a computer search for 'fatter' triangles having many common tangents, checking the number of tangents to 5 million different quadruples of triangles. Several had as many as 40 common tangents. This is discussed in Section 5.

We can improve the upper bound on N when the triangles are disjoint.

Theorem 3. *Four triangles in T admit at most 162 distinct common tangent lines. This number is at most 156 if the triangles are disjoint.*

When the four triangles are not in general position, the number of tangent lines can be infinite. In this case, we may group these tangents by connected components: two line tangents are in the same component if one may move continuously between the two lines while staying tangent to the four triangles. Each quadruple of edges may induce up to four components of tangent lines [3], giving a trivial upper bound of 324. This may be improved.

Theorem 4. *Four triangles have at most 162 connected components of common tangents. If the triangles are disjoint, then this number is at most 156.*

We believe that these upper bounds are far from optimal. Theorems 1, 2, 3, and 4 are proved in Sections 1, 2, 3, and 4, respectively. Section 5 discusses our search for 'fat' triangles with many common tangents.

1. A congruence

We prove Theorem 1 by showing that any two quadruples of triangles whose supporting lines are in general position are connected by a path such that common tangents are created and destroyed in pairs along that path. Thus the parity of $n(t_1, t_2, t_3, t_4)$ is constant for $(t_1, t_2, t_3, t_4) \in T$. The theorem follows as there are triangles in T with no common tangents.

We study the complement Σ of T in the set $\mathcal{T}$ of quadruples of all triangles. The reason is that the number $n(t_1, t_2, t_3, t_4)$ of common tangents is constant in each connected component of T and so we must pass through Σ to connect quadruples in T. Since the set of smooth points of Σ is open and dense in Σ, a path may be found which meets Σ only in its smooth points. We describe what happens near a smooth point of Σ.

This discriminant hypersurface Σ of $\mathcal{T}$ has 162 different algebraic components which come in two types. Recall that a quadruple (t_1, t_2, t_3, t_4) lies in T only if

(A) There are two lines in $\mathbb{CP}^3$ transversal to each quadruple $\ell_1, \ell_2, \ell_3, \ell_4$ of lines supporting one edge from each triangle, and

(B) the 162 such lines are distinct.

Lemma 5. *The discriminant hypersurface Σ has 162 algebraic components. Each component has an open dense set on which exactly one of (a) or (b) occurs.*

(a) *There is a unique transversal λ in $\mathbb{CP}^3$ to one quadruple of supporting lines.*

(b) *One of the lines λ meeting one quadruple of supporting lines $\ell_1, \ell_2, \ell_3, \ell_4$ meets one other supporting line ℓ' (and hence a vertex v of a triangle).*

Furthermore, in each case, the distinguished line λ is real.

Proof. We consider what happens when one of the conditions (A) or (B) fails, but the rest of the configuration remains generic. For (A), if there is a quadruple $\ell_1, \ell_2, \ell_3, \ell_4$ of supporting lines without two common transversals, then either there is only one transversal or there are infinitely many. Since we are considering generic such configurations, we may assume that ℓ_1, ℓ_2, and ℓ_3 are in general position in that they span a quadric Q as in Figure 1, and ask what happens as ℓ_4 moves out of general position. If ℓ_4 meets one of ℓ_1, ℓ_2, or ℓ_3, there still will be two lines, but if ℓ_4 becomes tangent to Q, then there will only be one, as the two lines λ_1 and λ_2 coalesce. Further degeneration is required for there to be infinitely many lines, and so we see that (a) describes what happens generically when (A) fails for a single quadruple of supporting lines.

For (B), we may assume that each quadruple of supporting lines has two transversals, but there are two quadruples with a common transversal. The generic way for this to occur is described in (b).

To see that it is possible for exactly one of (a) or (b) to occur, begin with a configuration of four triangles in T, and allow exactly one supporting line of one triangle to rotate about one vertex, remaining in the plane of the triangle. Perturbing the plane of this last triangle, if necessary, we see that only the configurations described in (a) or (b) can occur, each will occur finitely many times, and they will occur for distinct angles of rotation.

