

SOME PROBLEMS AT THE INTERFACE OF APPROXIMATION THEORY AND ALGEBRAIC GEOMETRY

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ABSTRACT. We sketch an assortment of problems that were posed—and not yet solved—during problem sessions at the conference “Approximation Theory and Numerical Analysis meet Algebra, Geometry, and Topology”, which was held at the Palazzone Cortona 5–9 September 2022.

INTRODUCTION AND BACKGROUND

Open problems are the bread and butter of mathematicians, and the congenial atmosphere at the Palazzone led several participants to share some of theirs, and make comments about problems others posed. This is an elaboration of a selection of those problems and comments, as understood by the author. We try to give references and to follow the conventions established in the overview contributions to this volume [LST23, MS23]. Any errors or misstatements are the fault of the author.

1. SPLINE SPACES

As explained in [LST23, Sect. 2.4], there are many open problems and directions of further study concerning the dimensions of spaces of splines, even for planar domains. Several specific problems along these lines arose during the problem sessions.

Given a triangulation or polyhedral subdivision Δ of a simply connected domain $D \subset \mathbb{R}^2$ and positive integers r, d , we let $S_d^r(\Delta)$ be the vector space of functions on D that have r continuous derivatives, and whose restriction to each two-dimensional cell of Δ is a polynomial of degree at most d . As Hal Schenck pointed out at the beginning of the problem sessions, the biggest open problem in this area, and a very hard one at that, is to determine the dimensions of every spline space $S_d^r(\Delta)$. That is, determine the Hilbert function of the spline module $S_\bullet^r(\Delta) := \bigoplus_{d \geq 0} S_d^r(\Delta)$. We discuss three related problems that are less of a Moonshot.

The $S_3^1(\Delta)$ problem: Michael Di Pasquale. This is a well-known outstanding problem in the theory of splines, which has been stated many times in the past. For a planar triangulation Δ , Schumaker [Sch79] gave a lower bound for the dimension of a spline space $S_d^r(\Delta)$ that depends only upon f_1° , the number of interior edges in Δ , f_0° , the number of interior vertices, and the number of distinct slopes at each vertex. This was later shown to be the dimension of $S_d^r(\Delta)$ when

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This research stay was partially supported by National Science Foundation through grant DMS-2201005 and the Simons Foundation and by the Mathematisches Forschungsinstitut Oberwolfach.

$d \geq 3r + 2$ [AS87, Hon91]. When $r = 1$ and $d \geq 4$, we have

$$(1) \quad \dim S_d^1(\Delta) = (1 - f_0^\circ) \binom{d+2}{2} + f_1^\circ \binom{d}{2} + 3f_0^\circ + \sigma,$$

where σ is the number of singular vertices, those interior vertices that have only two slopes among their incident edges. For $d \geq 5$, this is due to Morgan and Scott [MS75], and when $d = 4$, this is due to Alfeld, Schumaker, and Piper [APS87]. When $d = 2$, equality does not hold in (1) as Morgan and Scott showed [MS77]: Let Δ be the Schlegel diagram of a regular octahedron, displayed in Figure 1. Then $S_2^1(\Delta) = 7$, but if we perturb the configuration to a combinatorially

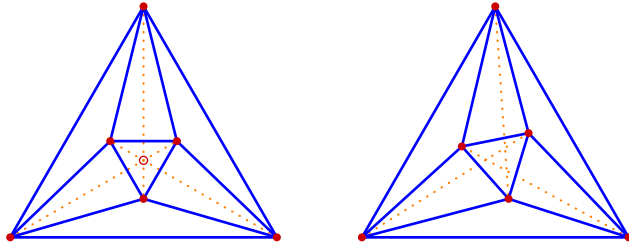


FIGURE 1. Morgan-Scott triangulation Δ on the left and a perturbation Δ' on the right.

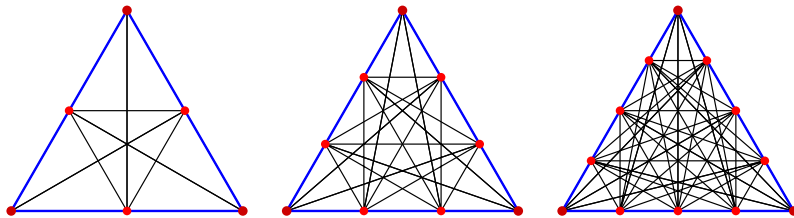
equivalent one Δ' in which the dotted lines (images of the three diagonals of the octahedron) do *not* meet, then $S_2^1(\Delta') = 6$. (This is also discussed in [LST23, Ex. 4].)

