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A GEOMETRIC PROOF OF AN EQUIVARIANT PIERI RULE FOR FLAG MANIFOLDS

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Abstract. We use geometry to give a short proof of an equivariant Pieri rule in the classical flag manifold. This rule is due to Robinson, who gave an algebraic proof.

Introduction

An important problem in Schubert calculus is to find a formula for Schubert structure constants for flag manifolds of general Lie type. For equivariant Schubert calculus, this is known in only two special cases, both in Lie type A: the Grassmannian, proved by Knutson and Tao [6], and two-step flag manifolds, proved by Buch [2]. For the manifold $\mathbb{F}\ell(n)$ of complete flags in $\mathbb{C}^n$, a special case of this problem is an equivariant Pieri rule that Robinson proved using algebra [10]. We give a short and direct proof of this Pieri rule, using geometric arguments.

Let $T \subset SL(n)$ be the diagonal torus. The T-equivariant cohomology ring $H^*_T(\mathbb{F}\ell(n))$ has an $H^*_T(\text{pt})$-additive basis of Schubert classes $[X_w]_T$ indexed by permutations w in the symmetric group S_n. The equivariant Schubert structure constants $c^u_{w,v}$ in the product $[X_w]_T \cdot [X_v]_T = \sum_u c^u_{w,v}[X_u]_T$ are Graham-positive [5]. The equivariant Pieri rule is a Graham-positive formula for $c^u_{w,v}$ when v is a special permutation, and it determines the multiplication in equivariant cohomology of any flag varety in type A. Its non-equivariant limit gives the classical Pieri rule first stated by Lascoux and Schützenberger [7]. Our proof uses an explicit description of the projected Richardson variety associated to w and u from [11], reducing $c^u_{w,v}$ to the restriction of a special Schubert class for a Grassmannian to a torus-fixed point as in [8].

The same arguments establish the same formula for torus equivariant Chow groups of flag varieties over an algebraically closed field [1].

1. Statement of Results

Let $E_\bullet$ be the standard flag in $\mathbb{C}^n$ where E_i is spanned by the first i standard basis vectors, and let $E'_\bullet$ be the standard opposite flag. The Schubert variety $X_w = X_wE_\bullet$ of $\mathbb{F}\ell(n)$, where $w \in S_n$, is defined with respect to the flag $E_\bullet$,

$$X_wE_\bullet \;:=\; \{F_\bullet \in \mathbb{F}\ell(n) \mid \dim F_i \cap E_{n+1-j} \geq \#\{k \leq i \mid w(k) \geq j\}\}\,. \tag{1}$$

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This has codimension $\ell(w)$, and equals the Schubert variety of dimension $\ell(w_0w)$ with index w_0w as defined in [3, p. 157]. Here $\ell(w)$ is the length of the permutation w, and w_0 is the longest permutation in S_n, so that $w_0(i) = n+1-i$. Each coefficient $c^u_{w,v}$ is either 0 or a homogeneous polynomial in $\mathbb{Z}_{\geq 0}[t_2-t_1,\ldots,t_n-t_{n-1}]$ of degree $\ell(w)+\ell(v)-\ell(u)$ (this is Graham-positivity [5]). Here, $t_i = c_1(\mathbb{C}_{\chi_i})$ is the first Chern class of the T-equivariant line bundle over a point induced by a one-dimensional representation $\mathbb{C}_{\chi_i}$ of T, where χ_i denotes the character that sends $\mathrm{diag}(z_1,\ldots,z_n)\in T$ to z_{n+1-i}.

Fix a positive integer $m<n$. The m-Bruhat order on S_n is defined by $w\leq_m u$ when we have $u = w\tau_{a_1b_1}\cdots\tau_{a_sb_s}$ and $\ell(w\tau_{a_1b_1}\cdots\tau_{a_ib_i}) = \ell(w)+i$ for $1\leq i\leq s$. Here, $\tau_{a_jb_j} = (a_j,b_j)$ is a transposition with $a_j\leq m<b_j$, for $j=1,\ldots,s$ where $s:=\ell(u)-\ell(w)$. Consequently, when $w\leq_m u$ and $a\leq m<b$, we have $w(a)\leq u(a)$ and $w(b)\geq u(b)$. We write $w\xrightarrow{r_m}u$ if in addition, the integers $b_1,\ldots,b_s$ are distinct.