Since the lines and vertices defining the special line λ are all real and λ is unique, it will also be real. Lastly, there are 81 different components of each type.

Proof of Theorem 1. Suppose now that we have two quadruples of triangles in T. A consequence of Lemma 5 is that there exists a path γ in $\mathcal{T}$ connecting them such that each time γ meets the discriminant hypersurface Σ, exactly one of (a) or (b) occurs. We need only show that the parity of the number of tangents does not change as we move along γ and one of (a) or (b) occurs.

If (a) occurs, the number of tangents changes only if the double line λ is tangent to the triangles. Approaching this configuration along the curve γ, either two real lines or two complex lines coalesce into λ. Thus the parity of $n(t_1, t_2, t_3, t_4)$ does not change when crossing Σ in a component of type (a).

For (b), we suppose that $\ell' = \ell'_4$ is a supporting line to the fourth triangle, t_4. Let C_4 be the conic which is the intersection of the hyperboloid spanned by ℓ_1, ℓ_2, and ℓ_3 with the plane π_4 spanned by t_4. Through every point of C_4 there is a unique line meeting ℓ_1, ℓ_2, and ℓ_3. In particular, the line λ corresponds to the vertex v of t_4 where ℓ_4 meets ℓ'_4. Figure 2 illustrates the two possibilities for the configuration of C_4 and t_4: Either (i) C_4 meets the

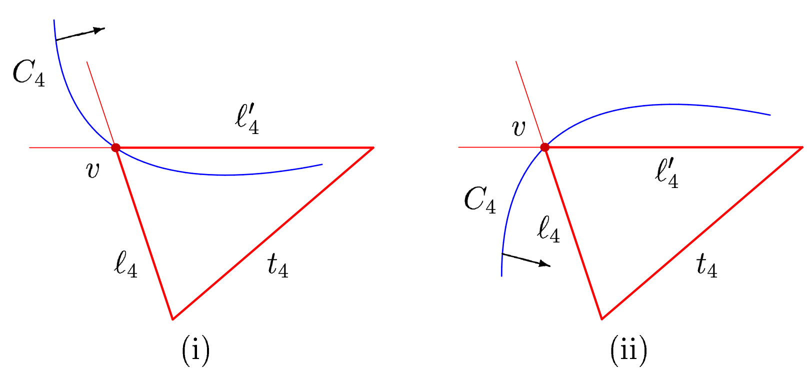

FIGURE 2. Configuration in plane π_4

interior of t_4 or (ii) it does not. Moving along the curve γ perturbs the configuration. Topologically, this corresponds to moving C_4 off the vertex v, which is suggested by the arrows in Figure 2. In (i), there will be one line near to λ meeting ℓ_1, ℓ_2, and ℓ_3, and t_4

both before and after the conic C_4 meets the vertex v (but these lines will meet different edges of t_4). In case (ii), two lines which meet the supporting lines outside of t_4 coalesce into λ, and then become two lines meeting t_4.

Thus the parity of the number of lines tangent to the four triangles does not change when crossing Σ, which completes the proof of Theorem 1.

2. A construction with 62 tangents

Consider the four triangles whose vertices are given in Table 1.

t_1	$(-10.5, 1, -10.5)$ $(.5628568345479573470378601, 1, .5628568345479573470378601)$ $(.56285683454726874605620706, .9999999999822994290647247, .56285683454726874605620706)$
t_2	$(-10.5, -1, 10.5)$ $(1.394218989475, -1, -1.394218989475)$ $(1.3942406911811439954597161, -1.0000237884694881275439271, -1.3942406911811439954597161)$
t_3	$(-9.5, -9.5, .25)$ $(.685825, .685825, .25)$ $(.69121730616063647303519136, .69121730616063647303519136, .26069756890079842876805653)$
t_4	$(9.5, 0, 0)$ $(-.511, 0, 0)$ $(-1.0873912730501133759642956, 0, -.51645811088049333541289247)$

TABLE 1. Four triangles with 62 common tangents

Theorem 2′. *There are exactly* 62 *lines tangent to the four triangles of Table* 1.