For $r = 1$ and $d = 3$, Schumaker's bound becomes $10 - 7f_0^\circ + 3f_1^\circ + \sigma$, for any planar triangulation Δ . The challenge/problem is either to prove that this is the dimension of $S_3^1(\Delta)$ for any triangulation Δ of a simply connected planar domain Δ , or to find a counterexample for which $S_3^1(\Delta)$ exceeds this number. Any counterexample must also have unexpected splines in degree 2 à la Morgan-Scott.

Wang-Shi splits WS_d of the triangle: Tom Lyche, Carla Manni, Hendrik Speleers. This is a simple question about plane geometry which has consequences for the dimension of spline spaces of a particular subdivision of a triangle that was studied in the proposers' paper [LMS22, Sect. 2.2].

Begin with a triangle and subdivide each edge into d uniform segments. These $3d$ segments have $3d$ endpoints along the perimeter of the triangle, which include the vertices. Draw all $3d(d-1)$ lines connecting these points. This gives a cross-cut partition of the triangle, called a *Wang-Shi split*, WS_d , as this was originally proposed by Wang and Shi [WS90].

When $d = 1$, this partition is just a triangle, and when $d = 2$, this is the Powell-Sabin split [PS77]. We display WS_2 , WS_3 , and WS_4 .



Define μ_d to be the maximum number of lines in WS_d incident on a point in the interior of the triangle. We have $\mu_2 = 3$, $\mu_3 = 3$, and $\mu_4 = 4$. Table 1 displays values of μ_d for $d \leq 18$. Figure 2

TABLE 1. Values of μ_d

d	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
μ_d	3	3	4	5	6	7	6	8	8	9	9	10	10	10	12	11	12

shows WS_7 and WS_8 . The circled vertices are those at which μ_7 and μ_8 achieve their maximum value. The website¹ gives vector graphics pictures of WS_d for $d \leq 18$. Note that μ_d is not a

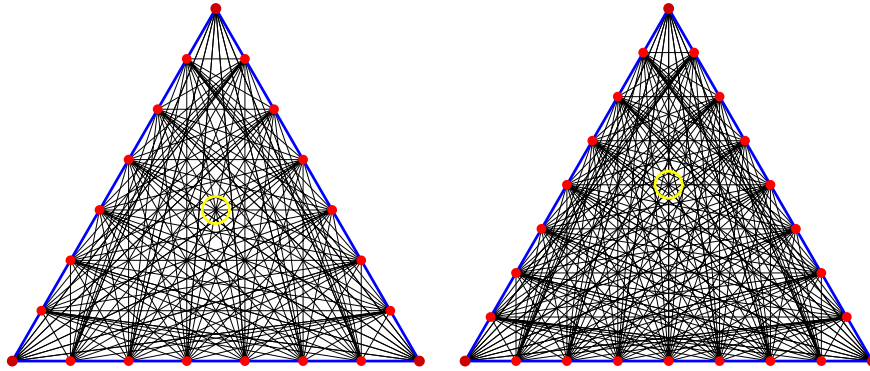


FIGURE 2. Wang-Shi splits WS_7 and WS_8 with vertices of maximal degrees circled.

monotone function of d . As the table shows, for d equal to 8 and 17, $\mu_d < \mu_{d-1}$.

The problem is to determine μ_d for all d . A subproblem is to show that $\mu_d \leq d+1$ for all d , which appears likely given the data. Other problems include determining the total number of polygonal regions in WS_d or the maximum number of sides of a polygon in WS_d (WS_{15} has a decagon—a polygonal cell with ten sides).

The motivation comes from the problem of determining the dimension of the spline spaces $S_k^r(WS_d)$. The dimension of spline spaces for cross-cut partitions were determined by Chui and Wang [CW83]. Their formula (given in Theorem 3.1 of [CW83]) is in terms of the number of cross-cuts (here, the number $3d(d-1)$ of lines), with a correction term that is the sum over the internal vertices of a function of the number of lines incident at that vertex.

