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NEW FEWNOMIAL UPPER BOUNDS FROM GALE DUAL POLYNOMIAL SYSTEMS

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We dedicate this paper to Askold Khovanskii on the occasion of his 60th birthday.

Abstract. We show that there are fewer than $\frac{e^2+3}{4}2^{\binom{k}{2}}n^k$ positive solutions to a fewnomial system consisting of n polynomials in n variables having a total of $n+k+1$ distinct monomials. This is significantly smaller than Khovanskii's fewnomial bound of $2^{\binom{n+k}{2}}(n+1)^{n+k}$. We reduce the original system to a system of k equations in k variables which depends upon the vector configuration Gale dual to the exponents of the monomials in the original system. We then bound the number of solutions to this Gale system. We adapt these methods to show that a hypersurface in the positive orthant of $\mathbb{R}^n$ defined by a polynomial with $n+k+1$ monomials has at most $C(k)n^{k-1}$ compact connected components. Our results hold for polynomials with real exponents.

Introduction

The zeroes of a (Laurent) polynomial f in n variables lie in the complex torus $(\mathbb{C}^\times)^n$ and the support of f is the set $\mathcal{W} \subset \mathbb{Z}^n$ of exponent vectors of its monomials. Kouchnirenko [11] showed that the number of non-degenerate solutions to a system of n polynomials with support $\mathcal{W}$ is at most the volume of the convex hull of $\mathcal{W}$, suitably normalized. This bound is sharp for every $\mathcal{W}$.

It is also a bound for the number of real solutions to real polynomial equations, but it is not sharp. In 1980, Khovanskii [9] gave a bound for the number of non-degenerate solutions in the positive orthant $\mathbb{R}^n_>$ to a system of polynomials which depends only on the cardinality $|\mathcal{W}|$ of $\mathcal{W}$. Multiplying this bound by the number 2^n of orthants in $(\mathbb{R}^\times)^n$ gives a bound for the total number of real solutions which is smaller than the Kouchnirenko bound when $|\mathcal{W}|$ is small relative to the volume of the convex hull of $\mathcal{W}$.

Khovanskii's fewnomial bound is quite large. If $|\mathcal{W}| = n+k+1$, then it is

$$2^{\binom{n+k}{2}}(n+1)^{n+k}\,.$$

When $n = k = 2$, this becomes $2^6 \cdot 3^4 = 5184$. The first concrete result showing that Khovanskii's bound is likely overstated was due to Li, Rojas, and Wang [12] who showed that two trinomials in two variables have at most 5 positive solutions. Since we may multiply the polynomials by monomials without changing their positive solutions, we can assume that they both have a constant term. Thus there are at most 5 monomials between them and so this is a fewnomial system with $n = k = 2$. While 5 is considerably smaller than 5184, two trinomials do not constitute a general fewnomial system with $n = k = 2$, and the problem of finding a bound tighter than 5184 in this case remained open.

Our main result is a new fewnomial upper bound, which is a special case of Theorem 3.2.

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Theorem. *A system of n polynomials in n variables having a total of $n+k+1$ distinct monomials has fewer than*

$$\frac{e^2+3}{4}\, 2^{\binom{k}{2}} n^k \tag{$*$}$$

non-degenerate solutions in the positive orthant.

When $n = k = 2$, this bound is 20. Like Khovanskii's fewnomial bound, the bound $(*)$ is for polynomials whose exponents in $\mathcal{W}$ may be real vectors. We will use real exponents to simplify our proof in a key way.

While this bound $(*)$ is significantly smaller than Khovanskii's bound, they both have the same quadratic factor of k in the exponent of 2. Despite that similarity, our method of proof is different from Khovanskii's induction. A construction based on [3] gives a lower bound of $(1+n/k)^k$ real solutions. Thus the dependence, n^k, on n in the bound $(*)$ cannot be improved. We believe that the term $2^{\binom{k}{2}}$ is considerably overstated. Consequently, we do not consider it important to obtain a smaller constant than $\frac{e^2+3}{4}$, even though there is room in our proof to improve this constant.

This constant comes from estimates for a smaller bound that we prove using polyhedral combinatorics and some topology of toric varieties. When $k = 2$, this smaller bound becomes $2n^2 + \lfloor\frac{(n+3)(n+1)}{2}\rfloor$, and when $n = 2$ it is 15. While 15 is smaller than 5184, we believe that the true bound is still smaller.

When k=1, Bihan [3] gave the tight bound of n+1. This improved an earlier bound of n^2+n by Li, Rojas, and Wang [12]. Bihan's work built upon previous work of Bertrand, Bihan, and Sottile [2] who gave a bound of $2n$+1 for all solutions in $(\mathbb{R}^\times)^n$ for Laurent polynomials with primitive support. For these results, a general polynomial system supported on $\mathcal{W}$ is reduced to a peculiar univariate polynomial, which was analyzed to obtain the bounds.

Our first step is to generalize that reduction. A system of n polynomials in n variables having a total of n+k+1 distinct monomials has an associated Gale system, which is a system of k equations in variables $y = (y_1, \ldots, y_k)$ having the form

$$\prod_{i=1}^{n+k} p_i(y)^{a_{i,j}} \;=\; 1 \qquad \text{for} \quad j = 1, 2, \ldots, k\,,$$

where each polynomial $p_i(y)$ is linear in $y \in \mathbb{R}^k_>$ and the exponents $a_{i,j} \in \mathbb{R}$ come from a vector configuration Gale dual to $\mathcal{W}$. Our bounds are obtained by applying the Khovanskii-Rolle Theorem [10, pp. 42–51] (following some ideas of [8]) and some combinatorics of the polyhedron defined by $p_i(y) > 0$, for $i = 1, \ldots, n$+k.

We also use Gale systems to show that the number of compact connected components of a hypersurface in $\mathbb{R}^n_>$ given by a polynomial with n+k+1 monomials is at most

$$\Bigl(\tfrac{k}{2}2^{\binom{k}{2}} \;+\; \tfrac{e^2+1}{8}k2^{\binom{k-1}{2}}\Bigr)\, n^{k-1} \;+\; \Bigl(\tfrac{e^2}{8}2^{\binom{k-2}{2}}\Bigr)\, n^{k-2}\,,$$

when $k \leq n$. When $k = 2$, we improve this to $\lfloor\frac{5n+1}{2}\rfloor$. These results improve earlier bounds. Khovanskii [9, §3.14] gives a bound having the order $2^{\binom{n+k}{2}}(2n^3)^{n-1}$ for the total Betti number of such a hypersurface, and Perrucci [15] gave the bound $2^{\binom{n+k}{2}}(n{+}1)^{n+k}$ for the number of all components of such a hypersurface.

We first recall some basics on polynomial systems in Section 1. Section 2 shows how to reduce a polynomial system to its Gale dual system. In Section 3, we establish bounds for the number of positive solutions to a system with $n+k+1$ monomials. Good bounds are known when $n=1$ [6] or $k=1$ [3], so we restrict attention to when $k,n\geq 2$. In Section 4 we give bounds on the number of compact components of a hypersurface in $\mathbb{R}^n_>$ defined by a polynomial with $n+k+1$ monomials.

1. Polynomial systems and vector configurations

Fix positive integers n and k. Let $\mathcal{W}=\{w_0,w_1,\ldots,w_{n+k}\}\subset\mathbb{Z}^n$ be a collection of integer vectors, which are exponents for monomials in $z\in(\mathbb{C}^\times)^n$. A linear combination of monomials with exponents from $\mathcal{W}$,

$$f(z)\;=\;c_0z^{w_0}\;+\;c_1z^{w_1}\;+\;c_2z^{w_2}\;+\;\cdots\;+\;c_{n+k}z^{w_{n+k}}\,, \tag{1.1}$$

is a (Laurent) polynomial with *support $\mathcal{W}$*. We will always assume that the coefficients c_i are real numbers. A *system with support $\mathcal{W}$* is a system

$$f_1(z)\;=\;f_2(z)\;=\;\cdots\;=\;f_n(z)\;=\;0\,, \tag{1.2}$$

of polynomials, each with support $\mathcal{W}$. Solutions to (1.2) are points in $(\mathbb{C}^\times)^n$. Multiplying each polynomial $f_i(z)$ by the monomial z^{-w_0} does not change the solutions to the system (1.2), but it translates $\mathcal{W}$ by $-w_0$ so that w_0 becomes the origin. We shall henceforth assume that $0\in\mathcal{W}$ and write $\mathcal{W}=\{0,w_1,\ldots,w_{n+k}\}$.

Kouchnirenko [11] gave a bound for the number of solutions to a polynomial system (1.2). Let $\mathrm{vol}(\mathcal{W})$ be the Euclidean volume of the convex hull of $\mathcal{W}$.

Kouchnirenko's Theorem. *The number of isolated solutions in $(\mathbb{C}^\times)^n$ to a system (1.2) with support $\mathcal{W}$ is at most $n!\mathrm{vol}(\mathcal{W})$. This bound is attained for generic systems with support $\mathcal{W}$.*

Here, generic means that there is a non-empty Zariski open subset consisting of systems where the maximum is acheived. Every solution z to a generic system is *non-degenerate* in that the differentials $df_1,\ldots,df_n$ are linearly independent at z.

For $w\in\mathbb{R}^n$, $z\mapsto z^w$ is a well-defined analytic function on the positive orthant $\mathbb{R}^n_>$. A *polynomial (with real exponents)* is a linear combination of such functions and its *support $\mathcal{W}\subset\mathbb{R}^n$* is the set of exponent vectors. While Kouchnirenko's Theorem does not apply to systems of polynomials with real exponents, Khovanskii's bound does. Our bounds are likewise for polynomials with real exponents.