Example 1.1. When $n=9$, if $w = 631594287$ in one-line notation, then $u = 839154267 = w\tau_{34}\tau_{18}\tau_{35}$ satisfies $w\leq_3 u$, and we also have $w\xrightarrow{r_3}u$ as $4,5,8$ are distinct. Note however that if $v = u\tau_{25} = 859134267$, then $w\leq_3 v$ but we do not have $w\xrightarrow{r_3}v$. ⋄

Fix a positive integer $p\leq n-m$. Let $r(m,p)\in S_n$ be the cyclic permutation

$$(m, m{+}p, m{+}p{-}1,\ldots,m{+}1)\,.$$

The Pieri rule involves the Schubert variety $X_{r(m,p)}$, which is defined by a single condition,

$$X_{r(m,p)} = \{F_\bullet\in\mathbb{F}\ell(n)\mid F_m\cap E_{n+1-m-p}\neq\{0\}\}\,.$$

This is the pullback of the codimension p special Schubert variety in the Grassmannian $G(m,n)$ under the natural forgetful map. As the equivariant cohomology ring of any type A flag manifold is a subring of that for $\mathbb{F}\ell(n)$, it suffices to establish the Pieri rule for $\mathbb{F}\ell(n)$.

Theorem 1.2. *For a permutation $w\in S_n$, we have the following formula in $H^*_T(\mathbb{F}\ell(n))$:*

$$[X_w]_T\cdot[X_{r(m,p)}]_T \;=\; \sum_{w\xrightarrow{r_m}u} c^u_{w,r(m,p)}[X_u]_T\,. \tag{2}$$

The coefficient $c^u_{w,r(m,p)}$ is nonzero if and only if $w\xrightarrow{r_m}u$ with $r(m,p)\leq u$ and $q:=p+\ell(w)-\ell(u)\geq 0$. When this holds, the subset $\nu:=\{n{+}1{-}w(1),\ldots,n{+}1{-}w(m)\}\cup\{n{+}1{-}w(b)\mid w(b)>u(b)\}$ consists of $m{+}p{-}q$ elements, and defines two increasing subsequences

$$\{a_1<\cdots<a_r\} \;:=\; \nu\cap\{1,\ldots,n{-}m{-}p{+}1\}$$

and

$$\{b_1<\cdots<b_{q+r-1}\} \;:=\; \{n{-}m{-}p{+}2,\ldots,n\}\smallsetminus\nu\,.$$

Then $c^u_{w,r(m,p)}$ is equal to 1 if $q=0$, or given by the following otherwise

$$c^u_{w,r(m,p)} \;=\; \sum\prod_{i=1}^{q}(t_{b_{c_i}}-t_{a_{c_i-i+1}})\;, \tag{3}$$

with the summation over increasing subsequences $\{c_1<\cdots<c_q\}$ in $\{1,2,\ldots,q{+}r{-}1\}$.

We remark that $c^u_{w,r(m,p)}$ is nonzero if and only if the ordinary cohomology class $[X_u]$ appears in the classical Pieri formula [7, 11] for the cup product $[X_w]\cup[X_{r(m,p-q)}]$, which is an important part of our proof. The transformation $n{+}1-x$ comes from the definition (1) of X_w.

Example 1.3. We compute the coefficients $c^u_{w,r(3,p)}$ for $p = 3, 4, 5, 6$, where w and u are from Example 1.1. First, the set ν is $\{4, 7, 9\} \cup \{5, 1, 2\}$, as $w = 631594287$ and $\{5, 9, 8\} = \{w(b) \mid w(b) > u(b)\}$. (Elements of ν have the form $n{+}1{-}w(i)$.) When $p = 3$, the coefficient $c^u_{w,r(3,3)} = 1$ by the classical Pieri rule. When $p = 4$, $q = 1$ and $n{-}m{-}p{+}1 = 3$, so that $\{a_1 < a_2\} = \{1 < 2\}$ and $\{b_1 < b_2\} = \{6 < 8\}$. There are $2 = q + r - 1$ choices for c_1, and so we have $c^u_{w,r(3,4)} = (t_6{-}t_1) + (t_8 - t_2)$. When $p = 5$, $q = 2$ and $n{-}m{-}p{+}1 = 2$, so that $\{a_1 < a_2\} = \{1 < 2\}$ and $\{b_1 < b_2 < b_3\} = \{3 < 6 < 8\}$. There are three choices for $c_1 < c_2$ among $\{1 < 2 < 3\}$ (namely $1 < 2$, $1 < 3$, and $2 < 3$), and they give

$$c^u_{w,r(3,5)} \;=\; (t_3{-}t_1)(t_6{-}t_1) + (t_3{-}t_1)(t_8 - t_2) + (t_6{-}t_2)(t_8 - t_2)\,.$$

