# Experimentation in the Schubert Calculus

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**Abstract.**

Many aspects of Schubert calculus are easily modeled on a computer. This enables large-scale experimentation to investigate subtle and ill-understood phenomena in the Schubert calculus. A well-known web of conjectures and results in the real Schubert calculus has been inspired by this continuing experimentation. A similarly rich story concerning intrinsic structure, or Galois groups, of Schubert problems is also beginning to emerge from experimentation. This showcases new possibilities for the use of computers in mathematical research.

## §1. Introduction

The Schubert calculus of enumerative geometry is a rich and well-understood class of enumerative-geometric problems that are readily modeled on a computer. It provides a laboratory in which to investigate poorly understood phenomena in enumerative geometry using supercomputers. Modern software tools and available computer resources allow us to test billions of instances of Schubert problems for the phenomena we wish to study. This is easily parallelized and therefore takes advantage of the current trend in computer architecture to increase computation power through increased parallelism. These computations have led to conjectures and new results and showcase new possibilities for the use of computers as a tool in mathematical research.

Of the solutions to a system of real polynomials or a geometric problem with real constraints, some may be real while the rest occur in

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complex conjugate pairs, and it is challenging to say anything meaningful about the distribution between the two types. Khovanskii showed that systems of polynomials with few monomials have upper bounds on their numbers of real solutions often far less than their numbers of complex solutions [34]. In contrast, geometric problems coming from the Schubert calculus on Grassmannians may have all their solutions be real [48, 49, 60], and while it is known only in some additional cases [51, 42], this reality is believed to hold for all flag manifolds.

The real-number phenomena that we discuss is of a different character than these results. The best-known involves the Shapiro Conjecture and its generalizations, where for certain classes of Schubert problems and natural choices of conditions, every solution is real. Less understood are Schubert problems whose numbers of real solutions possess further structure including lower bounds, congruences, and gaps.

Like field extensions, geometric problems have intrinsic structure encoded by Galois groups [32]. Unlike field extensions, little is known about such Galois groups. Work of Vakil [60], Billey and Vakil [3], and Leykin and Sottile [37] gives several avenues for studying Galois groups of Schubert problems on computers. Preliminary results suggest phenomena to study in future large-scale computational experiments. For example, most Schubert problems have a highly transitive Galois group that contains the alternating group, while the rest have only singly transitive Galois groups, and the intrinsic structure restricting their Galois groups also restricts their numbers of real solutions.

*Example* 1. The classical problem of four lines asks: "how many lines in $\mathbb{P}^3$ meet four general lines?" Three mutually skew lines $\ell_1$, $\ell_2$, and $\ell_3$ lie on a unique hyperboloid (Fig. 1). This hyperboloid has two

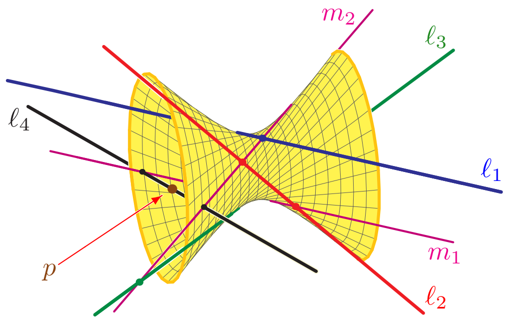



Fig. 1. Problem of four lines

rulings, one contains $\ell_1$, $\ell_2$, and $\ell_3$, and the second consists of the lines meeting these three. If the fourth line, $\ell_4$, is general, then it will meet the hyperboloid in two points, and through each of these points there is a unique line in the second ruling. These two lines, $m_1$ and $m_2$, are the solutions to this instance of the problem of four lines.

If the four lines are real, then $\ell_4$ either meets the hyperboloid in two real points (as in Fig. 1) giving two real solution lines, or in two complex conjugate points giving two complex conjugate solutions.

The Galois/monodromy group of this problem is the group of permutations of the solutions which arise by following the solutions over loops in the space of four-tuples $(\ell_1, \ell_2, \ell_3, \ell_4)$ of lines. A simple such loop is described by rotating $\ell_4$ 180° about the point $p$. Following the two solutions along the loop interchanges them and shows that the Galois/monodromy group of this problem is the full symmetric group $S_2$.

The Shapiro Conjecture asserts that if the lines $\ell_1, \ldots, \ell_4$ are tangent to a twisted cubic at real points, then the two solutions are real. Indeed, any three points on any twisted cubic are conjugate to any three points on another, so it suffices to consider the cubic curve given by

$$\gamma \;:\; t \;\longmapsto\; (12t^2 - 2\,,\, 7t^3 + 3t\,,\, 3t - t^3)\,,$$

and the first three lines to be $\ell(1)$, $\ell(0)$, and $\ell(-1)$, where $\ell(t)$ is the line tangent to $\gamma$ at the point $\gamma(t)$. As before, there is a unique hyperboloid

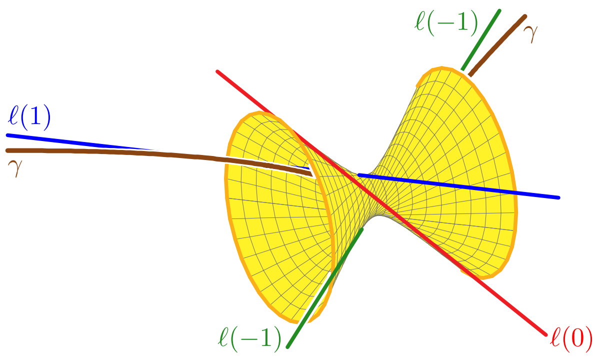



Fig. 2. Hyperboloid containing three lines tangent to $\gamma$.

ruled by the lines meeting all three, and the solutions correspond to points where the fourth tangent line meets the hyperboloid.

Consider the fourth line, $\ell(s)$, where $0 < s < 1$ (which we may assume as the three intervals between $\gamma(1)$, $\gamma(0)$, and $\gamma(-1)$ are projectively equivalent). In Fig. 3, we look down the throat of the hyperboloid at the interesting part of this configuration. As $\ell(s)$ is tangent to the branch of $\gamma$ between $\gamma(1)$ and $\gamma(0)$, it must meet the hyperboloid in two real points. Through each point, there is a real line in the second

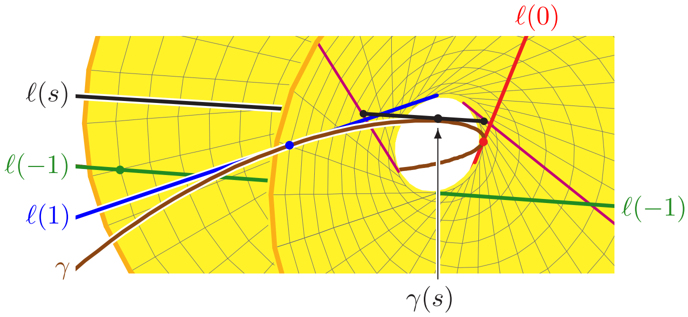



Fig. 3. $\ell(s)$ meets the hyperboloid in two real points.

ruling which meets all four tangent lines, and this proves the Shapiro Conjecture for this problem of four lines.

This paper is organized as follows. In Section 2 we give background on the Shapiro Conjecture and the Schubert calculus, and then explain how we may study Schubert problems on a computer. In Section 3 we discuss the Shapiro Conjecture more extensively, describing its generalizations and evidence that has been found for these generalizations. In Section 4 we discuss additional structure that has been observed and proven concerning the number of real solutions to the osculating Schubert calculus on Grassmannians. Finally, we close in Section 5 discussing several approaches to obtaining information about Galois groups of Schubert problems, and sketch how they were used to nearly determine the Galois groups of all Schubert problems in $\mathrm{Gr}(4,9)$.

## §2. Background

### 2.1. The Shapiro Conjecture

The *Wronskian* of univariate polynomials $f_1(t),\ldots,f_k(t)$ of degree at most $n-1$ is the determinant of the matrix of their derivatives,

$$\mathrm{Wr}(f_1,\ldots,f_k) \;=\; \det\left(f_j^{(i-1)}(t) \mid i,j=1,\ldots,k\right)\,.$$

Up to a scalar, this depends only on the linear span of the polynomials $f_1,\ldots,f_k$. Putting $f_1,\ldots,f_k$ into echelon form with respect to the basis of monomials shows that we may assume $\deg f_1 > \cdots > \deg f_k$, from which we see that their Wronskian has degree at most $k(n-k)$. The Wronskian gives a map

$$\text{Wr} : \text{Gr}(k, \mathbb{C}_{n-1}[t]) \longrightarrow \mathbb{P}(\mathbb{C}_{k(n-k)}[t]), \tag{2.1}$$

where $\mathbb{C}_d[t]$ is the space of univariate polynomials in $t$ of degree at most $d$, $\text{Gr}(k, \mathbb{C}_{n-1}[t])$ is the Grassmannian of $k$-dimensional linear subspaces (*$k$-planes*) in $\mathbb{C}_{n-1}[t]$, and $\mathbb{P}(\mathbb{C}_{k(n-k)}[t])$ is the projective space of 1-planes in $\mathbb{C}_{k(n-k)}[t]$. The map (2.1) is surjective of degree

$$\frac{[k(n-k)]!\, 1!2! \cdots (k-1)!}{(n-1)!(n-2)! \cdots (n-k)!}, \tag{2.2}$$

and each fiber consists of this number of points, counted with multiplicity [9]. The *inverse Wronski problem* asks for the $k$-planes of polynomials with a given Wronskian. This naturally arises in the theory of linear series on $\mathbb{P}^1$ [9], static output feedback control of linear systems [6], and mathematical physics [41].

The *Shapiro Conjecture* posited that if $\Psi(t) \in \mathbb{P}(\mathbb{R}_{k(n-k)}[t])$ had only real roots, then every $k$-plane of polynomials in $\text{Wr}^{-1}(\Psi)$ is real. This was first studied computationally [50, 62], then it was shown to be true if the roots of $\Psi$ were sufficiently clustered together [49]. When $\min\{k, n-k\} = 2$, it is equivalent to the statement that a rational function with only real critical points is essentially a quotient of real polynomials [13, 14]. Finally, when a connection to integrable systems was realized, Mukhin, Tarasov, and Varchenko exploited that to prove the conjecture in full generality, eventually using that a symmetric matrix has only real eigenvalues [38, 40]. See [53] for a full account.

### 2.2. Schubert calculus of enumerative geometry

The Schubert calculus of enumerative geometry consists of all problems of determining the linear subspaces that have specified positions with respect to other fixed, but general linear spaces. We broadly interpret it as the class of geometric problems which may be formulated as intersecting sufficiently general Schubert varieties in flag manifolds. We will only describe the Schubert calculus on the Grassmannian in full.

The *Grassmannian* $\text{Gr}(k, n)$ is the set of all $k$-planes of $\mathbb{C}^n$, which is an algebraic manifold of dimension $k(n-k)$. A *flag* $F_\bullet$ is a sequence of linear subspaces $F_\bullet : F_1 \subset F_2 \subset \cdots \subset F_n$, where $\dim F_i = i$. A *partition* $\lambda : (n-k) \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0$ is a weakly decreasing sequence of integers. A fixed flag $F_\bullet$ and a partition $\lambda$ determine a *Schubert variety*

$$X_\lambda F_\bullet := \{H \in \text{Gr}(k, n) \mid \dim H \cap F_{n-k+i-\lambda_i} \geq i, \text{ for } i = 1, \ldots, k\},$$

which has codimension $|\lambda| := \lambda_1 + \cdots + \lambda_k$ in $\mathrm{Gr}(k,n)$. We often denote a partition $\lambda$ by its Young diagram, thus $\square$ denotes the partition $(1,0,\ldots,0)$. As $\dim H\cap F_{n-k+i} \geq i$ for all $i$ and any $k$-plane $H$ and flag $F_\bullet$, the Schubert variety $X_{\square}F_\bullet$ consists of those $H$ with $H\cap F_{n-k} \supsetneq \{0\}$. As $|\square| = 1$, this is a hypersurface Schubert variety.

A *Schubert problem* is a list of partitions $\boldsymbol{\lambda} = (\lambda^1,\ldots,\lambda^r)$ such that $|\lambda^1| + \cdots + |\lambda^r| = k(n-k)$. Given general flags $F^1_\bullet,\ldots,F^r_\bullet$, Kleiman's Transversality Theorem [35] asserts that the intersection

$$X_{\lambda^1}F^1_\bullet \cap X_{\lambda^2}F^2_\bullet \cap \cdots \cap X_{\lambda^r}F^r_\bullet \tag{2.3}$$

is transverse and therefore if it is nonempty it is zero-dimensional as $|\lambda^1|+\cdots+|\lambda^r| = k(n-k)$. The number, $d(\boldsymbol{\lambda})$, of points in (2.3) is independent of the choice of general flags. A zero-dimensional intersection is an *instance* of the Schubert problem $\boldsymbol{\lambda}$. The points in (2.3) are the *solutions* to that instance. We may write Schubert problems multiplicatively. For example, we write $\square\cdot\square\cdot\square\cdot\square = \square^4$ for the Schubert problem $(\square,\square,\square,\square)$ in $\mathrm{Gr}(2,4)$—this is the problem of four lines in $\mathbb{P}^3$. Then the number (2.2) is $d(\square^{k(n-k)})$.

Any rational normal curve is projectively equivalent to the curve $\gamma(t) := (1, t, t^2/2, t^3/3!, \ldots, t^{n-1}/(n-1)!)$. For $t \in \mathbb{C}$ and any $1 \leq i \leq n$, the $i$-plane osculating the curve $\gamma$ at $\gamma(t)$ is

$$F_i(t) \ := \ \mathrm{span}\{\gamma(t), \gamma'(t), \ldots, \gamma^{(i-1)}(t)\}\,.$$

The flag *$F_\bullet(t)$ osculating* $\gamma$ at $\gamma(t)$ is the flag whose subspaces are the $F_i(t)$. The limit of $F_i(t)$ as $t \to \infty$ is the linear span of the last $i$ standard basis vectors, and these subspaces form the flag $F_\bullet(\infty)$. An instance of a Schubert problem $\boldsymbol{\lambda}$ given by flags osculating $\gamma$ is an *osculating instance* of $\boldsymbol{\lambda}$. Osculating flags are not general in the sense of Kleiman's Theorem [35], as shown in [43, § 2.3.6].