In [LMS22, Sect. 2.2], this formula is applied to the spline space $S_d^{d-1}(WS_d)$. For this space, the Chui-Wang formula simplifies to become

$$\dim S_d^{d-1}(WS_d) = \binom{d+2}{2} + 3d(d-1),$$

when $\mu_d \leq d+1$. Thus, the Wang-Shi splits are quite complicated polyhedral subdivisions which (conjecturally) support a relatively simple space of splines $S_d^{d-1}(WS_d)$.

¹<https://www.math.tamu.edu/~sottile/research/stories/WangShi/index.html>

Overlap Splines: Oleg Davydov. This problem arises in Davydov's contribution [Dav23] to this volume. Fix positive integers N, m, n and let $X \subset \mathbb{R}^n$ be a set of N points. Suppose also that we have a system $\{X_i \subset X \mid i = 1, \dots, m\}$ of subsets of X whose union is X (the sets form a cover of X). Let us write X for the set of points and the cover.

Let P_d be the space of polynomials on \mathbb{R}^n of degree at most d . An *overlap spline of degree d with respect to X* is a vector $s = \{p_i \in P_d \mid i = 1, \dots, m\}$ of polynomials of degree at most d which satisfy the overlap condition

$$(2) \quad p_i|_{X_i \cap X_j} = p_j|_{X_i \cap X_j} \quad \text{for all } i, j.$$

(The condition is vacuous when $X_i \cap X_j = \emptyset$.) Write $S_d(X)$ for the vector space of all such splines. The problem is to determine the dimension of $S_d(X)$ in general.

Since $S_d(X)$ is a linear subspace of P_d^m , we have the upper bound

$$\dim(S_d(X)) \leq m \cdot \dim(P_d) = m \binom{n+d}{n}.$$

Rewriting the conditions (2) in terms of $x \in X$ gives

$$p_i(x) = p_j(x) \quad \forall i, j \text{ with } x \in X_i \cap X_j,$$

which are $m_x - 1$ independent equations, where $m_x := \#\{i \mid x \in X_i\}$. As $\sum_{x \in X} m_x = \sum_{i=1}^m |X_i|$, we obtain the lower bound,

$$(3) \quad \dim(S_d(X)) \geq m \binom{n+d}{n} - \left(\sum_{i=1}^m |X_i| - N \right),$$

which is an equality when the equations imposed by different $x \in X$ are independent.

There is one fairly straightforward case in which this is known, and it suggests some of the ingredients of an answer to this problem, or some conditions that may be imposed on X for a uniform solution. An *interpolation set for P_d* is a finite set $Y \subset \mathbb{R}^n$ of exactly $\dim(P_d) = \binom{n+d}{n}$ points which impose independent conditions on polynomials in P_d . Equivalently, the map from P_d to \mathbb{R}^Y given by restricting polynomials in P_d to functions on Y is an isomorphism of vector spaces. Proposition 1 in [Dav23] states that when each X_i is an interpolation set for P_d , then $\dim(S_d(X)) = N$. This follows from (3) as $|X_i| = \binom{n+d}{n}$ when X_i is an interpolation space for P_d .

An audience member mentioned that this has the flavor of Čech cohomology. For each $1 \leq i < j \leq m$, let $X_{i,j} := X_i \cap X_j$ and write $\mathbb{R}^{X_{i,j}}$ for the vector space of functions on $X_{i,j}$. Then the overlap splines form the kernel of the map

$$(4) \quad \bigoplus_{i=1}^m P_d \longrightarrow \bigoplus_{1 \leq i < j \leq m} \mathbb{R}^{X_{i,j}},$$

in which a vector of polynomials $(p_i \mid i = 1, \dots, m)$ is mapped to the vector $(p_i|_{X_{i,j}} - p_j|_{X_{i,j}} \mid 1 \leq i < j \leq m)$ of functions on the $X_{i,j}$. The lower bound (3) is obtained by analyzing the image of the map in (4). This map (4) has the flavor of the first coboundary operator in a Čech complex [BT82].