The usual theory of discriminants for ordinary polynomial systems holds in this setting (see [14, §2] for more details). In the space $\mathbb{R}^{2n(n+k+1)}$ of parameters (coefficients and exponents) for systems of polynomials with real exponents having $n+k+1$ monomials there is a discriminant hypersurface Σ which is defined by the vanishing of an explicit analytic function, and has the following property: In every connected component of the complement of Σ the number of positive solutions is constant, and all solutions are non-degenerate. Perturbing the coefficients and exponents of such a system does not change the number of non-degenerate positive solutions, so we may assume that the exponents are rational numbers. After a monomial change of coordinates, we may assume that the exponents are integers. Thus the maximum number of non-degenerate positive solutions to a system of n polynomials in n variables having a total of $n+k+1$ monomials with real exponents is achieved by Laurent polynomials.

We sometimes write $V(f)$ for the set of zeroes of a function f on a given domain Δ, and $V(f_1,\ldots,f_j)$ for the common zeroes of functions $f_1,\ldots,f_j$ on Δ.

2. Gale systems

Given a system of n polynomials in n variables with common support $\mathcal{W}$ where $|\mathcal{W}| = n{+}k{+}1$, we use the linear relations on $\mathcal{W}$ (its Gale dual configuration) to obtain a special system of k equations in k variables, called a Gale system. Here, we treat the case when $\mathcal{W}$ is a configuration of real vectors, and prove an equivalence between positive solutions of the original system and solutions of the associated Gale system. There is another version for non-zero complex solutions which holds when $\mathcal{W}$ is a configuration of integer vectors. We will present that elsewhere [5].

Suppose that $\mathcal{W} = \{0, w_1, w_2, \ldots, w_{n+k}\}$ spans $\mathbb{R}^n$, for otherwise no system with support $\mathcal{W}$ has any non-degenerate solutions. We assume that the last n vectors are linearly independent. We consider a system with support $\mathcal{W}$ such that the coefficient matrix of $z^{w_1},\ldots,z^{w_n}$ is invertible, and then put it into diagonal form

$$z^{w_i} \;=\; p_i(z^{w_{n+1}}, z^{w_{n+2}}, \ldots, z^{w_{n+k}}) \;=:\; g_i(z) \qquad \text{for} \quad i = 1,\ldots,n\,, \tag{2.1}$$

where each p_i is linear, so that $g_i(z)$ is a polynomial with support in $\{0, w_{n+1}, \ldots, w_{n+k}\}$.

A vector $a \in \mathbb{R}^{n+k}$ with

$$0 \;=\; w_1a_1 + w_2a_2 + \cdots + w_{n+k}a_{n+k}$$

gives a monomial identity

$$1 \;=\; (z^{w_1})^{a_1}(z^{w_2})^{a_2}\cdots(z^{w_{n+k}})^{a_{n+k}}\,.$$

If we substitute the polynomials $g_i(z)$ of (2.1) into this identity, we obtain a consequence of the system (2.1) ,

$$1 \;=\; (g_1(z))^{a_1}\cdots(g_n(z))^{a_n}\cdot(z^{w_{n+1}})^{a_{n+1}}\cdots(z^{w_{n+k}})^{a_{n+k}}\,. \tag{2.2}$$

Let $A = (a_{i,j}) \in \mathbb{R}^{(n+k)\times k}$ be a matrix whose columns form a basis for the linear relations among the elements of $\mathcal{W}$. Under the substitution

$$y_j \;=\; z^{w_{n+j}} \qquad \text{for} \quad j = 1,\ldots,k\,, \tag{2.3}$$

the polynomials $g_i(z)$ become linear functions $p_i(y)$. Set $p_{n+i}(y) := y_i$ for $i = 1,\ldots,k$. Then the k equations of the form (2.2) given by the columns of A

$$1 \;=\; \prod_{i=1}^{n+k} p_i(y)^{a_{i,j}} \qquad \text{for} \quad j = 1,\ldots,k\,, \tag{2.4}$$

constitute a *Gale system* associated to the diagonal system (2.1). This system only makes sense for variables y in the polyhedron

$$\Delta \;:=\; \{y \mid p_i(y) > 0 \text{ for } i = 1,\ldots,n{+}k\} \;\subset \mathbb{R}^k_>\,.$$

Remark 2.1. While the system (2.4) appears to depend upon a choice of basis (the columns of A) for the linear relations among $\mathcal{W}$, in fact it does not as the invertible linear transformation between two bases gives an invertible multiplicative transformation between their corresponding Gale systems.

Theorem 2.2. *Set* $\mathcal{V} := \{w_{n+1}, \ldots, w_{n+k}\}$. *Then the association*

$$\varphi_{\mathcal{V}} \,:\, \mathbb{R}^n_> \ \ni\ z \ \longmapsto\ (z^{w_{n+1}}, z^{w_{n+2}}, \ldots, z^{w_{n+k}}) \ =:\ y \ \in\ \mathbb{R}^k_>$$

is a bijection between solutions $z \in \mathbb{R}^n_>$ *to the diagonal system* (2.1) *and solutions* $y \in \Delta$ *to the Gale system* (2.4) *which restricts to a bijection between their non-degenerate solutions.*

Proof. Since the vectors $\{w_{k+1}, \ldots, w_{n+k}\}$ are linearly independent, we may change coordinates and assume that for $1 \leq i \leq n$ we have $w_{k+i} = e_i$, the ith standard basis vector. The linear relations among the vectors in $\mathcal{W}$ then have a basis of the form

$$w_j \ =\ w_{j,1}e_1 + w_{j,2}e_2 + \cdots + w_{j,n}e_n \qquad \text{for} \quad j = 1, 2, \ldots, k\,. \tag{2.5}$$

Suppose first that $k > n$. Then the original system is

$$z^{w_j} \ =\ p_j(z^{w_{n+1}}, z^{w_{n+2}}, \ldots, z^{w_k},\ z_1, \ldots, z_n) \qquad \text{for} \quad j = 1, 2, \ldots, n\,. \tag{2.6}$$

Using the linear relations (2.5), the Gale system becomes

$$1 \ =\ p_j(y)^{-1} \prod_{i=1}^{n} y_{k-n+i}^{w_{j,i}} \qquad \text{for} \quad j = 1, 2, \ldots, n\,, \qquad \text{and} \tag{2.7}$$

$$1 \ =\ y_{j-n}^{-1} \prod_{i=1}^{n} y_{k-n+i}^{w_{j,i}} \qquad \text{for} \quad j = n{+}1, n{+}2, \ldots, k\,. \tag{2.8}$$

The image $\varphi_{\mathcal{V}}(\mathbb{R}^n_>) \subset \mathbb{R}^k_>$ is a graph defined by the subsystem (2.8). Given this, the subsystem (2.7) is just a restatement of the original system (2.6). This proves the theorem in the case $k > n$ as this equivalence of systems respects the multiplicity of solutions.

Suppose now that $k \leq n$. Then $y_i = z_{n+i}$ for $i = 1, \ldots, k$, and (2.6) becomes

$$z^{w_j} \ =\ p_j(z_{n-k+1}, z_{n-k+2}, \ldots, z_n) \qquad \text{for} \quad j = 1, 2, \ldots, k\,, \tag{2.9}$$

$$z_j \ =\ p_{j+k}(z_{n-k+1}, z_{n-k+2}, \ldots, z_n) \qquad \text{for} \quad j = 1, 2, \ldots, n{-}k\,. \tag{2.10}$$

Using the linear relations (2.5), the Gale system is

$$1 \ =\ p_j(y)^{-1} \cdot \prod_{i=1}^{n-k} p_{j+k}(y)^{w_{j,i}} \cdot \prod_{i=n-k+1}^{n} y_{k-n+i}^{w_{j,i}} \qquad \text{for} \quad j = 1, 2, \ldots, k\,. \tag{2.11}$$

Conversely, the Gale system (2.11) together with the subsystem (2.10) implies the subsystem (2.9). This proves the theorem when $k \leq n$. $\square$

3. Upper bounds for Gale systems

Let $n, k \geq 2$ be positive integers and $y_1, \ldots, y_k$ be indeterminates. Let $B = (b_{i,j})$ be a $(n{+}k) \times (k{+}1)$ matrix of real numbers whose columns are indexed by $0, 1, \ldots, k$. Define linear polynomials $p_i(y)$ by

$$p_i(y) \ :=\ b_{i,0} + b_{i,1}y_1 + \cdots + b_{i,k}y_k \qquad \text{for} \quad i = 1, 2, \ldots, n{+}k\,,$$

and define the (possibly unbounded) polyhedron

$$\Delta \ :=\ \{y \in \mathbb{R}^k \mid p_i(y) > 0 \text{ for } i = 1, 2, \ldots, n{+}k\}\,.$$

Suppose that $A = (a_{i,j})$ is a $(n+k) \times k$ matrix of real numbers, none of whose row vectors is zero. We consider the system of equations on Δ of the form

$$(3.1) \qquad 1 \;=\; f_j(y) \;:=\; \prod_{i=1}^{n+k} p_i(y)^{a_{i,j}}\,, \qquad \text{for} \quad j = 1, 2, \ldots, k\,.$$

Theorem 3.1. *The system* (3.1) *has fewer than* $\frac{e^2+3}{4}\, 2^{\binom{k}{2}} n^k$ *non-degenerate solutions in* Δ.