This can be verified by a direct computation. Software is provided on this paper's web page†. More illuminating perhaps is our construction. The idea is to perturb a configuration of four lines in $\mathbb{R}^3$ with two transversals such as in Figure 1. The resulting triangles of Theorem 2′ are very thin. In degrees, their smallest angles are

$$t_1 \colon 6.482 \times 10^{-12}, \quad t_2 \colon 8.103 \times 10^{-5}, \quad t_2 \colon 4.253 \times 10^{-2}, \quad \text{and} \quad t_4 \colon 2.793\,.$$

2.1. **The construction.** The lines given parametrically

$$\ell_1 \ : \ (t, 1, t)\,, \quad \ell_2 \ : \ (t, -1, -t)\,, \quad \ell_3 \ : \ (t, t, \tfrac{1}{4})\,, \quad \text{and} \quad \ell_4 \ : \ (t, 0, 0)\,,$$

have two transversals

$$\lambda_1 \ : \ (\tfrac{1}{2}, 2t, t) \quad \text{and} \quad \lambda_2 \ : \ (-\tfrac{1}{2}, 2t, -t)\,.$$

For each $i = 1, 2, 3, 4$, let Q_i be the hyperboloid spanned by the lines other than ℓ_i. For example, Q_3 has equation $z = xy$. The intersection of Q_i with a plane containing ℓ_i will be a conic which meets ℓ_i in two points (corresponding to the common transversals λ_1 and λ_2 at $t = \pm\frac{1}{2}$). We choose the plane π_i so that these two points lie in the same connected component of the conic. Here is one possible choice

$$\pi_1 \ : \ x = z\,, \quad \pi_2 \ : \ x = -z\,, \quad \pi_3 \ : \ x = y\,, \quad \text{and} \quad \pi_4 \ : \ y = 0\,.$$

For each i, let C_i be the conic $\pi_i \cap Q_i$, shown in the plane π_i in Figure 3. Here, the horizontal coordinate is t, the parameter of the line ℓ_i, while the vertical coordinate is $y-1$ for π_1, $y+1$ for π_2, $z-\frac{1}{4}$ for π_3, and z for π_4.

†`http://www.math.tamu.edu/~sottile/stories/4triangles/index.html`

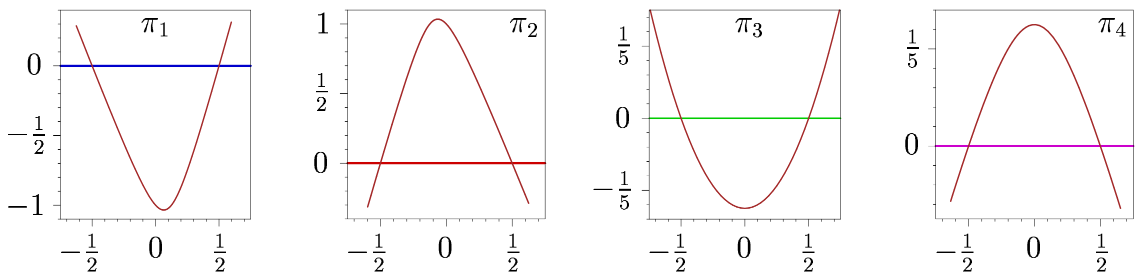

FIGURE 3. Conics in the planes π_i

Rotate each line ℓ_i very slightly about a point that is far from the conic C_i, obtaining a new line k_i which also meets C_i in two points. Consider now the transversals to $\ell_i \cup k_i$, for $i = 1, \dots, 4$. Because k_i is near to ℓ_i and there were two transversals to $\ell_1, \ell_2, \ell_3, \ell_4$, there will be 2 transversals to each of the 16 quadruples of lines obtained by choosing one of ℓ_i or k_i for $i = 1, \dots, 4$. By our choice of the point of rotation, all of these will meet ℓ_i and k_i in one of the two thin wedges they form. In this wedge, form a triangle by adding a third side so that the edges on ℓ_i and k_i contain all the points where the transversals meet the lines. The resulting triangles will then have at least 32 common tangents. We claim that by carefully choosing the third side (and tuning the rotations) we are able to get 30 additional tangents.

