# A New Approach to Hilbert's Theorem on Ternary Quartics

# Une nouvelle approche du théorème de Hilbert sur les quartiques ternaires

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**Abstract**

Hilbert proved that a non-negative real quartic form $f(x, y, z)$ is the sum of three squares of quadratic forms. We give a new proof which shows that if the plane curve $Q$ defined by $f$ is smooth, then $f$ has exactly 8 such representations, up to equivalence. They correspond to those real 2-torsion points of the Jacobian of $Q$ which are not represented by a conjugation-invariant divisor on $Q$.

**Résumé**

Hilbert a démontré qu'une forme réelle non négative $f(x, y, z)$ de degré 4 est la somme de trois carrés de formes quadratiques. Nous donnons une nouvelle démonstration qui montre que si la courbe plane $Q$ definie par $f$ est non singulière, alors $f$ a exactement 8 telles représentations, à equivalence près. Elles correspondent aux points de 2- torsion du jacobien de $Q$ qui ne sont pas représentés par un diviseur de $Q$ invariant par conjugaison.

## 1. Introduction

A *ternary quartic form* is a homogeneous polynomial $f(x, y, z)$ of degree 4 in three variables. If $f$ has real coefficients, then $f$ is *non-negative* if $f(x, y, z) \geq 0$ for all real $x$, $y$, $z$. Hilbert [5] showed that every non-negative real ternary quartic form is a sum of three squares of quadratic forms. His proof (see [8], [9] for modern expositions) was non-constructive: The map

$$\pi\colon\ (p, q, r)\ \longmapsto\ p^2 + q^2 + r^2$$

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from triples of real quadratic forms to non-negative quartic forms is surjective, as it is both open and closed when restricted to the preimage of the (dense) connected set of non-negative quartic forms which define a smooth complex plane curve. An elementary and constructive approach to Hilbert's theorem was recently begun by Pfister [6].

A *quadratic representation* of a complex ternary quartic form $f = f(x, y, z)$ is an expression

$$f = p^2 + q^2 + r^2 \tag{1}$$

where $p$, $q$, $r$ are complex quadratic forms. A representation $f = (p')^2 + (q')^2 + (r')^2$ is *equivalent* to this if $p, q, r$ and $p', q', r'$ have the same linear span in the space of quadratic forms.

Powers and Reznick [7] investigated quadratic representations computationally, using the Gram matrix method of [1]. In several examples of non-negative real ternary quartics, they always found 63 inequivalent representations as a sum of three squares of complex quadratic forms; 15 of these were sums or differences of squares of real forms. We explain these numbers, in particular the number 15, and show that precisely 8 of the 15 are sums of squares.

If the complex plane curve $Q$ defined by $f = 0$ is smooth, it has genus 3, and so the Jacobian $J$ of $Q$ has $2^6 - 1 = 63$ non-zero 2-torsion points. Coble [2, Chap 1,§14] showed that these are in one-to-one correspondence with equivalence classes of quadratic representations of $f$. If $f$ is real, then $Q$ and $J$ are defined over $\mathbb{R}$. The non-zero 2-torsion points of $J(\mathbb{R})$ correspond to *signed quadratic representations* $f = \pm p_1^2 \pm p_2^2 \pm p_3^2$, where $p_i$ are real quadratic forms. If $f$ is also non-negative, the real Lie group $J(\mathbb{R})$ has two connected components, and hence has $2^4 - 1 = 15$ non-zero 2-torsion points. We use Galois cohomology to determine which 2-torsion points give rise to sum of squares representations over $\mathbb{R}$.

**Theorem 1** *Suppose that $f(x, y, z)$ is a non-negative real quartic form which defines a smooth plane curve $Q$. Then the inequivalent representations of $f$ as a sum of three squares are in one-to-one correspondence with the eight 2-torsion points in the non-identity component of $J(\mathbb{R})$, where $J$ is the Jacobian of $Q$.*

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## 2. Quadratic representations of smooth ternary quartics

Let $f(x, y, z)$ be an irreducible quartic form over $\mathbb{C}$, and let $Q$ be the curve $f = 0$ in the complex projective plane. Assume that $Q$ is smooth. The Picard group $\mathrm{Pic}(Q)$ of $Q$ is the group of Weil divisors on $Q$, modulo divisors of rational functions. Let $J$ be the Jacobian of $Q$, so that $J$ is the identity component of $\mathrm{Pic}(Q)$. The following proposition is due to Coble [2, Chap 1,§14].