The osculating Schubert calculus naturally arises in the study of linear series on $\mathbb{P}^1$, where ramification at a point $t$ corresponds to membership in a Schubert variety $X_\lambda F_\bullet(t)$. An elementary consequence is the following useful proposition which provides a substitute for Kleiman's Theorem for osculating flags.

**Proposition 2.** *Let $\lambda^1,\ldots,\lambda^r$ be partitions and $t_1,\ldots,t_r$ be distinct points of $\mathbb{P}^1$. Then the intersection*

$$X_{\lambda^1}F_\bullet(t_1) \cap X_{\lambda^2}F_\bullet(t_2) \cap \cdots \cap X_{\lambda^r}F_\bullet(t_r), \tag{2.4}$$

*when nonempty, has dimension $k(n-k) - |\lambda^1| - \cdots - |\lambda^r|$, so that if $\lambda^1,\ldots,\lambda^r$ is a Schubert problem, it is zero-dimensional. Furthermore,* *if $H \in \mathrm{Gr}(k,n)$, then there is a unique Schubert problem $\lambda^1, \ldots, \lambda^r$ and unique points $t_1, \ldots, t_r \in \mathbb{P}^1$ such that $H$ lies in the intersection* (2.4)*.*

Schubert problems are efficiently represented on a computer through local coordinates and determinantal equations. The set of all $k$-planes $H \in \mathrm{Gr}(k,n)$ not in $X_{\square}F_\bullet(\infty)$ is identified with the space of $k \times (n-k)$ matrices $X$ via $X \mapsto$ row space$[I_k : X]$, where $I_k$ is the identity matrix. This forms a dense open subset of the Grassmannian and the entries of $X$ give local coordinates for $\mathrm{Gr}(k,n)$.

We formulate membership in Schubert varieties (and thus Schubert problems) in terms of determinantal equations. If we represent an $i$-plane $F_i$ as the row space of a full rank $i \times n$ matrix (also denoted by $F_i$) and $H$ by a $k \times n$ matrix, then

$$\dim H \cap F_i \geq j \iff \mathrm{rank}\begin{bmatrix} H \\ F_i \end{bmatrix} \leq k + i - j \ , \tag{2.5}$$

which is defined by the vanishing of all $(k{+}i{-}j{+}1) \times (k{+}i{-}j{+}1)$ minors. Therefore, when representing a flag $F_\bullet$ by a $n \times n$ matrix whose first $i$ rows span $F_i$, (2.5) (with $i$ replaced by $n{-}k{+}j{-}\lambda_j$) gives polynomial equations for the Schubert variety $X_\lambda F_\bullet$ in the affine patch $[I_k : X]$. When desired, we may use similar smaller coordinate patches for $X_\lambda F_\bullet(\infty)$ and $X_\lambda F_\bullet(\infty) \cap X_\mu F_\bullet(0)$.

### 2.3. Experimentation on a supercomputer

Equations for Schubert problems based on (2.5) may be solved in some sense using software tools, and information extracted that is relevant to the questions we are studying (e.g. real solutions or Galois groups), again using software tools. This ability to study individual instances of Schubert problems on a computer becomes a powerful method of investigation when automated, for literally billions of instances of thousands to millions of Schubert problems may be studied.

The challenge posed by scaling computations from the few score to the billions is two-fold—it requires careful organization *and* access to computational resources. The fundamental observation which allows this scale of investigation is that it is intrinsically parallel. Computing/studying one instance of a Schubert problem is independent of any other instance. This enables us to take advantage of current widely available computer resources—multiprocessor computational servers, established computer clusters, as well as *ad hoc* resources for our investigations. For example, most of the experimentation in [18, 24] was done on the Calclabs, which consists of over 200 Linux workstations that moonlight as a Beowulf cluster—their day job being calculus instruction, and that in [26] used the brazos cluster at Texas A&M University in which our research group controls 20 eight-core nodes.

The challenge of organizing a computational investigation on this scale, as well as ensuring that it is repeatable and robust, is met through modern software tools. These include organizing the computation with a database, monitoring it with web-based tools, running the computation using a job scheduler, as well as the core code itself, written in a scripting language to communicate with the database and organize the parts of the computation which are carried out by special purpose optimized software that is either widely available or written by our team.

The structure of these investigations is due to Chris Hillar. A detailed description of the experimental design and its execution is in [30], which explains our paradigm for large-scale experimentation using supercomputers and modern software tools. We give few details here, more may be found in the individual papers referenced.

## §3. History and generalizations of the Shapiro Conjecture

The Shapiro Conjecture of Subsection 2.1 may alternatively be formulated in terms of the osculating Schubert calculus.

**Shapiro Conjecture** (Theorem of Mukhin, Tarasov, and Varchenko [38, 40])**.** *Let* $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^r)$ *be a Schubert problem in* $\mathrm{Gr}(k,n)$ *and let* $t_1, \dots, t_r$ *be distinct real numbers. The intersection*

$$X_{\lambda^1} F_\bullet(t_1) \cap X_{\lambda^2} F_\bullet(t_2) \cap \cdots \cap X_{\lambda^r} F_\bullet(t_r) \tag{3.1}$$

*is transverse and consists of* $d(\boldsymbol{\lambda})$ *real points.*

The connection between this and the Wronskian formulation of Subsection 2.1 is given carefully in [53] and [55, Ch. 10]. The main idea is straightforward. A univariate polynomial $f(t)$ of degree at most $n-1$ is a linear form $\Lambda$ evaluated on the rational normal curve $\gamma(t)$. A linearly independent set $f_1, \dots, f_k$ of univariate polynomials of degree $n-1$ gives independent linear forms $\Lambda_1, \dots, \Lambda_k$. Thus $H := \ker(\Lambda_1, \dots, \Lambda_k)$ lies in $\mathrm{Gr}(n-k, n)$. The following is a calculation.

**Lemma 3.** *Let* $f_1, \dots, f_k$ *be polynomials in* $\mathbb{C}_{n-1}[t]$ *coming from independent linear forms* $\Lambda_1, \dots, \Lambda_k$ *and set* $H := \ker(\Lambda_1, \dots, \Lambda_k)$*. Then* $t$ *is a root of the Wronskian* $\mathrm{Wr}(f_1, \dots, f_k)$ *if and only if* $H \in X_{\square} F_\bullet(t)$*.*

### 3.1. The Shapiro Conjecture for other flag manifolds

Let $\eta$ be a regular nilpotent element of the Lie algebra of a group $G$ (the closure of its adjoint orbit contains all nilpotents). Then $t \mapsto$

$\exp(t\eta) \in G$ is a subgroup $\Gamma(t)$ of $G$. When $G = \mathrm{SL}_n\mathbb{C}$ and $\eta$ has all entries zero except for a 1 in each position $(i, i{+}1)$, the matrix $\Gamma(t)$ represents the flag $F_\bullet(t)$ with $\Gamma(t).F_\bullet(0) = F_\bullet(t)$. Then (3.1) becomes

$$\Gamma(t_1).X_{\lambda^1}F_\bullet(0) \cap \Gamma(t_2).X_{\lambda^2}F_\bullet(0) \cap \cdots \cap \Gamma(t_r).X_{\lambda^r}F_\bullet(0)\,.$$

This gives osculating instances of Schubert problems, and therefore a version of the Shapiro Conjecture, for any flag manifold.

Purbhoo proved that the Shapiro Conjecture holds for the orthogonal Grassmannian [42], but it is known to fail for other non-Grassmannian flag manifolds. For the Lagrangian Grassmannian and type $A$ flag manifolds, the conjecture may be repaired. For the Lagrangian Grassmannian, see [53, Sec. 7.1]. For flag manifolds of type $A$ a counterexample was found in [50]. We present the simplest counterexample.

*Example* 4. Let $\mathbb{F}\ell(2,3;4)$ be the manifold of flags $m \subset M$ in $\mathbb{P}^3$ where $m$ is a line and $M$ is a plane. Consider the problem in $\mathbb{F}\ell(2,3;4)$ where $m$ meets three lines $\ell_1$, $\ell_2$, and $\ell_3$ and $M$ contains two points $p, q$. Then $M$ contains the line $\overline{pq}$ and so $m$ meets $\overline{pq}$. Thus $m$ is one of two solutions to the problem of four lines given by $\ell_1$, $\ell_2$, $\ell_3$, and $\overline{pq}$, and $M$ is the span of $m$ and $\overline{pq}$.

Consider osculating instances of this Schubert problem where the lines are tangent to the rational normal curve $\gamma$ of Example 1 and the points lie on $\gamma$. Let $\ell(1)$, $\ell(0)$, and $\ell(-1)$ be the tangent lines and $\gamma(s), \gamma(t)$ the points. Consider the auxiliary problem of $m$ meeting these three tangent lines as well as the secant line $\ell(s,t) := \overline{\gamma(s)\gamma(t)}$.

When $0 < s < t < 1$ as in Fig. 4, the line $\ell(s,t)$ meets the hyper-

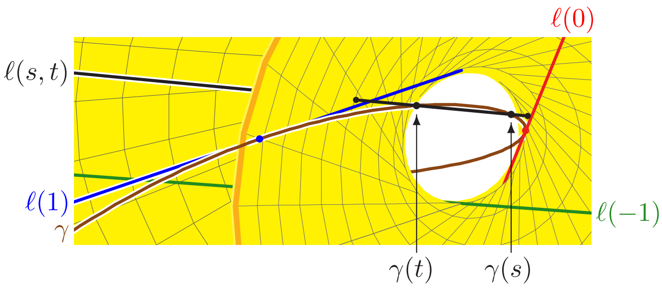



Fig. 4. A secant line meeting the hyperboloid.

boloid in two real points. As before, there are two real lines $m$ and two real solutions $m \subset M$ to our Schubert problem.

In contrast, Fig. 5 shows an example when $-1 < s < 0 < t < 1$ and the secant line $\ell(s,t)$ does not meet the hyperboloid in two real points. In this case, the two solutions $m$ to the auxiliary problem are

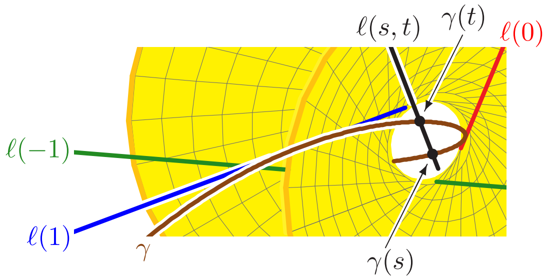



Fig. 5. A secant line not meeting the hyperboloid.

complex conjugates, and the same is true for the solutions $m \subset M$ to our Schubert problem. Thus the assertion of the Shapiro Conjecture does not hold for this Schubert problem.

This failure of the Shapiro Conjecture is particularly interesting. If we label the points $-1, 0, 1$ with 1 to indicate they are conditions on the line $m$, and the points $s, t$ with 2 to indicate they are conditions on the plane $M$, then these labels occur in the following orders along $\gamma$

(3.2) 11122 in Fig. 4 and 11212 in Fig. 5.

The first sequence is *monotone* and both solutions are always real, while the second sequence is not monotone and the two solutions are not necessarily real. This leads to a version of the Shapiro Conjecture for flag manifolds, and to two further extensions. We formalize these ideas.

Let $\boldsymbol{\alpha}_\bullet : 0 < a_1 < \cdots < a_p < n$ be a sequence of integers. A *flag $E_\bullet$ of type $\boldsymbol{\alpha}_\bullet$* is a sequence of linear subspaces $E_\bullet \colon E_{a_1} \subset E_{a_2} \subset \cdots \subset E_{a_p}$, were $\dim E_{a_i} = a_i$. The set of all such sequences is the *flag manifold $\mathbb{F}\ell(\boldsymbol{\alpha}_\bullet; n)$*, which has dimension $\dim(\boldsymbol{\alpha}_\bullet) := \sum_{i=1}^{p}(n-a_i)(a_i-a_{i-1})$, where $a_0 = 0$. When $p = 1$, this is the Grassmannian $\mathrm{Gr}(a_1, n)$.

Consider the projections $\pi_{a_i} \colon \mathbb{F}\ell(\boldsymbol{\alpha}_\bullet; n) \to \mathrm{Gr}(a_i, n)$ given by $E_\bullet \mapsto E_{a_i}$. A *Grassmannian Schubert variety* has the form $\pi_b^{-1} X_\lambda F_\bullet$, where $b \in \boldsymbol{\alpha}_\bullet$ and $\lambda$ is a partition for $\mathrm{Gr}(b, n)$. Write $X_{(\lambda,b)}F_\bullet$ for $\pi_b^{-1} X_\lambda F_\bullet$. A list $(\boldsymbol{\lambda}, \mathbf{b}) := \big((\lambda^1, b_1), (\lambda^2, b_2), \ldots, (\lambda^r, b_r)\big)$, with $|\lambda^1| + \cdots + |\lambda^r| = \dim \mathbb{F}\ell(\boldsymbol{\alpha}_\bullet; n)$ is a *Grassmannian Schubert problem*. A list of real numbers $t_1, \ldots, t_r \in \mathbb{R}$ is *monotone* with respect to the Grassmannian Schubert problem $(\boldsymbol{\lambda}, \mathbf{b})$, if $t_i < t_j$ whenever $b_i < b_j$. More generally, if $\prec$ is any cyclic order on $\mathbb{RP}^1$, then $t_1, \ldots, t_r \in \mathbb{RP}^1$ is monotone with respect to $(\boldsymbol{\lambda}, \mathbf{b})$, if $b_i < b_j \Rightarrow t_i \prec t_j$.