To begin, look at Figure 4 which displays the configuration in π_4 given by the four triangles from Table 1. Since the lines ℓ_i and k_i for $i = 1, 2$ are extremely close, the four conics given by transversals to them and to ℓ_3 cannot be resolved in these pictures. The same is true for the four conics given by k_3, so the apparent 2 conics are each clusters of four nearby conics. The picture on the left is a view of this configuration in the coordinates for π_4 of Figure 3. It includes a secant line m_4 to the conics. We choose coordinates on the right so that m_4 is vertical, but do not change the coordinates on ℓ_4. The horizontal scale has been accentuated to separate the two clusters of conics. The three lines, ℓ_4, k_4,

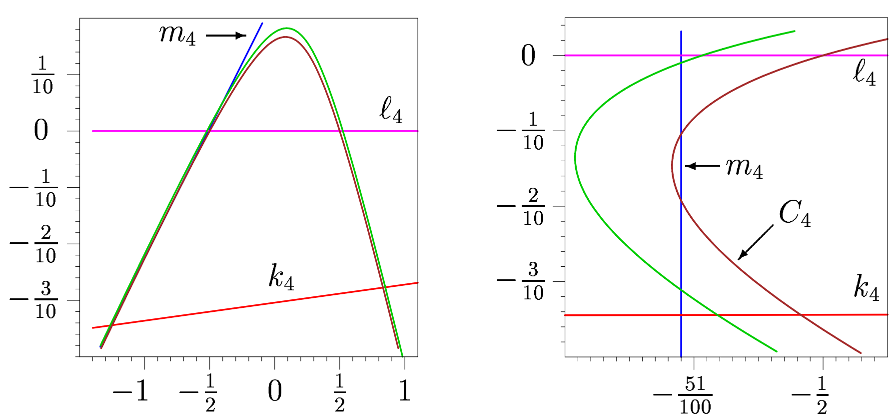

FIGURE 4. Configuration in plane π_4

and m_4 form the triangle t_4. Let its respective edges be e_4, f_4, and g_4. Each edge meets each of the 8 conics in two points and these 48 points of intersection give 48 lines tangent to the four triangles.

This last assertion that the 16 lines transversal to m_4 and to $\ell_i \cup k_i$ for $i = 1, 2, 3$ meet the edges of the triangles t_1, t_2, and t_3 needs justification. Consider for example the transversals to ℓ_1, ℓ_2, and ℓ_3. These form a ruling of the doubly-ruled quadric Q_4 and are parameterized by their point of intersection with ℓ_1. The intersection of Q_4 with π_4 is the conic C_4. Since the intersections of the conic C_4 with the segment g_4 supported on m_4 lie between its intersections with ℓ_4 and k_4, the corresponding transversals to ℓ_1, ℓ_2, ℓ_3, and g_4 meet ℓ_1 between points of ℓ_1 met by common transversals to $\ell_4 \cup k_4$ and ℓ_1, ℓ_2, and ℓ_3. The same argument for the other lines and for all 8 conics justifies the assertion.

Naïvely, we would expect that this same construction (the third side cutting all 8 conics in π_i) could work to select each of the remaining sides of the triangles g_3, g_2, and g_1, and that this would give four triangles having

$$32 + 16 + 16 + 16 + 16 \;=\; 96$$

common tangents. Unfortunately this is not the case. In π_4, the conics come in two clusters, depending upon whether or not they correspond to ℓ_3 or to k_3. In order for the edge g_4 to cut all conics, the angle between ℓ_4 and k_4 has to be large, in fact significantly larger than the angle between ℓ_3 and k_3. Thus in π_3, the conics corresponding to ℓ_4 are quite far from the conics corresponding to k_4, and the side g_3 can only be drawn to cut four of the conics, giving 8 additional common tangents. Similarly, g_2 can only cut two conics, and g_1 only 1. In this way, we arrive at four triangles having

$$32 + 16 + 8 + 4 + 2 \;=\; 62$$

common tangents, which we can verify by computer.