2. POSSIBLE STRUCTURES ON SPLINES

Another class of problems concerned structures in spaces of splines inspired by other mathematical disciplines.

A Chinese Remainder Theorem for Splines? Kiumars Kaveh. This question is about extensions of splines. Suppose that Δ is a triangulation (or subdivision) of a domain D and that $D' \subset D$ is a simply connected domain that is a union of some triangles or polyhedra in Δ . Write Δ' for the induced subdivision of D' , which we call a subtriangulation of Δ . Given a spline g' on Δ' , this question asks what are conditions on g' so that there is a spline g on Δ whose restriction to Δ' is g' . That is, under what conditions can a spline g' on Δ' be extended to a spline g on Δ ?

The motivation for the name of this question is that a spline is the solution to a system of congruences on vectors of polynomials. It is not hard to find examples for which the map $S_d^r(\Delta) \rightarrow S_d^r(\Delta')$ is not surjective for any d . Thus the pertinent question is to describe the image in general. Di Pasquale's contribution to this volume [DP23] explains the lack of surjectivity and studies the image in many cases when the extension of Δ' to Δ consists of adding a single simplex (in \mathbb{R}^2 , adding a single triangle).

More generally, we may have graph G whose edges are labeled by ideals in a ring, or a GKM [GKM98] graph G , and $G' \subset G$ is an induced subgraph. We may again ask the same question about the surjectivity or iage of restriction.

An audience member asked if there is a version of Maier-Vietoris for splines. That is, suppose that we have a triangulation or subdivision Δ of a domain that is the union of two subtriangulations A, B whose intersection $A \cap B$ is also a triangulation. Then we have restriction maps $S^r(\Delta) \rightarrow S^r(A) \oplus S^r(B) \rightarrow S^r(A \cap B)$, and a natural question would be under what conditions does this sequence have any interesting homological properties. This comes from the perspective that the space of splines on a subdivision is an analog of the global sections functor H^0 of that subdivision.

Spline Schemes: Frank Sottile. Let Δ be a polyhedral subdivision of a domain in \mathbb{R}^n . For any integer $r \geq 0$, the graded vector space $S_\bullet^r := \bigoplus_{d \geq 0} S_d^r(\Delta)$ and the filtered vector space $S^r(\Delta) = \bigcup_{d \geq 0} S_d^r(\Delta)$ are finitely generated modules over the rings $\mathbb{R}[x_0, \dots, x_n]$ and $\mathbb{R}[x_1, \dots, x_n]$, respectively. For $S_\bullet^r(\Delta)$, we use the homogenized versions of the spline spaces $S_d^r(\Delta)$, so that $S^r(\Delta)$ is a dehomogenization.

These modules of splines are also finitely generated \mathbb{R} -algebras. To see that they are rings, let I be an ideal and suppose that (f_1, f_2) and (g_1, g_2) are pairs whose differences lie in I . That is, $f_1 - f_2 \in I$ and $g_1 - g_2 \in I$. Then (f_1g_1, f_2g_2) is another pair whose difference lies in I . Indeed,

$$f_1g_1 - f_2g_2 = f_1g_1 - f_2g_1 + f_2g_1 - f_2g_2 = (f_1 - f_2)g_1 + f_2(g_1 - g_2) \in I.$$

The spline ring is finitely generated over \mathbb{R} by the generators of the ring of constant splines (those that are global polynomial functions, and thus take the same value at each n -dimensional cell in Δ) and the module generators.

In algebraic geometry, the functor Proj associates a projective variety (scheme) to a graded algebra that is finitely generated in degree 1, and Spec associates an affine variety (scheme) to a finitely generated algebra. These may be embedded into projective or affine space in such a way that the generators are the restrictions of coordinate functions. For example, $\text{Spec}(\mathbb{R}[x_1, \dots, x_N]/\langle f_1(x), \dots, f_m(x) \rangle)$

is the set $\{x \mid f_i(x) = 0, i = 1, \dots, m\}$. (Here $x \in \mathbb{P}^{N-1}$ when the ring is graded, otherwise $x \in \mathbb{R}^N$.) Thus we may define the associated *spline schemes*

$$\mathcal{X}^r(\Delta) := \text{Proj}(S_\bullet^r(\Delta)) \quad \text{and} \quad X^r(\Delta) := \text{Spec}(S^r(\Delta)).$$

We now restrict to $S^r(\Delta)$ and $X^r(\Delta)$; while the same questions may be asked of $S_\bullet^r(\Delta)$ and $\mathcal{X}^r(\Delta)$, we expect they will have similar properties. We will consider $X^r(\Delta)$ to be a subset of \mathbb{R}^N , implicitly choosing generators of the spline module.