**Monotone Conjecture.** *Let* $(\boldsymbol{\lambda}, \mathbf{b}) = \big((\lambda^1, b_1), \ldots, (\lambda^r, b_r)\big)$ *be a Grassmannian Schubert problem in* $\mathbb{F}\ell(\boldsymbol{\alpha}_\bullet; n)$*. If* $t_1, \ldots, t_r \in \mathbb{RP}^1$ *is monotone with respect to* $(\boldsymbol{\lambda}, \mathbf{b})$*, then intersection*

$$X_{(\lambda^1,b_1)}F_\bullet(t_1) \cap X_{(\lambda^2,r_2)}F_\bullet(t_2) \cap \cdots \cap X_{(\lambda^r,b_r)}F_\bullet(t_r)$$

*is transverse with all points of intersection real.*

This conjecture was first noted in [51]. A formulation for two- and three-step flags was given in [52] together with computational evidence supporting it. The general statement was made in [43], which reported on an experiment testing 1,140 Schubert problems in 29 flag manifolds, solving more than 525 million random instances and verifying the Monotone Conjecture in each of more than 158 million monotone instances. These computations took 15.76 gigaHertz-years.

We explain how the number of real solutions was determined. For a Grassmannian Schubert problem $\big((\lambda^1, b_1), \ldots, (\lambda^r, b_r)\big)$, select $r$ random points on the rational normal curve $\gamma$ and construct osculating flags. Using these flags, represent the Schubert problem as a system of polynomial equations given by the determinantal conditions in (2.5) in some system of local coordinates for $\mathbb{F}\ell(\boldsymbol{\alpha}_\bullet; n)$. Then eliminate all but one variable from the equations, obtaining an eliminant. When the eliminant is square-free and has degree equal to the expected number of complex solutions, the Shape Lemma guarantees that the number of real solutions to the Schubert problem equals the number of real roots of the eliminant, which may be computed using Sturm sequences.

A given set of $r$ points is permuted in each of a predetermined set of orders along $\mathbb{RP}^1$ (called *necklaces*) to give different orders along $\mathbb{RP}^1$ in which the conditions are evaluated, and for each the number of real solutions is determined. The result is stored in a frequency table for that Schubert problem which records how often a given number of real solutions was observed for a given necklace.

To illustrate the data obtained in this experiment, consider the Schubert problem $(\square, 2)^7 \cdot (\square\square, 3)^2$ in $\mathbb{F}\ell(2, 3; 6)$, which looks for the flags $m \subset M$ in $\mathbb{C}^6$, where $m$ is a 2-plane meeting seven 4-planes and $M$ is a 3-plane meeting two 2-planes. This problem has 14 solutions. Table 1 records data from 800000 random osculating instances of this problem. The columns are indexed by even integers from 0 to 14 for the possible numbers of real solutions. The rows are indexed by the possible necklaces, using the notation of (3.2). The first row labeled with 222222233 represents tests of the Monotone Conjecture, verifying it in 200000 instances as the only entries lie in the column for 14 real solutions.

Table 1. Frequency table for $(\square, 2)^7 \cdot (\square\!\square, 3)^2$ in $\mathbb{F}\ell(2,3;6)$.

| | Real Solutions | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | Total |
| 222222233 | | | | | | | | 200000 | 200000 |
| 222223223 | | | 22150 | 8705 | 34833 | 45439 | 39481 | 49392 | 200000 |
| 222222323 | | | 24773 | 10591 | 14377 | 11029 | 8033 | 131197 | 200000 |
| 222232223 | 5 | 52 | 3146 | 16758 | 42337 | 66967 | 50282 | 20453 | 200000 |
| Total | 5 | 52 | 50069 | 36054 | 91547 | 123435 | 97796 | 401042 | 800000 |

Theoretical evidence in support of the Monotone Conjecture was provided by Eremenko, Gabrielov, Shapiro, and Vainshtein [15] who proved the conjecture for all Schubert problems in $\mathbb{F}\ell(n-2, n-1; n)$ and $\mathbb{F}\ell(1,2;n)$. Their result can be formulated in $\mathrm{Gr}(n-2, n)$, where it becomes a statement about real points of intersection of Schubert varieties given by flags that are secant to a rational normal curve $\gamma$ in a specific way. This condition on the secant flags makes sense for any Grassmannian, and leads to a second generalization of the Shapiro Conjecture.

A flag $F_\bullet$ is *secant along an interval $I$* of a rational normal curve $\gamma$ if each subspace $F_i$ is spanned by its points of intersection with $I$. Secant flags are *disjoint* if the intervals of secancy are pairwise disjoint.

**Secant Conjecture.** *Let $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^r)$ be a Schubert problem in $\mathrm{Gr}(k,n)$. If $F^1_\bullet, \dots, F^r_\bullet$ are disjoint secant flags, then intersection*

$$X_{\lambda^1}F^1_\bullet \cap X_{\lambda^2}F^2_\bullet \cap \cdots \cap X_{\lambda^r}F^r_\bullet$$

*is transverse with all points of intersection real.*

The Secant Conjecture holds in two special cases beyond the Grassmannian $\mathrm{Gr}(n-2, n)$ that was shown in [15]. A family of secant flags becomes osculating in the limit as the intervals of secancy shrink to a point. In this way, the limit of the Secant Conjecture is the Shapiro Conjecture (Theorem of Mukhin, Tarasov, and Varchenko [38, 40]) and so the Secant Conjecture is true when the points of secancy are sufficiently clustered. The special case when the points of secancy form arithmetic sequences and the Schubert problem is $\square^{k(n-k)}$ was shown in [39].

The strongest evidence for the Secant Conjecture is an experiment that used 1.07 teraHertz-years of computation, testing more than 498 million instances of the Secant Conjecture in 703 Schubert problems in 13 Grassmannians. This is reported in [18]. As with the Monotone Conjecture, these computations relied upon counting the number of real roots of an eliminant. Table 2 displays the data obtained for the Schubert problem $(3,1)^4$ in $\mathrm{Gr}(4,8)$ with 9 solutions. Its rows are indexed by the odd numbers from 1 to 9 for the possible number of real solutions.

Table 2. Real solutions to $\yng(3,1)^4$ in $\mathrm{Gr}(4,8)$.

| Real Solutions \ Overlap Number | 0 | 1 | 2 | 3 | 4 | 5 | 6 | $\cdots$ | Total |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | 16 | $\cdots$ | 758 |
| 3 | | | | | 7 | 612 | 783 | $\cdots$ | 18276 |
| 5 | | | | 123 | 659 | 4541 | 4847 | $\cdots$ | 79173 |
| 7 | | | | 158 | 663 | 3804 | 4545 | $\cdots$ | 91536 |
| 9 | 141420 | | 4051 | 7937 | 11241 | 17310 | 15705 | $\cdots$ | 310257 |
| Total | 141420 | 0 | 4051 | 8218 | 12570 | 26267 | 25896 | $\cdots$ | 500000 |

The columns are indexed by the *overlap number*, which measures intersections between secant flags. The overlap number is 0 if and only if the flags are disjoint. Thus, the first column in Table 2 represents tests of the Secant Conjecture, verifying it in the 141420 instances computed.

The column corresponding to overlap number one is empty as this cannot be attained by the intervals of secancy for this problem. Another interesting feature in Table 2 is the column corresponding to overlap number two, which are flags that are very slightly non-disjoint. For this column, the solutions were also all real, while in the next column, at least five were real. It is only with overlap number six and beyond that we found instances with only one real solution.

The Monotone Conjecture and Secant Conjecture have a common generalization, the Monotone-Secant Conjecture. Disjoint flags have a naturally occurring order along $\mathbb{RP}^1$. A list $F_\bullet^1, \dots, F_\bullet^r$ of disjoint secant flags is *monotone* with respect to a Grassmannian Schubert problem $(\boldsymbol{\lambda}, \mathbf{b})$, if $F_\bullet^i$ proceeds $F_\bullet^j$ whenever $b_i < b_j$.

**Monotone-Secant Conjecture.** *Let* $(\boldsymbol{\lambda}, \mathbf{b}) = \big((\lambda^1, b_1), \dots, (\lambda^r, b_r)\big)$ *be a Grassmannian Schubert problem in* $\mathbb{F}\ell(\boldsymbol{\alpha}_\bullet; n)$*. If a list* $F_\bullet^1, \dots, F_\bullet^r$ *of disjoint secant flags is monotone with respect to* $(\boldsymbol{\lambda}, \mathbf{b})$*, then intersection*

$$X_{(\lambda^1,b_1)}F_\bullet^1 \cap X_{(\lambda^2,b_2)}F_\bullet^2 \cap \cdots \cap X_{(\lambda^r,b_r)}F_\bullet^r$$

*is transverse with all points of intersection real.*

This conjecture was studied on a supercomputer. By then end of 2013, we tested over 11 billion instances of 1300 Schubert problems taking 1.901 teraHertz-years. Of these, 256 million were instances of the Monotone-Secant Conjecture where the conjecture was verified. We also tested 263 million instances of the Monotone Conjecture for comparison. Table 3 displays the data obtained for Monotone-Secant instances of the Schubert problem $(\yng(1), 2)^7 \cdot (\yng(2), 3)^2$ in $\mathbb{F}\ell(2,3;6)$. This is the same problem studied in Table 1 and the notation is the same. These two tables are similar, except that the data in Table 3 suggest a lower bound

Table 3. Frequency table for $(\square, 2)^7 \cdot (\square\square, 3)^2$ in $\mathbb{F}\ell(2,3;6)$.

| | Real Solutions | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | Total |
| 222222233 | | | | | | | | 400000 | 400000 |
| 222223223 | | | 131815 | 51761 | 92849 | 73988 | 27054 | 22533 | 400000 |
| 222222323 | | | 142271 | 43847 | 36252 | 40595 | 22399 | 114636 | 400000 |
| 222232223 | | | 419 | 2881 | 27328 | 89208 | 195921 | 84243 | 400000 |
| Total | 0 | 0 | 274505 | 98489 | 156429 | 203791 | 245374 | 621412 | 1600000 |

of four for the number of real solutions. This is an illusion. The Monotone Conjecture is a limiting case of the Monotone-Secant Conjecture, and for any selection of osculating flags, there is are sufficiently nearby disjoint secant flags occuring in the same order. Thus, from the computations in the last row of Table 1, we know there exist disjoint secant flags with necklace 222232223 having no real solutions, and disjoint secant flags with two real solutions, even though these were not observed in the experiment.

This idea shows that there should be *fewer* restrictions on the numbers of real solutions for secant flags than for osculating flags. Typically, we observe that the tables for osculating and secant flags look basically the same, with a few exceptions. We do not understand why the tables are so similar, and why in some cases they differ slightly.

## §4. Lower bounds and gaps on the number of real solutions

By the Theorem of Mukhin, Tarasov, and Varchenko, any osculating instance of a Schubert problem in a Grassmannian with real osculation points has all solutions real. The set of solutions forms a real variety, but there are other ways for an osculating instance to define a real variety (e.g. some pairs of osculation points are complex conjugates). Work of Eremenko and Gabrielov [11] suggests that there may be lower bounds on the numbers of real solutions to such real osculating instances of Schubert problems. We explain the background, describe an experiment to study this question of additional structure, and give some results that have been inspired by this experimentation. This work formed part of the 2013 Ph.D. thesis of Nickolas Hein.

### 4.1. Topological lower bounds

Eremenko and Gabrielov [11] considered the Wronski map (2.1) restricted to spaces of real polynomials,

$$\mathrm{Wr}_{\mathbb{R}} \;:\; \mathrm{Gr}(k, \mathbb{R}_{n-1}[t]) \;\longrightarrow\; \mathbb{P}(\mathbb{R}_{k(n-k)}[t])\,. \qquad (4.1)$$

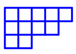
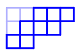
These are real manifolds of dimension $k(n-k)$. A point $\Psi \in \mathbb{P}(\mathbb{R}_{k(n-k)}[t])$ is a *regular value* if the differential of Wr is nonsingular at all points in the fiber above $\Psi$. If the manifolds in (4.1) were oriented, the Wronski map would have a well-defined topological degree which is computed on the fiber over any regular value $\Psi \in \mathbb{P}(\mathbb{R}_{k(n-k)}[t])$,

$$\deg \mathrm{Wr}_{\mathbb{R}} \;=\; \sum_{H \in \mathrm{Wr}^{-1}(\Psi)} \operatorname{sign} d\, \mathrm{Wr}_{\mathbb{R}}(H)\,,$$

where $\operatorname{sign} d\,\mathrm{Wr}_{\mathbb{R}}(H)$ is 1 if the orientations at $H$ and $\Psi$ agree and $-1$ if they do not. The point is that if the spaces in (4.1) were oriented so that $\deg \mathrm{Wr}_{\mathbb{R}}$ is defined, then $|\deg \mathrm{Wr}_{\mathbb{R}}|$ would be a lower bound on the number of real points in a fiber above a regular value $\Psi$.

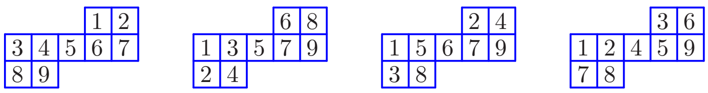
While the Grassmannian and projective space in (4.1) are often not orientable, they have orientable double covers (the oriented Grassmannian and the sphere, respectively), so the degree may be computed on the double cover and its absolute value gives a lower bound on the number of points of the fiber $\mathrm{Wr}^{-1}(\Psi)$. Eremenko and Gabrielov more generally considered the Wronski map (4.1) restricted to real Schubert varieties $X_\lambda F_\bullet(\infty)$. Soprunova and Sottile [46, Th. 6.4] extended this to Richardson varieties $X_\lambda F_\bullet(\infty) \cap X_\mu F_\bullet(0)$.

Given a partition $\lambda$, let $\lambda^c$ be $n-k-\lambda_k \geq \cdots \geq n-k-\lambda_1$, the complement of its Young diagram in the $k \times (n-k)$ rectangle. For $\lambda = (3,1)$ and $k = 3, n = 8$, we have $\lambda^c = (5,4,2)$. When $\mu \subset \lambda$, the *skew diagram* $\lambda/\mu$ is $\lambda$ with the boxes of $\mu$ removed. For example, if

$\lambda \;=$  and $\mu \;=$  then $\lambda/\mu \;=$  .

When $\mu = (0)$, we have $\lambda/\mu = \lambda$.