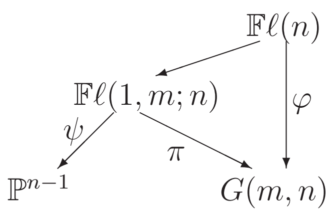
Finally, when $p = 6$, $q = 3$ and $n{-}m{-}p{+}1 = 1$, so that $a_1 = 1$ and $\{b_1 < b_2 < b_3\} = \{3 < 6 < 8\}$. Then $c^u_{w,r(3,6)} = (t_3{-}t_1)(t_6{-}t_1)(t_8 - t_1)$. $\diamond$

2. Proof of the equivariant Pieri rule

We prove Theorem 1.2 using the method of [8] and exploiting the explicit description of certain Richardson varieties and their projections in [11].

Let $\mathbb{F}\ell(1, m; n)$ denote the manifold of partial flags $F_1 \subset F_m \subset \mathbb{C}^n$, and $G(m, n)$ denote the Grassmannian of m-dimensional vector subspaces in $\mathbb{C}^n$. Notice that $\mathbb{P}^{n-1} = G(1, n)$. Let ψ, π, φ denote the natural projection (forgetful) maps among these spaces.

(4)



Let X^u denote the Schubert variety $X_{w_0u}E'_\bullet$, defined with respect to the flag $E'_\bullet$. Then $X^u_w := X_w \cap X^u$ is a Richardson variety of dimension $\ell(u){-}\ell(w)$ [9]. Denote by ρ^M the T-equivariant map from a T-space M to pt—a point equipped with a trivial T-action. Let $Y_{(p)}$ denote the codimension p special Schubert variety in $G(m, n)$. Noticing that $X_{r(m,p)} = \varphi^{-1}Y_{(p)} = \varphi^{-1}\pi\psi^{-1}(E_{n+1-m-p})$, we compute the coefficient $c^u_{w,r(m,p)}$ of $[X_u]_T$ in the product $[X_w]_T \cdot [X_{r(m,p)}]_T$ using the equivariant push forward to a point and the projection formula,

$$\begin{aligned} c^u_{w,r(m,p)} &= \rho^{\mathbb{F}\ell(n)}_*\big([X_w]_T \cdot [X_{r(m,p)}]_T \cdot [X^u]_T\big) \;=\; \rho^{\mathbb{F}\ell(n)}_*\big([X^u_w]_T \cdot [X_{r(m,p)}]_T\big)\\ &= \rho^{G(m,n)}_*\big(\varphi_*[X^u_w]_T \cdot [Y_{(p)}]_T\big)\\ &= \rho^{\mathbb{F}\ell(1,m;n)}_*\big(\pi^*\varphi_*[X^u_w]_T \cdot \psi^*[E_{n+1-m-p}]_T\big)\\ &= \rho^{\mathbb{P}^{n-1}}_*\left((\psi_*\pi^*\varphi_*[X^u_w]_T) \cdot [E_{n+1-m-p}]_T\right)\,. \end{aligned}$$

Let $Y^u_w := \varphi(X^u_w)$ be the image of X^u_w in $G(m, n)$ and $Z^u_w := \psi \circ \pi^{-1}(Y^u_w)$, which is the set of points in $\mathbb{P}^{n-1}$ that lie on some m-plane in a flag in X^u_w. We have that $\dim Y^u_w \le \dim X^u_w$

and $\dim Z_w^u \leq m + \dim Y_w^u$, since the general fiber of π has dimension m. Moreover, the class $\psi_*\pi^*\varphi_*[X_w^u]_T$ is zero if either inequality is strict.