Let Δ_n be the set of n -dimensional cells in Δ . Then $S^{-1}(\Delta)$, the module/ring of discontinuous splines on Δ , is

$$\mathbb{R}[x_1, \dots, x_n]^{\Delta_n} := \bigoplus_{\sigma \in \Delta_n} \mathbb{R}[x_1, \dots, x_n].$$

Then $X^{-1}(\Delta) = \text{Spec}(S^{-1}(\Delta)) = \coprod_{\sigma \in \Delta_n} \mathbb{R}^n$ is the disjoint union of affine spaces \mathbb{R}^n , one for each n -dimensional cell σ of Δ .

For each $-1 \leq r < s$, we have inclusions of spline rings $S^s(\Delta) \hookrightarrow S^r(\Delta)$, which induce surjections of spline schemes $X^r(\Delta) \twoheadrightarrow X^s(\Delta)$. The map $X^{-1}(\Delta) \twoheadrightarrow X^r(\Delta)$ is a resolution of singularities of $X^r(\Delta)$, and shows that $X^r(\Delta)$ has $|\Delta_n|$ irreducible components. We use the adjectival scheme for spline schemes, for in algebraic geometry, the term variety is often reserved for integral schemes, which are reduced (no nilpotent elements of their coordinate rings) and irreducible. Spline schemes are reduced (as spaces of splines are functions) but not irreducible.

The inclusion of $\mathbb{R}[x_1, \dots, x_n]$ of the ring of constant splines induces a surjection $\varphi: X^r(\Delta) \rightarrow \mathbb{R}^n$, which is a finite map as $S^r(\Delta)$ is a finite module over $\mathbb{R}[x_1, \dots, x_n]$. Also, while algebraic geometry often works best over \mathbb{C} , the maps $\coprod_{\sigma \in \Delta_n} \mathbb{R}^n \rightarrow X^r(\Delta) \rightarrow \mathbb{R}^n$ show that there is no loss restricting to \mathbb{R} .

It is worthwhile to investigate these when $n = 1$. Consider the subdivision Δ of the interval $[-1, 2]$ into three unit cells:

$$(5) \quad \begin{array}{c} \leftarrow \quad \sigma_0 \quad \sigma_1 \quad \sigma_2 \quad \rightarrow \\ \leftarrow \quad -1 \quad 0 \quad 1 \quad 2 \quad \rightarrow \end{array}$$

Then we have

$$S^0(\Delta) = \{(f_0(t), f_1(t), f_2(t)) \in \mathbb{R}[t]^3 \mid f_0(0) = f_1(0) \text{ and } f_1(1) = f_2(1)\}.$$

The spline function represented by (f_0, f_1, f_2) takes value $f_i(t)$ for $t \in \sigma_i$, and it is constant when $f_0 = f_1 = f_2$. Observe that $S^0(\Delta)$ is generated as an $\mathbb{R}[t]$ -module by $(1, 1, 1)$, $x := (t, 0, 0)$, and $z := (0, 0, t - 1)$. If we let $y := t(1, 1, 1) = (t, t, t)$, then $S^0(\Delta)$ is generated as a ring by x, y, z . These satisfy the following relations $x(x - y) = xz = z(z - y + 1) = 0$, and thus

$$S^0(\Delta) = \mathbb{R}[x, y, z] / \langle x(x - y), xz, z(z - y + 1) \rangle.$$

We may check that

$$\langle x(x - y), xz, z(z - y + 1) \rangle = \langle x - y, z \rangle \cap \langle x, z \rangle \cap \langle x, z - y + 1 \rangle,$$

and thus $X^0(\Delta)$ is the union of three lines. We display this on the left in Figure 3.