A *Young tableau* of *shape* $\lambda/\mu$ is a filling of the boxes of $\lambda/\mu$ with the integers $1, 2, \ldots, |\lambda|-|\mu|$ that increases across each row and down each column. The *standard filling* is when the numbers are in reading order. For example, here are four (of the 324) Young tableaux of shape $(5,5,2)/(3)$ and the first is the standard filling.



Let $\mathrm{YT}(\lambda/\mu)$ be the set of all Young tableaux of shape $\lambda/\mu$.

Given partitions $\lambda, \mu$, the Wronski map restricts to give a map

$$\mathrm{Wr}_{\lambda,\mu} \;:\; X_\lambda F_\bullet(\infty) \cap X_\mu F_\bullet(0) \;\longrightarrow\; \mathbb{P}\big(t^{|\mu|}\mathbb{C}_{k(n-k)-|\lambda|-|\mu|}[t]\big) \qquad (4.2)$$

whose degree is equal to the number of Young tableaux of shape $\lambda^c/\mu$. Thus when $k=3$, $n=8$, and $\lambda=\mu=\yng(3)$, the degree of $\mathrm{Wr}_{\lambda,\mu}$ is 324. Restricting to the real points, the map (4.2) becomes a map between the real Richardson variety and the real projective space. Lifting to double covers as before, it has a topological degree (the singularities of the Richardson variety cause no harm, as they are in codimension 2).

Every tableau $\mathcal{T}\in\mathrm{YT}(\lambda^c/\mu)$ has a sign, $\mathrm{sgn}(\mathcal{T})$, which is the sign of the permutation mapping the standard filling to $\mathcal{T}$. The *sign-imbalance* $\sigma(\lambda^c/\mu)$ of $\mathrm{YT}(\lambda^c/\mu)$ is defined to be

$$\sigma(\lambda^c/\mu) \ := \ \Bigl|\sum_{\mathcal{T}\in\mathrm{YT}(\lambda^c/\mu)} \mathrm{sgn}(\mathcal{T})\Bigr| .$$

For $\lambda=\mu=\yng(3)$, we have $\sigma((552)/(3))=\sigma(\young(:::\ \ ,\ \ \ \ \ ,\ \ ))=4$.

**Theorem 5** ([11, 46])**.** *The restriction* $\mathrm{Wr}_{\lambda,\mu}$ (4.2) *of the real Wronski map has topological degree whose absolute value is equal to the sign-imbalance* $\sigma(\lambda^c/\mu)$*.*

This sign-imbalance is thus a *topological lower bound* on the number of real points in a fiber of the real Wronski map. It is not known in general whether this lower bound is attained. Eremenko and Gabrielov showed that when $\lambda=\mu=\emptyset$, the topological lower bound is positive when $n$ is odd and it is zero when $n$ is even [11]. Later, they showed that when both $k$ and $n$ are even, the map $\mathrm{Wr}_{\mathbb{R}}$ is not surjective [12], so the topological lower bound of zero is attained when both $k$ and $n$ are even. All other cases remained open.

### 4.2. Real osculating instances of Schubert problems

An osculating instance of a Schubert problem

$$X \ = \ X_{\lambda^1}F_\bullet(t_1)\cap X_{\lambda^2}F_\bullet(t_2)\cap\cdots\cap X_{\lambda^r}F_\bullet(t_r) \tag{4.3}$$

is *real* if $X$ equals its complex conjugate $\overline{X}$. By Proposition 2, this means that for each $i=1,\dots,r$, either $t_i$ is real or there is a unique $j\neq i$ with $\lambda^j=\lambda^i$ and $t_j=\overline{t_i}$.

The *type* of a real osculating instance (4.3) of a Schubert problem $\lambda^1,\dots,\lambda^r$ is the list $(\rho_\lambda\mid\lambda\in\{\lambda^1,\dots,\lambda^r\})$ where $\rho_\lambda$ is the number of indices $i$ with $t_i$ real and $\lambda=\lambda^i$. For example, the Schubert problem

$$X_{\square}F_\bullet(0)\cap X_{\square}F_\bullet(\infty)\cap X_{\square}F_\bullet(1)\cap X_{\yng(2,1)}F_\bullet(\sqrt{-1})\cap X_{\yng(2,1)}F_\bullet(-\sqrt{-1})$$

in $\mathrm{Gr}(3,6)$ has type $(\rho_{\square},\rho_{\yng(2,1)})=(3,0)$. The Theorem of Mukhin, Tarasov, and Varchenko involves real osculating instances of maximal type, where $\rho_\lambda=\#\{i\mid\lambda^i=\lambda\}$.

The experimentation and results we describe shed light on structures in the possible numbers of real solutions to real osculating instances of Schubert problems which may depend on type.

### 4.3. Experiment

Hein, Hillar, and Sottile set up and ran a computational experiment using the framework of the projects [18, 24, 30] described in Section 3 to investigate structure in the numbers of real solutions to real osculating instances of Schubert problems that depend upon the osculation type of the instance. This followed the broad outline of those projects, with some differences. These differences included that it was mostly run on the brazos cluster at Texas A&M and did not use Maple, relying instead on Singular's [8] `nrroots` command from the `rootsur` [58] library which computes the numbers of real roots of a real univariate polynomial.

Since Singular (and symbolic software in general) does not perform efficiently over the field $\mathbb{Q}[\sqrt{-1}]$, this experiment used a slightly different formulation of Schubert problems than indicated in Subsection 2.2.

**Proposition 6.** *Suppose that $t \in \mathbb{C}$ is not real and $S$ is the collection of minors of matrices $\left[\begin{smallmatrix} I_k : X \\ F_i(t) \end{smallmatrix}\right]$ that define $X_\lambda F_\bullet(t)$ in the local coordinates $[I_k : X]$. The intersection $X_\lambda F_\bullet(t) \cap X_\lambda F_\bullet(\bar{t})$ is defined in $[I_k : X]$ by the real polynomials*

$$\{\Re(f) + \Im(f) \mid f \in S\}\,,$$

*the real and imaginary parts of polynomials in $S$.*

The data from the experiment are available on line [26] and are presented as before in frequency tables for each Schubert problem showing the observed numbers of real solutions to osculating intersections of a given type. For example, Table 4 shows the frequency table for the problem $\square\!\square\!\square\!\square\!\square \cdot \square^7$ with six solutions. Note that $\rho_{\square}-1$ is the apparent lower

Table 4. Frequency table for $\square\!\square\!\square\!\square\!\square \cdot \square^7 = 6$ in $\mathrm{Gr}(2,8)$.

| | Number of Real Solutions | | | | |
|---|---|---|---|---|---|
| $\rho_{\square}$ | 0 | 2 | 4 | 6 | Total |
| 1 | 8964 | 67581 | 22105 | 1350 | 100000 |
| 3 | | 47138 | 47044 | 5818 | 100000 |
| 5 | | | 77134 | 22866 | 100000 |
| 7 | | | | 100000 | 100000 |
| Total | 8964 | 114719 | 146283 | 130034 | 400000 |

bound for the minimal number of real solutions as a function of the type

$\rho_{\yng(1)}$. The observed lower bound of zero for this Schubert problem is the sign-imbalance of Theorem 5, as $\yng(5)^c$ has sign-imbalance zero.

This experiment studied 756 Schubert problems, 273 of which had a topological lower bound given by Theorem 5. These included the Wronski maps for Gr(2, 4), Gr(2, 6), and Gr(2, 8) for which Eremenko and Gabrielov had shown the lower bound of zero was sharp. For 264 of the remaining 270 cases, the sharpness of the topological lower bound was verified. There were six Schubert problems for which the topological lower bounds were not observed. These were

$$\left(\yng(3,3),\yng(1,1,1),\yng(1)^7\right),\ \left(\yng(2,2,2),\yng(3),\yng(1)^7\right),\ \left(\yng(3,3),\yng(2,2),\yng(1)^6\right),\ \left(\yng(2,2,2),\yng(2,2),\yng(1)^6\right),\ \left(\yng(3,3,2),\yng(1)^8\right),$$

all in Gr(4, 8), and $\yng(1)^9$ in Gr(3, 6). These have observed lower bounds of $3, 3, 2, 2, 2, 2$ and sign-imbalances of $1, 1, 0, 0, 0, 0$, respectively. There is not yet an explanation for the first four, but the last two are symmetric Schubert problems, which were observed to have a congruence modulo four on their numbers of real solutions. This congruence gives a lower bound of two for both problems $\yng(3,3,3)\cdot\yng(1)^8$ in Gr(4, 8) and $\yng(1)^9$ in Gr(3, 6). This will be discussed in Subsection 4.4.

There is another family of Schubert problems containing the problem $\yng(5)\cdot\yng(1)^7$ in Gr(2, 8) of Table 4. This family has one problem in each Grassmannian $\mathrm{Gr}(k, n)$—it involves a large rectangular partition and the partition $\yng(1)$ repeated $n-1$ times, such as $\yng(3,3)\cdot\yng(1)^6$ in Gr(3, 7). Each Schubert problem in this family has lower bounds which depend upon $\rho_{\yng(1)}$, as well as gaps in the possible numbers of real solutions. This is discussed in Subsection 4.5.

In addition to these families of Schubert problems, this experiment [26] found many Schubert problems with apparent additional structure to their numbers of real solutions to real osculating instances. However, the data did not suggest any other clear conjectures that would explain most of the observed structure. Table 5 shows another frequency table from this experiment.

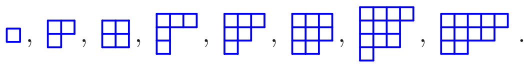
Table 5. Frequency table for $\yng(3,3)\cdot\yng(1,1)^5 = 10$ in Gr(4, 8).

Number of Real Solutions

| $\rho_{\yng(1,1)}$ | 0 | 2 | 4 | 6 | 8 | 10 | Total |
|---|---|---|---|---|---|---|---|
| 1 | | 138225 | 49674 | 2077 | 5404 | 4620 | 200000 |
| 3 | | | 163693 | 6458 | 8142 | 21707 | 200000 |
| 5 | | | | | | 200000 | 200000 |
| Total | | 138225 | 213367 | 8535 | 1356 | 226327 | 600000 |

### 4.4. Symmetric Schubert problems

Partitions for $\mathrm{Gr}(k, 2k)$ are subsets of a $k \times k$ square. A partition $\lambda$ is *symmetric* if it equals its matrix transpose. All except the last of the following partitions are symmetric,



For a symmetric partition $\lambda$, let $\ell(\lambda)$ be the number of boxes on its main diagonal, which is the maximum number $i$ with $i \leq \lambda_i$. Thus $\ell(\square) = \ell($$) = 1$ and $\ell($$) = 2$. A Schubert problem $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^r)$ is *symmetric* if each partition $\lambda^i$ is symmetric. The numbers of real solutions to real osculating instances of many symmetric Schubert problems obey a congruence modulo four.

**Theorem 7.** *Let $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^r)$ be a symmetric Schubert problem with $\sum_i \ell(\lambda^i) \geq k{+}4$. Then the number of real solutions to any real osculating instance of $\boldsymbol{\lambda}$ is congruent to $d(\boldsymbol{\lambda})$ modulo four.*

A weak version of Theorem 7 was proven in [28], where the full statement was conjectured, and finally proved in [27]. The fundamental idea is that the Grassmannian $\mathrm{Gr}(k, 2k)$ has an algebraic Lagrangian involution which restricts to an involution on the solutions to any osculating instance of the symmetric Schubert problem $\boldsymbol{\lambda}$. This involution commutes with complex conjugation on the set of solutions to a real osculating instance. When a codimension condition on the fixed points is satisfied, the interaction of these two involutions implies the congruence modulo four. Before sketching the main ideas in the proof of Theorem 7, we give an interesting Corollary.

**Corollary 8.** *Real osculating instances of the Schubert problem $\square^9$ in $\mathrm{Gr}(3, 6)$ with 42 solutions always have at least two real solutions, counted with multiplicity.*

Thus, the topological lower bound of zero for this Schubert problem is not sharp. A similar lack of sharpness holds for the symmetric Schubert problem ($, \square^8) = 90$ in $\mathrm{Gr}(4, 8)$.

Let $\langle \cdot, \cdot \rangle$ be a symplectic (non-degenerate and skew-symmetric) form on $\mathbb{C}^{2k}$, which we may assume is

$$\langle \mathbf{e}_i, \mathbf{e}_{2k+1-j} \rangle \;=\; (-1)^i \delta_{i,j}\,,$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_{2k}$ is the standard basis for $\mathbb{C}^{2k}$. Any subspace $V \subset \mathbb{C}^{2k}$ has an annihilator $V^{\perp}$ under $\langle \cdot, \cdot \rangle$,

$$V^{\perp} \;:=\; \{w \in \mathbb{C}^{2k} \mid \langle w, v \rangle = 0 \quad \forall v \in V\}\,.$$

As $\langle\cdot,\cdot\rangle$ is non-degenerate, we have $\dim V + \dim V^\perp = 2k$ and $(V^\perp)^\perp = V$, so that $\perp$ is an involution on the set of linear subspaces of $\mathbb{C}^{2k}$ which restricts to an involution on the Grassmannian $\mathrm{Gr}(k,2k)$. We have $(X_\lambda F_\bullet)^\perp = X_{\lambda^T} F_\bullet^\perp$, where $\lambda^T$ is the matrix transpose of $\lambda$. The *Lagrangian Grassmannian* $\mathrm{LG}(k)$ is the set of points of $\mathrm{Gr}(k,2k)$ that are fixed under this involution.

Given a flag $E_\bullet\colon E_{a_1}\subset\cdots\subset E_{a_p}$ its annihilators $E_{a_p}^\perp\subset\cdots\subset E_{a_1}^\perp$ form a flag $E_\bullet^\perp$. If $\mathbb{F}\ell(2k) := \mathbb{F}\ell(\{1,2,\ldots,2k{-}1\};2k)$ is the manifold of complete flags, then $F_\bullet \mapsto F_\bullet^\perp$ is an involution on $\mathbb{F}\ell(2k)$ whose fixed points are *symplectic flags*, the flag manifold for the symplectic group which preserves the form $\langle\cdot,\cdot\rangle$.