Given a set $\mathcal{W} = \{0, w_1, \ldots, w_{n+k}\} \subset \mathbb{R}^n$ wich spans $\mathbb{R}^n$, let N be the number of non-zero rows in a $(n+k) \times k$ matrix of linear relations among $\{w_1, \ldots, w_{n+k}\}$ and set $n(\mathcal{W}) := N - k$. Then $n(\mathcal{W}) \leq n$.

Theorem 3.2. *A polynomial system supported on* $\mathcal{W} = \{0, w_1, \ldots, w_{n+k}\} \subset \mathbb{R}^n$ *has fewer than*

$$\tfrac{e^2+3}{4}\, 2^{\binom{k}{2}} n(\mathcal{W})^k$$

non-degenerate positive solutions.

Proof of Theorem 3.2. Suppose that the vectors $\mathcal{W}$ are ordered so that $w_{k+1}, \ldots, w_{n+k}$ are linearly independent. Given a system supported on $\mathcal{W}$, we perturb its coefficients without decreasing its number of non-degenerate solutions, put it into diagonal form (2.1), and then consider an associated Gale system. By Theorem 2.2, the number of non-degenerate positive solutions to the diagonal system is equal to the number of non-degenerate solutions to the Gale system in the polyhedron Δ. This Gale system has the form (3.1) where the number of factors is $n(\mathcal{W}) + k$, and so the theorem follows from Theorem 3.1. $\square$

We will want to consider systems that are equivalent to (3.1). By Remark 2.1, we obtain an equivalent system from any matrix A' of size $(n+k) \times k$ with the same column span as A. Since we assumed that no row vector of A vanishes, we may thus assume that no entry of the matrix vanishes, and hence no exponent $a_{i,j}$ vanishes.

We may perturb the (real-number) exponents $a_{i,j}$ in (3.1) without decreasing the number of non-degenerate solutions. Doing this, if necessary, we may assume that every square submatrix of A is invertible, and that the exponents are all rational numbers. We will invoke this assumption to insure that solution sets to certain equations are smooth algebraic varieties and have the expected dimension. Similarly, we may perturb the coefficients B of the linear factors $p_i(y)$ so that they are in general position and thus the polyhedron Δ is *simple* in that every face of codimension j lies on exactly j facets.

To prove Theorem 3.1 we use Khovanskii's generalization of Rolle's Theorem to 1-manifolds (see §3.4 in [10]). Let $f_1, \ldots, f_k$ be smooth functions defined on $\Delta \subset \mathbb{R}^k$ which have finitely many common zeroes $V(f_1, \ldots, f_k)$ in Δ, where $V(f_1, \ldots, f_{k-1})$ is a smooth curve C in Δ. Let $\flat(C)$ denote the number of unbounded components of C in Δ, and let

$$\Gamma \;=\; J(f_1, f_2, \ldots, f_k) \;:=\; \det\left(\frac{\partial f_j}{\partial y_l}\right)_{j,l=1,\ldots,k}$$

be the Jacobian of $f_1, \ldots, f_k$. This vanishes when the differentials $df_1(y), \ldots, df_k(y)$ are linearly dependent.

Theorem 3.3 (Khovanskii-Rolle)**.** $|V(f_1, f_2, \ldots, f_k)| \;\leq\; \flat(C) + |V(f_1, f_2, \ldots, f_{k-1}, \Gamma)|$.

The simple idea behind this basic estimate is that along any arc of the curve C connecting two points, say a and b, where C meets the hypersurface $\{y \mid f_k(y) = 0\}$, there must be a point $c \in C$ between a and b where C is tangent to a level set of f_k, so that the normal df_k to this level set is normal to C. Since the normal directions to C are spanned by $df_1, \ldots, df_{k-1}$, this means that Γ vanishes at such a point c.

We can use the ordinary Rolle theorem to prove Theorem 3.3. Suppose that the arc along C from a to b is parametrized by a differentiable function $\varphi\colon [0,1] \to C$ with $\varphi(0) = a$ and $\varphi(1) = b$. By Rolle's Theorem applied to the function $f(t) := f_k(\varphi(t))$, we obtain a point $c \in (a,b)$ such that $f'(c) = 0$, and so Γ vanishes at $\varphi(c)$. Figure 1 illustrates the Khovanskii-Rolle Theorem when $k = 2$.

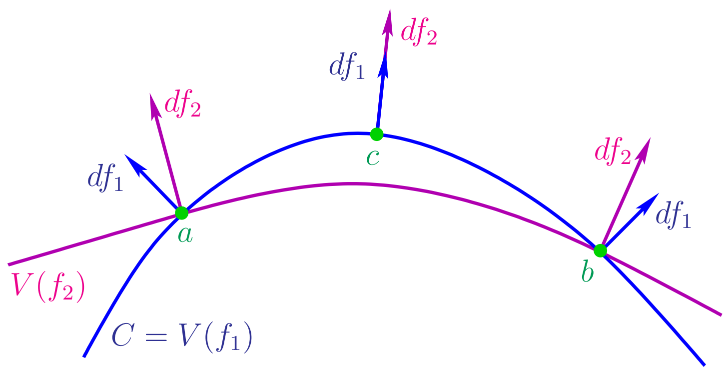

FIGURE 1. $df_1 \wedge df_2(a) > 0$, $df_1 \wedge df_2(c) = 0$, and $df_1 \wedge df_2(b) < 0$.

On Δ, the system (3.1) has an equivalent formulation in terms of logarithmic equations

$$0 \;=\; \psi_j(y) \;:=\; \log\Bigl(\prod_{i=1}^{n+k} p_i(y)^{a_{i,j}}\Bigr) \;=\; \sum_{i=1}^{n+k} a_{i,j}\log(p_i(y)) \qquad \text{for} \quad j = 1, 2, \ldots, k\,. \tag{3.2}$$

Define functions $\Gamma_k(y), \ldots, \Gamma_1(y)$ by the recursion

$$\Gamma_j \;:=\; J(\psi_1, \psi_2, \ldots, \psi_j,\, \Gamma_{j+1}, \Gamma_{j+2}, \ldots, \Gamma_k)\,.$$

The domain of Γ_j is Δ. Then, for each $j = 1, \ldots, k$, define the curve C_j by

$$C_j \;:=\; \{y \in \Delta \mid \psi_1(y) = \cdots = \psi_{j-1}(y) \;=\; \Gamma_{j+1}(y) = \cdots = \Gamma_k(y) \;=\; 0\}\,.$$

(The jth function ψ_j or Γ_j is omitted.) This is a smooth curve by our assumption that the exponents $a_{i,j}$ are sufficiently general. Since we also assumed that the exponents are rational, we shall see that C_j is even an algebraic curve.

If we iterate the Khovanskii-Rolle Theorem, we obtain

$$\begin{aligned} |V(\psi_1, \psi_2, \ldots, \psi_k)| \;&\le\; \flat(C_k) \,+\, |V(\psi_1, \psi_2, \ldots, \psi_{k-1},\, \Gamma_k)| \\ &\le\; \flat(C_k) \,+\, \flat(C_{k-1}) \,+\, |V(\psi_1, \ldots, \psi_{k-2},\, \Gamma_{k-1}, \Gamma_k)| \\ &\le\; \flat(C_k) \,+\, \cdots \,+\, \flat(C_1) \,+\, |V(\Gamma_1, \Gamma_2, \ldots, \Gamma_k)|\,. \end{aligned} \tag{3.3}$$

Lemma 3.4.

(1) $\Gamma_{k-j}(y) \cdot \bigl(\prod_{i=1}^{n+k} p_i(y)\bigr)^{2^j}$ *is a polynomial of degree* $2^j\, n$.

(2) C_j *is a smooth algebraic curve and* $\flat(C_j) \le \frac{1}{2}\, 2^{\binom{k-j}{2}} n^{k-j} \binom{n+k+1}{j}$.

(3) $|V(\Gamma_1, \Gamma_2, \ldots, \Gamma_k)| \le 2^{\binom{k}{2}} n^k$.

Lemma 3.5. *Let k and n be integers with $2 \leq k, n$. Then*

$$\sum_{j=1}^{k} 2^{\binom{k-j}{2}} n^{k-j} \tbinom{n+k+1}{j} \ \leq \ \tfrac{e^2-1}{2} \cdot 2^{\binom{k}{2}}\, n^k\,.$$

Proof of Theorem 3.1. By (3.3), Lemma 3.4 and Lemma 3.5, $|V(\psi_1,\ldots,\psi_k)|$ is at most

$$2^{\binom{k}{2}}\, n^k + \frac{1}{2}\sum_{j=1}^{k} 2^{\binom{k-j}{2}} n^{k-j} \tbinom{n+k+1}{j} \ < \ (1+\tfrac{e^2-1}{4})\, 2^{\binom{k}{2}}\, n^k\,,$$

which proves Theorem 3.1. □

Proof of Lemma 3.5. Set $a_j := 2^{\binom{k-j}{2}} n^{k-j}\binom{n+k+1}{j}$ for $j = 0,1,\ldots,k$, Note that $a_0 = 2^{\binom{k}{2}} n^k$. If $k = n = 2$, then

$$a_1 + a_2 \ = \ 20 \ \leq \ \tfrac{e^2-1}{2}\cdot 8 \ = \ \tfrac{e^2-1}{2}\cdot a_0\,.$$

(We have $3.1 < \frac{e^2-1}{2} < 3.2$.) For other values of (k,n), we will show that

$$a_j \ \leq \ \frac{2^{j-1}}{j!} a_0\,. \tag{3.4}$$

The lemma follows as

$$\begin{aligned}\sum_{j=1}^{k} a_j \quad &\leq \quad \Big(\sum_{j=1}^{k}\frac{2^{j-1}}{j!}\Big)\cdot 2^{\binom{k}{2}} n^k \\ &< \quad \Big(\sum_{j=1}^{\infty}\frac{2^{j-1}}{j!}\Big)\cdot 2^{\binom{k}{2}} n^k \ = \ \tfrac{e^2-1}{2}\cdot 2^{\binom{k}{2}} n^k\ .\end{aligned}$$

To show (3.4), consider the expression

$$E(j,k,n) \ := \ \frac{a_{j-1}}{a_j} \ = \ \frac{j\, n\, 2^{k-j}}{n{+}k{-}j{+}2}\,.$$

If we fix j and n, then this is an increasing function of k as $(n{+}k{-}j{+}2)\cdot \ln 2 > 1$. Similarly, fixing j and k, we obtain an increasing function of n, as $k{-}j{+}2 > 0$.