Let us now consider $S^1(\Delta)$, which is

$$\{(f_0(t), f_1(t), f_2(t)) \in \mathbb{R}[t]^3 \mid f_0 - f_1 \in \langle t^2 \rangle \text{ and } f_1 - f_2 \in \langle (t - 1)^2 \rangle\}.$$

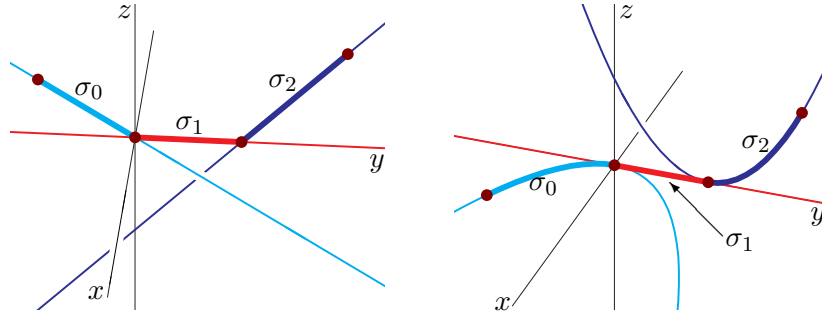


FIGURE 3. Spline Schemes $X^0(\Delta)$ and $X^1(\Delta)$ for the subdivision Δ of (5).

Observe that this is generated as a $\mathbb{R}[t]$ -module by $(1, 1, 1)$, $x := (t^2, 0, 0)$ and $z := (0, 0, (t-1)^2)$. If we let $y := (t, t, t)$, then $S^1(\Delta)$ is generated as an \mathbb{R} -algebra by x, y, z . These satisfy the following relations $x^2 - y^2x = xz = z^2 - (y-1)^2z$, and thus

$$S^1(\Delta) = \mathbb{R}[x, y, z] / \langle x(x - y^2), xz, z(z - (y-1)^2) \rangle .$$

We may check that

$$\langle x(x - y^2), xz, z(z - (y-1)^2) \rangle = \langle x - y^2, z \rangle \cap \langle x, z \rangle \cap \langle x, z - (y-1)^2 \rangle ,$$

and thus $X^1(\Delta)$ is the union of the y -axis and two parabolas, $x = y^2$ in the x, y plane and $z = (y - 1)^2$ in the y, z plane. We display this on the right in Figure 3.

Let $\sigma \in \Delta_n$ be a n -dimensional cell of Δ . Write $X_\sigma^r(\Delta)$ for the component of the spline scheme indexed by σ . The composition $X_\sigma^r(\Delta) \hookrightarrow X^r(\Delta) \twoheadrightarrow \mathbb{R}^n$ with the last map the finite map φ is an isomorphism. Let $\hat{\sigma} \subset X_\sigma^r(\Delta)$ be the inverse image of σ under this isomorphism. The union of these lifts over all $\sigma \in \Delta_n$ is a lift of Δ_n to $X^r(\Delta)$. This forms the positive part $X_+^r(\Delta)$ of the spline scheme $X^r(\Delta)$. In Figure 3, these positive parts are indicated by the thickenings of the components.

It is not hard to generalize these examples to arbitrary subdivisions of \mathbb{R}^1 . Spline schemes for arbitrary subdivisions in \mathbb{R}^d will have a strong combinatorial flavor. One advantage of this perspective is that spline functions become restrictions of polynomial functions in \mathbb{R}^N to the positive part of the spline scheme.

This can be asked for splines on graphs, or splines with varies smoothness conditions, such as supersmoothness [AS03], as long as one obtains a ring.

Splines and the pushforward to a point: David Anderson. As noted in [LST23, Sects. 1.2, 1.4], for some algebraic manifolds X , the (torus) equivariant cohomology ring $H_T^*(X)$ is equal to the space of C^0 -splines on a related subdivision Δ or labeled graph G . This is the case when X is a toric variety [Pay06] or a GKM-space [Tym16]. Equivariant cohomology has a natural map to the ring of polynomials, called integration or pushforward to a point. For the spaces whose equivariant cohomology rings are also spaces of splines, the Atiyah-Bott formula [AF23, Cor. 5.2.4] for this pushforward is nicely expressed in terms of splines. The question posed is to what extent can the Atiyah-Bott formula be generalized to arbitrary spline spaces?