The Lagrangian Grassmannian has Schubert varieties $Y_\lambda F_\bullet$ which are given by a symmetric partition $\lambda$ and a symplectic flag. The Schubert variety $Y_\lambda F_\bullet$ is the set of fixed points of the Lagrangian involution acting on $X_\lambda F_\bullet$. The dimension of $\mathrm{LG}(k)$ is $\binom{k+1}{2}$ and the codimension of $Y_\lambda F_\bullet$ is $\|\lambda\| := \frac{1}{2}(\ell(\lambda)+|\lambda|)$.

If we take our rational normal curve to be

$$\gamma(t) \ := \ (1\,,\, t\,,\, \tfrac{1}{2}t^2\,,\, \tfrac{1}{6}t^3\,,\,\ldots\,,\, \tfrac{1}{(2k-1)!}t^{2k-1})\,, \tag{4.4}$$

then the flags $F_\bullet(t)$ osculating $\gamma$ are symplectic. (This curve and flag come from a regular nilpotent as in Subsection 3.1.)

Theorem 7 is a consequence of a more general result. An instance

$$X_{\lambda^1}F_\bullet^1 \,\cap\, X_{\lambda^2}F_\bullet^2 \,\cap\, \cdots \,\cap\, X_{\lambda^r}F_\bullet^r$$

of a Schubert problem is *real* if for every $i=1,\ldots,r$ there is a $j$ with $\lambda^i=\lambda^j$ and $\overline{F_\bullet^i}=F_\bullet^j$. The following is proven in [27].

**Theorem 9.** *Suppose that $\boldsymbol{\lambda}$ is a symmetric Schubert problem with $\sum_i \ell(\lambda^i)\geq k{+}4$. Then for any real instance of the Schubert problem $\boldsymbol{\lambda}$*

$$X_{\lambda^1}F_\bullet^1 \,\cap\, X_{\lambda^2}F_\bullet^2 \,\cap\, \cdots \,\cap\, X_{\lambda^r}F_\bullet^r \tag{4.5}$$

*where the flags $F_\bullet^i$ are symplectic, the number of real solutions is congruent to $d(\boldsymbol{\lambda})$ modulo four.*

Theorem 7 is the special case of Theorem 9 when the symplectic flags are osculating. The proof rests on a simple lemma from [28].

Suppose that $f\colon Y\to Z$ is a proper dominant map between complex algebraic varieties with $Z$ smooth, and that $Y$, $Z$, and $f$ are all defined over $\mathbb{R}$. The degree $d$ of $f$ is the number of complex points of $Y$ above any regular value $z\in Z$ of $f$. If $z\in Z(\mathbb{R})$ is a real regular value, then the number of real points in $f^{-1}(z)$ is congruent to $d$ modulo two. Suppose that the variety $Y$ has an involution $\iota\colon Y \to Y$ that preserves the fibers of $f$, so that $f(y) = f(\iota(y))$. We have the following.

**Lemma 10.** *If the image $f(Y^\iota)$ of the fixed points of $Y$ under $\iota$ has codimension at least two in $Z$, then for any real regular values $z, z'$ of $f$ lying in the same connected component of $Z(\mathbb{R})$, the number of real points in $f^{-1}(z)$ is congruent to the number of real points in $f^{-1}(z')$, modulo four.*

Lemma 10 is applied to a universal family $Y_{\boldsymbol{\lambda}} \to Z_{\boldsymbol{\lambda}}$ of instances of a symmetric Schubert problems $\boldsymbol{\lambda}$. The base $Z_{\boldsymbol{\lambda}}$ consists of $r$-tuples of symplectic flags $(F^1_\bullet, \ldots, F^r_\bullet)$ where if $\lambda^i = \lambda^j$, then the order of $F^i_\bullet$ and $F^j_\bullet$ does not matter. Specifically, $Z_{\boldsymbol{\lambda}}$ is the quotient of the $r$-fold product of symplectic flag manifolds by the subgroup $G$ of the symmetric group $S_r$ consisting of permutations $\sigma$ where if $\sigma(i) = j$, then $\lambda^i = \lambda^j$. It is a product of symmetric products of symplectic flag manifolds.

The fiber of $Y_{\boldsymbol{\lambda}}$ over a point $z = (F^1_\bullet, \ldots, F^r_\bullet)$ of $Z_{\boldsymbol{\lambda}}$ is the intersection (4.5). The main result of [54] states that when the point $z \in Z_{\boldsymbol{\lambda}}$ is general, the intersection (4.5) is zero-dimensional, and thus $Y_{\boldsymbol{\lambda}}$ has the same dimension as $Z_{\boldsymbol{\lambda}}$. (This does not follow from Kleiman's Transversality Theorem [35] as general symplectic flags are not general flags.)

The points of $Y_{\boldsymbol{\lambda}}$ fixed by the Lagrangian involution are intersections of the corresponding Schubert varieties in the Lagrangian Grassmannian. Kleiman's Theorem applies to those Lagrangian Schubert varieties which implies that the fixed points $Y^\iota_{\boldsymbol{\lambda}}$ have codimension

$$\sum_{i=1}^{r} \|\lambda^i\| \;-\; \binom{k+1}{2} \;=\; \frac{1}{2}\Bigl(\sum_{i=1}^{r} \ell(\lambda^i) \;-\; k\Bigr)\,.$$

The condition $\sum_i \ell(\lambda^i) \geq k+4$ implies that this codimension is at least two, so Lemma 10 applies. Lastly, the real points of $Z_{\boldsymbol{\lambda}}$ are connected and the Theorem of Mukhin, Tarasov, and Varchenko gives points of $Z_{\boldsymbol{\lambda}}(\mathbb{R})$ with all $d(\boldsymbol{\lambda})$ solutions real, which implies Theorems 9 and 7.

### 4.5. Lower bounds and gaps

Table 6 shows the result of computing 800000 osculating instances of the symmetric Schubert problem $\boxplus \cdot \square^7$ in $\mathrm{Gr}(4,8)$ with 20 solutions, recording the number of observed real solutions to osculating instances of a given type. The ellipses $\cdots$ mark columns (numbers of real solutions) that were not observed. The hypotheses of Theorem 7 hold, so the number of real solutions is congruent to 20 modulo 4. The lack of instances with 12 and 16 real solutions, and the triangular shape of the rest of the table (similar to the triangular shape of Table 4) are additional structures which we explain.

Table 6. Gaps and lower bounds for $\boxplus \cdot \square^7 = 20$ in $\mathrm{Gr}(4,8)$

Number of Real Solutions

| $\rho_\square$ | 0 | 2 | 4 | 6 | 8 | $\cdots$ | 20 | Total |
|---|---|---|---|---|---|---|---|---|
| 1 | 37074 | | 47271 | | 14517 | $\cdots$ | 1138 | 100000 |
| 3 | | | 66825 | | 30232 | $\cdots$ | 2943 | 100000 |
| 5 | | | | | 85080 | $\cdots$ | 14920 | 100000 |
| 7 | | | | | | $\cdots$ | 100000 | 100000 |
| Total | 37074 | | 114096 | | 129829 | $\cdots$ | 119001 | 400000 |

These Schubert problems $\boxplus \cdot \square^7$ and $\square\square\square\square\square \cdot \square^7$ are members of a family of Schubert problems whose osculating instances we may solve completely and thereby determine all possibilities for their numbers of real solutions. Details are given in [25].

For $k, n$, let $\boxplus_{k,n}$ ($\boxplus$ for short) be the partition consisting of $k{-}1$ parts, each of size $n{-}k{-}1$. For example,

$$\boxplus_{2,8} = \square\square\square\square\square\,, \quad \boxplus_{3,7} = \boxplus\,, \quad \text{and} \quad \boxplus_{4,8} = \boxplus\,.$$

The osculating Schubert problems in this family have the form $\boldsymbol{\lambda} = (\boxplus, \square^{n-1})$ in $\mathrm{Gr}(k,n)$, and they all have the topological lower bounds of Theorem 5. The multinomial coefficient $\binom{n}{a,b}$ is zero unless $n = a{+}b$, and in that case it equals $\frac{n!}{a!b!}$.

**Lemma 11.** *For the Schubert problem $\boldsymbol{\lambda} = (\boxplus, \square^{n-1})$ in $\mathrm{Gr}(k,n)$, we have $d(\boldsymbol{\lambda}) = \binom{n-2}{k-1}$ and $\sigma(\boxplus^c) = \binom{\lfloor \frac{n-2}{2} \rfloor}{\lfloor \frac{k-1}{2} \rfloor, \lfloor \frac{n-k-1}{2} \rfloor}$, which is zero unless $n$ is even and $k$ is odd.*

We show that these Schubert problems reduce to finding ordered factorizations $f = gh$ of univariate polynomials where $f$ has distinct roots. For this, we will regard another factorization $f = g_1 h_1$ where $g_1$ is a scalar multiple of $g$ (and the same for $h_1$ and $h$), to be equivalent to $f = gh$, but $f = hg$ to be a different factorization.

**Theorem 12.** *For any $k, n$, the solutions to the osculating instance of the Schubert problem $(\boxplus, \square^{n-1})$ in $\mathrm{Gr}(k,n)$*

$$X_\square(t_1) \cap X_\square(t_2) \cap \cdots \cap X_\square(t_{n-1}) \cap X_\boxplus(\infty) \tag{4.6}$$

*may be identified with all ordered factorizations $f'(t) = g(t)h(t)$ where*

$$f(t) \;=\; \prod_{i=1}^{n-1} (t - t_i) \tag{4.7}$$

*with $\deg g = n{-}k{-}1$ and $\deg h = k{-}1$.*

Thus the number of real solutions to a real osculating instance of the Schubert problem $(\boxplus, \square^{n-1})$ with osculation type $\rho_{\square}$ is the number of real factorizations $f'(t) = g(t)h(t)$ where $f(t)$ has exactly $\rho_{\square}$ real roots, $\deg g = n{-}k{-}1$, and $\deg h = k{-}1$. This counting problem was studied in [46, Sect. 7], which we recall. Let $\rho$ be the number of real roots of $f'(t)$. By Rolle's Theorem, $\rho_{\square}{-}1 \leq \rho \leq n{-}2$. Then the number $\nu(k,n,\rho)$ of such factorizations is the coefficient of $x^{n-k-1}y^{k-1}$ in $(x+y)^{\rho}(x^2+y^2)^c$, where $c = \frac{n-2-\rho}{2}$, the number of irreducible quadratic factors of $f'(t)$.

**Corollary 13.** *The number of real solutions to a real osculating instance of the Schubert problem* $(\boxplus, \square^{n-1})$ (4.6) *with osculation type* $\rho_{\square}$ *is* $\nu(k,n,\rho)$*, where $r$ is the number of real roots of* $f'(t)$*, where $f$ is the polynomial* (4.7).

*Remark* 14. When $\rho < n{-}4$, we have that $\nu(k,n,\rho) \leq \nu(k,n,\rho{+}2)$, so $\nu(k,n,\rho_{\square}{-}1)$ is a lower bound for the number of real solutions to a real osculating instance of $(\boxplus, \square^{n-1})$ of osculation type $\rho_{\square}$. Since at most $\lfloor \frac{n}{2} \rfloor$ different values of $r$ may occur for the numbers of real roots of $f'(t)$, but the number $\nu(k,n,\rho)$ satisfies

$$\binom{\lfloor \frac{n-2}{2} \rfloor}{\lfloor \frac{k-1}{2} \rfloor, \lfloor \frac{n-k-1}{2} \rfloor} \;\leq\; \nu(k,n,\rho) \;\leq\; \binom{n-2}{k-1}\,,$$

there will in general be gaps in the possible numbers of real solutions, as we saw in Table 6. For example, the possible values of $\nu(5,13,\rho)$ are

$$10\,,\ 18\,,\ 38\,,\ 78\,,\ 162\,,\ \text{and } 330\,.$$

*Proof of Theorem* 12. The Schubert variety $X_{\boxplus}(\infty)$ consists of the $k$-planes $H$ with

$$\dim H \cap F_{i+1}(\infty) \;\geq\; i \qquad \text{for } i = 1, \ldots, k{-}1\,.$$

By Proposition 2, the solutions to (4.6) will be points in $X_{\boxplus}(\infty)$ that do not lie in any other smaller Schubert variety $X_\lambda(\infty)$. This is the Schubert cell of $X_{\boxplus}(\infty)$ [16], and it consists of the $k$-planes $H$ which are row spaces of matrices of the form

$$\begin{pmatrix} 1 & x_1 & \cdots & x_{n-k-1} & x_{n-k} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & x_{n-k+1} & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & x_{n-1} \end{pmatrix},$$

where $x_1, \ldots, x_{n-1}$ are indeterminates. If $x_{n-k} = 0$, then $H \in X_{\square}(0)$, but if one of $x_{n-k+1}, \ldots, x_{n-1}$ vanishes, then $H \in X_{\boxminus}(0)$, which cannot occur for a solution to (4.6), again by Proposition 2. Thus we may assume that $x_{n-k-1}, \dots, x_{n-1}$ are non-zero.

We use a scaled version of these coordinates. Let $(f_0, g, h)$ be the variables $(f_0, g_0, \dots, g_{n-k-2}, h_0, \dots, h_{k-2})$ with $h_0, \dots, h_{k-2}$ all non-zero. Define constants $c_i := (-1)^{n-k-i+1}(n-k-i)!$, $g_{n-k-1} := 1$, and $h_{k-1} := 1$. Let $H(f_0, g, h)$ be the set of matrices of the following form

$$\begin{pmatrix} c_1 g_{n-k-1} & \cdots & c_{n-k} g_0 & \frac{f_0}{h_0} & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -1 & \frac{h_0}{h_1} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & -2 & \ddots & \vdots & \vdots \\ \vdots & & \vdots & \vdots & & \ddots & & 0 \\ 0 & \cdots & 0 & 0 & \cdots & -(k-2) & \frac{h_{k-3}}{h_{k-2}} & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & -(k-1) & \frac{h_{k-2}}{h_{k-1}} \end{pmatrix},$$

which parameterizes the Schubert cell of $X_{\boxplus}(\infty)$. The following calculation is done in [25].