Note that $E(j,j,2) = \frac{j}{2}$ and also that

$$E(1,2,3) \ = \ 1 \quad \text{and} \quad E(1,3,2) \ = \ \tfrac{4}{3}\,.$$

Thus if $(k,n) \neq (2,2)$, we have $1 \leq E(1,k,n)$ and so $a_1 \leq a_0$ and if $j > 1$, then $\frac{j}{2} = E(j,j,2) \leq E(j,k,n)$ as $j \leq k$ and $2 \leq n$. Thus $a_j \leq \frac{2}{j} a_{j-1}$ for $j > 1$ and $a_1 \leq a_0$. This leads to

$$a_j \ \leq \ \frac{2^{j-1}}{j!} a_1 \ \leq \ \frac{2^{j-1}}{j!} a_0$$

which is (3.4). □

We prove Lemma 3.4(1) and (2) in the following subsections. Note that Lemma 3.4(3) follows from Lemma 3.4(1). Since $\prod_{i=1}^{n+k} p_i(y)$ does not vanish on Δ, if we set

$$F_j \ := \ \Gamma_j \cdot \Big(\prod_{i=1}^{n+k} p_i(y)\Big)^{2^{k-j}}\,, \tag{3.5}$$

then by Lemma 3.4(1) this is a polynomial of degree $2^{k-j}n$ and

$$V(F_1, F_2, \ldots, F_k) \;=\; V(\Gamma_1, \Gamma_2, \ldots, \Gamma_k)\,.$$

But then Bézout's Theorem gives the upper bound

$$\deg(F_1)\deg(F_2)\cdots\deg(F_k) \;=\; 2^{(k-1)+\cdots+2+1+0}n^k \;=\; 2^{\binom{k}{2}}n^k$$

for the number of isolated common zeroes of $F_1, \ldots, F_k$ in $\mathbb{R}^k$, which is a bound for $V(\Gamma_1, \ldots, \Gamma_k)$, the common zeroes of $\Gamma_1, \ldots, \Gamma_k$ in Δ.

3.1. **Proof of Lemma 3.4(1).** We first give a useful lemma, and then determine the form of the functions $\Gamma_i(y)$.

Let $m \geq k$ be integers. Then $\binom{[m]}{k}$ is the collection of subsets of $\{1, \ldots, m\}$ with cardinality k. Given a vector $c = (c_1, \ldots, c_m)$, and $I = \{i_1 < \cdots < i_k\} \in \binom{[m]}{k}$, set $c_I := c_{i_1} \cdots c_{i_k}$. If D is a $m \times k$ matrix and $I \in \binom{[m]}{k}$, let D_I be the determinant of the $k \times k$ submatrix of D formed by the rows indexed by I.

Lemma 3.6. *Let $c = (c_1, \ldots, c_m)$ be a vector of indeterminates and $D = (d_{i,j})$ and $E = (e_{i,j})$ be $m \times k$ matrices of indeterminates. Then*

$$\det\Bigl(\sum_{i=1}^{m} c_i d_{i,j} e_{i,l}\Bigr)_{j,l=1,\ldots,k} \;=\; \sum_{I \in \binom{[m]}{k}} c_I D_I E_I\,.$$

Proof. This follows from the Cauchy-Binet formula. □

We use Lemma 3.6 to determine the form of the functions Γ_j. Since

$$\frac{\partial \psi_j}{\partial y_l} \;=\; \sum_{i=1}^{n+k} \frac{a_{i,j} b_{i,l}}{p_i}\,,$$

Lemma 3.6 implies that

$$\Gamma_k \;=\; \sum_{I \in \binom{[n+k]}{k}} \frac{A_I B_I}{p_I}\,. \tag{3.6}$$

Lemma 3.4(1) is a consequence of the following computation. The set $\binom{[n+k]}{k}^{2^j}$ consists of lists $\mathcal{I}$ of 2^j elements of $\binom{[n+k]}{k}$. For each such list $\mathcal{I} \in \binom{[n+k]}{k}^{2^j}$, set

$$B(\mathcal{I}) \;:=\; \prod_{I \in \mathcal{I}} B_I \qquad \text{and} \qquad p(\mathcal{I}) \;:=\; \prod_{I \in \mathcal{I}} p_I\,.$$

Lemma 3.7. *There exist numbers $A(\mathcal{I})$ for each $\mathcal{I} \in \binom{[n+k]}{k}^{2^j}$, so that*

$$\Gamma_{k-j} \;=\; \sum_{\mathcal{I} \in \binom{[n+k]}{k}^{2^j}} \frac{A(\mathcal{I})B(\mathcal{I})}{p(\mathcal{I})}\,. \tag{3.7}$$

Proof. We prove this by induction on j. The case $j = 0$ is formula (3.6). We have

$$\Gamma_{k-j} \;:=\; J(\psi_1, \ldots, \psi_{k-j}, \Gamma_{k-j+1}, \ldots, \Gamma_k)\,.$$

Since the Jacobian is multilinear, we use the formulas (3.7) for $\Gamma_{k-j+1},\ldots,\Gamma_k$ to obtain

$$\Gamma_{k-j} \;=\; \sum_{\mathcal{I}^{(j-1)}\in\binom{[n+k]}{k}^{2^{j-1}}} \cdots \sum_{\mathcal{I}^{(0)}\in\binom{[n+k]}{k}^{2^0}} \prod_{r=0}^{j-1} A(\mathcal{I}^{(r)})B(\mathcal{I}^{(r)}) \;\cdot\; J(\mathcal{I}^{(j-1)},\ldots,\mathcal{I}^{(0)})\,,$$

where $J(\mathcal{I}^{(j-1)},\ldots,\mathcal{I}^{(0)}) := J\bigl(\psi_1,\ldots,\psi_{k-j},\frac{1}{p(\mathcal{I}^{(j-1)})},\;\ldots,\;\frac{1}{p(\mathcal{I}^{(0)})}\bigr)$, which equals

$$\Bigl(\prod_{r=0}^{j-1}\frac{1}{p(\mathcal{I}^{(r)})}\Bigr)\cdot J\Bigl(\psi_1,\ldots,\psi_{k-j},\,\log\Bigl(\tfrac{1}{p(\mathcal{I}^{(j-1)})}\Bigr),\ldots,\log\Bigl(\tfrac{1}{p(\mathcal{I}^{(0)})}\Bigr)\Bigr)\,.$$

Since

$$\frac{\partial}{\partial y_l}\log\Bigl(\frac{1}{p(\mathcal{I}^{(r)})}\Bigr) \;=\; \sum_{i=1}^{n+k}\frac{a'_{i,r}b_{i,l}}{p_i(y)}\,,$$

where $-a'_{i,l}$ counts the number of sets I in the list $\mathcal{I}^{(l)}$ which contain i, Lemma 3.6 implies that $J(\mathcal{I}^{(j-1)},\ldots,\mathcal{I}^{(0)})$ has the form

$$\sum_{I\in\binom{[n+k]}{k}}\frac{A'_I B_I}{p_I}\,, \tag{3.8}$$

where A' is the matrix whose entry in column r is $a_{i,r}$ if $r\le k-j$ and is $a'_{i,k-r}$ otherwise. If we substitute (3.8) into the previous formulas and write it as a single sum, we obtain an expression of the form (3.7). □

3.2. **Proof of Lemma 3.4(2).** Let $\mu_j\subset\Delta$ be defined by the equations

$$\psi_1(y) \;=\; \psi_2(y) \;=\; \cdots \;=\; \psi_{j-1}(y) \;=\; 1\,.$$

The polyhedron Δ is an open subset of $\mathbb{R}^k$ and we let $\overline{\Delta}$ be its closure in the real projective space $\mathbb{RP}^k$. If Δ is bounded, then $\overline{\Delta}$ is its closure, which is a polytope. If Δ is unbounded, then $\overline{\Delta}$ is combinatorially equivalent to the polytope

$$\Delta\cap\{y\mid v\cdot y\le r\}\,, \tag{3.9}$$

if $v+\Delta\subset\Delta$ and r is large enough so that all the vertices of Δ lie in (3.9).

A *PL-manifold* is a manifold in the category of piecewise-linear spaces. Given a polyhedral complex P, let $M_j(P)$ be the maximum number of j-dimensional faces (j-faces) whose union forms a j-dimensional PL-submanifold of P.