The Chevalley formula [4] expresses $[X_w]_T \cdot [X_{r(m,1)}]_T$ as a Graham-positive sum of classes $[X_u]_T$, where either $u = w$ or u covers w in the m-Bruhat order. Iterating the Chevalley formula shows that $[X_{r(m,p)}]_T$ is a term of $([X_{r(m,1)}]_T)^p$. Thus $[X_w]_T \cdot [X_{r(m,p)}]_T$ is a subsum of $[X_w]_T \cdot ([X_{r(m,1)}]_T)^p$, which is a Graham-positive combination of classes $[X_u]_T$ with $w \leq_m u$, again by the Chevalley formula. Thus $c^u_{w,r(m,p)} = 0$ unless $w \leq_m u$. Assuming this, $\varphi_*[X_w^u]_T$ is a positive multiple of $[Y_w^u]_T$. By [11, Lemma 10], Z_w^u is a subset of a linear subspace (witten there as Y) of $\mathbb{P}^{n-1}$ of dimension $m + \#\{m < b \mid w(b) > u(b)\}$. From the definition of the m-Bruhat order and $\xrightarrow{r_m}$, this is strictly less than $m + \dim X_w^u$, unless $w \xrightarrow{r_m} u$. Thus $c^u_{w,r(m,p)} \neq 0$ only if $w \xrightarrow{r_m} u$.

We recall Lemma 15 of [11], which identifies both Y_w^u and Z_w^u when $w \xrightarrow{r_m} u$ (and shows that the maps φ and ψ to them are birational). This differs from the statement in [11] in that our standard basis is different from the basis used there, with e_i here being e_{n+1-i} in [11].

Lemma 15 from [11]. *Suppose that $w \xrightarrow{r_m} u$ and $u = w\tau_{a_1b_1}\cdots\tau_{a_sb_s}$ with $a_i \leq m < b_i$ and $\ell(w\tau_{a_1b_1}\cdots\tau_{a_ib_i}) = \ell(w) + i$ for $1 \leq i \leq s = \ell(u) - \ell(w)$. Define*

$$\begin{aligned} L_j &= \langle e_{n+1-w(j)}, e_{n+1-w(b_i)} \mid a_i = j\rangle \qquad j = 1,\dots,m \\ M &= \langle e_{n+1-w(k)} \mid m < k \text{ and } w(k) = u(k)\rangle\,. \end{aligned}$$

Then $\dim L_j = 1 + \#\{i \mid a_i = j\}$, if $F_\bullet \in X_w^u$ then $\dim F_m \cap L_j = 1$ for $1 \leq j \leq k$. Consequently, the image Y_w^u is a Richardson variety in the Grassmannian with respect to different coordinate flags than $E_\bullet$ and $E'_\bullet$, and the map $\varphi\colon X_w^u \to Y_w^u$ has degree 1.

(The $e_{n+1-w(j)}$ in the definition of L_j corrects a typographic error in the published article.) As the map $\varphi\colon X_w^u \to Y_w^u$ has degree 1, we have $\varphi_*[X_w^u]_T = [Y_w^u]_T$. More can be said about Y_w^u, but the important consequence for now is that $[Z_w^u]_T = \psi_*\pi^*\varphi_*[X_w^u]_T$, and

$$Z_w^u = L_1 \oplus \cdots \oplus L_m = \langle e_{n+1-w(j)}, e_{n+1-w(b_i)} \mid j = 1,\dots,m\,,\ i = 1,\dots,r\rangle = \langle e_k \mid k \in \nu\rangle\,.$$