**Proposition 1** *The non-trivial 2-torsion points of $J$ are in one-to-one correspondence with the equivalence classes of quadratic representations of $f$.*

*Proof.* Given a quadratic representation (1), consider the map

$$\varphi\colon \mathbb{P}^2 \to \mathbb{P}^2, \quad x \mapsto \big(p(x) : q(x) : r(x)\big).$$

The image of $Q$ under $\varphi$ is the conic $C$ defined by the equation $y_0^2 + y_1^2 + y_2^2 = 0$. Let $y$ be any point in $C$, then $\varphi^*(y)$ is an effective divisor of degree 4 that is not the divisor of a linear form. Indeed, after a linear change of coordinates we can assume $y = (0 : 1 : i)$. A linear form vanishing on $\varphi^*(y)$ would divide each conic $\alpha p + \beta(q + ir)$ through $\varphi^*(y)$, and thus would divide

$$f = p^2 + (q + ir)(q - ir),$$

contradicting the irreducibility of $f$.

Fix a linear form $\ell$, then $L := \mathrm{div}(\ell)$ is an effective divisor of degree 4 on $Q$. Let $\zeta = [\varphi^*(y) - L]$. Since $2y$ is the divisor of a linear form (the tangent line to $C$ at $y$), $\varphi^*(2y)$ is the divisor on $Q$ of a quadratic form. Thus $2\zeta = 0$. Moreover, $\zeta \neq 0$ as $\varphi^*(y)$ is not the divisor of a linear form. The 2-torsion point $\zeta$ of $J$ depends only upon the map $\varphi$.

Conversely, suppose that $\zeta \in J(\mathbb{C})$ is a non-zero 2-torsion point. Let $D \neq D'$ be effective divisors which represent the class $\zeta + [L]$ in $\mathrm{Pic}(Q)$. As $Q$ has genus 3, the Riemann-Roch Theorem implies that there is a pencil of such divisors. Then $2D$, $2D'$ and $D + D'$ are effective divisors of degree 8, and are all linearly equivalent to $2L$, the divisor of a conic. Again from the Riemann-Roch Theorem it follows that there are quadratic forms $q_0$, $q_1$ and $q_2$ such that

$$\mathrm{div}(q_0) = 2D, \quad \mathrm{div}(q_1) = 2D' \quad \text{and} \quad \mathrm{div}(q_2) = D + D'.$$

Therefore, the rational function $g := q_0 q_1 / q_2^2$ on $Q$ is constant. Scaling $q_1$ and $q_2$ appropriately, we may assume that $g \equiv 1$ on $Q$ and also that $f = q_0 q_1 - q_2^2$. Diagonalizing the quadratic form $q_0 q_1 - q_2^2$ gives a quadratic representation for $f$. This defines the inverse of the previous map. □

## 3. Quadratic representations of real quartics

Suppose now that $f$ is a non-negative real quartic form defining a smooth real plane curve $Q$ with complexification $Q_\mathbb{C} = Q \otimes_\mathbb{R} \mathbb{C}$. The elements of $\mathrm{Pic}(Q)$ can be identified with those divisor classes in $\mathrm{Pic}(Q_\mathbb{C})$ that are represented by a conjugation-invariant divisor. Let $J$ be the Jacobian of $Q$.

If $\zeta \in J(\mathbb{C})$ is the 2-torsion point corresponding to a signed quadratic representation

$$f \;=\; \pm p^2 \pm q^2 \pm r^2$$

consisting of real polynomials $p$, $q$, $r$, then $\zeta = \overline{\zeta}$, i.e., $\zeta \in J(\mathbb{R})$.

Conversely, let $0 \neq \zeta \in J(\mathbb{R})$ with $2\zeta = 0$, and let $L$ be the divisor on $Q$ of a linear form $\ell$. We can choose an effective divisor $D \neq \overline{D}$ on $Q_\mathbb{C}$ representing the class $\zeta + [L]$. Then $2D$, $2\overline{D}$ and $D + \overline{D}$ are each equivalent to $2L$. Let $r$ be a real quadratic form with divisor $D + \overline{D}$, and let $g$ be a complex quadratic form with divisor $2D$ (both divisors taken on $Q_\mathbb{C}$).