Lemma 3.8. *The set μ_j is a smooth algebraic subset of Δ of dimension $k{+}1{-}j$. The intersection of its closure $\overline{\mu}_j$ with the boundary of $\overline{\Delta}$ is a union of $(k{-}j)$-faces of $\partial\overline{\Delta}$ which forms a PL-submanifold of $\partial\overline{\Delta}$.*

Since C_j is the subset of μ_j cut out by the polynomials $F_{j+1},\ldots,F_k$ (3.5), it is an algebraic subset of Δ and, as we already noted, a smooth curve. Our choice of generic exponents $a_{i,l}$ ensures that each unbounded component of C_j meets the boundary of Δ in two distinct points, and no two components meet the boundary at the same point.

Corollary 3.9. $\flat(C_j)\;\le\;\frac{1}{2}\,2^{\binom{k-j}{2}}n^{k-j}\,M_{k-j}(\overline{\Delta})$.

Corollary 3.9 implies the bound of Lemma 3.4(2). Indeed, if $\Phi_j(\overline{\Delta})$ is the number of j-faces of $\overline{\Delta}$, then

$$M_{k-j}(\overline{\Delta}) \ \leq\ \Phi_{k-j}(\overline{\Delta}) \ \leq\ \tbinom{n+k+1}{j}\,. \tag{3.10}$$

The last inequality follows as each $(k-j)$-face lies on j of the $n+k+1$ facets.

Proof of Corollary 3.9. Let $\overline{C_j}$ be the closure of C_j in $\overline{\Delta}$. First, $\flat(C_j) = \frac{1}{2}|\overline{C}_j \cap \partial\overline{\Delta}|$ as each unbounded component of C_j contributes two points to $\overline{C}_j \cap \partial\overline{\Delta}$. The points in $\overline{C}_j \cap \partial\overline{\Delta}$ are points of $\overline{\mu}_j \cap \partial\overline{\Delta}$ where the polynomials $\overline{F}_{j+1}, \ldots, \overline{F}_k$ vanish. For each $(k-j)$-dimensional face ϕ in $\overline{\mu}_j \cap \partial\overline{\Delta}$, there will be at most

$$2^{\binom{k-j}{2}} n^{k-j} \ =\ \deg(\overline{F}_{j+1}) \cdots \deg(\overline{F}_{k-1}) \cdot \deg(\overline{F}_k)$$

points of $\overline{C}_j \cap \phi$. Indeed, this is the Bézout bound for the number of isolated common zeroes of the polynomials $\overline{F}_{j+1}, \ldots, \overline{F}_k$ on the affine span of the face ϕ. Since these faces ϕ form a PL-submanifold of $\partial\overline{\Delta}$, we deduce the bound of the corollary. $\square$

Proof of Lemma 3.8. Since ψ_i and f_i define the same subset of Δ, μ_j is defined on Δ by

$$f_1(y) \ =\ f_2(y) \ =\ \cdots \ =\ f_{j-1}(y) \ =\ 1\,. \tag{3.11}$$

The exponents $a_{i,l}$ are rational numbers. Let N be their common denominator. As the functions f_i are positive on Δ, we see that $\mu_j \subset \Delta$ is defined by the rational functions

$$f_1^N(y) \ =\ f_2^N(y) \ =\ \cdots \ =\ f_{j-1}^N(y) \ =\ 1\,,$$

and is thus algebraic. Since the exponents are also general, we conclude that μ_j is smooth and has the expected dimension $k-j+1$.

Our arguments to prove the remaining statements in Lemma 3.8 are local, and in particular, we will work near points on the boundary of $\overline{\Delta} \subset \mathbb{RP}^k$. For this, we may need to homogenize the functions f_i and F_i. We first homogenize the linear polynomials $p_i(y)$ with respect to a new variable y_0,

$$\overline{p}_i(y) \ :=\ b_{i,0}y_0 + b_{i,1}y_1 + \cdots + b_{i,k}y_k\,.$$

Let $\overline{F}_j$ be the homogeneous version of F_j, where we replace p_i by $\overline{p}_i$. For the functions f_j, we first replace p_i by $\overline{p}_i$ and then multiply by an appropriate power of y_0,

$$\overline{f}_j \ := y_0^{-\sum_i a_{i,j}} \prod_{i=1}^{n+k} \overline{p}_i(y)^{a_{i,j}}\,.$$

We show that the intersection of μ_j with the boundary of $\overline{\Delta}$ is a union of faces of $\overline{\Delta}$ of dimension $k-j$. For this, select a face ϕ of $\overline{\Delta}$ of dimension $k-j$. Changing coordinates in $\mathbb{RP}^k$ if necessary, we may assume that the face lies along the coordinate plane defined by $y_1 = \cdots = y_j = 0$. We also reindex the forms p_i so that $p_i = y_i$ for $i = 1, \ldots, k$.

Since every square submatrix of the matrix A is invertible, we may suppose that its first $j-1$ rows and columns form the indentity matrix—without changing the column span of its first $j-1$ columns. By Remark 2.1, if we multiply every entry in A by -1, we may assume that the equations defining μ_j have the form

$$y_i \ =\ y_j^{a_{j,i}} \cdot \prod_{l=j+1}^{n+k} p_l(y)^{a_{l,i}}\,, \qquad \text{for} \quad i = 1, \ldots, j-1\,, \tag{3.12}$$

and no exponent $a_{j,i}$ vanishes.

We first observe that the face ϕ lies in the closure $\overline{\mu}_j$ of μ_j if and only if the exponents $a_{j,i}$ for $i=1,\ldots,j{-}1$ are all positive. Moreover, $\overline{\mu}_j$ cannot contain any point in the relative interior of a face which properly contains ϕ, as on that face we still have $p_l(y)>0$ for $l>j$, at least one of the variables $y_1,\ldots,y_j$ will vanish, and at least one is non-vanishing. But if one of the variables $y_1,\ldots,y_j$ vanishes, then the equations (3.12) imply that they all do, which is a contradiction.

Now consider the variety μ_j in the neighborhood of a vertex of the face ϕ. Applying a further affine linear change to our variables y, we may assume this vertex lies at the origin of $\mathbb{R}^k$. Then the equations defining μ_j have the form

$$y_i \;=\; y_j^{a_{j,i}} y_{j+1}^{a_{j+1,i}} \cdots y_k^{a_{k,i}} \cdot \prod_{l=k+1}^{n+k} p_l(y)^{a_{l,i}}\,, \qquad \text{for} \quad i=1,\ldots,j{-}1\,.$$

Since $p_l(0)>0$ for $l>k$, we see that μ_j is approximated in this neighborhood of the origin by the zero set of equations of the form

$$y_i \;=\; y_j^{a_{j,i}} y_{j+1}^{a_{j+1,i}} \cdots y_k^{a_{k,i}} \cdot b_i\,, \qquad \text{for} \quad i=1,\ldots,j{-}1\,.$$

where $b_i>0$. We may scale y_i so that $b_i=1$, and then these equations define a subtorus of the positive part of the real torus $\mathbb{R}^k_>$. Since the exponents are rational numbers, the closure of this subtorus in the non-negative orthant is the non-negative part of a (not necessarily normal) toric variety, which is homeomorphic to a polytope [7, § 4]. In particular, its boundary is homeomorphic to the boundary of a polytope and is thus a manifold.

This proves that, in a neighborhood of any vertex, the intersection of $\overline{\mu_j}$ with the boundary of Δ forms a manifold. But this implies that the union of faces of $\overline{\Delta}$ which are contained in $\overline{\mu}_j$ forms a PL-manifold, and completes the proof of the lemma. $\square$

3.3. **Concrete bounds for $k=2,3$.** When $k=2$, the estimate (3.3) becomes

$$|V(\psi_1,\psi_2)| \;\leq\; \flat(C_1) \;+\; \flat(C_2) \;+\; |V(\Gamma_1,\Gamma_2)|\,.$$

Theorem 3.10. *Suppose that $k=2$. Then*

(1) $\flat(C_2)\leq\lfloor\frac{n+3}{2}\rfloor$,
(2) $\flat(C_1)\leq\lfloor\frac{n(n+3)}{2}\rfloor$, *and*
(3) $|V(\Gamma_1,\Gamma_2)|\leq 2n^2$.

Thus a fewnomial system of n polynomials in n variables having $n+3$ monomials has at most $2n^2+\lfloor\frac{(n+1)(n+3)}{2}\rfloor$ positive solutions.

Proof. $\overline{\Delta}$ is a polygon with at most $n+3$ edges and $n+3$ vertices. By Corollary 3.9 and (3.10), we have $2\flat(C_2)\leq\Phi_0(\overline{\Delta})\leq n+3$ and $2\flat(C_1)\leq n\Phi_1(\overline{\Delta})\leq n(n+3)$. By Lemma 3.4(3), we have $|V(\Gamma_2,\Gamma_1)|\leq 2n^2$. $\square$

In particular, when $n=k=2$, this gives a bound of 15 for the number of positive solutions to a fewnomial system of 2 polynomials in 2 variables and with 5 monomials.

When $k=3$, the estimate (3.3) becomes

$$|V(\psi_1,\psi_2,\psi_3)| \;\leq\; \flat(C_1) \;+\; \flat(C_2) \;+\; \flat(C_3) \;+\; |V(\Gamma_1,\Gamma_2,\Gamma_3)|\,.$$

Theorem 3.11. *Suppose that $k=3$. Then*

(1) $\flat(C_3) \leq n+2$,
(2) $\flat(C_2) \leq n(n+2)$,
(3) $\flat(C_1) \leq n^2(n+4)$,
(4) $|V(\Gamma_1, \Gamma_2, \Gamma_3)| \leq 8n^3$.