Here, ν is the indexing set defined in the statement of Theorem 1.2. As in [8], Z_w^u equals the projected Richardson variety coming from the single point Richardson variety $\langle e_k \mid k \in \nu\rangle$ in the Grassmannian $G(m{+}p{-}q, n)$, via the standard maps (4) from $\mathbb{F}\ell(1, m{+}p{-}q; n)$. By [8, Proposition 4.2], $c^u_{w,r(m,p)}$ equals $N^\nu_{\nu,q}$, the localization of the special Schubert class of codimension q in the Grassmannian $G(m{+}p{-}q, n)$ at the torus-fixed point $\langle e_k \mid k \in \nu\rangle$. The indexing set ν is a Schubert symbol in [8], and the Grassmannian permutation associated to ν is obtained by sorting $\tilde{\nu} := \{w(1),\dots,w(m)\} \cup \{w(b) \mid w(b) > u(b)\}$ in increasing order. Thus $N^\nu_{\nu,q}$ is nonzero if and only if $q + m + p - q \leq \max\tilde{\nu} = \max\{u(1),\dots,u(m)\}$, namely $r(m,p) \leq u$ in the Bruhat order for S_n. Here the last equality holds due to the observations: (i) $u(a_i) = w(b_s) > u(b_s)$ with $s = \max\{k \mid a_i = a_k, k \leq r\}$ so that $u(j) \in \tilde{\nu}$ for any $1 \leq j \leq m$; (ii) $w(j) \leq u(j)$ for $1 \leq j \leq m$, and if $w(b) > u(b)$ then $b = b_i$ for a unique $1 \leq i \leq m$, implying that $u(a_i) \geq w(b)$ since $w \xrightarrow{r_m} u$. By [8, Appendix A] the nonvanishing localization is (3), which completes the proof.

From the proof of Lemma 15 in [11], there are partitions $\mu \subset \lambda$ for $G(m, n)$, and a permutation $\omega \in S_n$ such that $c^u_{w,r(m,p)} = \omega(c^\lambda_{\mu,(p)})$.

Example 2.1. We illustrate Lemma 15. An invertible matrix F gives a flag $F_\bullet$, where F_i is the row space of the first i rows of F. The two matrices below represent general elements of X_w and X^u, for w and u from Example 1.1. Here, $\cdot$ is zero and $* \in \mathbb{C}$ is arbitrary.

$$\begin{pmatrix} * & * & * & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ * & * & * & \cdot & * & * & 1 & \cdot & \cdot \\ * & * & * & \cdot & * & * & \cdot & * & 1 \\ * & * & * & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & * & * & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & * & * & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & 1 & * & * & * & * & * & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & * & * \\ 1 & \cdot & * & * & * & * & \cdot & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & * & \cdot & * & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & * & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

The Richardson variety X_w^u has a parametrization by triples (α, β, γ) of nonzero complex numbers. We show this in two equivalent ways. The matrix on the left lies in X_w and that on the right in X^u. For every $i = 1, \ldots, 9$ the first i rows of both matrices have the same span.

$$\left(\begin{array}{ccccccccc} \cdot & \beta & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \gamma & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & 1 \\ \hline \gamma & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \gamma & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right) \qquad \left(\begin{array}{ccccccccc} \cdot & \beta & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \gamma & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & 1 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

The row span of their first three rows parameterizes the projected Richardson variety Y_w^u. Since the remaining rows depend upon the first three, the map $X_w^u \to Y_w^u$ is birational. $\diamond$

References

[1] Michel Brion, Equivariant Chow groups for torus actions, Transform. Groups **2** (1997), no. 3, 225–267.

[2] Anders Skovsted Buch, Mutations of puzzles and equivariant cohomology of two-step flag varieties, Ann. of Math. (2) **182** (2015), no. 1, 173–220.

[3] William Fulton, Young tableaux, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997.

[4] William Fulton and Alain Lascoux, A Pieri formula in the Grothendieck ring of a flag bundle, Duke Math. J. **76** (1994), no. 3, 711–729.

[5] William Graham, Positivity in equivariant Schubert calculus, Duke Math. J. **109** (2001), no. 3, 599–614.

[6] Allen Knutson and Terence Tao, Puzzles and (equivariant) cohomology of Grassmannians, Duke Math. J. **119** (2003), no. 2, 221–260.

[7] Alain Lascoux and Marcel-Paul Schützenberger, Polynômes de Schubert, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), no. 13, 447–450.

[8] Changzheng Li and Vijay Ravikumar, Equivariant Pieri rules for isotropic Grassmannians, Math. Ann. **365** (2016), no. 1-2, 881–909.

[9] Roger Wolcott Richardson, Intersections of double cosets in algebraic groups, Indag. Math. (N.S.) **3** (1992), no. 1, 69–77.

[10] Shawn Robinson, A Pieri-type formula for $H_T^*(SL_n(\mathbb{C})/B)$, J. Algebra **249** (2002), no. 1, 38–58.
[11] Frank Sottile, Pieri's formula for flag manifolds and Schubert polynomials, Ann. Inst. Fourier (Grenoble) **46** (1996), no. 1, 89–110.

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