Since $D \sim \overline{D}$, there is a rational function $h$ on $Q_\mathbb{C}$ with $\mathrm{div}(h) = \overline{D} - D$. Let $c = h\overline{h}$, a nonzero real constant on $Q$. Since $\mathrm{div}(r) = \mathrm{div}(g) + \mathrm{div}(h)$, there is a complex number $\alpha \neq 0$ with $\frac{r}{g} = \alpha h$ on $Q$, which implies that

$$c|\alpha|^2 \;=\; \frac{r}{g} \cdot \frac{\overline{r}}{\overline{g}} \;=\; \frac{r^2}{p^2 + q^2}$$

on $Q$, where $p$ and $q$ are the real and imaginary parts of $g = p + iq$. So the quartic form

$$u \;:=\; r^2 - c\,|\alpha|^2 (p^2 + q^2)$$

vanishes identically on $Q$. Since $u \neq 0$, $f$ is a constant multiple of $u$. If $c > 0$, we get a signed quadratic representation of $f$, with both signs $\pm$ occuring. If $c < 0$, $f$ must be a positive multiple of $u$ since $f$ is non-negative, and we get a representation of $f$ as a sum of three squares of real forms.

We now calculate the sign of $c$. For this we use the well-known exact sequence

$$0 \to \mathrm{Pic}(Q) \to \mathrm{Pic}(Q_\mathbb{C})^G \xrightarrow{\partial} \mathrm{Br}(\mathbb{R}) \to \mathrm{Br}(Q).$$

It arises from the Hochschild-Serre spectral sequence for étale cohomology with coefficients $\mathbb{G}_m$. Here $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathrm{Pic}(Q_\mathbb{C})$ by conjugation, and $\mathrm{Pic}(Q_\mathbb{C})^G$ is the group of $G$-invariant divisor classes. Moreover, $\mathrm{Br}(\mathbb{R})$ is the Brauer group of $\mathbb{R}$, which is of order 2, and $\mathrm{Br}(Q)$, the Brauer group of $Q$, can be identified with the subgroup of $\mathrm{Br}\,\mathbb{R}(Q)$ consisting of all Brauer classes which are everywhere unramified. The map $\mathrm{Br}(\mathbb{R}) \to \mathrm{Br}(Q)$ is the restriction map.

It is easy to see that $c < 0$ if and only if $\partial(\zeta)$ is the non-trivial class in $\mathrm{Br}(\mathbb{R})$.

By a classical theorem of Witt [12], every non-negative rational function on a smooth projective curve over $\mathbb{R}$ is a sum of two squares of rational functions. Since $Q$ is smooth and $f$ is non-negative, this forces

$Q(\mathbb{R}) = \emptyset$. Hence $-1$ is a sum of two squares in $\mathbb{R}(Q)$. This means $(-1,-1) = 0$ in $\mathrm{Br}(Q)$, and hence the map $\partial$ is surjective.

Since the genus of $Q$ is odd (equal to 3), it follows from a classical theorem of Weichold [11,3] that all classes in $\mathrm{Pic}(Q_{\mathbb{C}})^G$ have even degree, and the real Lie group $J(\mathbb{R})$ has exactly two connected components. This implies that the sequence

$$0 \to J(\mathbb{R})^0 \to J(\mathbb{R}) \xrightarrow{\partial} \mathrm{Br}(\mathbb{R}) \to 0$$

is (split) exact. Since $J(\mathbb{R})^0 \cong (S^1)^3$ as a real Lie group, there exist $2^4 - 1 = 15$ non-zero 2-torsion classes in $J(\mathbb{R})$. The 8 that do not lie in $J(\mathbb{R})^0$, or equivalently, which cannot be represented by a conjugation-invariant divisor on $Q_{\mathbb{C}}$, are precisely those that give rise to sums of squares representations of $f$. This completes the proof of Theorem 1.

We close with a few remarks about the singular case. Wall [10] studies quadratic representations of (possibly singular) complex ternary quartic forms $f$. If $f$ is irreducible, the non-trivial 2-torsion points on the generalized Jacobian of the curve $Q = \{f = 0\}$ again give equivalence classes of quadratic representations of $f$. These representations are special in that they have no basepoints.

Quadratic representations with a given base locus $B \neq \emptyset$ are in one-to-one correspondence with all 2-torsion points on the Jacobian of a curve $\widetilde{Q}$, which is the image of $Q$ under the complete linear series of quadrics through $B$. By classifying all possibilities for $B$ one arrives at the number of inequivalent quadratic representations of $f$. If the form $f$ is real and non-negative, this classification, together with arguments from Galois cohomology, gives all inequivalent representations of $f$ as a sum of squares. If $f$ is reducible, different methods can be applied to complete the picture. This complete analysis will appear in an unabridged version.

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