Thus a fewnomial system of n polynomials in n variables having $n+4$ monomials has at most $9n^3+5n^2+3n+2$ positive solutions.

Proof. The polytope $\overline{\Delta}$ is a simple three-dimensional polytope and Corollary 3.9 gives
(1) $2\flat(C_3) \leq M_0(\overline{\Delta})$,
(2) $2\flat(C_2) \leq nM_1(\overline{\Delta})$,
(3) $2\flat(C_1) \leq 2n^2M_2(\overline{\Delta})$.

We have $M_j(\overline{\Delta}) \leq \Phi_j(\overline{\Delta})$ and McMullen's Upper Bound Theorem [13] applied to the dual of our polytope gives $\Phi_0(\overline{\Delta}) \leq 2(n+2)$, $\Phi_1(\overline{\Delta}) \leq 3(n+2)$, and $\Phi_2(\overline{\Delta}) \leq n+4$. This gives estimates for $M_j(\overline{\Delta})$. However, $M_1(\overline{\Delta}) \leq \Phi_0(\overline{\Delta})$, as a 1-dimensional PL-manifold is a union of closed polygons. Combining these bounds proves the result. $\square$

4. Number of compact components of hypersurfaces

Let f be a polynomial with $n+k+1$ monomials whose real exponents affinely span $\mathbb{R}^n$. Let $V(f) \subset \mathbb{R}^n_>$ be the set of zeroes of f in the positive orthant, which we assume is smooth, and let $\kappa(f)$ be the number of compact components of $V(f)$. When $n=1$, a bound of $n+k$ for $\kappa(f)$ is given by Descartes's rule of signs [6], and when $k=1$ the bound of 1 for $\kappa(f)$ is proven by Bihan, Rojas, and Stella [4]. We therefore assume that $n, k \geq 2$.

Theorem 4.1.
(1) *We have $\kappa(f) < \frac{e^2+3}{8} \cdot 2^{\binom{k}{2}} n^k$.*
(2) *We also have $\kappa(f) < C(k)n^{k-1}$, where $C(k)$ has order $O(k2^{\binom{k}{2}})$.*

The first assertion is a straightforward application of Theorem 3.2. When $k > n$, it implies the second assertion. The proof of the second assertion when $k \leq n$ exploits the sparsity of a Gale system.

After an analytic change of variables, we may assume that f has the form

$$f \;=\; \sum_{i=1}^{n} e_i z_i \;+\; c_1 z^{a_1} \;+\cdots+\; c_k z^{a_k} \;+\; e_0\,,$$

where $e_i \in \{\pm 1\}$, $a_j \in \mathbb{R}^n$ and the coefficients c_i are non-zero Its support is $\mathcal{A} := \{0, a_1, \ldots, a_k,\, e_1, \ldots, e_n\}$.

Since $V(f)$ is smooth, the coordinate function z_1 has at least 2 critical points on each bounded component of $V(f)$. These critical points are solutions to the system

$$f \;=\; z_2\frac{\partial f}{\partial z_2} \;=\; z_3\frac{\partial f}{\partial z_3} \;=\; \cdots \;=\; z_n\frac{\partial f}{\partial z_n} \;=\; 0\,. \tag{4.1}$$

(Since we work in $\mathbb{R}^n_>$, no coordinate z_i vanishes.) The resulting system has the form

$$\begin{aligned} -e_1 z_1 \;&=\; \sum_{i=2}^{n} e_i z_i \;+\; c_1 z^{a_1} \;+\cdots+\; c_k z^{a_k} \;+\; e_0 \\ -e_j z_j \;&=\; c_1 a_{1,j} z^{a_1} + \cdots + \; c_k a_{k,j} z^{a_k} \qquad \text{for} \quad j = 2, 3, \ldots, n\,. \end{aligned} \tag{4.2}$$

This is a system of n polynomials in n variables with support $\mathcal{A}$, which we may assume has only non-degenerate solutions by perturbing the parameters without altering the topology of $V(f)$, as $V(f)$ is smooth. Theorem 4.1(1) follows now from Theorem 3.2 as $2\kappa(f)$ is bounded by the number of solutions to (4.2).

By Theorem 2.2, the number of solutions in $\mathbb{R}^n_>$ to (4.1) is equal to the number of solutions to an associated Gale system in a polyhedron $\Delta \subset \mathbb{R}^k$.

Performing Gaussian elimination on (4.2), we obtain a system of the form

$$\begin{array}{rcllcl} z_1 &=& b_{1,1}z^{a_1} + \cdots + b_{1,k}z^{a_k} + b_{1,0} &=: & p_1(z^{a_1},\ldots,z^{a_k}) \\ z_2 &=& b_{2,1}z^{a_1} + \cdots + b_{2,k}z^{a_k} &=: & p_2(z^{a_1},\ldots,z^{a_k}) \\ \vdots & & & \vdots & \\ z_n &=& b_{n,1}z^{a_1} + \cdots + b_{n,k}z^{a_k} &=: & p_n(z^{a_1},\ldots,z^{a_k}) \end{array} \tag{4.3}$$

where $b_{i,j} \in \mathbb{R}$.

Set $y_i := z^{a_i}$, for $i = 1,\ldots,k$, so that the right hand sides of (4.3) are linear functions $p_i(y)$ of y. Then use the exponents $a_1,\ldots,a_k$ to get the Gale system

$$f_j(y) \ := \ y_j^{-1} \cdot (p_1(y))^{a_{1,j}} \cdot \prod_{i=2}^{n} p_i(y)^{a_{i,j}} \ = \ 1 \qquad \text{for} \quad j = 1,2,\ldots,k\,. \tag{4.4}$$

This has the form of (2.4), but the linear factors p_i for $i > 1$ are sparse. As in Section 3, we may assume that every square submatrix of $A = (a_{i,j})$ is invertible, and that the exponents $a_{i,j}$ are rational numbers.

Theorem 4.2. *Let f be a polynomial with real exponents in n variables having $n+k+1$ monomials with $k \leq n$. Then*

$$\kappa(f) \ \leq \ \left(\tfrac{k}{2}2^{\binom{k}{2}} \ + \ \tfrac{e^2+1}{8}\cdot k2^{\binom{k-1}{2}}\right)n^{k-1} \ + \ \left(\tfrac{e^2}{8}\cdot 2^{\binom{k-2}{2}}\right)n^{k-2}\,.$$

This implies Theorem 4.1(2).

We prove Theorem 4.2 with the methods of Section 3, taking into account the sparsity of the Gale system (4.4). Consider the polyhedron $\Delta \subset \mathbb{R}^k_>$ defined by

$$\Delta \ := \ \{y \in \mathbb{R}^k_> \mid p_i(y) > 0 \quad \text{for} \quad i = 1,\ldots,n\}\,. \tag{4.5}$$

For each $j = 1,\ldots,k$, set $p_{n+j} = y_j$. Since the linear polynomials $p_2,\ldots,p_{n+k}$ do not have a constant term but are otherwise general, they define a cone with vertex the origin and base a simple polytope ϕ^∞ with at most $n+k-1$ facets lying in the hyperplane at infinity. We obtain Δ from this cone by adding the single inequality $p_1(y) > 0$. This gives at most one new facet, ϕ^1. By our assumptions on the genericity of the polynomials p_i, every face of dimension $k-j$ lies on j facets, unless $j = k$ and that face is the origin.

Let $\overline{\Delta} \subset \mathbb{RP}^k$ be the closure of Δ. Let $\Phi^\ell_j(\overline{\Delta})$ count the *linear faces* of dimension j, those whose affine span contains the origin, and let $\Phi^{n\ell}_j(\overline{\Delta})$ count the j-faces of $\overline{\Delta}$ lying on either ϕ^∞ or ϕ^1.

For each $j = 1,\ldots,k$, let ψ_j be the logarithm of right hand side of the corresponding equation in (4.4). Define functions $\Gamma_k(y),\ldots,\Gamma_1(y)$ by the recursion

$$\Gamma_j \ := \ J(\psi_1,\psi_2,\ldots,\psi_j,\,\Gamma_{j+1},\Gamma_{j+2},\ldots,\Gamma_k)\,.$$

Then, for $j = 1,\ldots,k$, define the curve C_j by

$$C_j \ := \ \{y \in \Delta \mid \psi_1(y) = \cdots = \psi_{j-1}(y) \ = \ \Gamma_{j+1}(y) = \cdots = \Gamma_k(y) \ = \ 0\}\,.$$

Successive uses of the Khovanskii-Rolle theorem give an upper bound for $\kappa(f)$,

$$\kappa(f) \;\leq\; \tfrac{1}{2}\big(|V(\psi_1,\dots,\psi_k)| \;\leq\; \tfrac{1}{2}\big(\flat(C_k) \,+\, \cdots \,+\, \flat(C_1) \,+\, |V(\Gamma_1,\Gamma_2,\dots,\Gamma_k)|\big)\,. \tag{4.6}$$

For $j=1,\dots,k$, define F_j by formula (3.5). This polynomial of degree $n2^{k-j}$ defines the same subset of Δ as does Γ_j. The sparsity of p_i (for $i>2$) implies the sparsity of polynomials F_j.

Lemma 4.3.

(1) *Every monomial of F_j has degree in the interval* $[(n{-}1)2^{k-j}, n2^{k-j}]$.