3. Upper bound for triangles in T

Four triangles in T have at most 162 common tangents. If the triangles are disjoint, we slightly improve this upper bound to 156. Our method will be to show that not all $81 = 3^4$ quadruples of edges can give rise to a common tangent. Our proof follows that for the upper bound for the number of tangents to four polytopes [1], limiting the number of configurations for disjoint triangles in $\mathbb{R}^3$. We divide the proof into two lemmas, which do not assume that the triangles lie in T.

In order for a tangent to meet an edge e, the plane it spans with e must meet one edge from each of the other triangles. A triple of edges, one from each of the other triangles, is *contributing* if there is a plane containing e which meets the three edges. We say that an edge e *stabs* a triangle t if its supporting line meets the interior of t.

Lemma 6. *Let e be an edge of some triangle. If e stabs exactly one of the other triangles, then there are at most 26 contributing triples of edges. If e stabs no other triangle, then there are at most 25 contributing triples.*

It is not hard to see that if e stabs at least two of the other triangles, then each of the $27 = 3^3$ triples of edges can be contributing.

Proof. Suppose that e is an edge of some triangle. Let $\pi(\alpha)$ be the pencil of planes containing e. (This is parametrized by the angle α.) For each edge f of another triangle t, there is an interval of angles α for which $\pi(\alpha)$ meets f. Figure 5 illustrates the two possible configurations for these intervals, which depend upon whether or not e stabs the triangle t. The intervals are labeled 1, 2, and 3 for the three edges of t. When e stabs t,

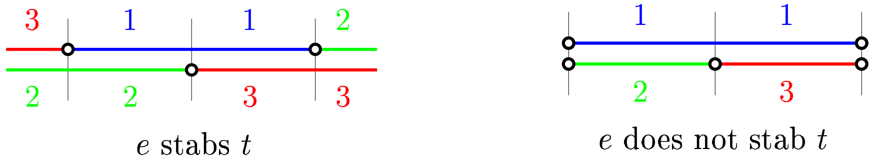

FIGURE 5. Stabbing and non-stabbing configurations

these intervals cover the entire range of α and the picture is actually wrapped. Call this a *stabbing diagram*. When the supporting line of e does not meet t, these intervals do not cover the entire range of α, and there are two endpoints and one *interior vertex* of the diagram. If the supporting line of e meets an edge of t, then the two endpoints of the non-stabbing diagram wrap around and coincide. Call either of these last two configurations a non-stabbing diagram.

To count contributing triples, we line up (overlay) diagrams from each of the three triangles not containing e and count how many of the 27 triples $\{1,2,3\}^3$, one from each triangle, occur at some value of α. For example, Figure 6 displays a configuration with 26 contributing triples (where e stabs a single triangle) and a configuration with 25 contributing triples (e stabs no other triangles). The configuration on the left is missing the triple

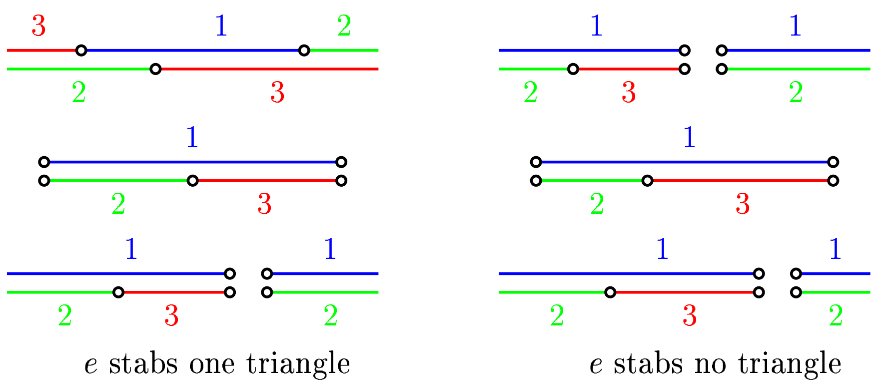

FIGURE 6. Configurations with 26 and 25 contributing triples

$(2,3,3)$, while the configuration on the right is missing the triples $(2,2,3)$ and $(3,3,2)$.