We present two examples of this pushforward for toric varieties. For these, the domain D of the spline space is all of \mathbb{R}^n (or a restriction to a neighborhood of the origin), and the triangulation Δ

is a complete rational fan. We begin with a one-dimensional example. Let Δ be the union of two intervals $\sigma_- = (-\infty, 0]$ and $\sigma_+ = [0, \infty)$, which form the 1-dimensional complete fan:

$$(6) \quad \begin{array}{c} \longleftarrow \sigma_- \quad \sigma_+ \longrightarrow \\ | \\ 0 \end{array}$$

The corresponding toric variety is the projective line, \mathbb{P}^1 . A C^0 -spline (element of $H_T^*(\mathbb{P}^1)$) is a pair $f = (f_-, f_+)$ of polynomials in $\mathbb{R}[t]$ such that $f_-(0) = f_+(0)$. Here f_- is the restriction of f to σ_- , and f_+ is the restriction of f to σ_+ . Let $f = (-x^2 + x, x^2 - 2x)$ be such a spline. Then

$$\int f = \frac{-x^2 + x}{-x} + \frac{x^2 - 2x}{x} = (x - 1) + (x - 2) = 2x - 3.$$

(Notice that this is simply $(f_+ - f_-)/x$.)

Let us now consider the projective plane \mathbb{P}^2 , which corresponds to the fan:

$$(7) \quad \begin{array}{c} \begin{array}{c} -x \longleftarrow \quad \longrightarrow x \quad \sigma_0 \\ \uparrow y \\ \sigma_2 \\ \downarrow -y \\ \begin{array}{c} y-x \\ \swarrow \quad \searrow \\ x-y \end{array} \quad \sigma_1 \end{array} \end{array}$$

A C^0 -spline is a triple $f = (f_0, f_1, f_2)$ of polynomials in $\mathbb{R}[x, y]$ that satisfies $f_0 - f_2 \in \langle x \rangle$, $f_0 - f_1 \in \langle y \rangle$, and $f_1 - f_2 \in \langle x - y \rangle$. The integral $\int f$ is defined to be the sum

$$\int f := \frac{f_0}{xy} + \frac{f_1}{-y(x-y)} + \frac{f_2}{-x(y-x)}.$$

The denominator of the summand corresponding to σ_i is the product of the linear forms defining the boundary of σ_i , with signs chosen so that they are positive on σ_i (this is indicated in (7)).

For example, if $f = (xy, x^2 - xy, -xy + y^2)$, then we have

$$\int f = \frac{xy}{xy} + \frac{-xy + y^2}{-y(x-y)} + \frac{x^2 - xy}{-x(y-x)} = 1 + 1 + 1 = 3,$$

and if $f = (x^2y + y^3, y^2x - x^2y, y^3 - xy^2)$, then

$$\int f = \frac{x^2y + y^3}{xy} + \frac{y^2x - x^2y}{-y(x-y)} + \frac{y^3 - xy^2}{-x(y-x)} = 2x.$$

This integral makes sense for splines on any complete rational fan in \mathbb{R}^d , as such a spline is a class in equivariant cohomology.

We may use this to define the integral for splines on the decomposition $\Delta \subset \mathbb{R}^1$ of the interval $[-1, 2]$ (5). Indeed, let $f = (f_0, f_1, f_2)$ be a spline on Δ . Then

$$\int f = \frac{f_0}{-t} + \frac{f_1}{t(1-t)} + \frac{f_2}{t-1} = \frac{f_1 - f_0}{t} + \frac{f_2 - f_1}{t-1},$$

as $\frac{f_1}{t(1-t)} = \frac{f_1}{t} + \frac{f_1}{1-t}$. Note that not only is this well-defined, but if f is homogeneous of degree d , then $\int f$ has degree $d-1$.

The question is if it is possible to generalize this pushforward to arbitrary splines, and if so, what are its properties? For example, if f is homogeneous of degree d , is $\int f$ a polynomial of degree $d-n$? Some hurdles may be the choice of a normalization when the decomposition is not rational (defined over \mathbb{Q}), and what to do if Δ is not a triangulation. We note that the pushforward to a point is an extremely important and natural map in equivariant cohomology.

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