**Lemma 15.** *We have*

$$\det \begin{pmatrix} H(f_0, g, h) \\ F_{n-k}(t) \end{pmatrix} = (-1)^{k(n-k)} \Big( \sum_{i=0}^{n-k-1} \sum_{j=0}^{k-1} \frac{t^{i+j+1}}{i+j+1}\, g_i\, h_j + f_0 \Big). \tag{4.8}$$

Call this polynomial $f(t)$. If $H$ lies in the intersection (4.6), then $f$ is the polynomial (4.7). If we set

$$\begin{aligned} g(t) &:= g_0 + t g_1 + \cdots + t^{n-k-1} g_{n-k-1} \quad \text{and} \\ h(t) &:= h_0 + t h_1 + \cdots + t^{k-1} h_{k-1}, \end{aligned}$$

then $f(0) = f_0$ and $f'(t) = g(t)h(t)$. Theorem 12 is immediate.

## §5. Galois groups of Schubert problems

Not only do field extensions have Galois groups, but so do problems in enumerative geometry, as Jordan explained in 1870 [32]. These algebraic Galois groups are identified with geometric monodromy groups. While the earliest reference we know is Hermite in 1851 [29], this point was eloquently expressed by Harris in 1979 [20]. Jordan's treatise included examples of geometric problems, such as the 27 lines on a cubic surface, whose (known) intrinsic structure prevents their Galois groups from being the full symmetric group on their set of solutions. In contrast, Harris's geometric methods enabled him to show that several classical enumerative problems had the full symmetric group as their Galois group, and therefore had no intrinsic structure. Despite this, Galois groups are known for very few enumerative problems.

The first non-trivial computation of a Galois group in the Schubert calculus is due to Byrnes and Stevens [7]. Interest in determining Galois groups of Schubert problems was piqued when Derksen (see [60]) discovered that the Schubert problem $\boxplus^4 = 6$ in $\mathrm{Gr}(4,8)$ has Galois group isomorphic to $S_4$ and is not the full symmetric group $S_6$. Ruffo et al. [43] exhibited a Schubert problem in the flag manifold $\mathbb{F}\ell(2,4;6)$ with six solutions whose Galois group was $S_3$ and not the full symmetric group $S_6$. In both of these problems the intrinsic structure implies restrictions on the numbers of real solutions. This is similar to the 27 lines on a real cubic surface, which may have either 3, 7, 15, or 27 real lines.

Vakil used the principle of specialization in enumerative geometry and group theory to give a combinatorial method to obtain information about Galois groups [60]. Together with his geometric Littlewood-Richardson rule [59] this gives a recursive procedure that can show the Galois group of a Schubert problem contains the alternating group on its set of solutions. This inspired Leykin and Sottile to show how numerical algebraic geometry can be used to compute Galois groups [37]. A third method based on elimination theory was proposed by Billey and Vakil [3]. We discuss these three methods, including preliminary results and potential experimentation, explain how they were used to nearly determine the Galois groups of all Schubert problems in $\mathrm{Gr}(4,8)$ and $\mathrm{Gr}(4,9)$, and close with a description of two Schubert problems in $\mathrm{Gr}(4,8)$ whose Galois groups are not the full symmetric group.

### 5.1. Galois groups

Let $f\colon Y \to Z$ be a proper, generically separable and finite morphism of degree $d$, where $Z$ and $Y$ are schemes (over an algebraically closed field) of the same dimension, with $Z$ smooth and $Y$ irreducible. A point $z \in Z$ is a *regular value* of $f$ when the fiber over $z$ consists of $d$ distinct points $\{y_1,\ldots,y_d\}$. Write $S_d$ for the symmetric group on $d$ letters. Let $Y^{(d)}$ be the subscheme

$$\overline{(\underbrace{Y\times_Z\cdots\times_Z Y}_{d})\setminus\Delta}\,,$$

of the fiber product, where $\Delta$ is the big diagonal. Fixing a regular value $z\in Z$ of $f$ with $f^{-1}(z)=\{y_1,\ldots,y_d\}$, the *Galois/monodromy group* $\mathcal{G}_{Y\to Z}$ is the group of permutations $\sigma\in S_d$ for which $(y_1,\ldots,y_d)$ and $(y_{\sigma(1)},\ldots,y_{\sigma(d)})$ lie on the same component of $Y^{(d)}$. The Galois group is well-defined up to conjugation in $S_d$. As $Y$ is irreducible, $\mathcal{G}_{Y\to Z}$ is transitive, and if $(Y\times_Z Y)\smallsetminus\Delta$ is irreducible, then it is doubly transitive.

Fix a Schubert problem $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^r)$ in $\mathrm{Gr}(k,n)$. Let $\mathbb{F}\ell(n)$ be the manifold of complete flags in $\mathbb{C}^n$ and set $Z_{\boldsymbol{\lambda}} := \prod_{i=1}^r \mathbb{F}\ell(n)$, the $r$-fold product of flag manifolds, which is smooth. Set

$$Y_{\boldsymbol{\lambda}} := \{(H; F_\bullet^1, \ldots, F_\bullet^r) \mid H \in X_{\lambda^i} F_\bullet^i\,, \quad \text{for } i = 1, \ldots, r\}\,,$$

the total space of the Schubert problem $\boldsymbol{\lambda}$. The projection $Y_{\boldsymbol{\lambda}} \to \mathrm{Gr}(k,n)$ exhibits it as a fiber bundle with fibers the product of Schubert varieties of $\mathbb{F}\ell(n)$ of codimensions $|\lambda^1|, \ldots, |\lambda^r|$. Thus $Y_{\boldsymbol{\lambda}}$ is irreducible, and, as $\boldsymbol{\lambda}$ is a Schubert problem, $\dim Y_{\boldsymbol{\lambda}} = \dim Z_{\boldsymbol{\lambda}}$.

The fiber of $Y_{\boldsymbol{\lambda}}$ over a point $(F_\bullet^1, \ldots, F_\bullet^r)$ of $Z_{\boldsymbol{\lambda}}$ is the instance of the Schubert problem $\boldsymbol{\lambda}$,

$$X_{\lambda^1} F_\bullet^1 \cap X_{\lambda^2} F_\bullet^2 \cap \cdots \cap X_{\lambda^r} F_\bullet^r\,. \tag{5.1}$$

When $(F_\bullet^1, \ldots, F_\bullet^r)$ is general, this is either empty or it consists of finitely many points, by Kleiman's Theorem [35]. Thus $Y_{\boldsymbol{\lambda}} \to Z_{\boldsymbol{\lambda}}$ has a Galois/monodromy group. We call the Galois group $\mathcal{G}_{Y_{\boldsymbol{\lambda}} \to Z_{\boldsymbol{\lambda}}}$ the *Galois group of the Schubert problem* $\boldsymbol{\lambda}$ and write $\mathcal{G}_{\boldsymbol{\lambda}}$ for it.

### 5.2. Vakil's combinatorial criterion

Vakil [60] described how the monodromy group of the restriction to a subscheme $U \subset Z$ affects the Galois group $\mathcal{G}_{Y \to Z}$. Let $U \hookrightarrow Z$ be a closed embedding of a Cartier divisor, with $Z$ smooth in codimension one along $U$. Consider the fiber diagram

$$\begin{array}{ccc} W & \hookrightarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ U & \hookrightarrow & Z \end{array} \tag{5.2}$$

where $f\colon W \to U$ is generically finite and separable of degree $d$. When $W$ is irreducible or has two components the following holds.

(a) If $W$ is irreducible, then $\mathcal{G}_{W \to U}$ includes into $\mathcal{G}_{Y \to Z}$.

(b) If $W$ has two components, $W_1$ and $W_2$, each of which maps dominantly to $U$ of respective degrees $d_1$ and $d_2$, then there is a subgroup $H$ of $\mathcal{G}_{W_1 \to U} \times \mathcal{G}_{W_2 \to U}$ which maps surjectively onto each factor $\mathcal{G}_{W_i \to U}$ and which includes into $\mathcal{G}_{Y \to Z}$.

A Galois group $\mathcal{G}_{Y \to Z}$ is *at least alternating* if it is either $S_d$ or its alternating subgroup. In the above situation, Vakil gave criteria for deducing that $\mathcal{G}_{Y \to Z}$ is at least alternating, based on purely group-theoretic arguments including Goursat's Lemma.

**Vakil's Criteria.** *Suppose we have a fiber diagram as in* (5.2). *The Galois group* $\mathcal{G}_{Y \to Z}$ *is at least alternating if one of the following holds.*

(i) *In Case (a), if* $\mathcal{G}_{W\to U}$ *is at least alternating.*
(ii) *In Case (b), if* $\mathcal{G}_{W_1\to U}$ *and* $\mathcal{G}_{W_2\to U}$ *are at least alternating and either* $d_1 \neq d_2$ *or* $d_1 = d_2 = 1$*.*
(iii) *In Case (b), if* $\mathcal{G}_{W_1\to U}$ *and* $\mathcal{G}_{W_2\to U}$ *are at least alternating, one of* $d_1$ *or* $d_2$ *is not* 6*, and* $\mathcal{G}_{Y\to Z}$ *is doubly transitive.*

Let $F_\bullet, E_\bullet$ be flags in general position and $\lambda, \mu$ be partitions. Vakil's geometric Littlewood-Richardson rule is a sequence of degenerations that convert the intersection $X_\lambda F_\bullet \cap X_\mu E_\bullet$ into a union of Schubert varieties $X_\nu F_\bullet$ with $|\nu| = |\lambda| + |\mu|$. We write this as a formal sum

$$X_\lambda F_\bullet \cap X_\mu E_\bullet \;\sim\; \sum_\nu c^\nu_{\lambda,\mu}\, X_\nu F_\bullet\,, \tag{5.3}$$

where $c^\nu_{\lambda,\mu}$ is the Littlewood-Richardson number. Each step from one degeneration to another is the specialization to a Cartier divisor $U$ in a family $Y \to Z$ representing the total space of the current degeneration as in (5.2). These geometric degenerations and Vakil's criteria lead to a recursive algorithm to show that the Galois group of a Schubert problem is at least alternating, but which is not a decision procedure—when the criteria fails, the Galois group may still be at least alternating. Vakil wrote a maple script to apply this procedure (with criteria (i) and (ii)) to all Schubert problems in a given Grassmannian.

Vakil's method has been used to show that the Galois group of any Schubert problem in $\mathrm{Gr}(2,n)$ is at least alternating. We begin with some general definitions. A *special Schubert condition* is a partition $\lambda$ with only one non-zero part. Write $a$ for the special Schubert condition $(a, 0, \dots, 0)$. A *special Schubert problem* in $\mathrm{Gr}(k,n)$ is a list $\mathbf{a}_\bullet := (a_1, \dots, a_r)$ where $a_i > 0$ and $|\mathbf{a}_\bullet| := a_1 + \cdots + a_r = k(n-k)$. Its number $d(\mathbf{a}_\bullet)$ of solutions is a Kostka number, which counts the number of Young tableaux of shape $(n-k, \dots, n-k) = (n-k)^k$ of content $\mathbf{a}_\bullet$.

A special Schubert problem $\mathbf{a}_\bullet$ in $\mathrm{Gr}(2,n)$ has $a_i \le n-2$ for all $i$. It is *reduced* if for all $i<j$ we have $a_i + a_j \le n-2$, which implies that $r \ge 4$. Any Schubert problem in $\mathrm{Gr}(2,n)$ is equivalent to a reduced special Schubert problem, possibly in a smaller Grassmannian.

In 1884 Schubert [44] gave a degeneration for special Schubert varieties in $\mathrm{Gr}(2,n)$ that is a particular case of the geometric Littlewood-Richardson rule and which may be used to decompose intersections in the same way as (5.3). Schubert's degeneration yields the recursion,

$$\begin{aligned} d(a_1,\dots,a_r) \;&=\; d(a_1,\dots,a_{r-2},a_{r-1}+a_r) \\ &\quad +\; d(a_1,\dots,a_{r-2},a_{r-1}-1,a_r-1)\,. \end{aligned} \tag{5.4}$$

Notice that the right hand side involves different Grassmannians. Vakil's criterion (ii) implies that if both Schubert problems

$$(a_1,\dots,a_{r-2},a_{r-1}+a_r) \quad \text{and} \quad (a_1,\dots,a_{r-2},a_{r-1}{-}1,a_r{-}1)$$

of (5.4) are at least alternating and if the Kostka numbers on the right hand side are either distinct or are both equal to one, then the Galois group of the Schubert problem $\mathbf{a}_\bullet$ is at least alternating. Vakil used his maple script to check that all Schubert problems in $\mathrm{Gr}(2,n)$ for $n \leq 16$ were at least alternating, and Brooks, et al. wrote their own script and extended Vakil's verification to $n \leq 40$. Buoyed by these observations, Brooks, et al. [5] proved the following theorem.

**Theorem 16.** *Every Schubert problem in* $\mathrm{Gr}(2,n)$ *has Galois group that is at least alternating.*

The proof of Theorem 16 is based on the following lemma.

**Lemma 17.** *Let* $\mathbf{a}_\bullet$ *be a reduced Schubert problem in* $\mathrm{Gr}(2,n)$*. When* $\mathbf{a}_\bullet \neq (1,1,1,1)$ *there is a rearrangement* $\mathbf{a}_\bullet = (a_1,\dots,a_r)$ *with*

$$d(a_1,\dots,a_{r-2},a_{r-1}{+}a_r) \;\neq\; d(a_1,\dots,a_{r-2},a_{r-1}{-}1,a_r{-}1)\,. \tag{5.5}$$

*When* $\mathbf{a}_\bullet = (1,1,1,1)$*, both terms of* (5.5) *are equal to* 1*.*

When some pair $a_i, a_j$ are unequal, the inequality (5.5) follows from a combinatorial injection of Young tableaux. The remaining cases use that the Kostka numbers $d(\mathbf{a}_\bullet)$ are coefficients in the decomposition of tensor products of irreducible representations of $SU(2)$. Then the Weyl integral formula gives

$$d(a_1,\dots,a_r) \;=\; \frac{2}{\pi}\int_0^\pi \Bigl(\prod_{i=1}^r \frac{\sin{(a_i{+}1)\theta}}{\sin\theta}\Bigr)\,\sin^2\theta\,d\theta\,. \tag{5.6}$$

Thus the inequality (5.5) is equivalent to showing that an integral is non-zero, which is done by estimation.