These configurations are the best possible. Indeed, begin with two non-stabbing diagrams in which all 9 pairs of edges occur. (If only 8 pairs occurred, there would be at most 24 contributing triples.) The unique way to do this up to relabeling the edges is given by the lower two diagrams in either picture in Figure 6. These two diagrams divide the domain of α into 6 intervals (the two at the ends are wrapped). The five pairs involving 1 occur in two intervals, but four exceptional pairs $\{(2,2),(2,3),(3,2),(3,3)\}$ occur uniquely in different intervals.

Consider now a third diagram. An exceptional pair extends to three contributing triples only if all three sides in the third diagram meet the interval corresponding to that pair. If the third diagram is stabbing, then one of its three vertices lies in that interval—thus there is at least one triple which does not contribute. If the third diagram is non-stabbing, then either the middle vertex or else both endpoints must lie in that interval—thus there are at least two triples which do not contribute.

Lemma 7. *At most 78 quadruples of edges of four disjoint triangles can lead to a common tangent.*

Proof. First consider the maximum number of stabbing edges between two triangles. If the triangles are disjoint, then there are at most three stabbing edges; one triangle could have three edges stabbing the other. Indeed, if at least two supporting lines of a triangle t meet another triangle t' which is disjoint from t, then t lies entirely on one side of the plane supporting t', and thus no supporting lines of t' can meet t. Figure 7 (a) shows a configuration in which all three supporting lines of t stab t'.

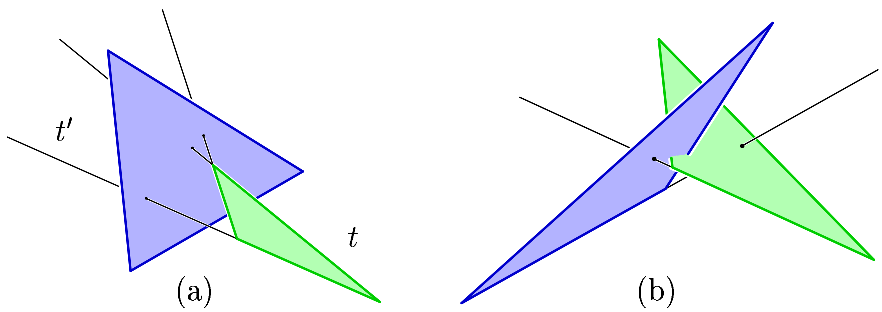

FIGURE 7. (a) Two disjoint triangles can have at most 3 stabbing lines. (b) Two intersecting triangles may have up to four.

Consider now the bipartite graph between 12 nodes representing the edges of the four triangles and 4 nodes representing the triangles. This graph has an arc between an edge e and a triangle t if the line supporting e stabs t. (We assume that e is not an edge of t.) We just showed that the edges of one triangle t can have at most 3 arcs incident on another triangle t', and so this graph has at most 18 edges.

Let the weight of a triangle be the number of arcs emanating from its edges in this graph. As the graph has at most 18 arcs, at least one triangle has weight less than 5. We argue that there is a triangle of weight at most 3. This is immediate if the graph has 15 or fewer edges. On the other hand, this graph has more structure. If it has 18 edges, then all pairs of triangles are in the configuration of Figure 7(a), and so every triangle has weight a multiple of 3, which implies that some triangle has weight at most 3. If the graph has 17 edges, then there is exactly one pair of triangles with only two stabbing edges, and so the possible weights less than 5 are 0, 2, and 3. If the graph has 16 edges, then there is one pair with only one edge stabbing, or two pairs with 2 edges stabbing. There can be at most 2 triangles of weight 4, and again we conclude that there is triangle with weight at most 3.

If a triangle has weight at most three, either all three edges stab a unique triangle, or else one edge stabs no triangles and another edge stabs at most one other triangle. We sum the number of contributing triples over the edges of this triangle. By Lemma 6, this sum will be at most 26+26+26=78 if all three edges stab a unique triangle and at most 27+26+25=78 if not. This proves the lemma.