(2) *C_j is a smooth algebraic curve,* $\flat(C_k) \leq \frac{1}{2}\big(1+\Phi_0^{n\ell}(\overline{\Delta})\big)$, *and for $j=1,\dots,k-1$ we have*

$$\flat(C_j) \;\leq\; \tfrac{1}{2}\Big(2^{\binom{k-j}{2}}\, n^{k-j}\Phi^{n\ell}_{k-j}(\overline{\Delta}) \;+\; 2^{\binom{k-j}{2}}\,\Big(n^{k-j}-(n{-}1)^{k-j}\Big)\;\Phi^{\ell}_{k-j}(\overline{\Delta})\Big)\,.$$

(3) $|V(\Gamma_1,\Gamma_2,\dots,\Gamma_k)| \leq 2^{\binom{k}{2}}\,\Big(n^k-(n{-}1)^k\Big)$.

Lemma 4.3 is proved in the next sections.

Lemma 4.4. *We have*

(1) $\Phi^{n\ell}_{k-j}(\overline{\Delta}) \leq 2\binom{n+k-1}{j-1}+\binom{n+k-1}{j-2}$, *for $j=1,\dots,k$,*

(2) $\Phi^{\ell}_{k-j}(\overline{\Delta}) \leq \binom{n+k-1}{j}$, *for $j=1,\dots,k{-}1$, and* $\Phi^{\ell}_0(\overline{\Delta})=1$.

Proof. The faces of $\overline{\Delta}$ arise in (possibly) three different ways. A $(k{-}j)$-face ϕ either (a) lies in the intersection of the facets ϕ^∞ and ϕ^1, or it (b) lies in exactly one of ϕ^∞ or ϕ^1, or else (c) its affine span contains the origin. Unless ϕ is the origin, it lies on j facets. Among those are some *linear facets* which are defined by one of $p_2,\dots,p_{n+k}$. In case (a) there are $j{-}2$ such facets, in case (b) there are $j{-}1$ facets, and in case (c) all j facets are linear. We may bound the $(k{-}j)$-faces of each type by enumerating the possible sets of linear facets on which they lie,

$$(a)\ \ \binom{n{+}k{-}1}{j{-}2}\,,\qquad (b)\ \ 2\binom{n{+}k{-}1}{j-1}\,,\qquad \text{and}\qquad (c)\ \ \binom{n{+}k{-}1}{j}\,.$$

The factor of 2 in the estimate (b) is because such faces lie on either ϕ^∞ or ϕ^1.

The lemma follows from this. For (1), note that $\Phi^{n\ell}_{k-j}(\overline{\Delta})$ counts faces of types (a) and (b). For (2), $\Phi^{\ell}_{k-j}(\overline{\Delta})$ counts faces of type (c), when $k{-}j>0$. ☐

Proof of Theorem 4.2. Substitute the estimates of Lemma 4.3 into (4.6) to obtain

$$\begin{aligned}\kappa(f) \;\leq\; \frac{1}{4} \;+\; \frac{1}{4}\sum_{j=1}^{k} 2^{\binom{k-j}{2}} n^{k-j}\Phi^{n\ell}_{k-j}(\overline{\Delta}) \qquad&\\ +\; \frac{1}{4}\sum_{j=1}^{k} 2^{\binom{k-j}{2}}(n^{k-j}-(n{-}1)^{k-j})\Phi^{\ell}_{k-j}(\overline{\Delta}) \;+\; \frac{1}{2}2^{\binom{k}{2}}(n^k-(n{-}1)^k)\,.&\end{aligned}$$

(We used $2^{\binom{k-k}{2}}n^{k-k}=1$ in the first sum and $n^{k-k}-(n{-}1)^{k-k}=0$ in the second.) Substituting the estimates of Lemma 4.4 and rearranging, we obtain the estimate

$$\kappa(f) \;\leq\; \frac{1}{4}\sum_{j=1}^{k} A_j \;+\; \frac{1}{4}\sum_{j=2}^{k} B_j \;+\; \frac{1}{4} \;+\; \frac{1}{2}2^{\binom{k}{2}}(n^k-(n-1)^k)\,,$$

where
$$A_j \;:=\; 2^{\binom{k-j}{2}}\left(\left(n^{k-j}-(n-1)^{k-j}\right)\tbinom{n+k-1}{j}+2n^{k-j}\tbinom{n+k-1}{j-1}\right)\ ,$$
and
$$B_j \;:=\; 2^{\binom{k-j}{2}}n^{k-j}\tbinom{n+k-1}{j-2}\,.$$

We have $n^{k-j}-(n-1)^{k-j}\leq(k-j)n^{k-j-1}$, as the function x^{k-j} is (weakly) convex and non-decreasing. Thus
$$A_j \;\leq\; 2^{\binom{k-j}{2}}n^{k-j}\left(\tfrac{k-j}{n}\tbinom{n+k-1}{j}+2\tbinom{n+k-1}{j-1}\right)\ .$$
Since $1\leq j\leq k\leq n$, we have
$$\tfrac{k-j}{n}\tbinom{n+k-1}{j} \;=\; \tfrac{k-j}{j}\,\tfrac{n+k-j}{n}\,\tfrac{n+k-j+1}{n+k}\tbinom{n+k}{j-1} \;<\; 2\tfrac{k-j}{j}\tbinom{n+k}{j-1}\ .$$
But we also have $\binom{n+k-1}{j-1}\leq\binom{n+k}{j-1}$, and so
$$A_j \;<2\cdot 2^{\binom{k-j}{2}}n^{k-j}\left(\tfrac{k-j}{j}\tbinom{n+k}{j-1}+\tbinom{n+k}{j-1}\right) \;=\; 2k\cdot 2^{\binom{k-j}{2}}n^{k-j}\tbinom{n+k}{j-1}\,.$$
Thus
$$\sum_{j=1}^{k}A_j \;\leq\; 2k\sum_{j=1}^{k}2^{\binom{k-j}{2}}\tfrac{n^{k-j}}{j}\tbinom{n+k}{j-1}\ .$$
If we set $i:=j-1$ and $l:=k{-}1$, then we obtain
$$\sum_{j=1}^{k}A_j \;\leq\; 2k\Bigl(2^{\binom{k-1}{2}}\,n^{k-1} \;+\; \sum_{i=1}^{l+1}2^{\binom{l-i}{2}}\tfrac{n^{l-i}}{i+1}\tbinom{n+l+1}{i}\Bigr)\ .$$
Arguing as in Lemma 3.5 bounds the second sum by
$$2^{\binom{l}{2}}n^l\sum_{i=1}^{\infty}\tfrac{2^{i-1}}{(i+1)!} \;=\; \tfrac{e^2-3}{4}\cdot 2^{\binom{k-1}{2}}\,n^{k-1}\,,$$
and thus we obtain
$$\sum_{j=1}^{k}A_j \;\leq\; \tfrac{e^2+1}{2}\,k\,2^{\binom{k-1}{2}}\,n^{k-1}\,. \tag{4.7}$$

Setting $l=k{-}2$ and $i=j{-}2$ gives
$$\sum_{j=2}^{k}B_j \;=\; \sum_{i=0}^{l}2^{\binom{l-i}{2}}n^{l-i}\tbinom{n+l+1}{i}\ .$$
Arguing as in Lemma 3.5 bounds this sum by
$$2^{\binom{l}{2}}n^l\sum_{i=0}^{\infty}\tfrac{2^{i-1}}{i!} \;=\; \tfrac{e^2}{2}\cdot 2^{\binom{k-2}{2}}\,n^{k-2}\ .$$

Finally, as $2\leq k\leq n$, we have $n^k-(n-1)^k\leq kn^{n-1}-1$, which gives
$$\tfrac{1}{2}+2^{\binom{k}{2}}(n^k-(n{-}1)^k) \;\leq\; k2^{\binom{k}{2}}n^{k-1}\,.$$
Collecting all these bounds gives the result. $\square$

4.1. **Proof of Lemma 4.3 (1).** By Lemma 3.7, we have

$$F_{k-j} \;=\; \Biggl(\sum_{\mathcal{I}\in\binom{[n+k]}{k}^{2^j}} \frac{A(\mathcal{I})B(\mathcal{I})}{p(\mathcal{I})}\Biggr)\cdot\Biggl(\prod_{i=1}^{n+k} p_i(y)\Biggr)^{2^j}\,, \tag{4.8}$$

where $A(\mathcal{I}), B(\mathcal{I})$ are real numbers, $\binom{[n+k]}{k}^{2^j}$ consists of lists $\mathcal{I}$ of 2^j elements of $\binom{[n+k]}{k}$, and $p(\mathcal{I})$ is the product over I in $\mathcal{I}$ of p_I, where if $I = i_1 < \cdots < i_k$ then $p_I := p_{i_1}\cdots p_{i_k}$.

Fix a list $\mathcal{I}$ and let r be the number of times that p_1 appears in $p(\mathcal{I})$. Then $0 \le r \le 2^j$ and the sparsity of $p_2,\ldots,p_{n+k}$ implies that the degree of each monomial of

$$\frac{1}{p(\mathcal{I})}\cdot\Biggl(\prod_{i=1}^{m} p_i(y)\Biggr)^{2^j}$$

lies in the interval $[(n-1)2^j + r\,,\, n2^j]$. Lemma 4.3(1) follows as any value of r in between 0 and 2^j can occur in a summand of (4.8). □

4.2. **Proof of Lemma 4.3(2) and (3).** Bernstein's Theorem generalizes Kouchnirenko's Theorem to the case when the polynomials have possibly different supports. We will only need a special case of Bernstein's Theorem. The *Newton polytope* $\mathrm{New}(f)$ of a polynomial f with integer exponents is the convex hull of its support.