Sottile and White [57] studied transitivity of Galois groups of Schubert problems, with an eye towards Vakil's Criterion (iii). They showed that many Schubert problems have doubly transitive Galois groups.

**Theorem 18.** *Every Schubert problem in* $\mathrm{Gr}(3,n)$*, and every special Schubert problem in* $\mathrm{Gr}(k,n)$ *has doubly transitive Galois group.*

They use the result for special Schubert problems and Vakil's criterion (iii) to give another proof of Theorem 16.

Our group plans to use Vakil's Criterion (ii) (and (iii) when double transitivity is known) to study all Schubert problems in all small

($k(n-k) \lesssim 30$) Grassmannians. The goal is to find Schubert problems whose Galois groups might not contain the alternating group, and then use other methods to determine the Galois groups. This approach has already been used for almost all Schubert problems in $\mathrm{Gr}(4,8)$ and $\mathrm{Gr}(4,9)$, as we explain in Subsection 5.5. This experiment may involve several billion Schubert problems, posing serious computer-science issues (such as data storage or memory usage) which must be resolved before it may begin.

### 5.3. Homotopy continuation

By definition, the Galois group $\mathcal{G}_{\boldsymbol{\lambda}}$ of a Schubert problem $\boldsymbol{\lambda}$ is the monodromy group of the family $f\colon Y_{\boldsymbol{\lambda}} \to Z_{\boldsymbol{\lambda}}$, which may be understood concretely as follows. Regular values of the map $f$ are $r$-tuples of flags $(F_\bullet^1, \ldots, F_\bullet^r)$ for which the intersection (5.1) is $d(\boldsymbol{\lambda})$ points. Given a path $\gamma\colon [0,1] \to Z_{\boldsymbol{\lambda}}$ consisting of regular values of $f$, we may lift $\gamma$ to $Y_{\boldsymbol{\lambda}}$ to obtain $d(\boldsymbol{\lambda})$ paths connecting points in the fiber $f^{-1}(\gamma(0))$ to those in $f^{-1}(\gamma(1))$, inducing a bijection between these fibers. When $\gamma$ is a loop based at a regular value $z$, we obtain a *monodromy permutation* of the fiber $f^{-1}(z)$, and the set of all such monodromy permutations is the Galois group $\mathcal{G}_{\boldsymbol{\lambda}}$. The computation of monodromy is feasible and is an elementary operation in the field of numerical algebraic geometry.

Numerical algebraic geometry [45] uses numerical analysis to study algebraic varieties on a computer. It is based on Newton's method for refining approximate solutions to a system of equations and its fundamental algorithm is path-continuation to follow solutions which depend upon a real parameter $t \in [0,1]$. Systems of equations are solved using *homotopy methods*, which start with known solutions to a system of equations and follow them along paths to obtain solutions to the desired system. *Parameter homotopy* is the most elementary; both the start and end systems have the same structure. An example is the fibers of the map $Y_{\boldsymbol{\lambda}} \to Z_{\boldsymbol{\lambda}}$ which are modeled by the determinantal equations of Subsection 2.2. There are more subtle and sophisticated homotopy methods that begin with solutions to simple systems of equations and bootstrap them to find all solutions to the desired equations.

This yields the following two-step procedure to compute monodromy permutations for a given Schubert problem $\boldsymbol{\lambda}$.

(1) Compute all solutions $(H_1, \ldots, H_{d(\boldsymbol{\lambda})})$ to a single instance of a Schubert problem for a regular value $z = (F_\bullet^1, \ldots, F_\bullet^r)$ of the map $Y_{\boldsymbol{\lambda}} \to Z_{\boldsymbol{\lambda}}$.

(2) Use parameter homotopy to follow these $d(\boldsymbol{\lambda})$ solutions over a loop $\gamma\colon [0,1] \to Z_{\boldsymbol{\lambda}}$ based at $z$ to compute a monodromy permutation.

Typically, (1) is quite challenging, while (2) is much easier.

Leykin and Sottile [37] used this method to compute Galois groups of some Schubert problems. For step (1), they implemented a simple version of the Pieri homotopy algorithm [31] to solve a single instance of a Schubert problem, then used off-the-shelf continuation software to compute monodromy permutations, and finally called GAP [17] to determine the group generated by these monodromy permutations. In every simple Schubert problem studied the Galois group was the full symmetric group. We explain this in more detail.

A Schubert problem $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^r)$ is *simple* if all but at most two partitions $\lambda^i$ are equal to the partition $\square$, i.e. $\boldsymbol{\lambda} = (\lambda^1, \lambda^2, \square, \ldots, \square)$. Leykin and Sottile only looked at simple Schubert problems, and only tried to determine if the monodromy was the full symmetric group. The restriction to simple Schubert problems is because there was no efficient algorithm to compute the numerical solutions to general Schubert problems, but the version of the Pieri homotopy algorithm for simple Schubert problems is efficient and easy to implement. Also, it is relatively easy to decide if a set of permutations generates the full symmetric group and this algorithm has a fast implementation in GAP.

Leykin and Sottile wrote a maple script[1] to study the Galois group of simple Schubert problems. It first sets up and runs the Pieri homotopy algorithm to compute all solutions to a general instance, calling PHCPack [61] for path-continuation. Then it starts computing monodromy permutations, again using PHCPack for path-continuation. When a new monodromy permutation is computed, it calls GAP to test if the permutations computed so far generate the symmetric group. If not, then it computes another monodromy permutation, and continues.

The largest Schubert problem studied was $(\yng(2,1), \yng(2), \square^{13}) = 17589$ in $\mathrm{Gr}(3,9)$. Solving one instance and computing seven monodromy permutations took 78.2 hours (wall time) on a single core. The maximum number of monodromy permutations needed in any computation to determine the full symmetric group was nine.

This verified that about two dozen simple Schubert problems have full symmetric Galois group, including the problems $(\square^{k(n-k)})$ in $\mathrm{Gr}(k,n)$ for $k = 2$ and $4 \leq n \leq 10$, $k = 3$ and $5 \leq n \leq 8$, and $(k,n)$ equal to $(4,6)$ and $(4,7)$, as well as three larger problems in $\mathrm{Gr}(3,9)$ and $\mathrm{Gr}(4,8)$. Table 7 records some data from the computations with $d(\boldsymbol{\lambda}) > 1000$. This suggests that the Galois group of any simple Schubert problem is the

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[1] `http://www.math.tamu.edu/~sottile/research/stories/Galois/HoG.tgz`

Table 7. Galois group computation (h := hours)

| $k, n$ | 2,10 | 3,8 | 3,9 | 3,9 | 4,8 |
|---|---|---|---|---|---|
| problem | $\square^{16}$ | $\square^{15}$ | $(\yng(2,1)^2, \square^{12})$ | $(\yng(2,1), \yng(2), \square^{13})$ | $(\yng(2,1), \square^{13})$ |
| solutions | 1430 | 6006 | 10329 | 17589 | 8580 |
| time | 2.6h | 18.6h | 49h | 78.2h | 44.5h |
| permutations | 7 | 6 | 7 | 7 | 9 |

full symmetric group on its set of solutions. These results are not mathematical proofs, as the computations did not come with certificates of validity.

Numerical methods directly computing monodromy permutations give a second approach to studying Galois groups of Schubert problems. The main bottleneck is the lack of efficient algorithms to compute all solutions to a single instance of a given Schubert problem.

In the same way that the geometric Pieri rule [47] led to the efficient Pieri homotopy algorithm [31], Vakil's geometric Littlewood-Richardson rule [59] leads to the efficient Littlewood-Richardson homotopy [56] to solve any Schubert problem in a Grassmannian. While this algorithm is proposed and described in [56], it lacks a practical implementation. There is one being written in Macaulay 2 [19] based on Leykin's NAG4M2 [36] package. When completed and optimized, our group plans an experiment along the lines proposed in Subsection 5.2 to use numerical algebraic geometry to compute Galois groups of Schubert problems. This is expected to be feasible for Schubert problems with up to 20000 solutions with a formulation having up to 25 local coordinates.

This work is affecting research in numerical algebraic geometry beyond the development and implementation of the Littlewood-Richardson homotopy algorithm. The method of regeneration [22] may yield practical algorithms to compute Schubert problems in other flag manifolds (Littlewood-Richardson homotopy is restricted to the Grassmannian). There are other possible continuation algorithms to develop and implement. We expect a broad numerical study of Galois groups of Schubert problems in other flag manifolds to result from these investigations.

The most significant impact of [37] on numerical algebraic geometry is that it has led to the incorporation of certification in software. As mentioned, numerical algebraic geometry computes approximations to solutions to systems of polynomial equations, and there are *a priori* no guarantees on the output. Smale studied the convergence of Newton's method and developed *$\alpha$-theory*, named after a constant $\alpha$ that may be computed at a point $x$ for a polynomial system $F$. When $\alpha(x, F) \lesssim 0.15$, Newton iterations starting at $x$ are guaranteed to converge quickly to a solution for $F = 0$. (This is explained in [4, Ch. 9].)

Certification was recently incorporated into software when Hauenstein and Sottile released alphaCertified [23], which certifies the output of a numerical solver. More fundamentally, Beltrán and Leykin [1, 2] extended $\alpha$-theory, giving an algorithm for certified path-tracking which has certified that the Schubert problem $\square^8 = 14$ in $\mathrm{Gr}(2,6)$ has Galois group equal to the full symmetric group $S_{14}$. Lastly, the (traditional) formulation of a Schubert problem in Section 2.2 typically involves far more equations than variables, and $\alpha$-theory is only valid when the number of equations is equal to the number of variables. Hauenstein, Hein, and Sottile [21] have shown how to reformulate any Schubert problem in a classical flag manifold as a system of $N$ bilinear equations in $N$ variables, enabling certification of general Schubert problems.

### 5.4. Frobenius method and elimination theory

A third method to study Galois groups on a computer exploits symbolic computation and the Chebotarev Density Theorem. Let $\boldsymbol{\lambda}$ be a Schubert problem. As $Z_{\boldsymbol{\lambda}} = \prod_{i=1}^{r} \mathbb{F}\ell(n)$ is a smooth rational variety, if $(F_\bullet^1, \ldots, F_\bullet^r) \in Z_{\boldsymbol{\lambda}}(\mathbb{Q})$ is a regular value of $Y_{\boldsymbol{\lambda}} \to Z_{\boldsymbol{\lambda}}$, then the smallest field of definition of the solutions to the corresponding instance of $\boldsymbol{\lambda}$,

$$X_{\lambda^1} F_\bullet^1 \cap X_{\lambda^1} F_\bullet^2 \cap \cdots \cap X_{\lambda^1} F_\bullet^r\,, \tag{5.7}$$

is a finite extension of $\mathbb{Q}$ whose Galois group is a subgroup of $\mathcal{G}_{\boldsymbol{\lambda}}$. These Galois groups coincide for a Zariski-dense subset of rational flags, by Hilbert's Irreducibility Theorem (see [3, p. 49]). This gives a probabilistic method to determine $\mathcal{G}_{\boldsymbol{\lambda}}$. Given a point $(F_\bullet^1, \ldots, F_\bullet^r) \in Z_{\boldsymbol{\lambda}}(\mathbb{Q})$, formulate the intersection (5.7) as a system of polynomials and compute an eliminant, $g(x)$. When $g$ is irreducible over $\mathbb{Q}$, the smallest field of definition of (5.7) is $\mathbb{Q}[x]/\langle g(x)\rangle$, and so its Galois group is the Galois group $\mathcal{G}_g$ of $g(x)$. Taking the largest group computed for different points in $Z_{\boldsymbol{\lambda}}(\mathbb{Q})$ determines $\mathcal{G}_{\boldsymbol{\lambda}}$ with high probability.

Unfortunately, this method is infeasible for the range of Schubert problems we are interested in. While eliminants $g(x)$ for a Schubert problem $\boldsymbol{\lambda}$ with $d(\boldsymbol{\lambda}) \lesssim 50$ may be computed, we know of no software to reliably compute $\mathcal{G}_g$ when $d(\boldsymbol{\lambda}) \gtrsim 12$. Also, the polynomials $g(x)$ are typically enormous with coefficients quotients of 1000-digit integers.

There is a feasible method that can either prove $\mathcal{G}_g = S_{d(\boldsymbol{\lambda})}$ or give very strong information about $\mathcal{G}_g$. Suppose that $g(x)$ has integer coefficients. Then for any prime $p$ that does not divide the discriminant of $g$, the reduction of $g$ modulo $p$ is square-free. The *Frobenius automorphism* $F_p\colon a \mapsto a^p$ acts on the roots of $g$, with one orbit for each irreducible factor of $g$ in $\mathbb{Z}/p\mathbb{Z}[x]$. Thus the cycle type of $F_p$ is given by the degrees of the irreducible factors of $g$ in $\mathbb{Z}/p\mathbb{Z}[x]$.

It turns out that $F_p$ lifts to a *Frobenius element* in the characteristic zero Galois group $\mathcal{G}_g$ with the same cycle type. Reducing $g(x)$ modulo different primes and factoring gives cycle types of many elements of $\mathcal{G}_g$. By the Chebotarev Density Theorem, these Frobenius elements are distributed uniformly at random in $\mathcal{G}_g$, for $p$ sufficiently large. This probabilistic method to understand the distribution of cycles types in $\mathcal{G}_g$ is often sufficient to determine $\mathcal{G}_g$, as we explain below.

What makes this method practical is that elimination commutes with reduction modulo $p$, computing eliminants modulo a prime $p$ is feasible for $d(\boldsymbol{\lambda}) \lesssim 500$, and factoring modulo a prime $p$ is also fast. The primary reason for this efficiency is that when working modulo primes $p < 2^{64}$, arithmetic operations on coefficients take only one clock cycle. A computation of a few hours in characteristic zero takes less than a second in characteristic $p$.