Remark 8. There exist four disjoint triangles whose bipartite graph has exactly 18 edges. Thus the previous argument cannot be improved without additional ideas. It is conceivable that further restrictions the bipartite graph may exist, leading to a smaller upper bound.

Remark 9. This proof does not enable us to improve the bound when the triangles are not disjoint. Two intersecting triangles can induce up to four arcs (see Figure 7(b)) and thus the total number of arcs is bounded above by 24. The minimal weight of a triangle is then 6, and the edges of such a triangle could all have degree 2, which leads to no restrictions.

4. Upper bound on the number of components

Let A be the number of quadruples of edges of the triangles whose supporting lines have infinitely many common transversals. Elementary geometry, or the arguments of [3], show that the set of common transversals to each quadruple of edges either is finite, consisting of 0, 1, or 2 lines, or it is infinite with at most four connected components. This gives the naïve bound of

$$4A + 2(81 - A) \;=\; 162 + 2A$$

for the number of connected components of common tangents to four triangles.

This may overcount the number of connected components, as any connected component containing a line which meets a vertex of one of the triangles is counted at least twice. Indeed, if a connected component of transversals to one quadruple of edges contains a line meeting a vertex, then that line is transversal to a different quadruple of edges involving the second edge incident to that vertex. We obtain an improved bound of

$$4A/2 + 2(81 - A) \;=\; 162\,.$$

This still may overcount the number of connected components of tangents, but further analysis is very delicate. Such complicated arguments are not warranted as we have already obtained the upper bound of 162 common tangents to four triangles in T. As in Section 3, if the triangles are disjoint, then not all quadruples of edges can contribute, which lowers this bound to 156.

5. Random triangles

We proved Theorem 2 by exhibiting four triangles having 62 common tangents. We do not know if that is the best possible. Since the geometric problem of determining the tangents to four triangles is computationally feasible—it is the disjunction of 81 problems with algebraic degree 2 and simple inequalities on the solutions—we investigated it experimentally.

For this, we generated 5 000 000 quadruples of triangles whose vertices were points with integral coordinates chosen uniformly at random from the cube $[-1000, 1000]^3$. For each, we computed the number of tangents. The resulting frequencies are recorded in Table 2. This search consumed 17 million seconds of CPU time on 1.2GHz processors at the MSRI and a DEC Alpha machine at the University of Massachusetts in 2004. It is archived on the web page[1] accompanying this article.

In this search, we found four different quadruples of triangles with 40 common tangents, and none with more. The vertices of one are given in Table 3. These triangles are rather 'fat', in that none have very small angles. Contrast that to the triangles of our construction in Section 3. In Figure 8 we compare these two configurations of triangles. On the left is the configuration of triangles from Figure 3, together with their 40 common tangents, while on the right is the configuration of triangles having 62 common tangents. The triangles are labeled in the second diagram, as they are hard to distinguish from the lines. As

[1]www.math.tamu.edu~sottile/stories/4triangles/index.html

Number	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
Frequency	1 515 706	331 443	64,150	403 679	637 202	327 159	358 312	238 913	253 396	114 046	80 199	44 870	27 726	12 426	5 796	2 016	813	111	30	3	4

TABLE 2. Number of triangles with a given number of tangents, out of 5 000 000 randoly constructed triangles

Triangle	Vertices		
t_1	$(-4, -731, -336)$	$(297, -507, 978)$	$(824, -62, -359)$
t_2	$(531, -631, -820)$	$(-24, -716, 713)$	$(807, 377, 177)$
t_3	$(586, -205, 952)$	$(861, -774, 235)$	$(-450, 758, 161)$
t_4	$(330, -141, -908)$	$(942, -920, 651)$	$(-226, 489, 968)$

TABLE 3. Four triangles with 40 common tangents

we remarked in Section 3, many of the lines are extremely close and cannot be easily distinguished; that is why one can only count 8 lines in this picture.