Theorem 4.5 (Bernstein [1])**.** *Let P be a fixed polytope in $\mathbb{R}^k$ and suppose that $f_1,\ldots,f_k$ are polynomials such that for each $i = 1,\ldots,k$, the Newton polytope of f_i is contained in $\lambda_i P$, where $\lambda_i \in \mathbb{N}$. Then the number of non-degenerate solutions in $(\mathbb{C}^\times)^k$ to the system $f_1(y) = \cdots = f_k(y) = 0$ is at most $\lambda_1\cdots\lambda_k\cdot k!\mathrm{vol}(P)$.*

We prove Lemma 4.3(3). By Lemma 4.3(1), if we set

$$P \;:=\; \{(i_1, i_2\ldots, i_k)\in\mathbb{R}^k_> \mid n-1 \le i_1+i_2+\cdots+i_k \le n\}$$

then the Newton polytope of F_j is contained in $2^{k-j}P$ for $j = 1,\ldots,k$. By Bernstein's Theorem the number of non-degenerate solutions to the system $F_1 = \cdots = F_k = 0$ in $\Delta \subset (\mathbb{R}^\times)^k \subset (\mathbb{C}^\times)^k$ is at most

$$\Biggl(\prod_{j=1}^{k} 2^{k-j}\Biggr) k!\mathrm{vol}(P) \;=\; 2^{\binom{k}{2}} k!\mathrm{vol}(P)\,.$$

To finish the proof of Lemma 4.3(3), we note that $\mathrm{vol}(P) = \frac{1}{k!}(n^k - (n-1)^k)$. □

We now prove Lemma 4.3(2), adapting the proof of Lemma 3.4(2) by taking into account of the sparsity of the polynomials $F_1,\ldots,F_k$. Define μ_j as in (3.11) by the equations $f_1(y) = \cdots = f_{j-1}(y) = 1$, where the functions f_i are defined by (4.4). Then the statement of Lemma 3.8 holds if we now claim that $\overline{\mu_j}\cap\partial\overline{\Delta}$ is a union of $(k-j)$-faces of $\overline{\Delta}$ which forms a PL-submanifold of $\overline{\Delta}$, except possibly at the origin. As before, $2\flat(C_j) = |\overline{C_j}\cap\partial\overline{\Delta}|$. Since $\overline{C_j}$ is the subset of $\overline{\mu_j}$ where we also have

$$\overline{F}_{j+1}(y) \;=\; \overline{F}_{j+2}(y) \;=\; \cdots \;=\; \overline{F}_k(y) \;=\; 0\,, \tag{4.9}$$

we estimate $|\overline{C_j}\cap\partial\overline{\Delta}|$ by estimating the number of points y in the union of $(k-j)$-faces of $\overline{\Delta}$ where (4.9) holds.

First, $\mu_k = C_k$, so $2\flat(C_k) \leq \Phi_0(\overline{\Delta})$. Otherwise, suppose that $1 \leq j \leq k{-}1$ and let ϕ be a $(k{-}j)$-face of $\overline{\Delta}$.

If ϕ is a linear face, then we may apply a linear transformation and assume that the facets supporting ϕ are defined by $y_i = 0$ for $i > k{-}j$, so that the face ϕ affinely spans the coordinate subspace $\mathbb{R}^{k-j}$ with coordinates $y_1, \ldots, y_{k-j}$. Thus the points of $\overline{C}_j \cap \phi$ are a subset of the points of (4.9) in $\mathbb{R}^{k-j}$ having non-zero coordinates. This homogeneous linear transformation does not alter the sparsity of the polynomials F_i or $\overline{F_i}$.

By Lemma 4.3(1), the substitution of $y_i = 0$ for $i > k{-}j$ into $\overline{F}_{k-l}$ gives a polynomial with Newton polytope a subset of $2^l P$, where

$$P \ := \ \{(i_1, \ldots, i_{k-j}) \in \mathbb{R}^{k-j}_{>} \mid n-1 \leq i_1 + i_2 + \cdots + i_{k-j} \leq n\}\,.$$

By Bernstein's Theorem, the number of points of $\overline{C}_j \cap \phi$ is at most

$$2^{\binom{k-j}{2}}(k{-}1)! \cdot \mathrm{vol}(P) \ = \ 2^{\binom{k-j}{2}} \left(n^{k-j} - (n-1)^{k-j}\right)\,.$$

If ϕ is not a linear face, then we simply apply Bézout's Theorem to bound the number of solutions to the system (4.9) lying in ϕ and obtain

$$\left(\prod_{i=1}^{k-j} n 2^{k-j-i}\right) \ = \ 2^{\binom{k-j}{2}} n^{k-j}\,. \tag{4.10}$$

This completes the proof of Lemma 4.3(2). □

4.3. **Concrete bounds when $k = 2, 3$.** When $k = 2$, $\overline{\Delta}$ is a polygon in $\mathbb{R}^2$ and the estimate (3.3) becomes

$$2\kappa(f) \ \leq \ |V(\psi_1, \psi_2)| \ \leq \ \flat(C_2) \ + \ \flat(C_1) \ + \ |V(\Gamma_2, \Gamma_1)|\,.$$

Lemma 4.3 provides the bounds

$$\begin{array}{lcll} 2\flat(C_2) & \leq & \Phi_0(\overline{\Delta})\,, & \\ 2\flat(C_1) & \leq & \Phi_1^{\ell}(\overline{\Delta}) + n\Phi_1^{n\ell}(\overline{\Delta})\,, & \text{and} \\ |V(\Gamma_2, \Gamma_1)| & \leq & 4n - 2\,. & \end{array}$$

Theorem 4.6. *Let f be a polynomial with $n{+}3$ monomials in n variables with $2 \leq n$. Then $\overline{\Delta}$ is a polygon with $\Phi_1^{\ell}(\overline{\Delta}) \leq 2$, $\Phi_1^{n\ell}(\overline{\Delta}) \leq 2$ and $\Phi_0(\overline{\Delta}) \leq 4$. Thus $\kappa(f) \leq \lfloor \frac{5n+1}{2} \rfloor$.*

When $k = 3$, $\overline{\Delta}$ is a polytope in $\mathbb{R}^3$ and the estimate (3.3) becomes

$$2\kappa(f) \ \leq \ |V(\psi_1, \psi_2, \psi_3)| \ \leq \ \flat(C_3) + \flat(C_2) \ + \ \flat(C_1) \ + \ |V(\Gamma_3, \Gamma_2, \Gamma_1)|\,.$$

Lemma 4.3 provides the bounds

$$\begin{array}{rcll} 2\flat(C_3) & \leq & \Phi_0(\overline{\Delta})\,, & \\ 2\flat(C_2) & \leq & \Phi_1^{\ell}(\overline{\Delta}) + n\Phi_1^{n\ell}(\overline{\Delta})\,, & \\ 2\flat(C_1) & \leq & 2(2n-1)\Phi_2^{\ell}(\overline{\Delta}) + 2n^2\Phi_2^{n\ell}(\overline{\Delta})\,, & \text{and} \\ |V(\Gamma_3, \Gamma_2, \Gamma_1)| & \leq & 24n^2 - 24n + 8\,. & \end{array}$$

Theorem 4.7. *Let f be a polynomial with $n + 4$ monomials and $n \geq 2$ variables. Then $\overline{\Delta}$ is a 3-polytope with*

- $\Phi_0(\overline{\Delta}) \leq 4n + 4$
- $\Phi_1^{\ell}(\overline{\Delta}) \leq n + 2$, $\ \Phi_1^{n\ell}(\overline{\Delta}) \leq 2n + 5$,

- $\Phi_2^\ell(\overline{\Delta}) \leq n+2,\ \ \Phi_2^{n\ell}(\overline{\Delta}) \leq 2.$

Thus

$$\kappa(f) \ \leq\ \tfrac{29}{2}n^2 - 8n + \tfrac{9}{2}\,.$$

Proof. We need only to establish the bounds on the number of faces. Let K be the cone in $\mathbb{R}^3$ defined by $p_i(y) \geq 0$ for $2 \leq i \leq n+3$. It is combinatorially equivalent to the cone over the face ϕ^∞ at infinity, which is a polygon with at most n+2 edges. Thus

$$\Phi_0(\overline{K}) \ \leq\ n+3\,, \qquad \Phi_1^\ell(\overline{K})\,,\ \Phi_1^{n\ell}(\overline{K})\,,\ \Phi_2^\ell(\overline{K}) \ \leq\ n+2\,, \qquad \text{and} \qquad \Phi_2^{n\ell}(\overline{K}) \ \leq\ 1\,.$$

We cut K with the half space $p_1(y) > 0$ to obtain Δ. This can only decrease the number of linear faces, so we consider the other faces. This adds at most one facet, so $\Phi_2^{n\ell}(\overline{\Delta}) \leq 2$. We can assume that $\Delta \neq K$ and let ϕ^1 be the new facet.

Since ϕ^1 cannot meet more than $2n$+2 of the edges of K, and must cut off at least one vertex of K, we have $\Phi_0(\overline{\Delta}) \leq 4n+4$.

Since K has $n+3$ facets, $\Phi_1^{n\ell}(\overline{\Delta}) \leq \Phi_2^\ell(\overline{K}) + n + 3$, which completes the proof. ☐

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