The method we use to show that a Galois group $\mathcal{G}_g$ is the full symmetric group is based on a theorem of Jordan. A subgroup $G$ of $S_n$ is *primitive* if it preserves no non-trivial partition of $\{1,\ldots,n\}$.

**Theorem 19** (Jordan [33])**.** *If a primitive permutation group $G \subset S_n$ contains an $\ell$-cycle for some prime number $\ell < n-2$, then $G$ is either $S_n$ or its alternating subgroup $A_n$.*

One way that a subgroup $G$ could be primitive would be if $G$ contains cycles of lengths $n$ and $n-1$, for then it is doubly transitive. Since one of $n$ or $n-1$ is even, $G$ is not a subgroup of $A_n$. This gives the following algorithm to show that a Galois group is the full symmetric group, which was suggested to us by Kiran Kedlaya.

**Frobenius Algorithm.** Suppose that $\boldsymbol{\lambda}$ is a Schubert problem with $n = d(\boldsymbol{\lambda})$ solutions.

0. Set $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 := 0$.
1. Choose a sufficiently general point $(F^1_\bullet,\ldots,F^r_\bullet) \in Z_{\boldsymbol{\lambda}}(\mathbb{Q})$ and a prime $p$. Compute an eliminant $g(x) \in \mathbb{Z}/p\mathbb{Z}[x]$ modulo $p$ for the corresponding instance of the Schubert problem $\boldsymbol{\lambda}$.
2. Factor $g(x)$ modulo $p$,

$$g(x) \;=\; h_1(x)\cdots h_s(x) \quad \text{in} \quad \mathbb{Z}/p\mathbb{Z}[x]\,.$$

   If $g$ is not square-free, return to step 1, otherwise do the following.
   (i) If $s = 1$ so that $g$ is irreducible, set $\varepsilon_1 := 1$.
   (ii) If $s = 2$ and one of $h_1, h_2$ has degree $n-1$, set $\varepsilon_2 := 1$.

(iii) If one of the $h_i$ has degree a prime number $\ell$ with $n/2 < \ell < n{-}2$, set $\varepsilon_3 := 1$.

3. If $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ then proclaim "$\mathcal{G}_{\boldsymbol{\lambda}} = S_n$", otherwise return to step 1.

Steps 2(i) and 2(ii) establish that $G_{\boldsymbol{\lambda}}$ contains cycles of lengths $n$ and $n{-}1$. For Step 2(iii), the Frobenius element has a prime cycle of length $\ell$ and all other cycles are shorter as $n/2 < \ell$, so raising the Frobenius element to the power $(\ell{-}1)!$ will result in an $\ell$-cycle in $\mathcal{G}_{\boldsymbol{\lambda}}$.

Assuming this samples elements of $\mathcal{G}_{\boldsymbol{\lambda}}$ uniformly at random (as appears to be the case in practice (see Table 8), and is true for $p$ large enough [10]), that $\mathcal{G}_{\boldsymbol{\lambda}} = S_n$, and that $n > 6$, then 2(i) will occur with probability $\frac{1}{n}$, 2(ii) with probability $\frac{1}{n-1}$, and 2(iii) with the higher probability

$$\sum_{n/2<\ell<n-2} \tfrac{1}{\ell}\ ,$$

(the sum is over primes $\ell$). In our experience, in 95% of the time the Frobenius algorithm took fewer than $2n$ steps to verify that $\mathcal{G}_{\boldsymbol{\lambda}} = S_n$.

When $\mathcal{G}_{\boldsymbol{\lambda}} \neq S_n$, factoring eliminants modulo $p$ gives cycle types in $\mathcal{G}_{\boldsymbol{\lambda}}$ (and can give their distribution), which may be used to help identify $\mathcal{G}_{\boldsymbol{\lambda}}$. This method has been used for both these tasks, and it appears to be feasible for Schubert problems with $d(\boldsymbol{\lambda}) \lesssim 500$.

### 5.5. Galois groups for $\mathrm{Gr}(4,8)$ and $\mathrm{Gr}(4,9)$

We explain how the methods of Subsections 5.2 and 5.4, together with geometric arguments, were used to nearly determine the Galois groups of all Schubert problems in $\mathrm{Gr}(4,8)$ and $\mathrm{Gr}(4,9)$. A careful write up of this is in progress.

Vakil used a maple script based upon his combinatorial criteria (i) and (ii) to show that all Schubert problems in $\mathrm{Gr}(2,n)$ for $n \leq 16$ and $\mathrm{Gr}(3,n)$ for $n \leq 9$ had at least alternating Galois groups. By Grassmannian duality, the smallest Grassmannians not yet studied were $\mathrm{Gr}(4,8)$ and $\mathrm{Gr}(4,9)$. Vakil also used his maple script to study the Schubert problems in $\mathrm{Gr}(4,8)$, and we altered his program to study the Schubert problems in $\mathrm{Gr}(4,9)$ which do not reduce to a smaller Grassmannian. This script determined that 3468 of the 3501 such Schubert problems in $\mathrm{Gr}(4,8)$ have at least alternating Galois group. For $\mathrm{Gr}(4,9)$, it determined that 36534 of the 36767 such Schubert problems have at least alternating Galois group.

Two of the remaining $33 = 3501 - 3468$ problems in $\mathrm{Gr}(4,8)$, $\square^{16} = 24024$ and $\boxplus\cdot\square^{12} = 2460$, were too large to compute modulo any prime $p$, but they are doubly transitive by Theorem 18 (for $\boxplus\cdot\square^{12} = 2460$ this uses an *ad hoc* argument) and Vakil's Criterion (iii) implies that their Galois groups are at least alternating. Computing cycle types of Frobenius elements showed that 17 of the remaining 31 Schubert problems have full symmetric Galois groups, and suggested that the remaining fourteen had Galois group either $S_4$ (for Derksen's example $\yng(2,2)^4 = 6$) or $D_4$, and this second type fell into two classes according to their underlying geometry, represented by

$$\yng(2,2)^2 \cdot \yng(1,1) \cdot \yng(3) \cdot \yng(1)^2 \;=\; 4 \qquad \text{and} \qquad \yng(1,1)^2 \cdot \yng(3)^2 \cdot \yng(1)^4 \;=\; 4\,, \tag{5.8}$$

respectively. We verified the predicted Galois groups for these fourteen using geometric arguments. These three classes generalize to give infinite families of Schubert problems whose Galois groups are not the full symmetric group.

We then applied the methods from Section 5.4 to study Galois groups of most (209) of the $233 = 36767 - 36534$ problems in $\mathrm{Gr}(4,9)$ for which Vakil's criteria were inconclusive. Computing cycles of Frobenius elements showed that 58 have full symmetric Galois groups. The remaining 151 did not have full symmetric Galois groups and they fell into classes having related geometry. All except one class are generalizations of those classes from $\mathrm{Gr}(4,8)$. For example, the problems

$$\yng(1,1)^2 \cdot \yng(4)^2 \cdot \yng(1)^6 \;=\; 10 \quad \text{and} \quad \yng(1,1)^2 \cdot \yng(4)^2 \cdot \yng(2) \cdot \yng(1)^4 \;=\; 6$$

are among the generalizations of the second problem in (5.8), and do not have full symmetric Galois groups.

Among the Schubert problems shown to have full symmetric Galois groups were the problems $\yng(1,1) \cdot \yng(3) \cdot \yng(1)^{10} = 420$ and $\yng(2,2)^2 \cdot \yng(1)^8 = 280$ in $\mathrm{Gr}(4,8)$, which gives some idea of the size of Schubert problems that may be studied with this method.

Before giving two examples from $\mathrm{Gr}(4,8)$, we remark that Galois groups of Schubert problems appear to either be highly transitive (e.g. full symmetric) or they act imprimitively, failing to be doubly transitive.

5.5.1. *The Schubert problem* $\yng(2,2)^4 = 6$ *in* $\mathrm{Gr}(4,8)$. Derksen determined this Galois group. While an instance of this problem is given by four complete flags in general position in $\mathbb{C}^8$, only their 4-planes $L_1,\dots,L_4$ matter. Its solutions will be those $H \in \mathrm{Gr}(4,8)$ for which $\dim H \cap L_i \geq 2$ for each $i = 1,\dots,4$.

To understand this problem, consider the *auxiliary* problem $\yng(3)^4$ in $\mathrm{Gr}(2,8)$ given by $L_1,\dots,L_4$. This asks for the $h \in \mathrm{Gr}(2,8)$ with $\dim h \cap L_i \geq 1$ for $i = 1,\dots,4$. There are four solutions $h_1,\dots,h_4$ to this problem, and its Galois group is the full symmetric group $S_4$. We also have that $h_1,\dots,h_4$ are in direct sum and they span $\mathbb{C}^8$.

The 4-planes $H_{i,j} := h_i \oplus h_j$ satisfy $\dim H_{i,j} \cap L_k = 2$, and so they are solutions to the original problem. In fact, they are the only solutions. It follows that the Galois group of $\yng(2,2)^4 = 6$ is $S_4$ acting on the pairs $\{h_i, h_j\}$. This is an imprimitive permutation group.

The structure of this problem shows that if the $L_i$ are real, then either two or all six of the solutions will be real. Indeed, if all four solutions $h_i$ to the auxiliary problem are real, than all six solutions $H_{i,j}$ will also be real. If however, two or four of the $h_i$ occur in complex conjugate pairs, then exactly two of the $H_{i,j}$ will be real.

5.5.2. *The Schubert problem* $\yng(1,1,1)^2 \cdot \yng(3)^2 \cdot \yng(1)^4 = 4$ *in* $\mathrm{Gr}(4,8)$. This has Galois group $D_4$, the group of symmetries of a square, which acts imprimitively on the solutions. An instance of this problem is given by the choice of two 6-planes $L_1, L_2$, two 2-dimensional linear subspaces $\ell_1, \ell_2$ and four 4-planes $K_1, \ldots, K_4$, all in general position. Solutions are those $H \in \mathrm{Gr}(4,8)$ such that

$$\dim H \cap L_i \ \geq\ 3\,,\ \dim H \cap \ell_i \ \geq\ 1\,,\ \text{and}\ \dim H \cap K_j \ \geq\ 1\,, \tag{5.9}$$

for $i = 1, 2$ and $j = 1, \ldots, 4$.

Consider the first four conditions in (5.9). Let $\Lambda := \langle \ell_1, \ell_2 \rangle$, the linear span of $\ell_1$ and $\ell_2$, which is isomorphic to $\mathbb{C}^4$. Then $h := H \cap \Lambda$ is two-dimensional. If we set $\ell_3 := \Lambda \cap L_1$ and $\ell_4 := \Lambda \cap L_2$, then $\dim h \cap \ell_3 = \dim h \cap \ell_4 = 1$, and so $h \in \mathrm{Gr}(2, \Lambda) \simeq \mathrm{Gr}(2,4)$ meets each of the four 2-planes $\ell_1, \ldots, \ell_4$. In particular, $h$ is a solution to the instance of the problem of four lines (realized in $\mathrm{Gr}(2, \Lambda)$) given by $\ell_1, \ldots, \ell_4$, and therefore there are two solutions, $h_1$ and $h_2$.

Now set $\Lambda' := L_1 \cap L_2$, which is four-dimensional, and fix one of the solutions $h_a$ to the problem of the previous paragraph. For each $j = 1, \ldots, 4$, set $\mu_j := \langle h_a, K_j \rangle \cap \Lambda'$, which is two-dimensional. These four 2-planes are in general position and therefore give a problem of four lines in $\mathbb{P}(\Lambda')$. Let $\eta_{a,1}$ and $\eta_{a,2}$ be the two solutions to this problem, so that $\dim \eta_{a,b} \cap \mu_j \geq 1$ for each $j$.

Then the four subspaces $H_{ab} := \langle h_a, \eta_{a,b} \rangle$ are solutions to the original Schubert problem. Indeed, since $\dim H_{ab} \cap \ell_j = 1$ for $j = 1, \ldots, 4$ and $\eta_{a,b} \subset L_1 \cap L_2$, we have $\dim H_{ab} \cap L_i = 3$, and so $H_{ab}$ satisfies the first four conditions of (5.9). Since $\Lambda \cap \Lambda' = \{0\}$, $h_a$ does not meet $\mu_j$ for $j = 1, \ldots, 4$, so $\dim H_{ab} \cap \langle h_a, K_j \rangle = 3$, which implies that $\dim H_{ab} \cap K_j \geq 1$, and shows that $H_{ab}$ is a solution.

The Galois group $\mathcal{G}_{\boldsymbol{\lambda}}$ acts imprimitively as it preserves the partition $\{H_{11}, H_{12}\} \sqcup \{H_{21}, H_{22}\}$ of the solutions. Since $\mathcal{G}_{\boldsymbol{\lambda}}$ is transitive, it is either $\mathbb{Z}_2 \times \mathbb{Z}_2$, or the dihedral group $D_4$. Computing cycle types of Frobenius elements shows that $\mathcal{G}_{\boldsymbol{\lambda}}$ contains cycles of type $(4)$, $(2,2)$,

$(2,1,1)$, and $(1,1,1,1)$, which shows that it is $D_4$. Computing 100000 eliminants modulo 11311, 99909 were square free and therefore gave Frobenius elements in $\mathcal{G}_{\boldsymbol{\lambda}}$. We record the observed frequencies of each cycle type in the following table. This agrees with [10] in that the

Table 8. Observed frequencies of Frobenius elements

| cycle type | $(4)$ | $(2,2)$ | $(2,1,1)$ | $(1,1,1,1)$ |
|---|---|---|---|---|
| frequency | 25014 | 37384 | 25145 | 12366 |
| fraction | 0.2504 | 0.3742 | 0.2517 | 0.1238 |

Frobenius elements appear to be uniformly distributed. We close with the observation that when we computed real osculating instances of this Schubert problem (as part of the experiment described in Section 4), we only found either zero or four real solutions, and never two. In all other Schubert problems with imprimitive Galois groups that we computed, we found similar interesting structure in their numbers of real solutions.

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