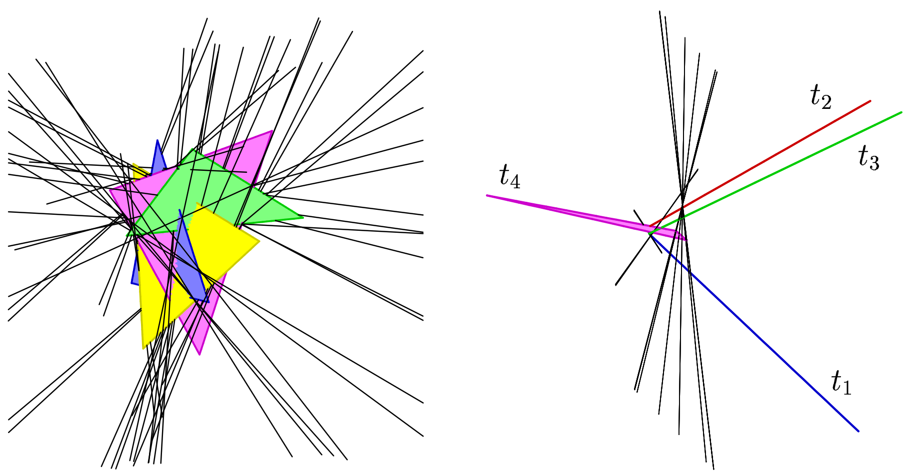

FIGURE 8. Triangles with many common tangents

ACKNOWLEDGMENTS

This research was initiated at the Second McGill-INRIA Workshop on Computational Geometry in Computer Graphics, February 7–14, 2003, co-organized by H. Everett, S. Lazard, and S. Whitesides, and held at the Bellairs Research Institute of McGill University. We would like to thank the other participants of the workshop for useful discussions.

References

1. H. Brönnimann, O. Devillers, V. Dujmovic, H. Everett, M. Glisse, X. Goaoc, S. Lazard, H.-S. Na, and S. Whitesides. On the number of lines tangent to four convex polyhedra. *Proc. Fourteenth Canad. Conf. Comput. Geom.*, pp. 113–117, 2002.
2. H. Brönnimann, O. Devillers, V. Dujmovic, H. Everett, M. Glisse, X. Goaoc, S. Lazard, H.-S. Na, and S. Whitesides. On the number of lines tangent to arbitrary polytopes in $\mathbb{R}^3$. *Proc. 20th Annu. ACM Symp. Comput. Geom.*, pp. 46–55, 2004.
3. H. Brönnimann, H. Everett, S. Lazard, F. Sottile, and S. Whitesides. The number of transversals to line segments in $\mathbb{R}^3$. *Proc. Fifteenth Canad. Conf. Comput. Geom.*, pp. 174–177, 2003. Preprint available: `math.MG/0306401`. Accepted for publication in *Discrete & Comput. Geom.*
4. F. Durand, G. Drettakis, and C. Puech. The 3D visibility complex. *ACM Trans. Graphics* 21(2):176–206, 2002.
5. H. Edelsbrunner. *Algorithms in Combinatorial Geometry.* Springer-Verlag, Heidelberg, Germany, 1987.
6. J. E. Goodman, R. Pollack, and R. Wenger. Geometric transversal theory. In *New Trends in Discrete and Computational Geometry*, (J. Pach, ed.), Springer Verlag, Heidelberg, pp. 163-198, 1993.
7. H. Pottman and J. Wallner. *Computational Line Geometry.* Springer-Verlag, Berlin, 2001.
8. R. Wenger. Progress in geometric transversal theory. In *Advances in Discrete and Computational Geometry*, (B. Chazelle, J. E. Goodman, and R. Pollack, eds.), Amer. Math. Soc., Providence, pp. 375–393, 1998.

CIS Dept, Polytechnic University, Six Metrotech, Brooklyn NY 11201, USA
E-mail address: hbr@poly.edu

INRIA, BP 93, 06902 Sophia-Antipolis, France
E-mail address: olivier.devillers@sophia.inria.fr

LORIA (INRIA, Université Nancy 2), 54506 Vandœuvre-lès-Nancy Cedex, France
E-mail address: sylvain.lazard@loria.fr

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA
E-mail address: sottile@math.tamu.edu